

Article

Biharmonic Identity Maps Between Euclidean Spaces and Warped Product Spaces

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Abstract

In this paper, we study biharmonic identity maps between Euclidean spaces and warped product spaces of the form $\mathbb{R}^r \times_{f^2} \mathbb{R}^{n-r}$. Our main results characterize both orientations of the identity map in terms of partial differential equations: For the map from Euclidean space to the warped space, biharmonicity is equivalent to the warping function satisfying a stationary Hamilton–Jacobi-type equation. While the only global solution is constant, we construct infinitely many explicit local solutions. Conversely, for the map from the warped space to Euclidean space, biharmonicity corresponds to a logarithmic transformation of the warping function satisfying this same PDE. This equation admits abundant explicit nonconstant global solutions and can be reduced to a Liouville-type equation via a suitable transformation.

Keywords: biharmonic maps; identity maps; Euclidean spaces; warped product spaces

MSC: 58E20; 53C42

1. Introduction

Let (M^m, g) and (N^n, h) be two Riemannian manifolds of dimensions m and n , respectively, and let $\varphi : (M^m, g) \rightarrow (N^n, h)$ be a smooth map. The map φ is called a harmonic map if its tension field vanishes identically (see e.g., [1]):

$$\tau(\varphi) := \text{Trace}_g \nabla d\varphi \equiv 0.$$

From a variational perspective, harmonic maps are the critical points of the energy functional:

$$E(\varphi) = \frac{1}{2} \int_M |d\varphi|^2 dv_g.$$

The map φ is said to be a biharmonic map if its bitension field satisfies the PDEs (see e.g., [2]):

$$\tau_2(\varphi) := \text{Trace}_g (\nabla^\varphi \nabla^\varphi - \nabla_{\nabla_M}^\varphi) \tau(\varphi) - \text{Trace}_g R^N(d\varphi, \tau(\varphi))d\varphi = 0, \quad (1)$$

where R^N is the curvature operator of (N, h) , defined by

$$R^N(X, Y)Z = [\nabla_X^N, \nabla_Y^N]Z - \nabla_{[X, Y]}^N Z.$$



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Variationally, biharmonic maps are the critical points of the bienergy functional

$$E_2(\varphi) = \frac{1}{2} \int_M |\tau(\varphi)|^2 dv_g.$$

Since $\tau(\varphi) \equiv 0$ implies $\tau_2(\varphi) \equiv 0$, all harmonic maps are trivially biharmonic. The nontrivial case occurs when $\tau_2(\varphi) \equiv 0$ but $\tau(\varphi) \neq 0$; such maps are called proper biharmonic maps.

The study of proper biharmonic maps reveals rich geometric properties (see, e.g., [3]). Several intriguing results are listed as follows:

- **Construction Methods & Existence:** In [4], proper biharmonic maps were constructed by first taking a harmonic map and then conformally deforming the source metric. Methods for building examples in n -spheres S^n were provided in [5], while reference [6] investigated the existence and stability of rotationally symmetric proper biharmonic maps between model spaces.
- **Conformal Maps & Special Manifolds:** For conformal maps between manifolds of dimension $n \geq 3$, reference [7] established a necessary and sufficient condition involving a second-order elliptic PDE for the dilation. On a 4-dimensional Einstein manifold, reference [8] proved that the problem reduces to a Yamabe-type equation. Related work included the study of conformal biharmonic immersions in [9], a complete classification of local and global conformal biharmonic maps between any two space forms in [10], and an analysis of biharmonic maps on doubly warped product manifolds in [11,12].
- **Submanifolds & Rigidity Results:** A rigidity result from [13] stated that spherical biharmonic submanifolds in Euclidean space are minimal. Reference [14] derived the necessary and sufficient condition for constant mean curvature hypersurfaces to be proper biharmonic in specific warped product spaces, and reference [15] established non-existence theorems for biharmonic isometric immersions into Euclidean spaces.
- **General Theory & Relationships:** The connection between biharmonicity and conformality was studied in [16], the relationship between biharmonic and k -harmonic maps was explored in [17], and biharmonic maps on principal bundles were studied in [12].

In this paper, We study biharmonic identity maps between n -dimensional Euclidean spaces $\mathbb{R}^n$ and warped product spaces of the form $\mathbb{R}^r \times_{f^2} \mathbb{R}^{n-r}$, equipped with the warped product metric

$$g(f, r) = \sum_{j=1}^r dx_j^2 + f^2(x_1, \dots, x_r) \sum_{l=r+1}^n dx_l^2,$$

where $f : \mathbb{R}^r \subset \mathbb{R}^n \rightarrow (0, +\infty)$. Warped products were first introduced by O'Neill and Bishop in [18]. In [4], new examples of proper biharmonic maps between Riemannian manifolds were constructed by first taking a harmonic map $\varphi : B \rightarrow F$ (automatically biharmonic) and then conformally deforming the metric on B to make φ proper biharmonic. The study of biharmonic maps in warped product manifolds was further advanced in [19], where the authors carried out the following:

- (1) Established conditions for the biharmonicity of the inclusion map $M_2 \rightarrow M_1 \times_{f^2} M_2$ and of the projection $M_1 \times_{f^2} M_2 \rightarrow M_1$;
- (2) Constructed two new classes of proper biharmonic maps by combining harmonic maps $\varphi = 1_{M_1} \times \phi : M_1 \times M_2 \rightarrow M_1 \times M_2$ and warping the metric on either the domain or codomain;
- (3) Investigated three classes of axially symmetric biharmonic maps using the warped product setting.

Later, reference [11] extended these results to doubly warped product manifolds, characterizing non-harmonic biharmonic maps through product of harmonic maps and warping metric.

Collectively, references [19] and [11] respectively focus on general discussions of biharmonicity for special maps (including identity maps) between warped product manifolds and product manifolds, and between doubly warped product manifolds and warped product manifolds. In contrast, we specifically concentrate on the biharmonic identity maps between n -dimensional Euclidean spaces $\mathbb{R}^n$ and Euclidean warped product spaces, and fully characterize such maps via the Hamilton–Jacobi equation.

We now consider the special case in [19] where M_1 , M_2 and φ in their construction are replaced by $\mathbb{R}^r$, $\mathbb{R}^{n-r}$, and the identity map ι , respectively. We then, with a method different from [19], study the biharmonicity of:

- (i) the identity map $\mathbb{R}^n \rightarrow (\mathbb{R}^n, g(f, r))$, and
- (ii) the map $(\mathbb{R}^n, g(f, r)) \rightarrow \mathbb{R}^n$.

Notably, we find that:

- The identity map $\mathbb{R}^n \rightarrow (\mathbb{R}^n, g(f, r))$ is biharmonic iff $f^2 = u$ satisfies the stationary Hamilton–Jacobi equation:

$$\Delta u - \frac{(n-r)}{4} |\nabla u|^2 = K \quad (K \in \mathbb{R}). \quad (2)$$

While the only global solution on $\mathbb{R}^n$ is the function $f^2 \equiv \text{constant}$, we construct infinitely many explicit local solutions f^2 .

