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Using the Embedding Theorem to Solve Interval-Valued Optimization Problems

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Abstract

The space of all bounded closed intervals cannot form a vector space because the concept of an additive inverse cannot be considered. Therefore, this paper presents an embedding theorem to show that the space of all bounded closed intervals can be embedded into a Banach space. In this case, the partial orders among the space of all bounded closed intervals can be proposed via the embedding theorem by considering the ordering cones in that Banach space. After a partial order is introduced in the interval-valued optimization problem, the solution concepts of interval-valued optimization problem can be naturally defined. On the other hand, using the embedding theorem, an auxiliary vector optimization problem is introduced such that solving the original interval-valued optimization problem is equivalent to solving the auxiliary vector optimization problem. The technique of scalarization is proposed to solve the auxiliary vector optimization problem. Three practical ordering cones are considered to study the linear type of interval-valued optimization problem for the purpose of presenting practical applications.

Keywords: embedding theorem; minimal element; ordering cone; partial order; scalarization

MSC: 90C29; 03E71

1. Introduction

Modeling an optimization problem is sometimes needed for studying the engineering or economics problems by properly assigning the coefficients of objective functions and constraint functions. In this case, we should collect the real-valued data from the related environment to set up the required coefficients. Frequently, the data are uncertain because of fluctuation. Therefore, we find it difficult to pick the proper real numbers to represent the coefficients of objective and constraint functions for an optimization problem. Therefore, there are many ways to tackle this kind of uncertainty. For example, the uncertainty can be treated as randomness or fuzziness. When the decision-makers decide to follow the random uncertainty, the coefficients of objective and constraint functions for an optimization problem can be taken to be the random variables with known probability distributions. Alternatively, when the fuzzy uncertainty is taken into account by the decision-makers, the coefficients can be assumed to be fuzzy numbers with known membership functions. However, determining the appropriate probability distributions or membership functions may result in a dilemma for the decision-makers. Usually, the decision-makers objectively assume the coefficients are the normal distributions for the probabilistic approach or the bell-shaped membership functions for the fuzzy approach by intuitively assigning the values of related parameters. Another simple way to describe the data uncertainty is to



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assume the coefficients are the bounded closed intervals. In most situations, the decision-makers can make sure that the desired coefficients will be located in some ranges, where the ranges can be assumed to be bounded closed intervals. In this case, the desired coefficients can be assumed to be bounded closed intervals instead of assuming them to be random variables or fuzzy numbers. Determining the upper and lower bounds of bounded closed intervals is easier than determining the suitable probabilistic distribution functions and the bell-shaped membership functions. In this case, the objective and constraint functions turn into interval-valued functions such that we should solve the interval-valued optimization problems. The arithmetics of intervals presented in Moore [1,2] may be useful in solving interval-valued optimization problems.

Presenting the partial orders among the space of all bounded and closed intervals in $\mathbb{R}$ is a significant issue in solving interval-valued optimization problems. Under this consideration, Ishibuchi and Tanaka [3], Jiang et al. [4], and Chanas and Kuchta [5] adopted three ordering relations to propose solution concepts for interval-valued optimization problems in which the possibility degree was considered and interval-valued optimization problem was transformed into a bi-objective optimization problem. Costa et al. [6] also presented an ordering relation to claim that the many existing relations presented in the literature are particular cases.

On the other hand, studying the optimality conditions of interval-valued optimization problems is also an interesting topic. The well-known Karush–Kuhn–Tucker optimality conditions for interval-valued optimization problem was studied by Wu [7], where the concept of the Hukuhara derivative of interval-valued functions was proposed. Chalco-Cano et al. [8] considered the generalized Hukuhara derivative of interval-valued functions to study the Karush–Kuhn–Tucker optimality conditions of interval-valued optimization problems. Osuna-Gomez et al. [9] studied the necessary and sufficient optimality conditions for unconstrained interval-valued optimization problems. Jayswal et al. [10] considered the invexity of interval-valued functions to study the Karush–Kuhn–Tucker optimality conditions of interval-valued optimization problems. Recently, Peng et al. [11] studied the optimality conditions for preinvex interval-valued vector optimization problems by considering generalized Hukuhara subdifferentiability.

The general type of interval-valued nonlinear programming problems was studied by Wu [12]. The inexact programming problem is a similar kind of interval-valued optimization problem, which was studied by Soyster [13–15], Pomerol [16], and Thuente [17] and Falk [18]. The solution concepts for interval-valued optimization problems and inexact programming problems are completely different. The inexact programming problem considers the conventional solution concept adopted in the usual optimization problems. However, the solution concept of interval-valued optimization problem follows the similar solution concept for the multiobjective optimization problems. On the other hand, some different methodologies were also proposed to solve interval-valued optimization problems. The concept of a null set was proposed by Wu [19], and the concept of an ordering cone based on an equivalence relation was proposed by Wu [20] to solve interval-valued optimization problems.

In Section 2, we provide an embedding theorem to show that the space of all bounded closed intervals can be embedded into a Banach space. In Section 3, we are going to introduce a partial order among the space of all bounded closed intervals via the embedding theorem. Using this partial order, the solution concepts for the interval-valued optimization problem are proposed in Section 4. Also, an auxiliary vector optimization problem is introduced such that solving the original interval-valued optimization problem is equivalent to solving the auxiliary vector optimization problem. In Section 5, using the technique of scalarization to solve the auxiliary vector optimization problem, the original interval-valued

optimization problem is equivalently solved. In Section 6, the linear type of interval-valued optimization problems is studied by considering three practical ordering cones.

2. Embedding Theorem

We consider the space of all continuous real-valued functions defined on $[0, 1]$. This space is also denote by $C[0, 1]$. Given any $f, g \in C[0, 1]$ and $\lambda \in [0, 1]$, the addition $f + g$ and scalar multiplication λf are usually defined by

$$(f + g)(x) = f(x) + g(x) \text{ and } (\lambda f)(x) = \lambda f(x) \text{ for all } x \in [0, 1].$$

It is clear to see that $C[0, 1]$ is a vector space. Now, we define a norm of $f \in C[0, 1]$ by

$$\|f\| = \sup_{x \in [0, 1]} |f(x)|. \quad (1)$$

Then, the vector space $C[0, 1]$ turns into a normed space $(C[0, 1], \|\cdot\|)$.

Given any sequence $\{f_n\}_{n=1}^{\infty}$ in $C[0, 1]$, we say that $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence when, given any $\epsilon > 0$, there exists an integer N such that

$$m, n > N \text{ implies } \|f_n - f_m\| < \epsilon.$$

For any Cauchy sequence $\{f_n\}_{n=1}^{\infty}$ in $C[0, 1]$, it is not difficult to show that there exists $f \in C[0, 1]$ satisfying

$$\lim_{n \rightarrow \infty} \|f_n - f\| = 0.$$

In other words, sequence $\{f_n\}_{n=1}^{\infty}$ converges to f with respect to norm $\|\cdot\|$. This shows that normed space $(C[0, 1], \|\cdot\|)$ is also a Banach space, which means that every Cauchy sequence in normed space $(C[0, 1], \|\cdot\|)$ is convergent.

Let $\mathcal{I}$ be the space of all bounded closed intervals in $\mathbb{R}$. Given any two bounded closed intervals

$$A = [a^{(l)}, a^{(u)}] \text{ and } B = [b^{(l)}, b^{(u)}],$$

the addition $A + B$ is a bounded closed interval defined by

$$A + B = [a^{(l)} + b^{(l)}, a^{(u)} + b^{(u)}].$$

Given any $\lambda \in \mathbb{R}$, the scalar multiplication is defined as follows,

- For $\lambda \geq 0$, the scalar multiplication λA is a bounded closed interval defined by

$$\lambda A = [\lambda a^{(l)}, \lambda a^{(u)}].$$

- For $\lambda < 0$, the scalar multiplication λA is a bounded closed interval defined by

$$\lambda A = [\lambda a^{(u)}, \lambda a^{(l)}].$$

However, under the above addition and scalar multiplication, the space $\mathcal{I}$ cannot form a vector space. The main reason is that the inverse element does not exist, which can be realized by looking at the subtraction

$$A - A = [a^{(l)}, a^{(u)}] - [a^{(l)}, a^{(u)}] = [a^{(l)} - a^{(u)}, a^{(u)} - a^{(l)}].$$

In other words, the difference $A - A$ is not a zero element, which means that A cannot have an additive inverse element.

However, the space $\mathcal{I}$ of all bounded closed intervals can be a metric space by considering the Hausdorff distance between A and B , defined by

$$d_H(A, B) = \max\left\{\left|a^{(l)} - a^{(u)}\right|, \left|b^{(l)} - b^{(u)}\right|\right\}.$$

Although the metric space $(\mathcal{I}, d_H)$ of bounded closed intervals cannot be a vector space, we can embed it into a Banach space $(C[0, 1], \|\cdot\|)$ as a convex cone isometrically and isomorphically, which will be shown below.

Theorem 1 (Embedding Theorem). *The embedding function*

$$\pi : (\mathcal{I}, d_H) \rightarrow (C[0, 1], \|\cdot\|) \text{ is defined by } [a, b] \mapsto f,$$

where the function $f \in C[0, 1]$ is defined by

$$f(x) = xa + (1 - x)b \text{ for all } x \in [0, 1].$$

Then, we have the following properties.

