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Input-to-Output Stability for Stochastic Complex Networked Control Systems

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Abstract

In this article, input-to-output stability (IOS) for stochastic complex networked control systems (SCNCS) is investigated. By applying Kirchhoff's matrix tree theorem in graph theory, an appropriate Lyapunov function is established which is related to topological structure and the Lyapunov function of each node system of SCNCS. Combining Lyapunov method and stochastic analysis skills, some sufficient criteria are provided to ensure SCNCS to satisfy IOS. In order to further analyze and verify the validity of our theoretical results, the results are applied to a class of stochastic Lurie coupled control systems on networks (SLCCSN) and the numerical test is performed.

Keywords: input-to-output stability; stochastic complex networked control systems; Lyapunov method

MSC: 45R05

1. Introduction

In recent years, complex networks have been a hot topic in contemporary scientific research. They play an important role in practice and are closely related to many high-complexity systems such as Internet networks, neural networks, and social networks [1–3]. As a network structure composed of a large number of nodes and intricate relationships between nodes, complex networks continue to develop at a rapid rate and are applied to many fields, such as mathematics, physics, power systems, transportation, finance, biology, climatology, computer science, sociology, and epidemiology [3–10]. However, there is almost interference in these different disciplines and fields in the real world. For example, the aircraft will have a certain bump due to the interference of unknown airflow and the crop harvest will be affected by seed activity and external climate. Similarly, for complex network, due to the influence of various network environmental factors such as white noise and other colored noise, network nodes and topological structure will inevitably be disturbed. Therefore, when using complex networks to solve practical problems, stochastic interference needs to be considered, i.e., stochastic complex networks. In order to better understand the structure and properties of stochastic complex networks, scholars have given full play to their pioneering abilities and actively explored the dynamical analysis of stochastic complex networks. Especially in the field of mathematics, many scholars focus on the dynamical behavior of stochastic complex networks, such as robustness, synchronization, consistency, and stability [11–17]. In addition, from the perspective of network

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science, these dynamical characteristics will influence the topological structure of stochastic complex networks, which in turn affects their dynamical processes. In this context, we can naturally expect that the topological structure of stochastic complex networks can affect their controllability. Therefore, relying on the dynamical behavior of stochastic complex networks, we introduce control systems [12,18], which are implemented to make the controlled object reach a predetermined ideal state. In summary, in this paper, we study the following stochastic complex networked control systems (SCNCS):

$$\begin{cases} \mathrm{d}x_k(t) = \left[f_k(x_k(t), u_k(t)) + \sum_{h=1}^{n} a_{kh} H_{kh}(x_h(t)) \right] \mathrm{d}t \\ \qquad\qquad + g_k(x_k(t), u_k(t)) \mathrm{d}B(t), \\ y_k(t) = h_k(|x_k(t)|), \quad k \in \mathbb{N}, \end{cases} \tag{1}$$

where positive integer n ($n \geq 2$) represents the number of the node systems in SCNCS (1), $x_k \in \mathbb{R}^{n_k}$ denote the system state and $y_k \in \mathbb{R}^{l_k}$ denote the system output. Control or input $u_k \in \mathbb{R}^{m_k}$ are measurable locally essentially bounded functions. Functions $f_k : \mathbb{R}^{n_k} \times \mathbb{R}^{m_k} \to \mathbb{R}^{n_k}$ of the k-th node system are embodied as drift coefficient and functions $g_k : \mathbb{R}^{n_k} \times \mathbb{R}^{m_k} \to \mathbb{R}^{n_k}$ of the k-th node system are embodied as diffusion coefficient, such that the solution of SCNCS (1) exists and is unique. Functions $H_{kh} : \mathbb{R}^{n_h} \to \mathbb{R}^{n_k}$ represent the coupling form from the h-th node system to the k-th node system. The coupling configuration matrix $\mathcal{A} = (a_{kh})_{n\times n}$ ($a_{kh} \geq 0$) is irreducible (i.e. it cannot be transformed via row and column permutations into a block upper triangular form with at least one zero block on the diagonal). Functions $h_k : \mathbb{R}^+ \to \mathbb{R}^{p_k}$ are locally Lipschitz continuous.