- Conversely, the identity map $(\mathbb{R}^n, g(f, r)) \rightarrow \mathbb{R}^n$ is biharmonic iff $\ln f^{-2} = u$ satisfies the original PDE $(\Delta u - \frac{(n-r)}{4} |\nabla u|^2 = K)$, which admits abundant explicit nonconstant global solutions f .

A key observation is that the transformation $v = e^{-\frac{n-r}{4}u}$ reduces this to a Liouville-type equation

$$\Delta v + \frac{(n-r)K}{4} v = 0. \quad (3)$$

Thus, we characterize biharmonic identity maps $\mathbb{R}^n \rightleftharpoons (\mathbb{R}^n, g(f, r))$ by analyzing the Hamilton–Jacobi Equation (2) (or equivalently, the Liouville-type Equation (3)) under the geometric constraints.

2. Biharmonic Identity Maps Between Euclidean Spaces and Warped Product Spaces

This section establishes a characterization of biharmonic identity maps from Euclidean spaces to warped product manifolds of the form $\mathbb{R}^r \times_{f^2} \mathbb{R}^{n-r}$. Our approach requires two key technical lemmas.

Lemma 1 (see [20]). *Let $\mathbb{R}^m$ denote the Euclidean space with the orthonormal frame $\{e_i = \frac{\partial}{\partial x^i}\}_{i=1}^m$. Let (N^n, h) be a Riemannian manifold with a globally defined orthonormal frame E_α . Let $\varphi : \mathbb{R}^m \rightarrow (N^n, h)$ be a map with $d\varphi(e_i) = B_i^\beta(E_\beta \circ \varphi)$. Let $\tau(\varphi) = \tau^\sigma(E_\sigma \circ \varphi)$. Then φ is biharmonic if and only if*

$$\begin{aligned} \Delta \tau^\gamma + [2\langle \nabla \tau^\sigma, B^\beta \rangle + \sum_{i=1}^m \tau^\sigma e_i(B_i^\beta)] \langle \nabla_{E_\beta}^N E_\sigma, E_\gamma \rangle \\ + \tau^\sigma B^\alpha \cdot B^\beta \left(\langle \nabla_{E_\alpha}^N \nabla_{E_\beta}^N E_\sigma, E_\gamma \rangle - \langle \tilde{R}(E_\alpha, E_\sigma) E_\beta, E_\gamma \rangle \right) = 0 \end{aligned} \quad (4)$$

for $\gamma = 1, 2, \dots, n$, where $\Delta\tau^\gamma$, $\nabla\tau^\sigma$ and $B^\alpha \cdot B^\beta$ denote, respectively, the Laplacian, the gradient, and the inner product on Euclidean space $\mathbb{R}^m$.

The following lemma provides a general orthonormal-frame formulation of the biharmonic equation for maps between Riemannian manifolds.

Lemma 2. Let (M^m, g) and (N^n, h) be two Riemannian manifolds with the orthonormal frames $\{e_i\}_{i=1}^m$ and $\{E_\alpha\}_{\alpha=1}^n$, respectively. A smooth map $\varphi : (M^m, g) \longrightarrow (N^n, h)$ with the differential $d\varphi(e_i) = B_i^\beta(E_\beta \circ \varphi)$ and the tension field $\tau(\varphi) = \tau^\sigma(E_\sigma \circ \varphi)$ is biharmonic if and only if

$$\sum_{i=1}^m (e_i e_i \tau^\gamma - \Gamma_{ii}^\delta e_\delta(\tau^\gamma)) + \left[\sum_{i=1}^m \left((2e_i \tau^\sigma) B_i^\beta + \tau^\sigma e_i(B_i^\beta) - \Gamma_{ii}^\delta B_\delta^\beta \tau^\sigma \right) \langle \nabla_{E_\beta}^N E_\sigma, E_\gamma \rangle \right. \\ \left. + \tau^\sigma B^\alpha \cdot B^\beta \left(\langle \nabla_{E_\alpha}^N \nabla_{E_\beta}^N E_\sigma, E_\gamma \rangle - \langle R^N(E_\alpha, E_\sigma) E_\beta, E_\gamma \rangle \right) \right] = 0 \quad (5)$$

for $\gamma = 1, 2, \dots, n$. Here, $\Delta\tau^\gamma$, $\nabla\tau^\sigma$ and ∇ denote, respectively, the Laplacian, the gradient, and the Levi-Civita connection on M^m , $\Gamma_{ij}^\delta = \langle \nabla_{e_i} e_j, e_\delta \rangle$, and $B^\alpha \cdot B^\beta$ represents the inner product on Euclidean space $\mathbb{R}^m$.

Proof. Hereafter, we adopt the abbreviation $E_\beta \circ \varphi$ for E_β , with the understanding that it represents the vector field along the map φ . A straightforward computation gives

$$\nabla_{e_i}^\varphi \tau(\varphi) = \nabla_{e_i}^\varphi (\tau^\sigma E_\sigma) = (e_i \tau^\sigma) E_\sigma + \tau^\sigma \nabla_{e_i}^\varphi E_\sigma \\ = (e_i \tau^\sigma) E_\sigma + \tau^\sigma \nabla_{d\varphi(e_i)}^N E_\sigma = (e_i \tau^\sigma) E_\sigma + \tau^\sigma B_i^\beta \nabla_{E_\beta}^N E_\sigma, \quad (6)$$

$$\nabla_{\nabla_{e_i} e_i}^\varphi \tau(\varphi) = \nabla_{\Gamma_{ii}^\delta e_\delta}^\varphi (\tau^\sigma E_\sigma) = \Gamma_{ii}^\delta (e_\delta \tau^\sigma) E_\sigma + \Gamma_{ii}^\delta \tau^\sigma \nabla_{e_\delta}^\varphi E_\sigma \\ = \Gamma_{ii}^\delta (e_\delta \tau^\sigma) E_\sigma + \Gamma_{ii}^\delta \tau^\sigma \nabla_{d\varphi(e_\delta)}^N E_\sigma = \Gamma_{ii}^\delta (e_\delta \tau^\sigma) E_\sigma + \Gamma_{ii}^\delta \tau^\sigma B_\delta^\beta \nabla_{E_\beta}^N E_\sigma, \quad (7)$$

and

$$\nabla_{e_i}^\varphi \nabla_{e_i}^\varphi \tau(\varphi) = \nabla_{e_i}^\varphi \left((e_i \tau^\sigma) E_\sigma + \tau^\sigma B_i^\beta \nabla_{E_\beta}^N E_\sigma \right) \\ = (e_i e_i \tau^\sigma) E_\sigma + [(2e_i \tau^\sigma) B_i^\beta + \tau^\sigma e_i(B_i^\beta)] \nabla_{E_\beta}^N E_\sigma + \tau^\sigma B_i^\beta \nabla_{E_\beta}^N \nabla_{E_\alpha}^N E_\sigma. \quad (8)$$

Additionally,

$$\text{Trace}_g R^N(d\varphi, \tau(\varphi)) d\varphi = \sum_{i=1}^m R^N(d\varphi(e_i), \tau^\sigma E_\sigma) d\varphi(e_i) \\ = \sum_{i=1}^m \tau^\sigma (B_i^\alpha B_i^\beta) \bar{R}(E_\alpha, E_\sigma) E_\beta = \tau^\sigma (B^\alpha \cdot B^\beta) R^N(E_\alpha, E_\sigma) E_\beta. \quad (9)$$