- (i) The function π is a one-to-one function.
- (ii) We have

$$\pi\left([a^{(l)}, a^{(u)}] + [b^{(l)}, b^{(u)}]\right) = \pi\left([a^{(l)}, a^{(u)}]\right) + \pi\left([b^{(l)}, b^{(u)}]\right)$$

and

$$\pi\left(x[a^{(l)}, a^{(u)}]\right) = x \cdot \pi\left([a^{(l)}, a^{(u)}]\right) \text{ for } x \geq 0.$$

- (iii) We have

$$d_H\left([a^{(l)}, a^{(u)}], [b^{(l)}, b^{(u)}]\right) = \left\|\pi\left([a^{(l)}, a^{(u)}]\right) - \pi\left([b^{(l)}, b^{(u)}]\right)\right\|. \quad (2)$$

This says that the embedding function π and its inverse function $\pi^{-1} : \pi(C[0, 1]) \rightarrow \mathcal{I}$ are continuous. Also, the image $\pi(\mathcal{I})$ is a convex cone in the Banach space $(C[0, 1], \|\cdot\|)$.

In other words, the space $(\mathcal{I}, d_H)$ can be embedded as a convex cone into the Banach space $(C[0, 1], \|\cdot\|)$ isometrically and isomorphically.

Proof. It is clear to see that the function π is a one-to-one function. Now, we have

$$\pi\left([a^{(l)}, a^{(u)}] + [b^{(l)}, b^{(u)}]\right) = \pi\left([a^{(l)} + b^{(l)}, a^{(u)} + b^{(u)}]\right) \equiv g,$$

where

$$\begin{aligned} g(x) &= x(a^{(l)} + b^{(l)}) + (1 - x)(a^{(u)} + b^{(u)}) \\ &= (xa^{(l)} + (1 - x)a^{(u)}) + (xb^{(l)} + (1 - x)b^{(u)}). \end{aligned}$$

Let

$$f_1(x) = xa^{(l)} + (1 - x)a^{(u)} \text{ and } f_2(x) = xb^{(l)} + (1 - x)b^{(u)}. \quad (3)$$

Then, we have

$$\pi\left([a^{(l)}, a^{(u)}] + [b^{(l)}, b^{(u)}]\right) = g = f_1 + f_2 = \pi\left([a^{(l)}, a^{(u)}]\right) + \pi\left([b^{(l)}, b^{(u)}]\right).$$

For $x \geq 0$, we have

$$\pi\left(x\left[a^{(l)}, a^{(u)}\right]\right) = \pi\left(\left[xa^{(l)}, xa^{(u)}\right]\right) \equiv g,$$

where

$$g(x) = xa^{(l)} + (1-x)xa^{(u)} = x\left(xa^{(l)} + (1-x)a^{(u)}\right) = xf(x)$$

with

$$f(x) = xa^{(l)} + (1-x)a^{(u)}.$$

Then, we have

$$\pi\left(x\left[a^{(l)}, a^{(u)}\right]\right) = g = xf = x\pi\left(\left[a^{(l)}, a^{(u)}\right]\right).$$

By referring to (3), we have

$$\pi\left(\left[a^{(l)}, a^{(u)}\right]\right) = f_1 \text{ and } \pi\left(\left[b^{(l)}, b^{(u)}\right]\right) = f_2.$$

Using (1), we obtain

$$\begin{aligned} \left\| \pi\left(\left[a^{(l)}, a^{(u)}\right]\right) - \pi\left(\left[b^{(l)}, b^{(u)}\right]\right) \right\| &= \|f_1 - f_2\| = \sup_{x \in [0,1]} |(f_1 - f_2)(x)| \\ &= \sup_{x \in [0,1]} \left| x\left(a^{(l)} - b^{(l)}\right) + (1-x)\left(a^{(u)} - b^{(u)}\right) \right| \end{aligned}$$

Since the set

$$\left\{ x\left(a^{(l)} - b^{(l)}\right) + (1-x)\left(a^{(u)} - b^{(u)}\right) : 0 \leq x \leq 1 \right\}$$

can be realized as a bounded closed interval joining the endpoints $a^{(l)} - b^{(l)}$ and $a^{(u)} - b^{(u)}$, it is clear to see

$$\begin{aligned} \sup_{x \in [0,1]} \left| x\left(a^{(l)} - b^{(l)}\right) + (1-x)\left(a^{(u)} - b^{(u)}\right) \right| \\ &= \max \left\{ \left| a^{(l)} - b^{(l)} \right|, \left| a^{(u)} - b^{(u)} \right| \right\} \\ &= d_H\left(\left[a^{(l)}, a^{(u)}\right], \left[b^{(l)}, b^{(u)}\right]\right). \end{aligned}$$

Therefore, we obtain (2), which shows that the function π is isometric. It also says that function π and its inverse function $\pi^{-1} : \pi(C[0,1]) \rightarrow \mathcal{I}$ are continuous.

Given any $[a^{(l)}, a^{(u)}], [b^{(l)}, b^{(u)}] \in \mathcal{I}$ and $x \in (0,1)$, we have

$$x\left[a^{(l)}, a^{(u)}\right] + (1-x)\left[b^{(l)}, b^{(u)}\right] = \left[xa^{(l)} + (1-x)b^{(l)}, xa^{(u)} + (1-x)b^{(u)}\right] \in \mathcal{I}.$$

Since

$$x\pi\left(\left[a^{(l)}, a^{(u)}\right]\right) + (1-x)\pi\left(\left[b^{(l)}, b^{(u)}\right]\right) = \pi\left(x\left[a^{(l)}, a^{(u)}\right] + (1-x)\left[b^{(l)}, b^{(u)}\right]\right) \in \pi(\mathcal{I}),$$

it follows that $\pi(\mathcal{I})$ is a convex cone in $C[0,1]$. This completes the proof. $\square$

3. Partial Orders

We are going to apply the Embedding Theorem 1 to define a partial order of $\mathcal{I}$ by referring to the embedding function $\pi : \mathcal{I} \rightarrow C[0,1]$. Given a subset $\mathfrak{C}$ of $C[0,1]$, we consider the following concepts

- We say that $\mathfrak{C}$ is a convex set when, given any $f, g \in \mathfrak{C}$, we have

$$\lambda f + (1-\lambda)g \in \mathfrak{C} \text{ for all } \lambda \in [0,1].$$

- We say that $\mathfrak{C}$ is a cone when

$$\lambda f \in \mathfrak{C} \text{ for all } \lambda \geq 0.$$

- We say that $\mathfrak{C}$ is a convex cone when it is both a convex set and a cone.

Let $\mathfrak{C}$ be a convex cone in $C[0, 1]$. We define a binary relation $\preccurlyeq$ on $C[0, 1]$ as follows. For any $f, g \in C[0, 1]$, we define

$$f \preccurlyeq g \text{ if and only if } g - f \in \mathfrak{C}. \quad (4)$$

It is not difficult to show that the binary relation $\preccurlyeq$ on $C[0, 1]$ is reflexive and transitive, which also means that the binary relation $\preccurlyeq$ is a partial order among $C[0, 1]$.

Using this partial order $\preccurlyeq$ and the embedding function π , we can define a partial order among $\mathcal{I}$ as follows. Given any $A, B \in \mathcal{I}$ with $A \neq B$, we have $\pi(A) \neq \pi(B)$, since the embedding function π is a one-to-one function by referring to Theorem 1 again.

Definition 1. Given any $A, B \in \mathcal{I}$, we define a binary relation $\preceq$ among $\mathcal{I}$ by

$$A \preceq B \text{ if and only if } \pi(A) \preccurlyeq \pi(B).$$

Equivalently, by referring to (4), we see that

$$A \preceq B \text{ if and only if } \pi(B) - \pi(A) \in \mathfrak{C}.$$

Since $\preccurlyeq$ is a partial order among $C[0, 1]$, we can also show that $\preceq$ is a partial order among $\mathcal{I}$.

Remark 1. Since the bounded closed interval $[a^{(l)}, a^{(u)}]$ can be written as

$$[a^{(l)}, a^{(u)}] = \{xa^{(l)} + (1-x)a^{(u)} : 0 \leq x \leq 1\},$$

we have the following observations:

- $xa^{(l)} + (1-x)a^{(u)} \geq 0$ for all $x \in [0, 1]$ if and only if $a^{(l)} \geq 0$.
- $xa^{(l)} + (1-x)a^{(u)} \leq 0$ for all $x \in [0, 1]$ if and only if $a^{(u)} \leq 0$.
- $xa^{(l)} + (1-x)a^{(u)} \leq 0$ for all $x \in [0, 0.5]$ and $xa^{(l)} + (1-x)a^{(u)} \geq 0$ for all $x \in (0.5, 1]$ if and only if $a^{(l)} < 0$ and $a^{(u)} = -a^{(l)}$.
- for some $x_0 \in (0, 1)$ satisfying $xa^{(l)} + (1-x)a^{(u)} \leq 0$ for all $x \in [0, x_0]$ and $xa^{(l)} + (1-x)a^{(u)} \geq 0$ for all $x \in (x_0, 1]$ if and only if $a^{(l)} < 0$ and $a^{(u)} > 0$.

In order to clearly realize the partial order $\preceq$ on $\mathcal{I}$, by referring to Remark 1, we consider some interesting convex cones in $C[0, 1]$ as follows.