With the continuous research on the dynamical behavior of SCNCS (1), we know that stability is the basic requirement and important guarantee to ensure the normal operation of SCNCS (1). Reviewing the research history of the stability of nonlinear systems, many scholars have studied the properties of input-to-state stability (ISS) introduced in [19]. Then they extended the concept of ISS to deal with output stability, and have obtained some concepts related to input-to-output stability (IOS) [20,21]. We can see [20–23] that the process of studying IOS is mainly to explore whether the output signal generated by the system is also bounded for a bounded input signal, or whether an impact stimulus will produce a result that decays to zero. From the initial theoretical research to now, IOS has been widely used in many fields such as engineering, automatic control systems, computer science, and aerospace [24–27]. For example, in [27], a stability formula for formation flying spacecraft is proposed based on IOS. It is these meaningful and interesting applications of IOS in real-world applications that stimulate our interest in exploring IOS. Therefore, in this paper, we study the IOS for SCNCS (1).

There are many theories and methods to study SCNCS (1). In the initial study of ISS, the ISS control design was largely dependent on the Lyapunov characteristic inspired by [19]. And in the case of $y = x$, the concept of IOS can be attributed to the properties of ISS [21]. Therefore, on the basis of [19,21], in the process of exploring IOS for SCNCS (1), we also reasonably hope to use Lyapunov method to solve the problem and many good results have been shown in [4,12–15,28,29]. However, due to the influence of stochastic interference, the complexity of SCNCS (1), and the intricate relationships between the topological structure of the network and the dynamical behavior of each node system, using a single method is hard to establish an appropriate Lyapunov function successfully to study IOS for SCNCS (1). So inspired by Li and Shuai [30], we try to use Kirchhoff's matrix tree theorem in graph theory [4,31–33] to link the dynamical behavior and topological structure of SCNCS (1). In summary, we combine Kirchhoff's matrix tree theory in graph theory with Lyapunov method and stochastic analysis skills to investigate IOS for SCNCS (1).

Compared with the existing articles [4,34,35], our contributions in this paper are as follows:

1. Combining Kirchhoff's matrix tree theorem in graph theory, Lyapunov method and stochastic analysis skills, we construct an appropriate Lyapunov function for SCNCS (1), which is closely related to topological structure and Lyapunov function of each node system, and propose some sufficient criteria for SCNCS (1) to satisfy IOS.
2. We apply the theoretical results to verify IOS for a class of stochastic Lurie coupled control systems on networks (SLCCSN), and the numerical test is carried out to verify the validity of our results.

The following is the structure of this article. In Section 2, model description and preliminaries are presented. Section 3 provides main results. In Section 4, there exists an application to a class of SLCCSN. Section 5 presents a numerical test to verify the feasibility. Finally, the conclusions of the article are illustrated in Section 6.

2. Model Description and Preliminaries

In this section, in order to better describe SCNCS (1), we use the related knowledge of graph theory. The network can be expressed as a directed graph $\mathcal{G}$, which is composed of a vertex set and a directed arc set. And from SCNCS (1), we can obtain that $\mathcal{A} = (a_{kh})_{n\times n}$ ($a_{kh} \geq 0$) is the coupling configuration matrix of SCNCS (1) and each a_{kh} represents the weight of arc(h,k). Here we introduce the weighted matrix $\mathcal{A}$ into the directed digraph $\mathcal{G}$ and use a weighted digraph $(\mathcal{G},\mathcal{A})$ consisting of $n(n \geq 2)$ vertices to describe SCNCS (1). Each vertex of the weighted digraph $(\mathcal{G},\mathcal{A})$ is regarded as a node system and the directed arc between two vertices represents the interaction between two node systems. Besides, it is noteworthy that if $\mathcal{A}$ is irreducible, then the weighted digraph $(\mathcal{G},\mathcal{A})$ is strongly connected. Readers can learn more from [36,37]. In summary, we can get the k-th node system of SCNCS (1) shown below:

$$\begin{cases} \mathrm{d}x_k(t) = f_k(x_k(t), u_k(t))\mathrm{d}t + g_k(x_k(t), u_k(t))\mathrm{d}B(t), \\ y_k(t) = h_k(|x_k(t)|), \end{cases} \tag{2}$$

where vector $x_k = (x_k^{(1)}, \ldots, x_k^{(m)})^{\mathrm{T}} \in \mathbb{R}^m$ and vector $y_k = (y_k^{(1)}, \ldots, y_k^{(n)})^{\mathrm{T}} \in \mathbb{R}^n$.