Thus, using (7), (8), and (9) we obtain the bitension field of the map φ as

$$\tau^2(\varphi) = \sum_{i=1}^m \{ \nabla_{e_i}^\varphi \nabla_{e_i}^\varphi (\tau(\varphi)) - \nabla_{\nabla_{e_i} e_i}^\varphi (\tau(\varphi)) - R^N(d\varphi(e_i), \tau(\varphi)) d\varphi(e_i) \} \\ = \sum_{i=1}^m \left((e_i e_i \tau^\sigma) E_\sigma - \Gamma_{ii}^\delta (e_\delta \tau^\sigma) E_\sigma + [(2e_i \tau^\sigma) B_i^\beta + \tau^\sigma e_i(B_i^\beta) - \Gamma_{ii}^\delta B_\delta^\beta \tau^\sigma] \nabla_{E_\beta}^N E_\sigma \right. \\ \left. + \tau^\sigma B^\alpha \cdot B^\beta \nabla_{E_\alpha}^N \nabla_{E_\beta}^N E_\sigma - \tau^\sigma (B^\alpha \cdot B^\beta) \bar{R}(E_\alpha, E_\sigma) E_\beta \right) \\ = \sum_{i=1}^m \left(e_i e_i \tau^\sigma - \Gamma_{ii}^\delta e_\delta(\tau^\sigma) \right) E_\sigma + \left[\sum_{i=1}^m \left((2e_i \tau^\sigma) B_i^\beta + \tau^\sigma e_i(B_i^\beta) - \Gamma_{ii}^\delta B_\delta^\beta \tau^\sigma \right) \nabla_{E_\beta}^N E_\sigma \right. \\ \left. + \tau^\sigma B^\alpha \cdot B^\beta \left(\nabla_{E_\alpha}^N \nabla_{E_\beta}^N E_\sigma - \bar{R}(E_\alpha, E_\sigma) E_\beta \right) \right].$$

If we denote $\tau^2(\varphi) = F^\gamma E_\gamma$, then it follows that

$$\begin{aligned} F^\gamma &= \langle \tau^2(\varphi), E_\gamma \rangle \\ &= \left\langle \sum_{i=1}^m (e_i e_i \tau^\sigma - \Gamma_{ii}^\delta e_\delta(\tau^\sigma)) E_\sigma + \left[\sum_{i=1}^m \left((2e_i \tau^\sigma) B_i^\beta + \tau^\sigma e_i (B_i^\beta) - \Gamma_{ii}^\delta B_\delta^\beta \tau^\sigma \right) \right] \nabla_{E_\beta}^N E_\sigma \right. \\ &\quad \left. + \tau^\sigma B^\alpha \cdot B^\beta \left(\nabla_{E_\alpha}^N \nabla_{E_\beta}^N E_\sigma - \bar{R}(E_\alpha, E_\sigma) E_\beta \right), E_\gamma \right\rangle \\ &= \sum_{i=1}^m (e_i e_i \tau^\gamma - \Gamma_{ii}^\delta e_\delta(\tau^\gamma)) + \left[\sum_{i=1}^m \left((2e_i \tau^\sigma) B_i^\beta + \tau^\sigma e_i (B_i^\beta) - \Gamma_{ii}^\delta B_\delta^\beta \tau^\sigma \right) \right] \langle \nabla_{E_\beta}^N E_\sigma, E_\gamma \rangle \\ &\quad + \tau^\sigma B^\alpha \cdot B^\beta \left(\langle \nabla_{E_\alpha}^N \nabla_{E_\beta}^N E_\sigma, E_\gamma \rangle - \langle R^N(E_\alpha, E_\sigma) E_\beta, E_\gamma \rangle \right), \end{aligned}$$

from which we conclude the lemma. $\square$

Remark 1. Note that Lemma 1 follows immediately from Lemma 2 by taking $M^m = \mathbb{R}^m$ with the standard orthonormal frame $\{e_i = \frac{\partial}{\partial x_i}\}_{i=1}^m$.

For convenience, from now on we assume that r and n are positive integers satisfying $r < n$.

Let $\mathbb{R}^n$ denote the Euclidean space with the standard orthonormal frame $\{\frac{\partial}{\partial x^i}\}_{i=1}^n$. Consider $\mathbb{R}^n$ equipped with the warped product metric

$$g(f, r) = \sum_{j=1}^r dx_j^2 + f^2(x_1, \dots, x_r) \sum_{l=r+1}^n dx_l^2.$$

This space is denoted by $(\mathbb{R}^n, g(f, r))$ or $\mathbb{R}^r \times_{f^2} \mathbb{R}^{n-r}$ and can be viewed as a warped product manifold.

It is easy to verify that an orthonormal basis for $(\mathbb{R}^n, g(f, r))$ is given by

$$E_i = \frac{\partial}{\partial x_i} \quad (i = 1, 2, \dots, r), \text{ and } E_l = \frac{1}{f} \frac{\partial}{\partial x_l} \quad (l = r+1, \dots, n). \quad (10)$$

In terms of this basis, the Levi-Civita connection ∇^N and the Lie brackets can be computed as follows:

$$\begin{aligned} \nabla_{E_l}^N E_i &= \frac{f_i}{f} E_l, \quad [E_i, E_l] = -[E_l, E_i] = -\frac{f_i}{f} E_l, \text{ for } i = 1, \dots, r, \text{ and } l = r+1, \dots, n, \\ \nabla_{E_l}^N E_l &= \sum_{i=1}^r -\frac{f_i}{f} E_i, \text{ for } l = r+1, \dots, n, \text{ all other } \nabla_{E_k}^N E_j = 0, \quad [E_k, E_j] = 0, \end{aligned} \quad (11)$$

where $f_i = \frac{\partial f}{\partial x_i}$.

Computing the Riemann curvature tensor R^N with respect to the orthonormal basis $\{E_i\}$, we find that the only non-zero components are:

$$\begin{aligned} R_{iljl}^N &= R^N(E_i, E_l, E_j, E_l) = g(R^N(E_j, E_l) E_l, E_i) = -\frac{f_{ij}}{f}, \\ R_{\eta l \eta l}^N &= R^N(E_\eta, E_l, E_\eta, E_l) = g(R^N(E_\eta, E_l) E_l, E_\eta) = -\sum_{i=1}^r \frac{f_i^2}{f^2}, \\ &\text{for } i, j = 1, 2, \dots, r, \text{ and } l \neq \eta \geq r+1, \end{aligned} \quad (12)$$

where $f_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$.

We now state our first main result:

Theorem 1. The identity map

$$\iota : \mathbb{R}^n \longrightarrow (\mathbb{R}^n, g(f, r)), \quad \iota(x_1, \dots, x_n) = (x_1, \dots, x_n)$$

with the metric

$$g(f, r) = \sum_{j=1}^r dx_j^2 + f^2(x_1, \dots, x_r) \sum_{l=r+1}^n dx_l^2$$

is biharmonic if and only if the warping function f satisfies the PDE system:

$$\Delta(f^2) - \frac{(n-r)}{4} |\nabla f^2|^2 = K = \text{constant}, \quad (13)$$

where Δ , ∇ and $|\cdot|$ denote the Laplacian, the gradient and the Euclidean norm on $\mathbb{R}^n$, respectively. Furthermore, the system (13) admits no nonconstant global solutions whatsoever.