3.1. The First Convex Cone

We first consider a subset $\mathfrak{C}_1$ of $C[0, 1]$ given by

$$\mathfrak{C}_1 = \{f \in C[0, 1] : f(x) \geq 0 \text{ for all } 0 \leq x \leq 1\}.$$

For any $f, g \in \mathfrak{C}_1$, it is clear to see

$$\lambda f(x) + (1-\lambda)g(x) \geq 0 \text{ for all } 0 \leq \lambda \leq 1 \text{ and } 0 \leq x \leq 1$$

and

$$\lambda f(x) \geq 0 \text{ for all } \lambda \geq 0 \text{ and } 0 \leq x \leq 1,$$

which say that $\mathfrak{C}_1$ is indeed a convex cone in $C[0, 1]$. Therefore, we can obtain a partial order $\preccurlyeq_1$ using $\mathfrak{C}_1$ by

$$f \preccurlyeq_1 g \text{ if and only if } g - f \in \mathfrak{C}_1.$$

According to the previous discussion, we can define a partial order $\preceq_1$ among $\mathcal{I}$ by

$$A \preceq_1 B \text{ if and only if } \pi(A) \preccurlyeq_1 \pi(B).$$

Let

$$\pi(A) = \pi\left([a^{(l)}, a^{(u)}]\right) = f_1 \text{ and } \pi(B) = \pi\left([b^{(l)}, b^{(u)}]\right) = f_2,$$

where

$$f_1(x) = xa^{(l)} + (1-x)a^{(u)} \text{ and } f_2(x) = xb^{(l)} + (1-x)b^{(u)} \text{ for all } x \in [0, 1].$$

We also have

$$[a^{(l)}, a^{(u)}] \preceq_1 [b^{(l)}, b^{(u)}] \text{ if and only if } f_2 - f_1 \in \mathfrak{C}_1.$$

Equivalently, we have

$$[a^{(l)}, a^{(u)}] \preceq_1 [b^{(l)}, b^{(u)}] \text{ if and only if } f_2(x) - f_1(x) \geq 0 \text{ for all } x \in [0, 1].$$

Since

$$f_2(x) - f_1(x) = x(b^{(l)} - a^{(l)}) + (1-x)(b^{(u)} - a^{(u)}),$$

using the first observation of Remark 1, we have

$$f_2(x) - f_1(x) \geq 0 \text{ for all } x \in [0, 1] \text{ if and only if } b^{(l)} - a^{(l)} \geq 0 \text{ and } b^{(u)} - a^{(u)} \geq 0,$$

which implies

$$[a^{(l)}, a^{(u)}] \preceq_1 [b^{(l)}, b^{(u)}] \text{ if and only if } a^{(l)} \leq b^{(l)} \text{ and } a^{(u)} \leq b^{(u)}. \quad (5)$$

3.2. The Second Convex Cone

We consider a subset $\mathfrak{C}_2$ of $C[0, 1]$ given by

$$\mathfrak{C}_2 = \{f \in C[0, 1] : f(x) \leq 0 \text{ for all } 0 \leq x \leq 1\}.$$

It is not difficult to show that $\mathfrak{C}_2$ is a convex cone in $C[0, 1]$. Therefore, we can define a partial order $\preceq_2$ among $C[0, 1]$ by

$$A \preceq_2 B \text{ if and only if } \pi(A) \preccurlyeq_2 \pi(B).$$

Using the second observation of Remark 1, we can similarly obtain

$$[a^{(l)}, a^{(u)}] \preceq_2 [b^{(l)}, b^{(u)}] \text{ if and only if } a^{(l)} \geq b^{(l)} \text{ and } a^{(u)} \geq b^{(u)}. \quad (6)$$

3.3. The Third Convex Cone

For some $x_0 \in (0, 1)$, we consider a subset $\mathfrak{C}_{x_0}$ of $C[0, 1]$ given by

$$\mathfrak{C}_{x_0} = \{f \in C[0, 1] : f(x) \leq 0 \text{ for all } x \in [0, x_0] \text{ and } f(x) \geq 0 \text{ for all } x \in (x_0, 1]\}.$$

For any $f, g \in \mathfrak{C}_{x_0}$, it is clear to see

$$\lambda f(x) + (1 - \lambda)g(x) \leq 0 \text{ for all } 0 \leq \lambda \leq 1 \text{ and } x \in [0, x_0]$$

and

$$\lambda f(x) + (1 - \lambda)g(x) \geq 0 \text{ for all } 0 \leq \lambda \leq 1 \text{ and } x \in (x_0, 1],$$

which says $\lambda f + (1 - x)g \in \mathfrak{C}_{x_0}$. We also have

$$\lambda f(x) \geq 0 \text{ for all } \lambda \geq 0 \text{ and } x \in [0, x_0]$$

and

$$\lambda f(x) \leq 0 \text{ for all } \lambda \geq 0 \text{ and } x \in (x_0, 1],$$

which says $\lambda f \in \mathfrak{C}_{x_0}$. Therefore, the subset $\mathfrak{C}_{x_0}$ is indeed a convex cone in $C[0, 1]$, which can also induce a partial order $\preceq_3$ among $C[0, 1]$, defined by

$$A \preceq_3 B \text{ if and only if } \pi(A) \preceq_3 \pi(B).$$

Using the fourth observation of Remark 1, we can similarly obtain

$$f_2(x) - f_1(x) \geq 0 \text{ for all } x \in [0, 1] \text{ if and only if } b^{(l)} - a^{(l)} \leq 0 \text{ and } b^{(u)} - a^{(u)} \geq 0,$$

which implies

$$\begin{aligned} [a^{(l)}, a^{(u)}] \preceq_3 [b^{(l)}, b^{(u)}] & \text{ if and only if } a^{(l)} \geq b^{(l)} \text{ and } a^{(u)} \leq b^{(u)} \\ & \text{ if and only if } [a^{(l)}, a^{(u)}] \subseteq [b^{(l)}, b^{(u)}]. \end{aligned} \quad (7)$$

We conclude the usage of cones $\mathfrak{C}_1$, $\mathfrak{C}_2$, and $\mathfrak{C}_{x_0}$ as follows.

- Regarding cone $\mathfrak{C}_1$, from (5), we see that the interval $B = [b^{(l)}, b^{(u)}]$ dominates the interval $A = [a^{(l)}, a^{(u)}]$ if and only if the lower and upper bounds of interval B are simultaneously greater than the lower and upper bounds of interval A . In other words, the preferable interval must simultaneously increase the bounds.
- Regarding cone $\mathfrak{C}_2$, from (6), we see that the interval $B = [b^{(l)}, b^{(u)}]$ dominates the interval $A = [a^{(l)}, a^{(u)}]$ if and only if the lower and upper bounds of interval B are simultaneously less than the lower and upper bounds of interval A . In other words, the preferable interval must simultaneously decrease the bounds.
- Regarding cone $\mathfrak{C}_{x_0}$, from (7), we see that the interval $B = [b^{(l)}, b^{(u)}]$ dominates the interval $A = [a^{(l)}, a^{(u)}]$ if and only if the interval B contains the interval A . In other words, the preferable interval must have wider ranges.

4. Interval-Valued Optimization Problems

From Theorem 1, we see that the embedding function $\pi : \mathcal{I} \rightarrow C[0, 1]$ is an additive and nonnegatively homogeneous function. According to the above discussion, given a convex cone $\mathfrak{C}$ in $C[0, 1]$, we can induce a partial order $\preceq$ among $C[0, 1]$ by defining

$$f \preceq g \text{ if and only if } g - f \in \mathfrak{C}, \quad (8)$$

and a partial order $\preceq$ among $\mathcal{I}$ by defining

$$A \preceq B \text{ if and only if } \pi(A) \preceq \pi(B). \quad (9)$$

Let $\mathcal{I}^\diamond$ be a subset of $\mathcal{I}$. We are going to define the concepts of minimal elements of $\mathcal{I}^\diamond$ and $\pi(\mathcal{I}^\diamond)$ as follows.

- Using the partial order $\preceq$ given in (9), an element $I^* \in \mathcal{I}^\diamond$ is called a *minimal element* of $\mathcal{I}^\diamond$ when $I \preceq I^*$ for $I \in \mathcal{I}^\diamond$ implies $I^* \preceq I$. We denote by $\text{MIN}(\mathcal{I}^\diamond)$ the set of all minimal elements of $\mathcal{I}^\diamond$ based on $\preceq$.
- Using the partial order $\preccurlyeq$ given in (8), an element $f^* \in \pi(\mathcal{I}^\diamond)$ is called a *minimal element* of $\pi(\mathcal{I}^\diamond)$ when $f \preccurlyeq f^*$ for $f \in \pi(\mathcal{I}^\diamond)$ implies $f^* \preccurlyeq f$. We denote by $\text{MIN}(\pi(\mathcal{I}^\diamond))$ the set of all minimal elements of $\pi(\mathcal{I}^\diamond)$ based on $\preccurlyeq$.

We say that the partial order $\preccurlyeq$ induced by $\mathfrak{C}$ is antisymmetric when $f \preccurlyeq g$ and $g \preccurlyeq f$ imply $f = g$. Let θ be the zero element of space $C[0, 1]$, which means

$$\theta(x) = 0 \text{ for all } 0 \leq x \leq 1.$$

Given a convex cone $\mathfrak{C}$ in $C[0, 1]$, we say that $\mathfrak{C}$ is pointed when

$$\mathfrak{C} \cap (-\mathfrak{C}) = \{\theta\}.$$

It is not difficult to show that if the convex cone $\mathfrak{C}$ is pointed then the partial order $\preccurlyeq$ induced by $\mathfrak{C}$ is antisymmetric.

Proposition 1. Suppose that $\mathfrak{C}$ is a pointed convex cone in $C[0, 1]$. Then, we have the following properties.

- f^* is a minimal element of set $\pi(\mathcal{I}^\diamond)$ if and only if $f \preccurlyeq f^*$ for $f \in \pi(\mathcal{I}^\diamond)$ implies $f = f^*$.
- $I^* \in \mathcal{I}^\diamond$ is a minimal element of set $\mathcal{I}^\diamond$ if and only if $I \preceq I^*$ for $I \in \mathcal{I}^\diamond$ implies $I = I^*$.

Proof. Since the convex cone $\mathfrak{C}$ in $C[0, 1]$ is pointed, it means that the partial order $\preccurlyeq$ is antisymmetric. By referring to (9), we also see that the partial order $\preceq$ among $\mathcal{I}$ is antisymmetric.