And we also give the definition of Laplacian matrix of $(\mathcal{G},\mathcal{A})$ as follows:

$$L = \begin{pmatrix} \sum\limits_{k\neq 1} a_{1k} & -a_{12} & \cdots & -a_{1n} \\ -a_{21} & \sum\limits_{k\neq 2} a_{2k} & \cdots & -a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & -a_{n2} & \cdots & \sum\limits_{k\neq n} a_{nk} \end{pmatrix}.$$

We will give the definition of SCNCS (1) satisfying IOS and two lemmas in graph theory.

Definition 1. *If for every $\varepsilon \in (0,1)$, every $t \in \mathbb{R}^+$, the initial value x_0 and the control or input u_k, there exist $\beta(\cdot,\cdot) \in \mathcal{KL}$ and $\rho_k(\cdot) \in \mathcal{K}_\infty$ such that*

$$\mathbb{P}\Big\{|y(t)| < \beta\big(|x_0|, t\big)\Big\} \geq 1 - \varepsilon,$$

when $|u_k| \leq \rho_k\big(|x_k|\big)$, then SCNCS (1) is IOS.

Lemma 1. *([36] Kirchhoff's matrix tree theorem) Assume that $n \geq 2$. Let ℓ_k denote the cofactor of the k-th diagonal element of the Laplacian matrix L of the weighted digraph $(\mathcal{G}, \mathcal{A})$. Then*

$$\ell_k = \sum_{\mathcal{T} \in \mathbb{T}_k} \mathcal{W}(\mathcal{T}), \quad k \in \mathbb{N},$$

where $\mathbb{T}_k$ is the set of all spanning trees $\mathcal{T}$ of the weighted digraph $(\mathcal{G}, \mathcal{A})$ that are rooted at vertex k, and $\mathcal{W}(\mathcal{T})$ is the weight of $\mathcal{T}$. Particularly, if the weighted digraph $(\mathcal{G}, \mathcal{A})$ is strongly connected, then $\ell_k > 0$.

Lemma 2. *([30]) Assume that $n \geq 2$. Let ℓ_k $(k = 1, 2, \ldots, n)$ denote the cofactor of the k-th diagonal element of the Laplacian matrix L of the weighted digraph $(\mathcal{G}, \mathcal{A})$. Then the following identity holds*

$$\sum_{k,h=1}^{n} \ell_k a_{kh} F_{kh}(x_k, x_h) = \sum_{\mathcal{Q} \in \mathbb{Q}} W(\mathcal{Q}) \sum_{(s,r) \in E(\mathcal{C}_\mathcal{Q})} F_{rs}(x_r, x_s),$$

where $F_{kh}(x_k, x_h), 1 \leq k, h \leq n$ are arbitrary functions, $\mathcal{Q}$ is the set of all spanning unicyclic graphs of the weighted digraph $(\mathcal{G}, \mathcal{A})$, $W(\mathcal{Q})$ is the weighted of $\mathcal{Q}$, and $\mathcal{C}_\mathcal{Q}$ is the directed cycle of $\mathcal{Q}$.

3. Main Results

In this section, we will provide some sufficient criteria for SCNCS (1) to meet IOS, which will be shown below.

Theorem 1. *If there exists functions $V_k(x_k, t) \in C^{2,1}(\mathbb{R}^{n_k} \times \mathbb{R}^+; \mathbb{R}^+), k \in \mathbb{N}$ such that each node system of SCNCS (1) satisfies the following conditions C1, C2 and C3, then SCNCS (1) is IOS.*