Proof. Let $\iota : \mathbb{R}^n \rightarrow (\mathbb{R}^n, g(f, r))$ be the identity map between Euclidean space and the warped product manifold. Consider the orthonormal frame $\{e_i = \frac{\partial}{\partial x_i}\}$ on $\mathbb{R}^n$ and the frame $\{E_i\}$ of $(\mathbb{R}^n, g(f, r))$ as in (10). The differential $d\iota$ maps the frames as:

$$\begin{aligned} di(e_i) &= E_i = B_i^\beta E_\beta \quad (i = 1, 2, \dots, r), \\ di(e_l) &= f E_l = B_l^\beta E_\beta \quad (l = r+1, \dots, n), \end{aligned} \quad (14)$$

where

$$\begin{aligned} B^i &= (0, 0, \dots, 0, \underset{(i)}{1}, 0, \dots, 0) \quad (i = 1, 2, \dots, r), \\ B^l &= (0, \dots, 0, \underset{(r)}{\dots}, \underset{(l)}{f}, \dots, 0) \quad (l = r+1, \dots, n). \end{aligned} \quad (15)$$

The tension field $\tau(\iota)$ is calculated as:

$$\tau(\iota) = \sum_{i=1}^n \{ \nabla_{e_i}^\iota d\iota(e_i) - d\iota(\nabla_{e_i}^{\mathbb{R}^n} e_i) \} = \sum_{i=1}^n \nabla_{e_i}^\iota B_i^\beta E_\beta.$$

Expanding the covariant derivative, we have

$$\tau(\iota) = \sum_{i=1}^n [e_i(B_i^\beta) E_\beta + B_i^\alpha B_i^\beta \nabla_{E_\alpha}^N E_\beta] = -(n-r) \sum_{i=1}^r f f_i E_i,$$

where $f_i = \frac{\partial f}{\partial x_i}$.

Thus the components are given by

$$\tau^\gamma = \begin{cases} -(n-r) f f_\gamma & (\gamma = 1, 2, \dots, r), \\ 0 & (\gamma = r+1, \dots, n). \end{cases} \quad (16)$$

Using Lemma 1 and Equations (11), (12) and (14)–(16), the bitension field components can be calculated as:

$$\begin{aligned} F^\gamma &= \langle \tau^2(\iota), E_\gamma \rangle \\ &= \Delta \tau^\gamma + [2 \langle \nabla \tau^\sigma, B^\beta \rangle + \sum_{i=1}^n \tau^\sigma e_i(B_i^\beta)] \langle \nabla_{E_\beta}^N E_\sigma, E_\gamma \rangle \\ &\quad + \tau^\sigma B^\alpha \cdot B^\beta \langle \nabla_{E_\alpha}^N \nabla_{E_\beta}^N E_\sigma, E_\gamma \rangle + \tau^\sigma (B^\alpha \cdot B^\beta) R_{\alpha\sigma\beta\gamma}^N \\ &= \Delta \tau^\gamma - \frac{f_\gamma}{f} \left[\sum_{\sigma=r+1}^n \{ 2 \langle \nabla \tau^\sigma, B^\sigma \rangle + \sum_{i=1}^n \tau^\sigma e_i(B_i^\sigma) \} \right] - \sum_{l=r+1}^n \sum_{\sigma=1}^r \frac{f_\sigma f_\gamma}{f^2} \tau^\sigma B^l \cdot B^l \\ &\quad + \sum_{\sigma=r+1}^n \tau^\sigma B^\alpha \cdot B^\sigma \langle \nabla_{E_\alpha}^N \nabla_{E_\sigma}^N E_\sigma, E_\gamma \rangle - \sum_{l=r+1}^n \sum_{\sigma=1}^r \frac{f f_\sigma}{f^2} \tau^\sigma B^l \cdot B^l \\ &= -(n-r) \Delta(f f_\gamma) + (n-r)^2 \sum_{\sigma=1}^r f f_\sigma (f f_{\sigma\gamma} + f_{\sigma} f_\gamma) \quad (\gamma = 1, \dots, r), \end{aligned} \quad (17)$$

and

$$\begin{aligned}
 F^l &= \langle \tau^2(1), E_l \rangle \\
 &= \Delta \tau^l + [2 \langle \nabla \tau^\sigma, B^\beta \rangle + \sum_{i=1}^n \tau^\sigma e_i(B_i^\beta)] \langle \nabla_{E_\beta}^N E_\sigma, E_l \rangle \\
 &\quad + \tau^\sigma B^\alpha \cdot B^\beta \langle \nabla_{E_\alpha}^N \nabla_{E_\beta}^N E_\sigma, E_l \rangle + \tau^\sigma (B^\alpha \cdot B^\beta) R_{\alpha\sigma\beta l}^N \\
 &= \Delta \tau^l + \sum_{\sigma=1}^r \frac{f_\sigma}{f} \left(2 \langle \nabla \tau^\sigma, B^l \rangle + \sum_{i=1}^n \tau^\sigma e_i(B_i^l) \right) \\
 &\quad + \sum_{\sigma=1}^r \sum_{\beta=r+1}^n \tau^\sigma B^\beta \cdot B^\beta \langle \frac{f_\sigma}{f} \nabla_{E_\beta}^N E_\beta, E_l \rangle + \sum_{\sigma=r+1}^n \tau^\sigma B^\sigma \cdot B^\sigma \langle \nabla_{E_\sigma}^N \nabla_{E_\sigma}^N E_\sigma, E_l \rangle \\
 &\quad + \sum_{\sigma=1}^r \tau^\sigma (B^\beta \cdot B^\beta) R_{\beta\sigma\beta l}^N = 0 \quad (l = r+1, \dots, n).
 \end{aligned} \tag{18}$$

From the bitension field computation (17), (18) and Lemma 1, the identity map ι is biharmonic if and only if the warping function $f = f(x_1, \dots, x_r)$ satisfies the following PDEs

$$-\Delta(ff_\gamma) + (n-r) \sum_{\sigma=1}^r ff_\sigma(ff_{\sigma\gamma} + f_\sigma f_\gamma) = 0 \quad \text{for } \gamma = 1, 2, \dots, r. \tag{19}$$

Let $y = \frac{1}{2}f^2(x_1, \dots, x_r)$. Then

$$y_\gamma = \frac{\partial y}{\partial x_\gamma} = ff_\gamma, \quad y_{\gamma\sigma} = \frac{\partial^2 y}{\partial x_\gamma \partial x_\sigma} = \frac{\partial(ff_\sigma)}{\partial x_\gamma} = (ff_\sigma)_\gamma = (ff_\gamma)_\sigma,$$

where the last equality follows from the symmetry of mixed partial derivatives.