To prove part (i), suppose that f^* is a minimal element of $\pi(\mathcal{I}^\diamond)$. The definition says $f \preccurlyeq f^*$ for $f \in \pi(\mathcal{I}^\diamond)$ implies $f^* \preccurlyeq f$. Since $\preccurlyeq$ is antisymmetric, it follows $f = f^*$. Part (ii) can be similarly realized by the fact of the antisymmetry of partial order $\preceq$ among $\mathcal{I}$. This completes the proof. $\square$

Proposition 2. I^* is a minimal element of set $\mathcal{I}^\diamond$ if and only if $\pi(I^*)$ is a minimal element of set $(\pi(\mathcal{I}^\diamond))$. In other words, we have that $I^* \in \text{MIN}(\mathcal{I}^\diamond)$ if and only if $\pi(I^*) \in \text{MIN}(\pi(\mathcal{I}^\diamond))$.

Proof. For $I^* \in \text{MIN}(\mathcal{I}^\diamond)$, assume that there exists $f \in \pi(\mathcal{I}^\diamond)$ satisfying $f \preccurlyeq \pi(I^*)$, where $f = \pi(I)$ for some $I \in \mathcal{I}^\diamond$, which also means $\pi(I) \preccurlyeq \pi(I^*)$. Then, we want to claim $\pi(I^*) \preccurlyeq \pi(I) = f$. The definition says that the order $\pi(I) \preccurlyeq \pi(I^*)$ is equivalent to the order $I \preceq I^*$, which also means $I^* \preceq I$ because of $I^* \in \text{MIN}(\mathcal{I}^\diamond)$. Therefore, we obtain $\pi(I^*) \preccurlyeq \pi(I) = f$, which shows that $\pi(I^*)$ is a minimal element in $\pi(\mathcal{I}^\diamond)$.

To prove the converse, for $\pi(I^*) \in \text{MIN}(\pi(\mathcal{I}^\diamond))$, assume that there exists $I \in \mathcal{I}^\diamond$ satisfying $I \preceq I^*$. Then, we want to claim $I^* \preceq I$. The definition says that the order $I \preceq I^*$ is equivalent to the order $\pi(I) \preccurlyeq \pi(I^*)$. Since $\pi(I^*) \in \text{MIN}(\pi(\mathcal{I}^\diamond))$, it follows $\pi(I^*) \preccurlyeq \pi(I)$, i.e., $I^* \preceq I$, which shows that I^* is a minimal element of $\mathcal{I}^\diamond$. This completes the proof. $\square$

Let U be a vector space. The function $F : U \rightarrow \mathcal{I}$ defined on vector space U is called an interval-valued function. Two kinds of interval-valued functions are presented below.

- Let $f^{(l)}$ and $f^{(u)}$ be two real-valued functions defined on U satisfying $f^{(l)}(x) \leq f^{(u)}(x)$ for all $x \in U$. Then, we can consider the interval-valued function F given by

$$F(x) = [f^{(l)}(x), f^{(u)}(x)] \text{ for all } x \in U.$$

- Assume $U = \mathbb{R}^n$. We consider the function with interval-valued coefficients as follows

$$f(x_1, \dots, x_n) = [a_1^{(l)}, a_1^{(u)}]x_1 + \dots + [a_n^{(l)}, a_n^{(u)}]x_n.$$

When we further assume that $x_i \geq 0$ for all $i = 1, \dots, n$, it is clear to see

$$F(x_1, \dots, x_n) = [f^{(l)}(x_1, \dots, x_n), f^{(u)}(x_1, \dots, x_n)] \text{ for all } (x_1, \dots, x_n) \in \mathbb{R}_+^n,$$

where

$$f^{(l)}(x_1, \dots, x_n) = a_1^{(l)}x_1 + \dots + a_n^{(l)}x_n$$

and

$$f^{(u)}(x_1, \dots, x_n) = a_1^{(u)}x_1 + \dots + a_n^{(u)}x_n.$$

Now, we consider the following interval-valued optimization problem

$$\begin{aligned} \text{(IOP)} \quad & \min \quad F(u) \\ & \text{subject to} \quad u \in G, \end{aligned}$$

where G is a subset of U . Let

$$\mathcal{I}^\diamond \equiv F(G) = \{F(u) : u \in G\} \subseteq \mathcal{I}$$

be the set of all objective values of problem (IOP). The solution concept of the interval-valued optimization problem (IOP) is based on the partial order $\preceq$, which is defined below.

Definition 2. We say that $u^* \in G$ is an optimal solution (minimal solution or minimizer) of the problem (IOP) when

$$F(u^*) \in \text{MIN}(F(G)) = \text{MIN}(\mathcal{I}^\diamond);$$

that is, the objective value $F(u^*)$ at u^* is the minimal element of the set of all objective values $F(G) = \mathcal{I}^\diamond$.

In order to solve problem (IOP), we are going to use the embedding function π to formulate an auxiliary vector optimization problem such that it is solvable by the well-known techniques. Recall the embedding function $\pi : \mathcal{I} \rightarrow C[0, 1]$. We can consider the following composite function

$$\pi \circ F : U \rightarrow C[0, 1] \text{ defined by } (\pi \circ F)(u) = \pi(F(u)),$$

which means that $\pi \circ F$ is a vector-valued function defined on U . In other words, each $(\pi \circ F)(u)$ is a vector in the vector space $C[0, 1]$ (as a matter of fact, it is a Banach space with the norm defined in (1)). In this case, we consider the following vector optimization problem

$$\begin{aligned} \text{(VOP)} \quad & \min \quad (\pi \circ F)(u) \\ & \text{subject to} \quad u \in G. \end{aligned}$$

It is clear that the set of all objective values of the vector optimization problem (VOP) is given by

$$(\pi \circ F)(G) = \pi(F(G)) = \pi(\mathcal{I}^\diamond).$$

Then, the solution concept of the vector optimization problem (VOP) is based on the partial order $\preceq$, which is defined below.

Definition 3. Under the above settings, we say that $u^* \in U$ is an optimal solution of the vector optimization problem (VOP) when

$$(\pi \circ F)(u^*) \in \text{MIN}(F(G)) = \text{MIN}(\pi(\mathcal{I}^\diamond));$$

that is, the objective value $(\pi \circ F)(u^*)$ at u^* is the minimal element of the set of all objective values $(\pi \circ F)(G) = \pi(\mathcal{I}^\diamond)$.

Proposition 3. u^* is an optimal solution of the interval-valued optimization problem (IOP) if and only if u^* is an optimal solution of the vector optimization problem (VOP).

Proof. We first note that the feasible sets of the problems (IOP) and (VOP) are identical to G . We assume $I^* = F(u^*)$. Proposition 2 says that

$$F(u^*) = I^* \in \text{MIN}(\mathcal{I}^\diamond) \text{ if and only if } (\pi \circ F)(u^*) = \pi(I^*) \in \text{MIN}(\pi(\mathcal{I}^\diamond)).$$

The definition of optimal solutions for problems (IOP) and (VOP) says that u^* is an optimal solution of problem (IOP) if and only if u^* is an optimal solution of problem (VOP). This completes the proof. $\square$

In order to solve the original interval-valued optimization problem (IOP), Proposition 3 says that it suffices to solve the vector optimization problem (VOP). In the sequel, we can apply the well-known techniques for the vector optimization problem to solve problem (VOP). The scalarization technique is frequently used for solving the vector optimization problem, which will be adopted in this paper.

5. Scalarization

Recall that the embedding function $\pi : \mathcal{I} \rightarrow C[0, 1]$ is additive and nonnegatively homogeneous from $\mathcal{I}$ into the Banach space $C[0, 1]$. We say that the real-valued function $\Gamma : C[0, 1] \rightarrow \mathbb{R}$ is a linear functional on $C[0, 1]$ when Γ is a linear function on $C[0, 1]$. More precisely, the real-valued function Γ satisfies

$$\Gamma(f + g) = \Gamma(f) + \Gamma(g) \text{ and } \Gamma(\lambda f) = \lambda \Gamma(f)$$

for any $f, g \in C[0, 1]$ and $\lambda \in \mathbb{R}$. The collection of all linear functionals on $C[0, 1]$ is denoted by $C'[0, 1]$.

Example 1. We present some practical linear functionals as follows.

- Given any $f \in C[0, 1]$, we can take the functional $\Gamma : C[0, 1] \rightarrow \mathbb{R}$ defined by

$$\Gamma(f) = \int_0^1 f(x) dx.$$

Then, it is clear to see that the functional Γ is linear.

- Given any $f \in C[0, 1]$, the functional $\Gamma_\lambda : C[0, 1] \rightarrow \mathbb{R}$ depending on parameter $\lambda \in [0, 1]$ is defined by

$$\Gamma_\lambda(f) = \lambda f(0) + (1 - \lambda)f(1).$$

Then, it is clear to see that each functional Γ_λ is linear for $\lambda \in [0, 1]$.

- Given any $f \in C[0, 1]$, the functional $\Gamma_\lambda : C[0, 1] \rightarrow \mathbb{R}$ depending on parameter $\lambda \in [0, 1]$ is defined by

$$\Gamma_\lambda(f) = f(\lambda).$$

Then, it is clear to see that each functional Γ_λ is linear for $\lambda \in [0, 1]$.

Let $\mathfrak{C}$ be a convex cone in $C[0, 1]$. The dual cone of $\mathfrak{C}$ is denoted and defined by

$$\mathfrak{C}' = \{\Gamma \in C'[0, 1] : \Gamma(f) \geq 0 \text{ for all } f \in \mathfrak{C}\}.$$

The scalarization technique will be based on the concept of a dual cone. Recall that $\mathfrak{C}$ is a pointed convex cone if and only if $\mathfrak{C} \cap -\mathfrak{C} = \{\theta\}$, where θ is the zero element of $C[0, 1]$. The partial order $\preceq$ induced by the pointed convex cone $\mathfrak{C}$ is antisymmetric. Now, we are in a position to present the scalarization.