C1. There exists functions $\alpha_k^{(1)}(\cdot)$, $\alpha_k^{(2)}(\cdot) \in \mathcal{K}_\infty$ such that

$$\alpha_k^{(1)}(|y_k|) \leq V_k(x_k, t) \leq \alpha_k^{(2)}(|x_k|).$$

C2. For any control or input u_k, there exist real number $\mu_k > 0$ and $\gamma_k(\cdot) \in \mathcal{K}_\infty$ such that

$$\mathcal{L}^{(2)} V_k(x_k, t) \leq -\mu_k V_k(x_k, t) + \gamma_k(|u_k|),$$

where

$$\begin{aligned} &\mathcal{L}^{(2)} V_k(x_k, t) \\ &\triangleq \frac{\partial V_k(x_k, t)}{\partial t} + \frac{\partial V_k(x_k, t)}{\partial x_k} f_k(x_k, u_k) \\ &+ \frac{1}{2} \mathrm{trace} \left\{ g_k(x_k, u_k)^{\mathrm{T}} \frac{\partial^2 V_k(x_k, t)}{\partial x_k^2} g_k(x_k, u_k) \right\}. \end{aligned}$$

C3. There exists real number $\sigma_k > 0$ and functions P_k, P_h (k, h = 1, 2, . . ., n) such that

$$\frac{\partial V_k(x_k, t)}{\partial x_k} H_{kh}(x_h) \leq P_h(x_h) - P_k(x_k) + \sigma_k \alpha_k^{(1)}(|y_k|),$$

where $\frac{3}{2} \max_{1 \leq k \leq n} \{ \sum_{h=1}^{n} a_{kh} \sigma_k \} < \min_{1 \leq k \leq n} \{ \mu_k \}$.

Proof. Consider an appropriate Lyapunov function $V(x,t) = e^{\delta t}\sum_{k=1}^{n}\ell_k V_k(x_k,t)$ where $\delta = \frac{2}{3}\mu - \max\limits_{1\le k\le n}\Big\{\sum\limits_{h=1}^{n} a_{kh}\sigma_k\Big\} > 0, \mu = \min\limits_{1\le k\le n}\{\mu_k\}$, and $\ell_k > 0$ $(k = 1,2,\ldots,n)$ in light of Lemma 1. Based on condition C1, we can obtain the following inequality derivation:

$$\begin{aligned} V(x,t) &\le e^{\delta t}\sum_{k=1}^{n}\ell_k\alpha_k^{(2)}(|x_k|) \\ &\le e^{\delta t}\sum_{k=1}^{n}\ell_k \max_{1\le k\le n}\{\alpha_k^{(2)}\}(|x|) \\ &\triangleq e^{\delta t}\overline{\alpha}(|x|), \end{aligned} \tag{3}$$

$$\begin{aligned} V(x,t) &\ge e^{\delta t}\sum_{k=1}^{n}\ell_k\alpha_k^{(1)}(|y_k|) \\ &\ge e^{\delta t}\min_{1\le k\le s}\{\ell_k\}\min_{1\le k\le n}\{\alpha_k^{(1)}\}\left(\frac{1}{n}\sum_{k=1}^{n}|y_k|\right) \\ &\ge e^{\delta t}\min_{1\le k\le n}\{\ell_k\}\min_{1\le k\le n}\{\alpha_k^{(1)}\}\left(\frac{|y|}{n}\right) \\ &\triangleq e^{\delta t}\underline{\alpha}(|y|), \end{aligned} \tag{4}$$

where $\overline{\alpha}(\cdot) = \sum\limits_{k=1}^{n}\ell_k \max\limits_{1\le k\le n}\{\alpha_k^{(2)}\}(\cdot) \in \mathcal{K}_\infty$ and $\underline{\alpha}(\cdot) = \min\limits_{1\le k\le n}\{\ell_k\}\min\limits_{1\le k\le n}\{\alpha_k^{(1)}\}(\frac{\cdot}{n}) \in \mathcal{K}_\infty$. Thus, combining (3) and (4), we can get the following inequality:

$$e^{\delta t}\underline{\alpha}(|y|) \le V(x,t) \le e^{\delta t}\overline{\alpha}(|x|). \tag{5}$$