A straightforward computation gives

$$\begin{aligned}
 &\Delta(ff_\gamma) - (n-r) \sum_{\sigma=1}^r ff_\sigma(ff_{\sigma\gamma} + f_\sigma f_\gamma) \\
 &= \sum_{\sigma=1}^r \frac{\partial^2(ff_\gamma)}{\partial x_\sigma \partial x_\sigma} - (n-r) \sum_{\sigma=1}^r ff_\sigma \frac{\partial}{\partial x_\gamma} (ff_\sigma) \\
 &= \sum_{\sigma=1}^r \frac{\partial}{\partial x_\gamma} \left[\frac{\partial(ff_\sigma)}{\partial x_\sigma} \right] - (n-r) \sum_{\sigma=1}^r \frac{\partial}{\partial x_\gamma} \left[\left(\frac{ff_\sigma}{2} \right)^2 \right] \\
 &= \sum_{\sigma=1}^r \frac{\partial}{\partial x_\gamma} \left[\frac{\partial^2 y}{\partial x_\sigma \partial x_\sigma} \right] - (n-r) \sum_{\sigma=1}^r \frac{\partial}{\partial x_\gamma} \left[\left(\frac{\frac{\partial y}{\partial x_\sigma}}{2} \right)^2 \right] \\
 &= \frac{\partial}{\partial x_\gamma} \left(\sum_{\sigma=1}^r \frac{\partial^2 y}{\partial x_\sigma \partial x_\sigma} - (n-r) \left(\frac{\frac{\partial y}{\partial x_\sigma}}{2} \right)^2 \right) \quad \text{for } \gamma = 1, 2, \dots, r.
 \end{aligned} \tag{20}$$

Because $y = \frac{1}{2}f^2(x_1, \dots, x_r)$ depends only on the first r variables, $\frac{\partial y}{\partial x_\sigma} = 0$ for $\sigma > r$. Therefore,

$$\sum_{\sigma=1}^r \frac{\partial^2 y}{\partial x_\sigma \partial x_\sigma} = \sum_{\sigma=1}^n \frac{\partial^2 y}{\partial x_\sigma \partial x_\sigma} = \Delta y, \quad \sum_{\sigma=1}^r \left(\frac{\partial y}{\partial x_\sigma} \right)^2 = \sum_{\sigma=1}^n \left(\frac{\partial y}{\partial x_\sigma} \right)^2 = |\nabla y|^2,$$

where Δ , ∇ and $|\cdot|$ denote the Laplacian, the gradient and the Euclidean norm on $\mathbb{R}^n$, respectively.

Using (19) and (20), we have $\frac{\partial}{\partial x_\gamma} \left(\Delta y - \frac{(n-r)}{2} |\nabla y|^2 \right) = 0$, $\gamma = 1, 2, \dots, r$. This shows that $\Delta y - \frac{n-r}{2} |\nabla y|^2$ is independent of $x_1, \dots, x_r$. Since y does not depend on $x_{r+1}, \dots, x_n$, it follows that $\Delta y - \frac{n-r}{2} |\nabla y|^2$ is constant on $\mathbb{R}^n$. Hence,

$$\Delta y - \frac{(n-r)}{2} |\nabla y|^2 = K \quad (\text{constant}), \tag{21}$$

Obviously, (21) can be rewritten as

$$\Delta(f^2) - \frac{(n-r)}{4} |\nabla f^2|^2 = \frac{K}{2} \quad (\text{constant}). \tag{22}$$

This completes the proof of the first part of the theorem. We now prove the second part. By the transformation $u = e^{-\frac{(n-r)y}{2}}$ (i.e., $y = -\frac{2}{n-r} \ln u$), the Equation (21) becomes

$$\Delta u = -\frac{(n-r)K}{2}u, \quad (23)$$

where Δ denotes the Laplacian on $\mathbb{R}^n$. Since $u = e^{-\frac{(n-r)y}{2}} = e^{-\frac{(n-r)f^2(x_1, \dots, x_r)}{2}}$, we have $0 < u < 1$. From (23) and $0 < u < 1$, the function $u = u(x_1, \dots, x_r)$ is a bounded subharmonic (harmonic) function on $\mathbb{R}^n$. By Liouville-type theorem (see e.g., [21]), if u is a bounded subharmonic (harmonic) function on $\mathbb{R}^n$, then u must be constant, i.e., f is a constant.

This completes the proof of the theorem. $\square$

Conversely, our second main result characterizes biharmonic identity maps from the warped product space $(\mathbb{R}^n, g(f, r))$ to Euclidean space $\mathbb{R}^n$.

Theorem 2. *The identity map*

$$\iota : (\mathbb{R}^n, g(f, r)) \longrightarrow \mathbb{R}^n, \iota(x_1, \dots, x_n) = (x_1, \dots, x_n)$$

with the warped metric

$$g(f, r) = \sum_{j=1}^r dx_j^2 + f^2(x_1, \dots, x_r) \sum_{l=r+1}^n dx_l^2$$

is a biharmonic map if and only if the warping function f satisfies the PDE

$$\Delta(\ln f^{-2}) - \frac{(n-r)}{4} |\nabla \ln f^{-2}|^2 = K = \text{constant}, \quad (24)$$

where Δ , ∇ and $|\cdot|$ denote the Laplacian, the gradient and the Euclidean norm on $\mathbb{R}^n$, respectively. Furthermore, the system (24) admits nonconstant global solutions.

Proof. Let $\iota : (\mathbb{R}^n, g(f, r)) \rightarrow \mathbb{R}^n$ from the warped product space to Euclidean space be the identity map. The orthonormal frame on $(\mathbb{R}^n, g(f, r))$ consists of $\{e_i = \frac{\partial}{\partial x_i} = E_i, e_l = \frac{1}{f(x_1, \dots, x_r)} \frac{\partial}{\partial x_l} = E_l\}$ with $i = 1, 2, \dots, r$, and $l = r+1, \dots, n$. For Euclidean space $\mathbb{R}^n$, the standard frame takes the form $\{\bar{E}_i = \frac{\partial}{\partial x_i}\}$. The differential $d\iota$ acts on the frames as follows

$$\begin{aligned} di(e_i) &= \bar{E}_i = B_i^\beta \bar{E}_\beta \text{ for } i = 1, 2, \dots, r, \\ di(e_l) &= \frac{1}{f} \bar{E}_l = B_l^\beta \bar{E}_\beta \text{ for } l = r+1, \dots, n, \end{aligned} \quad (25)$$

where

$$\begin{aligned} B^i &= (0, \dots, 0, \underset{(i)}{1}, 0, \dots, 0) \text{ for } i = 1, 2, \dots, r, \\ B^l &= (0, \dots, 0, \underset{(l)}{\frac{1}{f}}, 0, \dots, 0) \text{ for } l = r+1, \dots, n. \end{aligned} \quad (26)$$

Using (11), (25) and (26), together with the fact that $\nabla_{\bar{E}_\alpha}^{\mathbb{R}^n} \bar{E}_\beta = 0$, the tension field of ι is given by

$$\begin{aligned}
 \tau(\iota) &= \sum_{j=1}^n \{ \nabla_{e_j}^t \mathrm{d}\iota(e_j) - \mathrm{d}\iota(\nabla_{e_j} e_j) \} = \sum_{j=1}^n [\nabla_{e_j}^t B_j^\beta \bar{E}_\beta - \Gamma_{jj}^\delta B_j^\beta \bar{E}_\beta] \\
 &= \sum_{j=1}^n [e_j (B_j^\beta) \bar{E}_\beta - \Gamma_{jj}^\delta B_j^\beta \bar{E}_\beta + B_j^\alpha B_j^\beta \nabla_{\bar{E}_\alpha}^{\mathbb{R}^n} \bar{E}_\beta] \\
 &= \sum_{j=1}^n [e_j (B_j^\beta) \bar{E}_\beta - \Gamma_{jj}^\delta B_j^\beta \bar{E}_\beta] = (n-r) \sum_{j=1}^r \frac{f_j}{f} \bar{E}_j,
 \end{aligned} \tag{27}$$

where $f_j = \frac{\partial f}{\partial x_j}$ for $j = 1, \dots, r$.