Theorem 2. *Let $\mathfrak{C}$ be a pointed convex cone in the Banach space $C[0, 1]$. Suppose that there exist a linear functional Γ in the dual cone $\mathfrak{C}'$ and an element u^* in G satisfying the following strict inequality*

$$\Gamma((\pi \circ F)(u^*)) < \Gamma((\pi \circ F)(u)) \text{ for all } u \in G \text{ with } u \neq u^*. \quad (10)$$

Then u^* is an optimal solution of the interval-valued optimization problem (IOP).

Proof. We first note that the partial order $\preceq$ induced by the pointed convex cone $\mathfrak{C}$ is antisymmetric; that is, $f \preceq g$ and $g \preceq f$ imply $f = g$. Assume that u^* is not an optimal solution of the vector optimization problem (VOP). We are going to claim there is contradiction. According to the definition of the optimal solution of the vector optimization problem (VOP), since u^* is assumed to be not an optimal solution of problem (VOP), we must have $(\pi \circ F)(u^*) \notin \text{MIN}(\pi(\mathcal{I}^\diamond))$. In other words, the objective value $f^* \equiv (\pi \circ F)(u^*)$ at u^* cannot be a minimal element of the set of all objective values $\pi(\mathcal{I}^\diamond)$ with respect to the partial order $\preceq$. Using the contrapositive statement of part (i) of Proposition 1, there exists $u \in G$ with $f \equiv (\pi \circ F)(u)$ satisfying

$$(\pi \circ F)(u) = f \neq f^* = (\pi \circ F)(u^*) \text{ and } (\pi \circ F)(u) = f \preceq f^* = (\pi \circ F)(u^*),$$

which also says

$$u \neq u^* \text{ and } (\pi \circ F)(u^*) - (\pi \circ F)(u) = f^* - f \in \mathfrak{C}.$$

Since Γ is in the dual cone $\mathfrak{C}'$ and is linear, it follows that

$$\Gamma((\pi \circ F)(u^*)) - \Gamma((\pi \circ F)(u)) = \Gamma((\pi \circ F)(u^*) - (\pi \circ F)(u)) = \Gamma(f^* - f) \geq 0,$$

which contradicts the strict inequality (10). This contradiction shows that u^* must be an optimal solution of the vector optimization problem (VOP). Finally, using Proposition 3, we conclude that u^* is also an optimal solution of the interval-valued optimization problem (IOP). This completes the proof. $\square$

It is clear to see that the convex cones $\mathfrak{C}_1$, $\mathfrak{C}_2$, and $\mathfrak{C}_{x_0}$ presented above are all pointed convex cones. Therefore, Theorem 2 is applicable when the pointed convex cones $\mathfrak{C}_1$, $\mathfrak{C}_2$, and $\mathfrak{C}_{x_0}$ are taken into account.

Given a linear functional Γ in the dual cone $\mathfrak{C}'$, inspired by Theorem 2, we consider a real-valued function

$$\psi : U \rightarrow \mathbb{R} \text{ defined by } \psi(u) = \Gamma((\pi \circ F)(u)).$$

In this case, a scalar optimization problem can be formulated as follows

$$\begin{aligned} \text{(SOP)} \quad & \min \quad \psi(u) \\ & \text{subject to} \quad u \in G. \end{aligned}$$

Then, we have the following result.

Theorem 3. Let $\mathfrak{C}$ be a pointed convex cone in $C[0, 1]$, and let u^* be a unique optimal solution of the scalar optimization problem (SOP). Then u^* is an optimal solution of the interval-valued optimization problem (IOP).

Proof. Since u^* is a unique optimal solution of problem (SOP), the uniqueness says $\psi(u^*) < \psi(u)$ for all $u \in G$ with $u \neq u^*$, which also means

$$\Gamma((\pi \circ F)(u^*)) < \Gamma((\pi \circ F)(u)) \text{ for all } u \in G \text{ with } u \neq u^*$$

Using Theorem 2, it follows that u^* is an optimal solution of the interval-valued optimization problem (IOP). This completes the proof. $\square$

Without considering the uniqueness of the optimal solution in Theorem 3, we introduce the concept of the quasi-interior. Let $\mathfrak{C}$ be a convex cone in $C[0, 1]$. The quasi-interior of the dual cone of $\mathfrak{C}$ is defined by

$$\mathfrak{C}^\circ = \{\Gamma^\circ \in C'[0, 1] : \Gamma^\circ(f) > 0 \text{ for all } f \in \mathfrak{C} \text{ with } f \neq \theta\},$$

where θ is the zero element of $C[0, 1]$. The different types of scalarization are presented below.

Theorem 4. Let $\mathfrak{C}$ be a pointed convex cone in $C[0, 1]$. Suppose that there exist a linear functional Γ° in the quasi-interior $\mathfrak{C}^\circ$ and an element $u^* \in G$ satisfying

$$\Gamma^\circ((\pi \circ F)(u^*)) \leq \Gamma^\circ((\pi \circ F)(u)) \text{ for all } u \in G. \quad (11)$$

Then u^* is an optimal solution of the interval-valued optimization problem (IOP).

Proof. Assume that u^* is not an optimal solution of the vector optimization problem (VOP). This is going to lead to a contradiction. The proof of Theorem 2 has shown that there exists $u \in G$ satisfying

$$(\pi \circ F)(u) \neq (\pi \circ F)(u^*) \text{ and } (\pi \circ F)(u) \preceq (\pi \circ F)(u^*).$$

Therefore, we have

$$\theta \neq (\pi \circ F)(u^*) - (\pi \circ F)(u) \in \mathfrak{C}.$$

Since Γ° is in $\mathfrak{C}^\circ$, the definition of the quasi-interior says

$$\Gamma^\circ((\pi \circ F)(u^*) - (\pi \circ F)(u)) > 0.$$

Since Γ° is also linear, it follows that

$$\Gamma^\circ((\pi \circ F)(u^*)) - \Gamma^\circ((\pi \circ F)(u)) = \Gamma^\circ((\pi \circ F)(u^*) - (\pi \circ F)(u)) > 0,$$

which contradicts the inequality (11). This contradiction says that u^* must be an optimal solution of the vector optimization problem (VOP). Using Proposition 3, we also see that u^*

is an optimal solution of the interval-valued optimization problem (IOP). This completes the proof. $\square$

Given a linear functional Γ° in the quasi-interior $\mathfrak{C}^\circ$, inspired by Theorem 4, we consider a real-valued function,

$$\psi^\circ : U \rightarrow \mathbb{R} \text{ defined by } \psi^\circ(u) = \Gamma^\circ((\pi \circ F)(u)).$$

In this case, a scalar optimization problem can be formulated as follows

$$\begin{aligned} (\text{SOP}^\circ) \quad & \min \quad \psi^\circ(u) \\ & \text{subject to } u \in G. \end{aligned}$$

Then, we have the following results.

Theorem 5. *Let $\mathfrak{C}$ be a pointed convex cone in $C[0, 1]$, and let u^* be an optimal solution of scalar optimization problem (SOP°) . Then u^* is an optimal solution of problem (IOP).*

Proof. Since u^* is an optimal solution of problem (SOP°) , we have $\psi^\circ(u^*) \leq \psi^\circ(u)$ for all $u \in G$, which also means

$$\Gamma^\circ((\pi \circ F)(u^*)) \leq \Gamma^\circ((\pi \circ F)(u)) \text{ for all } u \in G.$$

Using Theorem 4, it follows that u^* is an optimal solution of the interval-valued optimization problem (IOP). This completes the proof. $\square$

Given a pointed convex cone $\mathfrak{C}$ in the Banach space $C[0, 1]$, Theorems 2 and 4 consider the different linear functionals Γ and Γ° , respectively, which can also be realized from the following example.

Example 2. *Let us take the pointed convex cone*

$$\mathfrak{C} = \{f \in C[0, 1] : f(x) \geq 0 \text{ for all } 0 \leq x \leq 1\}.$$

Continuing from Example 1, the functional $\Gamma^\circ : C[0, 1] \rightarrow \mathbb{R}$ defined by

$$\Gamma^\circ(f) = \int_0^1 f(x) dx \text{ for } f \in C[0, 1].$$

is clearly linear with $\Gamma^\circ \in \mathfrak{C}'$. Since f is a nonnegative continuous function on $[0, 1]$, we see that $f \neq \theta$ implies $\Gamma_1^\circ(f) \neq 0$. Therefore, we also have $\Gamma^\circ \in \mathfrak{C}^\circ$.

Now, we take the functionals $\Gamma_\lambda^\circ : C[0, 1] \rightarrow \mathbb{R}$ defined by

$$\Gamma_\lambda^\circ(f) = \lambda f(0) + (1 - \lambda)f(1).$$

It is clear to see $\Gamma^\circ \in \mathfrak{C}'$. For a non-zero function f , i.e., $f \neq 0$, it is possible that $f(0) = f(1) = 0$. In this case, we have $\Gamma_\lambda^\circ(f) = 0$, which means $\Gamma_\lambda^\circ \notin \mathfrak{C}^\circ$. The same situation can also apply to the functional Γ_λ defined by $\Gamma_\lambda(f) = f(\lambda)$.

6. Linear Interval-Valued Optimization Problems

Given any fixed $A_i \in \mathcal{I}$ for $i = 1, \dots, n$, we consider the following linear interval-valued optimization problem

$$\begin{aligned} \text{(LIVOP)} \quad & \min \quad F(u_1, \dots, u_n) = A_1 u_1 + A_2 u_2 + \dots + A_n u_n \\ & \text{subject to} \quad (u_1, \dots, u_n) \in G \subset \mathbb{R}^n \text{ and } u_i \geq 0 \text{ for all } i = 1, \dots, n. \end{aligned}$$

where G is a feasible set consisting of real-valued linear constraints, or a feasible set consisting of interval-valued linear constraints which are ordered by the partial order $\preceq$ among $\mathcal{I}$. The objective function F is an interval-valued function such that its coefficients are taken to be the bounded closed intervals in $\mathcal{I}$.