Next, in line with conditions C2, C3, Lemma 2 and the fact $W(\mathcal{Q}) \ge 0$, the diffusion operator $\mathcal{L}^{(1)}V$ of SCNCS (1) can be deduced that

$$\begin{aligned} \mathcal{L}^{(1)}V(x,t) &\le e^{\delta t}\sum_{k=1}^{n}\ell_k\left[\mathcal{L}^{(2)}V_k(x_k(t),t) + \frac{\partial V_k(x_k,t)}{\partial x_k}\sum_{h=1}^{n}a_{kh}H_{kh}(x_h) + \delta V_k(x_k(t),t)\right] \\ &\le e^{\delta t}\sum_{k=1}^{n}\ell_k\Bigg[\delta V_k(x_k(t),t) - \mu_k V_k(x_k(t),t) + \gamma_k(|u_k(t)|) \\ &\quad + \sum_{h=1}^{n}a_{kh}\Big(P_h(x_h) - P_k(x_k) + \sigma_k\alpha_k^{(1)}(|y_k(t)|)\Big)\Bigg] \\ &\le \left(\delta - \frac{2}{3}\mu + \max_{1\le k\le n}\Big\{\sum_{h=1}^{n}a_{kh}\sigma_k\Big\}\right)V(x,t) + e^{\delta t}\sum_{k,h=1}^{n}\ell_k a_{kh}(P_h(x_h) - P_k(x_k))) \\ &\le \left(\delta - \frac{2}{3}\mu + \max_{1\le k\le n}\Big\{\sum_{h=1}^{n}a_{kh}\sigma_k\Big\}\right)V(x,t) + e^{\delta t}\sum_{\mathcal{Q}\in\mathbb{Q}}W(\mathcal{Q})\sum_{(k,h)\in \mathrm{E}(\mathcal{C}_\mathcal{Q})}(P_h(x_h) - P_k(x_k))) \\ &\le \left(\delta - \frac{2}{3}\mu + \max_{1\le k\le n}\Big\{\sum_{h=1}^{n}a_{kh}\sigma_k\Big\}\right)V(x,t) = 0, \end{aligned} \tag{6}$$

where $\gamma_k(|u_k|) \le \gamma_k(|\rho_k(|x_k|)|) = \frac{\mu}{3}\alpha_k^{(1)}(|y_k|)$, $\rho(\cdot) = \gamma_k^{-1}\Big(\frac{\mu}{3}\alpha_k^{(1)}(h(\cdot))\Big) \in \mathcal{K}_\infty$.

Here, a stopping time sequence $\{\tau_r\}_{r=1}^{\infty}$ needs to be established, where $\tau_r = \inf\{t \ge 0 : |x(t)| \ge r, r = 1,2,\ldots\}$ and it satisfies that $\tau_r \to \infty$ as $r \to \infty$. Based on Itô's formula, taking the expectation of both sides of equality in $[0, t\wedge\tau_r)$, $r \to \infty$, according to (5) and (6), we can obtain that

$$\begin{aligned} e^{\delta t}\mathbb{E}\underline{\alpha}\big(|y(t)|\big) &\le \mathbb{E}V(x(t),t) \\ &= V(x_0,0)+\mathbb{E}\int_0^{t\wedge\tau_r}\mathcal{L}^{(1)}V(x(s),s)\mathrm{d}s \\ &\quad +\mathbb{E}\int_0^{t\wedge\tau_r}\left[e^{\delta t}\sum_{k=1}^{n}\ell_k\frac{\partial V_k(x_k(s),s)}{\partial x_k(s)}\times g_k(x_k(s),u_k(s))\right]\mathrm{d}B(s) \\ &= V(x_0,0)+\mathbb{E}\int_0^{t\wedge\tau_r}\mathcal{L}^{(1)}V(x(s),s)\mathrm{d}s \\ &\le V(x_0,0) \\ &\le \overline{\alpha}\big(|x_0|\big). \end{aligned} \tag{7}$$

Eventually, from (7), we can deduce that

$$\mathbb{E}\big(|y(t)|\big) \le \underline{\alpha}^{-1}\Big(e^{-\delta t}\overline{\alpha}\big(|x_0|\big)\Big) \triangleq \beta_1\big(|x_0|,t\big) \in \mathcal{KL}.$$

For any $\varepsilon \in (0,1)$, by applying Chebyshev's inequality [38], we can easily get that

$$\mathbb{P}\Big\{|y(t)| \ge \beta\big(|x_0|,t\big)\Big\} \le \frac{\mathbb{E}\big(|y(t)|\big)}{\beta\big(|x_0|,t\big)} < \varepsilon, \quad t \in \mathbb{R}^+, \tag{8}$$

where $\beta\big(|x_0|,t\big) = \frac{1}{\varepsilon}\beta_1\big(|x_0|,t\big) \in \mathcal{KL}$.