This leads to

$$\begin{aligned}
 \tau^j &= \frac{(n-r)f_j}{f} \quad \text{for } j = 1, \dots, r, \\
 \tau^l &= 0 \quad \text{for } l = r+1, \dots, n.
 \end{aligned} \tag{28}$$

By Lemma 2, Equations (11), (25), (26) and (28), and the fact that the target manifold $\mathbb{R}^n$ has zero sectional curvature and $\nabla_{\bar{E}_\alpha}^{\mathbb{R}^n} \bar{E}_\beta = 0$, the components of the bitension field of ι are computed as

$$\begin{aligned}
 F^\gamma &= \langle \tau^2(\iota), E_\gamma \rangle = \sum_{j=1}^n [e_j e_j \tau^\gamma - \Gamma_{jj}^\delta e_\delta (\tau^\gamma)] \\
 &= (n-r) \sum_{i=1}^r \frac{\partial^2}{\partial x_i \partial x_i} \left(\frac{f_\gamma}{f} \right) + (n-r)^2 \sum_{\alpha=1}^r \frac{f_\alpha (f f_{\alpha\gamma} - f_\alpha f_\gamma)}{f^3} \quad (\gamma = 1, \dots, r),
 \end{aligned} \tag{29}$$

and

$$F^l = \langle \tau^2(\iota), E_l \rangle = \sum_{j=1}^n e_j e_j \tau^l - \Gamma_{jj}^\delta e_\delta (\tau^l) = 0 \quad (l = r+1, \dots, n). \tag{30}$$

By Lemma 2 and Equations (29) and (30), the identity map ι is biharmonic for $n \geq 2$ if and only if the function f (which only depends on $x_1, x_2, \dots, x_r$ and is independent of $x_{r+1}, \dots, x_n$) solves the following PDE

$$\Delta \left(\frac{f_\gamma}{f} \right) + (n-r) \sum_{\alpha=1}^r \frac{f_\alpha (f f_{\alpha\gamma} - f_\alpha f_\gamma)}{f^3} = 0, \quad \gamma = 1, 2, \dots, r, \tag{31}$$

where Δ denotes the Laplacian on $\mathbb{R}^n$.

A straightforward computation shows that Equation (31) can be rewritten as

$$\frac{\partial}{\partial x_\gamma} \left(\Delta(\ln f) + \frac{(n-r)}{2} |\nabla \ln f|^2 \right) = 0 \quad \text{for } \gamma = 1, 2, \dots, r, \tag{32}$$

where ∇ and $|\cdot|$ denote the gradient and the Euclidean norm on $\mathbb{R}^n$, respectively.

Since $f = f(x_1, \dots, x_r)$ (and is independent of $x_{r+1}, \dots, x_n$), (32) reduces to

$$\Delta(\ln f) + \frac{(n-r)}{2} |\nabla \ln f|^2 = K = \text{constant}, \tag{33}$$

which is equivalent to

$$\Delta(\ln f^{-2}) - \frac{(n-r)}{4} |\nabla \ln f^{-2}|^2 = 2K = \text{constant}. \tag{34}$$

Thus, we obtain the theorem. $\square$

Remark 2. Theorems 1 and 2 reveal that the characterization of dual biharmonic identity maps $\mathbb{R}^n \hookrightarrow (\mathbb{R}^n, g(f, r))$ essentially relies on a single type of equation:

(i) Hamilton–Jacobi equation

$$\Delta u - \frac{(n-r)}{4} |\nabla u|^2 = K \quad (K \in \mathbb{R}),$$

where for Theorem 1 (the identity map $\mathbb{R}^n \rightarrow (\mathbb{R}^n, g(f, r))$), $u = f^2$; for Theorem 2 (the identity map $(\mathbb{R}^n, g(f, r)) \rightarrow \mathbb{R}^n$), $u = \ln f^{-2}$;

or equivalently,

(ii) its transformed Liouville-type version

$$\Delta v + \frac{(n-r)K}{4} v = 0,$$

where $v = e^{-\frac{n-r}{4}u}$.

Remark 3. When the warping function is constant, the identity map is a trivial biharmonic map (i.e., harmonic map). Introducing a non-constant warping function changes the curvature, gradient, and Laplacian on the Euclidean space. These modified operators then act on the identity map and can make it genuinely biharmonic yet non-harmonic.

3. Examples of Biharmonic Identity Maps Between Euclidean Spaces and Warped Product Spaces

In this section, we construct explicit examples of biharmonic identity maps between Euclidean spaces and warped product spaces.

By applying Theorem 1 to specific warping functions f , we obtain the following explicit family of biharmonic identity maps $\mathbb{R}^n \rightarrow (\mathbb{R}^n, g(f, r))$:

Proposition 1. Let $C_1^\mu \neq 0$, C_2^μ , C_3^μ , $b_1^\mu \neq 0$ and b_2^μ be constants, and for each $\mu \leq r < n$, take $f_\mu(x_\mu)$ to be one of the following positive functions:

$$\begin{aligned} f_\mu(x_\mu) &= b_1^\mu x_\mu + b_2^\mu, \\ f_\mu(x_\mu) &= -\ln(x_\mu + C_2^\mu)^{\frac{2}{n-r}} + C_3^\mu, \\ f_\mu(x_\mu) &= -\ln(\cos(C_1^\mu x_\mu + C_2^\mu))^{\frac{2}{n-r}} + C_3^\mu, \\ f_\mu(x_\mu) &= -\ln(\sinh(C_1^\mu x_\mu + C_2^\mu))^{\frac{2}{n-r}} + C_3^\mu, \\ f_\mu(x_\mu) &= -\ln(\cosh(C_1^\mu x_\mu + C_2^\mu))^{\frac{2}{n-r}} + C_3^\mu. \end{aligned} \quad (35)$$

Then, for $f = \sqrt{2 \sum_{\mu=1}^r f_\mu(x_\mu)}$, the identity map

$$\iota : \mathbb{R}^n \rightarrow (\mathbb{R}^n, g(f, r)), \quad \iota(x_1, \dots, x_n) = (x_1, \dots, x_n),$$

where the metric $g(f, r) = \sum_{j=1}^r dx_j^2 + f^2 \sum_{l=r+1}^n dx_l^2$, is proper biharmonic. Here, each f represents a local function on $\mathbb{R}^n$.