We write

$$A_i = [a_i^{(l)}, a_i^{(u)}] \text{ for } i = 1, \dots, n.$$

Therefore, the interval-valued function F can also be written as

$$F(u_1, \dots, u_n) = [F^{(l)}(u_1, \dots, u_n), F^{(u)}(u_1, \dots, u_n)], \quad (12)$$

where $F^{(l)}$ and $F^{(u)}$ are real-valued functions with the same domain of interval-valued function F . Since each u_i is nonnegative for $i = 1, \dots, n$, it is clear to see

$$F^{(l)}(u_1, \dots, u_n) = a_1^{(l)} u_1 + \dots + a_n^{(l)} u_n \quad (13)$$

and

$$F^{(u)}(u_1, \dots, u_n) = a_1^{(u)} u_1 + \dots + a_n^{(u)} u_n. \quad (14)$$

Recall the embedding function $\pi : \mathcal{I} \rightarrow C[0, 1]$. We consider the composite function $\pi \circ F$ given by

$$(\pi \circ F)(u_1, \dots, u_n) \in C[0, 1].$$

Given a pointed convex cone $\mathfrak{C}$ in $C[0, 1]$, in order to apply Theorem 5 to find the optimal solutions of problem (LIVOP), we need to take some suitable functionals, $\Gamma^\circ : C[0, 1] \rightarrow \mathbb{R}$, satisfying $\Gamma^\circ \in \mathfrak{C}^\circ$. In this paper, we consider the pointed convex cones $\mathfrak{C}_1$, $\mathfrak{C}_2$ and $\mathfrak{C}_{x_0}$ presented above.

Example 3. Recall that the pointed convex cone $\mathfrak{C}_1$ is given by

$$\mathfrak{C}_1 = \{f \in C[0, 1] : f(x) \geq 0 \text{ for all } 0 \leq x \leq 1\}.$$

In this case, the functional $\Gamma_1^\circ : C[0, 1] \rightarrow \mathbb{R}$ is defined by

$$\Gamma_1^\circ(f) = \int_0^1 f(x) dx \text{ for } f \in C[0, 1].$$

It is clear to see that Γ_1° is linear. Since f is a nonnegative continuous function on $[0, 1]$, we see that $f \neq \theta$ implies $\Gamma_1^\circ(f) \neq 0$. Therefore, we have $\Gamma_1^\circ \in \mathfrak{C}_1^\circ$.

Given any $A = [a^{(l)}, a^{(u)}] \in \mathcal{I}$, we have $\pi(A) \equiv f \in C[0, 1]$ given by

$$f(x) = xa^{(l)} + (1-x)a^{(u)} \text{ for } x \in [0, 1]. \quad (15)$$

Therefore, we obtain

$$\Gamma_1^\circ(\pi(A)) = \Gamma_1^\circ(f) = \int_0^1 xa^{(l)} dx + \int_0^1 (1-x)a^{(u)} dx = \frac{1}{2}(a^{(l)} + a^{(u)}) \quad (16)$$

By referring to (12), we also obtain

$$\begin{aligned}\Gamma_1^\circ((\pi \circ F)(u_1, \dots, u_n)) &= \frac{1}{2} \left(F^{(l)}(u_1, \dots, u_n) + F^{(u)}(u_1, \dots, u_n) \right) \text{ (using (16))} \\ &= \frac{1}{2} \left[(a_1^{(l)} + a_1^{(u)})u_1 + \dots + (a_n^{(l)} + a_n^{(u)})u_n \right] \text{ (using (13) and (14)).}\end{aligned}$$

Let

$$\xi_i = \frac{1}{2} (a_i^{(l)} + a_i^{(u)}) \text{ for } i = 1, \dots, n. \quad (17)$$

Then, we also have

$$\Gamma_1^\circ((\pi \circ F)(u_1, \dots, u_n)) = \xi_1 u_1 + \dots + \xi_n u_n.$$

Now, we consider the following linear programming problem

$$\begin{aligned}(\text{LP}^\circ) \quad & \min \quad \Gamma_1^\circ((\pi \circ F)(u_1, \dots, u_n)) = \xi_1 u_1 + \dots + \xi_n u_n \\ & \text{subject to} \quad (u_1, \dots, u_n) \in G \subset \mathbb{R}^n \text{ and } u_i \geq 0 \text{ for all } i = 1, \dots, n.\end{aligned}$$

Theorem 5 says that if $(u_1^*, \dots, u_n^*)$ is an optimal solution of the linear programming problem (LP°) , then it is also an optimal solution of the original linear interval-valued optimization problem (LIVOP).

Now, we take the following linear functional

$$\Gamma_\lambda(f) = \lambda f(0) + (1 - \lambda)f(1).$$

Then, we have $\Gamma_\lambda \in \mathfrak{C}'_1$. From (15), we obtain

$$\Gamma_\lambda(\pi(A)) = \Gamma_\lambda(f) = \lambda a^{(u)} + (1 - \lambda)a^{(l)}.$$

By referring to (12), we also obtain

$$\begin{aligned}\Gamma_\lambda((\pi \circ F)(u_1, \dots, u_n)) &= \lambda F^{(u)}(u_1, \dots, u_n) + (1 - \lambda)F^{(l)}(u_1, \dots, u_n) \text{ (using (16))} \\ &= \left(\lambda a_1^{(u)} + (1 - \lambda)a_1^{(l)} \right) u_1 + \dots + \left(\lambda a_n^{(u)} + (1 - \lambda)a_n^{(l)} \right) u_n \text{ (using (13) and (14)).}\end{aligned}$$

Let

$$\xi_i^{(\lambda)} = \lambda a_i^{(u)} + (1 - \lambda)a_i^{(l)} \text{ for } i = 1, \dots, n. \quad (18)$$

Then, we also have

$$\Gamma_\lambda((\pi \circ F)(u_1, \dots, u_n)) = \xi_1^{(\lambda)} u_1 + \dots + \xi_n^{(\lambda)} u_n.$$

Now, given a fixed $\lambda \in [0, 1]$, we consider the following linear programming problem

$$\begin{aligned}(\text{LP}_\lambda) \quad & \min \quad \Gamma_\lambda((\pi \circ F)(u_1, \dots, u_n)) = \xi_1^{(\lambda)} u_1 + \dots + \xi_n^{(\lambda)} u_n \\ & \text{subject to} \quad (u_1, \dots, u_n) \in G \subset \mathbb{R}^n \text{ and } u_i \geq 0 \text{ for all } i = 1, \dots, n.\end{aligned}$$

Theorem 3 says that if $(u_1^{(\lambda*)}, \dots, u_n^{(\lambda*)})$ is a unique optimal solution of the linear programming problem (LP_λ) , then it is also an optimal solution of the original linear interval-valued optimization problem (LIVOP).

We present a numerical example.

Example 4. Continuing from Example 3, we consider the following linear interval-valued optimization problem

$$\begin{aligned} \min \quad & F(u_1, u_2) = [2, 10]u_1 \oplus [1, 21]u_2 \\ \text{subject to} \quad & 2u_1 + 6u_2 - 27 \leq 0 \\ & 8u_1 + 6u_2 - 45 \leq 0 \\ & 3u_1 + u_2 - 15 \leq 0 \\ & u_1, u_2 \geq 0, \end{aligned}$$

where

$$A_1 = [a_1^{(l)}, a_1^{(u)}] = [2, 10] \text{ and } A_2 = [a_2^{(l)}, a_2^{(u)}] = [1, 21].$$

Using (17), we obtain

$$\xi_1 = \frac{1}{2}(a_1^{(l)} + a_1^{(u)}) = 12 \text{ and } \xi_2 = \frac{1}{2}(a_2^{(l)} + a_2^{(u)}) = 22.$$

Therefore, we need to solve the following linear programming problem

$$\begin{aligned} \min \quad & \psi^\circ(u_1, u_2) = 12u_1 + 22u_2 \\ \text{subject to} \quad & 2u_1 + 6u_2 - 27 \leq 0 \\ & 8u_1 + 6u_2 - 45 \leq 0 \\ & 3u_1 + u_2 - 15 \leq 0 \\ & u_1, u_2 \geq 0, \end{aligned}$$

Using the simplex method, we obtain that $(u_1^*, u_2^*) = (3, 3.5)$ is an optimal solution, which is also an optimal solution of the original linear interval-valued optimization by Theorem 5.

Using (18), we obtain

$$\xi_1^{(\lambda)} = \lambda a_1^{(u)} + (1 - \lambda)a_1^{(l)} = 8\lambda + 2 \text{ and } \xi_2^{(\lambda)} = \lambda a_2^{(u)} + (1 - \lambda)a_2^{(l)} = 20\lambda + 1.$$

Therefore, given any fixed $\lambda \in [0, 1]$, we need to solve the following linear programming problem

$$\begin{aligned} \min \quad & \psi^\circ(u_1, u_2) = (8\lambda + 2)u_1 + (20\lambda + 1)u_2 \\ \text{subject to} \quad & 2u_1 + 6u_2 - 27 \leq 0 \\ & 8u_1 + 6u_2 - 45 \leq 0 \\ & 3u_1 + u_2 - 15 \leq 0 \\ & u_1, u_2 \geq 0, \end{aligned}$$

If $(u_1^{(\lambda*)}, u_2^{(\lambda*)})$ is a unique optimal solution, then Theorem 3 says that it is also an optimal solution of the original linear interval-valued optimization. Since the coefficients of the objective functions are all positive and the optimal solution is one of the vertices of the feasible region, we believe that $(u_1^{(\lambda*)}, u_2^{(\lambda*)}) = (3, 3.5)$ for most of values of parameter $\lambda \in [0, 1]$, where the details are left for the readers.