For every $t \in \mathbb{R}^+$, from (8), it is obvious that

$$\mathbb{P}\Big\{|y(t)| < \beta\big(|x_0|,t\big)\Big\} \ge 1-\varepsilon, \tag{9}$$

which means SCNCS (1) is IOS in conformity with Definition 1. The proof is completed. □

Remark 1. *In fact, if we weaken the conditions in Theorem 1 by using* $V_k(x_k,t) \ge \alpha_k^{(1)}\big(|y_k|\big)$ *in C1 to further reduce C2 to*

$$\mathcal{L}^{(2)}V_k(x_k,t) \le -\mu_k\alpha_k^{(1)}\big(|y_k|\big)+\gamma_k\big(|u_k|\big),$$

then obviously our conclusion is also available.

Moreover, if $y = x$*, we can get the definition of SCNCS* (1) *satisfying ISS from Definition 1, and it can be further deduced that SCNCS* (1) *is ISS on the basis of Theorem 1. That is to say, when* $y = x$*, if SCNCS* (1) *satisfies IOS, then SCNCS* (1) *also satisfies ISS.*

Remark 2. *We consider SCNCS* (1) *disturbed by white noise. However, in the real world, SCNCS* (1) *inevitably be disturbed by various types of noise in practical applications. Therefore, the associated problems of colour noise perturbation, should be introduced into SCNCS* (1) *to simulate a more realistic system. How to use our method to study the SCNCS* (1) *is the topic of our next research step.*

4. An Application to a Class of Stochastic Lurie Coupled Control Systems on Networks

Lurie coupled control systems on networks have been widely used in various fields and many excellent results have been achieved [33,34,39,40]. However, the dynamical behavior of Lurie coupled control systems on networks will inevitably be interfered by various noise. So in this part, in order to further analyze and verify the effectiveness of our theoretical results, considering the interference of white noise, we give an application to a class of stochastic lurie coupled control systems on networks (SLCCSN):

$$\begin{cases} \mathrm{d}\theta_k(t) = \Big[-p_k\theta_k(t) + q_k\xi_k(t) \\ \qquad\qquad + \sum_{h=1}^{n} m_{kh}\big(\theta_h(t) - \theta_k(t)\big)\Big]\mathrm{d}t + \zeta_k\theta_k(t)\mathrm{d}B(t), \\ \mathrm{d}\xi_k(t) = \Big[r_k\theta_k(t) - s_k\xi_k(t) - \phi_k\big(\xi_k(t)\big) + \tilde{u}_k(t)\Big]\mathrm{d}t \\ \qquad\qquad + \omega_k\xi_k(t)\mathrm{d}B(t), \\ z_k(t) = h_k^{(1)}(|\theta_k(t)|) + h_k^{(2)}(|\xi_k(t)|), \quad k \in \mathbb{N}, \end{cases} \tag{10}$$

where $\theta_k, \xi_k \in R^1$ are the state of SLCCSN (10), $\tilde{u}_k \in R^1$ represent control or input, $z_k \in \mathbb{R}^1$ denote the system output, ζ_k and ω_k represent linear disturbance intensity, p_k, q_k, r_k, s_k are positive constants, $\theta_h(t) - \theta_k(t)$ represents the linear coupling form. Functions $h_k^{(1)}(\cdot), h_k^{(2)}(\cdot)$ are locally Lipschitz continuous. The coupling configuration matrix $M = (m_{kh})_{n\times n}$ is irreducible. Nonlinear functions $\phi_k\big(\xi_k\big)$ are monotonically increasing which satisfy the sector property $\xi_k\phi_k(\xi_k) > 0$ and $\lim\limits_{|\xi_k|\to\infty} \frac{\phi_k(\xi_k)}{\xi_k} = +\infty$, $\forall \xi_k > 0$ (see [33,34,40]).