Proof. By Theorem 1 with $\frac{f^2}{2} = \sum_{\mu=1}^r f_\mu(x_\mu)$ for $\mu \leq r < n$, the identity map ι is biharmonic if and only if

$$\sum_{\mu=1}^r \left\{ f_\mu'''(x_\mu) - (n-r) f_\mu'(x_\mu) f_\mu''(x_\mu) \right\} = 0. \quad (36)$$

We construct solutions by requiring each component in (36) to vanish separately:

$$f_{\mu}'''(x_{\mu}) - (n-r)f_{\mu}'(x_{\mu})f_{\mu}''(x_{\mu}) = 0. \quad (37)$$

Setting $y_{\mu} = f_{\mu}'(x_{\mu})$, the equation (37) transforms into:

$$y_{\mu}'' - (n-r)y_{\mu}y_{\mu}' = 0. \quad (38)$$

Applying the substitution $\frac{dy_{\mu}}{dx_{\mu}} = p$, $\frac{d^2y_{\mu}}{dx_{\mu}^2} = p \frac{dp}{dy_{\mu}}$ to (38) yields

$$p \frac{dp}{dy_{\mu}} - (n-r)y_{\mu} p = 0, \quad (39)$$

which admits solutions:

$$p = 0, \quad (40)$$

or

$$p = \frac{(n-r)}{2}y_{\mu}^2 + C, \quad (41)$$

where C is a constant.

Case I: $p = \frac{dy_{\mu}}{dx_{\mu}} \equiv 0$. This leads to the local solution as

$$f_{\mu}(x_{\mu}) = b_1x_{\mu} + b_2,$$

where b_1, b_2 are constants.

Case II: $p = \frac{dy_{\mu}}{dx_{\mu}} = \frac{(n-r)}{2}y_{\mu}^2 + C$. Rewriting as

$$\frac{dy_{\mu}}{\frac{(n-r)}{2}y_{\mu}^2 + C} = dx_{\mu}, \quad (42)$$

we obtain through integration:

$$y_{\mu} = \begin{cases} -\frac{2}{(n-r)x_{\mu} + c_1}, & \text{for } C = 0, \\ \sqrt{\frac{2C}{n-r}} \tan\left(\sqrt{\frac{C(n-r)}{2}} x_{\mu} + c_1\right), & \text{for } C > 0, \\ \sqrt{\frac{-2C}{n-r}} \frac{1 + c_2 e^{2\sqrt{\frac{-C(n-r)}{2}} x_{\mu}}}{1 - c_2 e^{2\sqrt{\frac{-C(n-r)}{2}} x_{\mu}}}, & \text{for } C < 0, \end{cases} \quad (43)$$

where c_1 and $c_2 \neq 0$ are constants.

Since $y_{\mu} = f_{\mu}'$, we perform further integration on (43) to derive

$$\begin{aligned} f_{\mu}(x_{\mu}) &= -\ln(x_{\mu} + C_2^{\mu})^{\frac{2}{n-r}} + C_3^{\mu}, \text{ or,} \\ f_{\mu}(x_{\mu}) &= -\ln(\cos(C_1^{\mu}x_{\mu} + C_2^{\mu}))^{\frac{2}{n-r}} + C_3^{\mu}, \text{ or,} \\ f_{\mu}(x_{\mu}) &= -\ln(\sinh(C_1^{\mu}x_{\mu} + C_2^{\mu}))^{\frac{2}{n-r}} + C_3^{\mu}, \text{ or,} \\ f_{\mu}(x_{\mu}) &= -\ln(\cosh(C_1^{\mu}x_{\mu} + C_2^{\mu}))^{\frac{2}{n-r}} + C_3^{\mu}, \end{aligned} \quad (44)$$

where $C_1^{\mu} \neq 0$, C_2^{μ} and C_3^{μ} are constants. Each solution represents a local function on $\mathbb{R}^n$.

Combining all cases, we complete the proof of the proposition. $\square$

Example 1. Let $\mathbb{R}_+^n = \{(x_1, x_2, \dots, x_n) \in \mathbb{R}^n; x_j > 0 \text{ for all } j = 1, 2, \dots, n\}$. For constants $c_j = 1$ ($j \leq r < n$) and $f = \sqrt{x_1 + x_2 + \dots + x_r}$, consider the identity map

$$\iota : \mathbb{R}_+^n \longrightarrow (\mathbb{R}_+^n, g(f, r)), \quad \iota(x_1, \dots, x_n) = (x_1, \dots, x_n)$$

equipped with the metric

$$g(f, r) = \sum_{j=1}^r dx_j^2 + f^2 \sum_{l=r+1}^n dx_l^2.$$

Then, the map ι is proper biharmonic.

In fact, for the function $f = \sqrt{\sum_{j=1}^r x_j} > 0$, one can directly check that the tension field

$\tau(\iota) = -\frac{1}{2} \sum_{i=1}^r E_i \neq 0$. By Proposition 1, ι is a non-harmonic biharmonic map (and thus a proper biharmonic map).

The proof of Proposition 1 yields the following corollary as a special case:

Corollary 1. The identity map

$$\iota : \mathbb{R}^n \rightarrow (\mathbb{R}^n, g(f(x_\mu), r)), \quad \iota(x_1, \dots, x_n) = (x_1, \dots, x_n),$$

where $g(f(x_\mu), r) = \sum_{j=1}^r dx_j^2 + f^2(x_\mu) \sum_{l=r+1}^n dx_l^2$, is biharmonic for $\mu \leq r < n$ if and only if the warping function f satisfies the ODE

$$(ff')'' - (n-r)ff'(ff')' = 0. \quad (45)$$

This ODE admits no nonconstant global solutions, and all its explicit solutions are as follows:

$$\begin{aligned} f(x_\mu) &= \sqrt{b_1 x_\mu + b_2}, \\ f(x_\mu) &= \sqrt{-\ln(x_\mu + C_2)^{\frac{4}{n-r}} + C_3}, \\ f(x_\mu) &= \sqrt{-\ln(\cos(C_1 x_\mu + C_2))^{\frac{4}{n-r}} + C_3}, \\ f(x_\mu) &= \sqrt{-\ln(\sinh(C_1 x_\mu + C_2))^{\frac{4}{n-r}} + C_3}, \\ f(x_\mu) &= \sqrt{-\ln(\cosh(C_1 x_\mu + C_2))^{\frac{4}{n-r}} + C_3} \end{aligned} \quad (46)$$

with $C_1 \neq 0$, C_2, C_3, b_1, b_2 being constants. Here, the first solution reduces to a constant when $b_1 = 0$, and the rest are local.