Example 5. Recall that the pointed convex cone $\mathfrak{C}_2$ is given by

$$\mathfrak{C}_2 = \{f \in C[0, 1] : f(x) \leq 0 \text{ for all } 0 \leq x \leq 1\}.$$

In this case, the functional $\Gamma_2^\circ : C[0, 1] \rightarrow \mathbb{R}$ is defined by

$$\Gamma_2^\circ(f) = - \int_0^1 f(x)dx \text{ for } f \in C[0, 1].$$

It is clear to see that Γ_2° is linear, satisfying $\Gamma_2^\circ(f) > 0$ for $f \in \mathfrak{C}_2$ with $f \neq \theta$. Therefore, we have $\Gamma_2^\circ \in \mathfrak{C}_2^\circ$.

Given any $A \in \mathcal{I}$, by referring to (16), we similarly have

$$\Gamma_2^\circ(\pi(A)) = -\frac{1}{2}(a^{(l)} + a^{(u)}).$$

Therefore, we obtain

$$\Gamma_2^\circ((\pi \circ F)(u_1, \dots, u_n)) = -\frac{1}{2}[(a_1^{(l)} + a_1^{(u)})u_1 + \dots + (a_n^{(l)} + a_n^{(u)})u_n].$$

Let

$$\xi_i = -\frac{1}{2}(a_i^{(l)} + a_i^{(u)}) \text{ for } i = 1, \dots, n. \quad (19)$$

Then, we also have

$$\Gamma_2^\circ((\pi \circ F)(u_1, \dots, u_n)) = \xi_1 u_1 + \dots + \xi_n u_n.$$

Theorem 5 says that if $(u_1^*, \dots, u_n^*)$ is an optimal solution of the linear programming problem (LP°) , then it is also an optimal solution of the original linear interval-valued optimization problem (LIVOP).

We present a numerical example.

Example 6. Continuing from Example 5, we consider the following linear interval-valued optimization problem

$$\begin{aligned} \min \quad & F(u_1, u_2) = [2, 10]u_1 \oplus [1, 21]u_2 \\ \text{subject to} \quad & 2u_1 + 6u_2 - 27 \leq 0 \\ & 8u_1 + 6u_2 - 45 \leq 0 \\ & 3u_1 + u_2 - 15 \leq 0 \\ & u_1, u_2 \geq 0, \end{aligned}$$

where

$$A_1 = [a_1^{(l)}, a_1^{(u)}] = [2, 10] \text{ and } A_2 = [a_2^{(l)}, a_2^{(u)}] = [1, 21].$$

Using (19), we obtain

$$\xi_1 = -\frac{1}{2}(a_1^{(l)} + a_1^{(u)}) = -12 \text{ and } \xi_2 = -\frac{1}{2}(a_2^{(l)} + a_2^{(u)}) = -22.$$

Therefore, we need to solve the following linear programming problem

$$\begin{aligned} \min \quad & \psi^\circ(u_1, u_2) = -12u_1 - 22u_2 \\ \text{subject to} \quad & 2u_1 + 6u_2 - 27 \leq 0 \\ & 8u_1 + 6u_2 - 45 \leq 0 \\ & 3u_1 + u_2 - 15 \leq 0 \\ & u_1, u_2 \geq 0, \end{aligned}$$

Using the simplex method, we obtain that $(u_1^*, u_2^*) = (3, 3.5)$ is an optimal solution, which is also an optimal solution of the original linear interval-valued optimization by Theorem 5.

Example 7. Recall that the pointed convex cone $\mathfrak{C}_{x_0}$ is given by

$$\mathfrak{C}_{x_0} = \{f \in C[0, 1] : f(x) \leq 0 \text{ for all } x \in [0, x_0] \text{ and } f(x) \geq 0 \text{ for all } x \in (x_0, 1]\}.$$

In this case, the functional $\Gamma_3^\circ : C[0, 1] \rightarrow \mathbb{R}$ is defined by

$$\Gamma_3^\circ(f) = - \int_0^{x_0} f(x)dx + \int_{x_0}^1 f(x)dx \text{ for } f \in C[0, 1].$$

It is clear to see that Γ_3° is linear, satisfying $\Gamma_3^\circ(f) > 0$ for $f \in \mathfrak{C}_{x_0}$ with $f \neq \theta$. Therefore, we have $\Gamma_3^\circ \in \mathfrak{C}_{x_0}^\circ$.

Given any $A \in \mathcal{I}$, by referring to (16), we have

$$\begin{aligned} \Gamma_3^\circ(\pi(A)) &= \Gamma_1^\circ(f) \\ &= - \int_0^{x_0} [xa^{(l)}dx + (1-x)a^{(u)}]dx + \int_{x_0}^1 [xa^{(l)}dx + (1-x)a^{(u)}]dx \\ &= \frac{1}{2}(a^{(l)} + a^{(u)}) - 2x_0a^{(u)} - x_0^2(a^{(l)} - a^{(u)}) \end{aligned} \quad (20)$$

Therefore, we obtain

$$\begin{aligned} \Gamma_3^\circ((\pi \circ F)(u_1, \dots, u_n)) &= \frac{1}{2}(F^{(l)}(u_1, \dots, u_n) + F^{(u)}(u_1, \dots, u_n)) \\ &\quad - 2x_0F^{(u)}(u_1, \dots, u_n) - x_0^2(F^{(l)}(u_1, \dots, u_n) - F^{(u)}(u_1, \dots, u_n)) \\ &= \xi_1u_1 + \dots + \xi_nu_n, \end{aligned}$$

where

$$\xi_i = \frac{1}{2}(a_i^{(l)} + a_i^{(u)}) - 2x_0a_i^{(u)} - x_0^2(a_i^{(l)} - a_i^{(u)}) \text{ for } i = 1, \dots, n. \quad (21)$$

Then, we also have

$$\Gamma_3^\circ((\pi \circ F)(u_1, \dots, u_n)) = \xi_1u_1 + \dots + \xi_nu_n.$$

Theorem 5 says that if $(u_1^*, \dots, u_n^*)$ is an optimal solution of the linear programming problem (LP°) , then it is also an optimal solution of the original linear interval-valued optimization problem (LIVOP).

Now, we take the following linear functional

$$\Gamma_\lambda(f) = -\lambda f(0) + (1-\lambda)f(1).$$

For $f \in \mathfrak{C}_{x_0}$, we have $f(0) \leq 0$ and $f(1) \geq 0$, which says $\Gamma_\lambda(f) \geq 0$. Therefore, we have $\Gamma_\lambda \in \mathfrak{C}_{x_0}'$. From (15), we obtain

$$\Gamma_\lambda(\pi(A)) = \Gamma_\lambda(f) = \lambda a^{(u)} + (1-\lambda)a^{(l)}.$$

By referring to (12), we also obtain

$$\begin{aligned} \Gamma_\lambda((\pi \circ F)(u_1, \dots, u_n)) &= -\lambda F^{(u)}(u_1, \dots, u_n) + (1-\lambda)F^{(l)}(u_1, \dots, u_n) \text{ (using (16))} \\ &= (-\lambda a_1^{(u)} + (1-\lambda)a_1^{(l)})u_1 + \dots + (-\lambda a_n^{(u)} + (1-\lambda)a_n^{(l)})u_n \text{ (using (13) and (14)).} \end{aligned}$$

Let

$$\xi_i^{(\lambda)} = -\lambda a_i^{(u)} + (1-\lambda)a_i^{(l)} \text{ for } i = 1, \dots, n. \quad (22)$$

Then, we also have

$$\Gamma_\lambda((\pi \circ F)(u_1, \dots, u_n)) = \xi_1^{(\lambda)}u_1 + \dots + \xi_n^{(\lambda)}u_n.$$

Now, given a fixed $\lambda \in [0, 1]$, we consider the following linear programming problem

$$(LP_\lambda) \quad \min \quad \Gamma_\lambda((\pi \circ F)(u_1, \dots, u_n)) = \xi_1^{(\lambda)} u_1 + \dots + \xi_n^{(\lambda)} u_n \\ \text{subject to} \quad (u_1, \dots, u_n) \in G \subset \mathbb{R}^n \text{ and } u_i \geq 0 \text{ for all } i = 1, \dots, n.$$

Theorem 3 says that if $(u_1^{(\lambda^*)}, \dots, u_n^{(\lambda^*)})$ is a unique optimal solution of the linear programming problem (LP_λ) , then it is also an optimal solution of the original linear interval-valued optimization problem (LIVOP).

We present a numerical example.

Example 8. Continuing from Example 7 by taking $x_0 = 0.5$, we consider the following linear interval-valued optimization problem

$$\min \quad F(u_1, u_2) = [2, 10]u_1 \oplus [1, 21]u_2 \\ \text{subject to} \quad 2u_1 + 6u_2 - 27 \leq 0 \\ 8u_1 + 6u_2 - 45 \leq 0 \\ 3u_1 + u_2 - 15 \leq 0 \\ u_1, u_2 \geq 0,$$

where

$$A_1 = [a_1^{(l)}, a_1^{(u)}] = [2, 10] \text{ and } A_2 = [a_2^{(l)}, a_2^{(u)}] = [1, 21].$$

Since $x_0 = 0.5$, using (21), we obtain

$$\xi_1 = \frac{1}{2} (a_1^{(l)} + a_1^{(u)}) - 2x_0 a_1^{(u)} - x_0^2 (a_1^{(l)} - a_1^{(u)}) = -2$$

and

$$\xi_2 = \frac{1}{2} (a_2^{(l)} + a_2^{(u)}) - 2x_0 a_2^{(u)} - x_0^2 (a_2^{(l)} - a_2^{(u)}) = -5.$$

Therefore, we need to solve the following linear programming problem

$$\min \quad \psi^\circ(u_1, u_2) = -2u_1 - 5u_2 \\ \text{subject to} \quad 2u_1 + 6u_2 - 27 \leq 0 \\ 8u_1 + 6u_2 - 45 \leq 0 \\ 3u_1 + u_2 - 15 \leq 0 \\ u_1, u_2 \geq 0,$$

Using the simplex method, we obtain that $(u_1^*, u_2^*) = (3, 3.5)$ is an optimal solution, which is also an optimal solution of the original linear interval-valued optimization by Theorem 5.