The k-th node system of SLCCSN (10) is displayed as follows:

$$\begin{cases} \mathrm{d}X_k(t) = I_k(X_k(t), \tilde{u}_k(t))\mathrm{d}t \\ \qquad\qquad + J_k(X_k(t), \tilde{u}_k(t))\mathrm{d}B(t), \\ z_k(t) = h_k^{(1)}(|x_k(t)|) + h_k^{(2)}(|y_k(t)|), \end{cases} \tag{11}$$

where $X_k = (x_k, y_k)^{\mathrm{T}}$, $I_k = (-p_kx_k + q_ky_k, r_kx_k - s_ky_k - \phi_k\big(y_k\big) + \tilde{u}_k)^{\mathrm{T}}$, $J_k = (\zeta_kx_k, \omega_ky_k)^{\mathrm{T}}$, and $T_{kh}(X_h) = (x_h(t) - x_k(t), 0)^{\mathrm{T}}$.

Next, we will consider some conditions such that SLCCSN (10) satisfies IOS and give the following theorem and its proof.

Theorem 2. *If there exist control or input $\tilde{u}_k = \eta_k x_k(t)$, where gain coefficient $\eta_k > 0$ and $\frac{3}{2}\eta_k^2 \le \min_{1\le k\le n}\{\tilde{\mu}_k\}$ $(k = 1, 2, \ldots, n)$, and satisfies the following condition:*

$$\max_{1\le k\le n}\Big\{\frac{1}{2}\zeta_k^2 + \frac{1}{2}q_k + \frac{1}{2}r_k - p_k, \frac{1}{2}\omega_k^2 + \frac{1}{2}q_k + \frac{1}{2}r_k + 1 - s_k\Big\} < -1, \tag{12}$$

then SLCCSN (10) is IOS.

Proof. For the k-th node system (11) in SLCCSN (10), we consider the Lyapunov function $V_k(X_k, t) = \frac{1}{2}|X_k|^2$. Next, consider $\alpha_k^{(1)}(|Y_k|) = \frac{1}{2k}|X_k|^2 \in \mathcal{K}_\infty$ where $Y_k = h_k(|X_k|)$ and $\alpha_k^{(2)}\big(|X_k|\big) = 2k|X_k|^2 \in \mathcal{K}_\infty$ $(k = 1, 2, \ldots, n)$, and then we can get the inequality $\alpha_k^{(1)}(|Y_k|) \le V_k(X_k, t) \le \alpha_k^{(2)}\big(|X_k|\big)$ such that condition C1 is satisfied. Based on the properties $y_k\phi_k(y_k) > 0$ and system (11), we can obtain the following derivation by applying basic inequality formula:

$$\begin{aligned} &\mathcal{L}^{(11)}V_k(X_k, t) \\ &= (x_k, y_k)\begin{pmatrix} -p_kx_k + q_ky_k \\ r_kx_k - s_ky_k - \phi_k\big(y_k\big) + \tilde{u}_k \end{pmatrix} \\ &\quad + \frac{1}{2}(\zeta_kx_k, \omega_ky_k)\begin{pmatrix} \zeta_kx_k \\ \omega_ky_k \end{pmatrix} \\ &\le (\frac{1}{2}\zeta_k^2 - p_k)x_k^2 + (\frac{1}{2}\omega_k^2 - s_k)y_k^2 + (q_k + r_k + \eta_k)x_ky_k \\ &\le (\frac{1}{2}\zeta_k^2 + \frac{1}{2}q_k + \frac{1}{2}r_k - p_k)x_k^2 \\ &\quad + (\frac{1}{2}\omega_k^2 + \frac{1}{2}q_k + \frac{1}{2}r_k + 1 - s_k)y_k^2 + \frac{1}{4}\tilde{u}_k^2 \\ &\le -\tilde{\mu}_kV_k(X_k, t) + \tilde{\gamma}_k(|\tilde{u}_k|), \end{aligned}$$

where $\tilde{\mu}_k = \min\left\{|\frac{1}{2}\zeta_k^2 + \frac{1}{2}q_k + \frac{1}{2}r_k - p_k|, |\frac{1}{2}\omega_k^2 + \frac{1}{2}q_k + \frac{1}{2}r_k + 1 - s_k|\right\}$ and $\tilde{\gamma}_k(|\tilde{u}_k|) = \frac{1}{4}|\tilde{u}_k|^2$. It is obvious that $\tilde{\gamma}_k(\cdot) \in \mathcal{K}_\infty$, which means condition C2 in Theorem 1 is satisfied.