Theorem 2 enables us to generate biharmonic identity maps $(\mathbb{R}^n, g(f, r)) \rightarrow \mathbb{R}^n$ for specific choices of f . A concrete realization is given by:

Proposition 2. Let $b_1^j \neq 0, b_2^j, C_1^j > 0, C_2^j$, and C_3^j be constants. For each $j \leq r < n$, let $f_j(x_j)$ be any positive function of the form:

$$\begin{aligned} f_j(x_j) &= b_1^j x_j + b_2^j, \\ f_j(x_j) &= -\ln\left(C_1^j(x_j + C_3^j)^{\frac{2}{n-r}}\right), \\ f_j(x_j) &= -\ln\left(C_1^j(\cos(C_2^j x_j + C_3^j))^{\frac{2}{n-r}}\right), \\ f_j(x_j) &= -\ln\left(C_1^j(\sinh(C_2^j x_j + C_3^j))^{\frac{2}{n-r}}\right), \\ f_j(x_j) &= -\ln\left(C_1^j(\cosh(C_2^j x_j + C_3^j))^{\frac{2}{n-r}}\right). \end{aligned} \quad (47)$$

Set $f = e^{-\sum_{j=1}^r f_j(x_j)}$. Then the identity map

$$\iota : (\mathbb{R}^n, g(f, r)) \longrightarrow \mathbb{R}^n, \iota(x_1, \dots, x_n) = (x_1, \dots, x_n),$$

where the metric $g(f, r)$ is

$$dx_1^2 + \dots + dx_r^2 + f^2(dx_{r+1}^2 + \dots + dx_n^2),$$

is a proper biharmonic map from the warped product space to Euclidean space.

Proof. Applying Theorem 2 with $\ln f = -\sum_{j=1}^r f_j(x_j)$, the identity map ι is biharmonic if and only if

$$\sum_{j=1}^r \left((f_j(x_j))''' - \frac{(n-r)}{4} (f_j(x_j))' (f_j(x_j))'' \right) = 0. \quad (48)$$

We find special solutions to Equation (48) where each $f_j(x_j)$ satisfies independently:

$$(f_j(x_j))''' - \frac{(n-r)}{4} (f_j(x_j))' (f_j(x_j))'' = 0 \text{ for } j = 1, 2, \dots, r. \quad (49)$$

By setting $y_j = (f_j(x_j))'$, we reduce (49) to:

$$y_j'' - (n-r)y_j y_j' = 0 \text{ for } j = 1, 2, \dots, r. \quad (50)$$

Following the method of (38)–(43), the solutions of (50) are:

$$y_j = \begin{cases} b_1^j, \\ -\frac{2}{(n-r)x_j + c_1^j}, \\ \sqrt{\frac{2}{n-r}} a^j \tan\left(\sqrt{\frac{n-r}{2}} a^j x_j + c_1^j\right), \\ \sqrt{\frac{2}{n-r}} b^j \frac{1+e^{2\sqrt{\frac{n-r}{2}} b^j x_j + 2c_1^j}}{1-e^{2\sqrt{\frac{n-r}{2}} b^j x_j + 2c_1^j}}, \\ \sqrt{\frac{2}{n-r}} b^j \frac{1-e^{2\sqrt{\frac{n-r}{2}} b^j x_j + 2c_1^j}}{1+e^{2\sqrt{\frac{n-r}{2}} b^j x_j + 2c_1^j}}, \end{cases} \quad (51)$$

where $a^j > 0$, $b^j > 0$, and b_1^j, c_1^j are constants for each $j \leq r$.

Since $y_j = f_j'(x_j)$, integrating (51) yields:

$$\begin{aligned} f_j(x_j) &= b_1^j x_j + b_2^j, \text{ or,} \\ f_j(x_j) &= -\ln\left(C_1^j (x_j + C_3^j)^{\frac{2}{n-r}}\right), \text{ or,} \\ f_j(x_j) &= -\ln\left(C_1^j (\cos(C_2^j x_j + C_3^j))^{\frac{2}{n-r}}\right), \text{ or,} \\ f_j(x_j) &= -\ln\left(C_1^j (\sinh(C_2^j x_j + C_3^j))^{\frac{2}{n-r}}\right), \text{ or,} \\ f_j(x_j) &= -\ln\left(C_1^j (\cosh(C_2^j x_j + C_3^j))^{\frac{2}{n-r}}\right), \end{aligned}$$

where $b_1^j, b_2^j, C_1^j > 0$, C_2^j and C_3^j are constants for $j \leq r$. Note that the first and last solutions are globally defined on $\mathbb{R}^n$.

Combining these results, we obtain the proposition. $\square$

Proposition 2 enables explicit constructions of infinitely many proper biharmonic maps. As a concrete illustration, consider the following example:

Example 2. Define the warping function

$$f = e^{(x_1+x_2+\dots+x_r)}$$

on $\mathbb{R}^n$ which is a nonconstant global function. Then the identity map

$$\iota : (\mathbb{R}^n, g(f, r)) \longrightarrow \mathbb{R}^n, \iota(x_1, \dots, x_n) = (x_1, \dots, x_n)$$

with the metric

$$g(f, r) = \sum_{j=1}^r dx_j^2 + f^2 \sum_{l=r+1}^n dx_l^2$$

is a proper biharmonic map.

A technical byproduct of the proof of Proposition 2 is the following:

Corollary 2. The identity map

$$\iota : (\mathbb{R}^n, g(f(x_\mu), r)) \longrightarrow \mathbb{R}^n, \iota(x_1, \dots, x_n) = (x_1, \dots, x_n)$$

from the warped product space to Euclidean space with the metric

$$g(f(x_\mu), r) = \sum_{j=1}^r dx_j^2 + f^2(x_\mu) \sum_{l=r+1}^n dx_l^2$$

is a biharmonic map when $\mu \leq r$ if and only if the warping function f solves the ODE

$$\left(\frac{f'}{f}\right)'' + (n-r)\frac{f'}{f}\left(\frac{f'}{f}\right)' = 0. \quad (52)$$

All solutions to this equation are explicitly given by:

$$\begin{aligned} f(x_\mu) &= e^{-b_1 x_\mu - b_2}, \\ f(x_\mu) &= C_1(x_\mu + C_3)^{\frac{2}{n-r}}, \\ f(x_\mu) &= C_1(\cos(C_2 x_\mu + C_3))^{\frac{2}{n-r}}, \\ f(x_\mu) &= C_1(\sinh(C_2 x_\mu + C_3))^{\frac{2}{n-r}}, \\ f(x_\mu) &= C_1(\cosh(C_2 x_\mu + C_3))^{\frac{2}{n-r}}, \end{aligned} \quad (53)$$

where $b_1, b_2, C_1 > 0$, $C_2 \neq 0$, and C_3 are constants. The first and last solutions in (53) are globally defined on $\mathbb{R}^n$.

4. Conclusions

The duality between the forward and reverse identity maps offers a notable insight: one direction admits only trivial global solutions, whereas the other allows rich families of global solutions. These conclusions are well supported by the presented computations and examples.

This work contributes explicit examples to the catalogue of proper biharmonic maps, a valuable addition to geometric analysis.

The equations studied (a nonlinear Hamiltonian–Jacobi-type elliptic equation: $\Delta u - \frac{n-r}{4}|\nabla u|^2 = K$ ($K \in \mathbb{R}$) and its transformed Liouville-type version: $\Delta v + \frac{(n-r)K}{4}v = 0$, where $v = e^{-\frac{n-r}{4}u}$) have direct links to conformal geometry. Their solutions correspond

to curvature-controlling factors for conformal deformations of manifolds, characterizing geometric rigidity under metric transformations. Notably, the Liouville-type equation derived herein can be regarded as a special case of Yamabe-type equations. These equations also hold potential connections to general relativity, as they are somewhat analogous to curvature evolution equations for certain spacetime slices. Future research may be extended to the geometric analysis of higher-order polyharmonic maps between Euclidean spaces and doubly warped product spaces.

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