Using (22), we obtain

$$\xi_1^{(\lambda)} = -\lambda a_1^{(u)} + (1 - \lambda) a_1^{(l)} = -12\lambda + 2 \text{ and } \xi_2^{(\lambda)} = -\lambda a_2^{(u)} + (1 - \lambda) a_2^{(l)} = -22\lambda + 1.$$

Therefore, given any fixed $\lambda \in [0, 1]$, we need to solve the following linear programming problem

$$\min \quad \psi^\circ(u_1, u_2) = (-12\lambda + 2)u_1 + (-22\lambda + 1)u_2 \\ \text{subject to} \quad 2u_1 + 6u_2 - 27 \leq 0 \\ 8u_1 + 6u_2 - 45 \leq 0 \\ 3u_1 + u_2 - 15 \leq 0 \\ u_1, u_2 \geq 0,$$

If $(u_1^{(\lambda^*)}, u_2^{(\lambda^*)})$ is a unique optimal solution, then Theorem 3 says that it is also an optimal solution of the original, which depends on the values of parameter $\lambda \in [0, 1]$.

7. Conclusions

The advantage of following the interval approach requires determining the lower and upper bounds, which is unlikely for the fuzzy or stochastic approaches that need to hardly determine the membership functions or probability distributions. This adoption makes the problems that involve uncertainty simpler to be formulated and solved than that of considering the fuzziness and randomness.

This paper proposes a new methodology to solve the interval-valued optimization problem by proving an embedding theorem. The space $\mathcal{I}$ of all bounded closed intervals cannot form a vector space because the concept of an additive inverse cannot be considered. However, it can be embedded into a Banach space $(C[0, 1], \|\cdot\|)$ as a convex cone isometrically and isomorphically, in which the embedding function

$$\pi : (\mathcal{I}, d_H) \rightarrow (C[0, 1], \|\cdot\|) \text{ is defined by } [a, b] \mapsto f,$$

where the function $f \in C[0, 1]$ is defined by

$$f(x) = xa + (1 - x)b \text{ for all } x \in [0, 1].$$

In this case, the partial order among the space $\mathcal{I}$ can be proposed by providing a convex cone $\mathfrak{C}$ in $C[0, 1]$.

Three practical convex cones are provided in this case, which are also summarized below.

- The first convex cone is given by

$$\mathfrak{C}_1 = \{f \in C[0, 1] : f(x) \geq 0 \text{ for all } 0 \leq x \leq 1\}.$$

Then, given any $A, B \in \mathcal{I}$, we can define a partial order $\preceq_1$ among $\mathcal{I}$ by

$$A \preceq_1 B \text{ if and only if } \pi(A) \preceq_1 \pi(B).$$

More precisely, we have

$$[a^{(l)}, a^{(u)}] \preceq_1 [b^{(l)}, b^{(u)}] \text{ if and only if } a^{(l)} \leq b^{(l)} \text{ and } a^{(u)} \leq b^{(u)}. \quad (23)$$

- The second convex is given by

$$\mathfrak{C}_2 = \{f \in C[0, 1] : f(x) \leq 0 \text{ for all } 0 \leq x \leq 1\}.$$

Then, we can define a partial order $\preceq_2$ among $C[0, 1]$ by

$$A \preceq_2 B \text{ if and only if } \pi(A) \preceq_2 \pi(B).$$

More precisely, we have

$$[a^{(l)}, a^{(u)}] \preceq_2 [b^{(l)}, b^{(u)}] \text{ if and only if } a^{(l)} \geq b^{(l)} \text{ and } a^{(u)} \geq b^{(u)}. \quad (24)$$

- For some $x_0 \in (0, 1)$, the third convex cone is given by

$$\mathfrak{C}_{x_0} = \{f \in C[0, 1] : f(x) \leq 0 \text{ for all } x \in [0, x_0] \text{ and } f(x) \geq 0 \text{ for all } x \in (x_0, 1]\}.$$

Then, we can define a partial order $\preceq_3$ among $C[0, 1]$ by

$$A \preceq_3 B \text{ if and only if } \pi(A) \preceq_3 \pi(B).$$

More precisely, we have

$$[a^{(l)}, a^{(u)}] \preceq_3 [b^{(l)}, b^{(u)}] \text{ if and only if } a^{(l)} \geq b^{(l)} \text{ and } a^{(u)} \leq b^{(u)}. \quad (25)$$

Given any two bounded closed intervals

$$A = [a^{(l)}, a^{(u)}] \text{ and } B = [b^{(l)}, b^{(u)}],$$

three frequently used orders are given in (23)–(25), which says that the general convex cone $\mathfrak{C}$ adopted in this paper covers the frequently used ordering relations as the special cases. This general setting for convex cones allows the decision-makers to choose the appropriate partial orders among the set of all bounded closed intervals to conquer their difficulty in comparing any two bounded closed intervals. In other words, the choice of convex cones may be problem-dependent, which can be determined by the decision-makers for the purpose of obtaining a better decision.

After a partial order among $\mathcal{I}$ is introduced, the solution concepts of the interval-valued optimization problem (IOP) are naturally defined by considering the concept of a minimal element. Using the embedding theorem, we can formulate a vector optimization problem (VOP) and show that u^* is an optimal solution of the vector optimization problem (VOP) if and only if u^* is also an optimal solution of the interval-valued optimization problem (IOP). In other words, solving the interval-valued optimization problem (IOP) is equivalent to solving the vector optimization problem (VOP).

In order to solve the interval-valued optimization problem (IOP), this paper proposes a new methodology using the embedding theorem to transform the original problem into the scalar optimization problem (SOP) such that solving this problem (SOP) is sufficient to obtain the optimal solutions of original problem (IOP). As we can see that the problem (SOP) is the conventional optimization problem, the complexity of solving the original problem (IOP) is the same as solving the conventional optimization problem (SOP), which means that the issue of discussing the complexity of solving problem (SOP) in the existing literature is still valid for the original problem (IOP). For example, when we aim to solve a linear interval-valued optimization problem, it suffices to solve a conventional linear programming problem that is transformed from the original linear interval-valued optimization problem by using the embedding theorem. In this case, the simplex method can be used to solve this conventional linear programming problem. It is well-known that the complexity of simplex method is categorized as the algorithm of polynomial time even for the large-scale problems.

By introducing the concepts of a dual cone and the quasi-interior of a dual cone, four different types of scalarizations are obtained to solve the vector optimization problem (VOP), which equivalently solves the original interval-valued optimization problem (IOP). Finally, the above three practical convex cones $\mathfrak{C}_1$, $\mathfrak{C}_2$, and $\mathfrak{C}_{x_0}$ are considered to solve the linear type of the interval-valued optimization problem for the purpose of presenting the practical applications.

As we can see from the theorems presented in this paper, the scalarization technique depends on the choice of linear functionals Γ and Γ° . This kind of situation complies with the similar situation for solving the (conventional) multiobjective optimization problem. In order to resolve the confliction of several objective functions, the so-called Pareto optimal solution is adopted to be the solution concept of multiobjective optimization problem. The

well-known technique for solving the multiobjective optimization problem is to formulate its corresponding weighting problem, which means that the objective function of the weighting problem is a real-valued function obtained by adding those weighting objective functions together. In this case, the optimal solutions of this weighting problem can be shown to be the Pareto optimal solutions. It also means that the different choice of weights will result in the different Pareto optimal solution. Therefore, the linear functionals Γ and Γ° presented in this paper play the same role of weights adopted for solving the multiobjective optimization problem. In other words, the different choices of linear functionals will obtain the different optimal solutions of interval-valued optimization problems.

The limitations of the approach proposed in this paper and considerations for future research are addressed below.

- This paper proposes a simple way of using a bounded closed interval to formulate the uncertainty of optimization problems in which the coefficients of objective functions are assumed to be bounded closed intervals. However, when the decision-makers finally find that the uncertainty should be inevitably formulated using the randomness or fuzziness, the techniques for solving the stochastic optimization problem of the fuzzy optimization problem should be taken into account. The technique proposed in this paper for solving the interval-valued optimization problem can only assume that the data are roughly estimated by determining the lower and upper bounds.
- The concept of the optimal solution proposed in this paper depends on the choices of the convex cones and linear functionals, which may result in the dilemma for the decision-makers to pick the suitable convex cones and linear functionals. Although this paper suggests some suitable cones like $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_{x_0}$, it is possible to have some other suitable cones that can be better adopted. On the other hand, although the choice of linear functionals has also been demonstrated in many practical examples, it still has more suitable linear functionals that can be used.
- For the conventional multiobjective optimization problems, different settings of weights will generate different Pareto optimal solutions. It means that we should have a lot of Pareto optimal solutions depending on the determination of the weights. In order to obtain the “best” Pareto optimal solution, the evolutionary algorithm can be used by evolving the weights. Since the choices of convex cones and linear functionals are similar to the determination of weights in the conventional multiobjective optimization problems, in future research, we may try to apply evolutionary computation to design an evolutionary algorithm to obtain a “best” optimal solution from the population of optimal solutions that are obtained from the different choices of convex cones and linear functionals. The main strategy is to design a method to evolve the convex cones and linear functionals by considering the related parameters that occur in the convex cones and linear functionals. In this case, the decision-maker can get rid of the dilemma of choosing the convex cones and linear functionals as described.

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