Meantime, in addition to the process expressed above, we have the following derivations:

$$\begin{aligned}
&\frac{\partial V_k(X_k,t)}{\partial X_k} T_{kh}(X_h) \\
&= (x_k, y_k)\begin{pmatrix} x_h - x_k \\ 0 \end{pmatrix} \\
&\leq \frac{1}{2}x_h^2 - \frac{1}{2}x_k^2 + \frac{1}{3k\max\limits_{1\leq k\leq n}\left\{\sum\limits_{h=1}^{n} a_{kh}\right\}}(x_k^2 + y_k^2) \\
&= P_h(X_h) - P_k(X_k) + \tilde{\sigma}_k \alpha_k^{(1)}(|Y_k|),
\end{aligned}$$

where $P_k(X_k) = \frac{1}{2}x_k^2$, $\tilde{\sigma}_k\alpha_k^{(1)}(|Y_k|) = \left(1/\left(3k\max\limits_{1\leq k\leq n}\left\{\sum\limits_{h=1}^{n} a_{kh}\right\}\right)\right)|X_k|^2$. Then based on condition (12), we can derive $\frac{3}{2}\max_{1\leq k\leq n}\left\{\sum\limits_{h=1}^{n} a_{kh}\tilde{\sigma}_k\right\} \leq \min_{1\leq k\leq n}\{\tilde{\mu}_k\}$ such that condition C3 is satisfied. Therefore, according to our above derivation, all the conditions of Theorem 1 are met, which means SLCCSN (10) is IOS. □

Remark 3. *In Theorem 2, some sufficient criteria are proposed by using the coefficients in system* (11)*. Compared with Theorem 1, the conditions in Theorem 2 are easier to verify and more suitable for application in actual production and life. Therefore, it is more intuitive and convenient to use Theorem 2 to explore IOS for SLCCSN* (10)*.*

5. Numerical Test

In this part, we will give the following numerical test to verify the validity and applicability of our results. Next, we select a function that meets the conditions of Theorem 1 to verify the conclusion. Consider nonlinear functions $\phi_k(y_k) = y_k^3$ and the locally Lipschitz continuous functions $h_k^{(1)}(|x|) = h_k^{(2)}(|x|) = \frac{1}{2}x^3$. We choose $X_1 = (0.10, 0.10)^{\mathrm{T}}$, $X_2 = (0.20, 0.20)^{\mathrm{T}}$, $X_3 = (0.30, 0.30)^{\mathrm{T}}$, $X_4 = (0.40, 0.40)^{\mathrm{T}}$, $X_5 = (0.50, 0.50)^{\mathrm{T}}$ as the initial values and $\tilde{u}_1 = 0.73x_1, \tilde{u}_2 = 0.74x_2, \tilde{u}_3 = 0.77x_3, \tilde{u}_4 = 0.79x_4, \tilde{u}_5 = 0.83x_5$ as the control or input. The complete data is shown in Table 1. And we set the coupling configuration matrix

$$M = \begin{pmatrix} 0 & 0.04 & 0.02 & 0.06 & 0.04 \\ 0.09 & 0 & 0.04 & 0.08 & 0.04 \\ 0.08 & 0.04 & 0 & 0.05 & 0.05 \\ 0.06 & 0.05 & 0.01 & 0 & 0.02 \\ 0.09 & 0.07 & 0.05 & 0.06 & 0 \end{pmatrix}.$$

Table 1. Corresponding parameters of SLCCSN (10).

No.	p_k	q_k	r_k	s_k	ζ_k	ω_k
1	2.19	0.01	0.02	2.15	1.02	0.01
2	2.21	0.03	0.02	2.18	1.03	0.03
3	2.24	0.05	0.04	2.22	1.05	0.04
4	2.28	0.06	0.05	2.31	1.07	0.06
5	2.33	0.08	0.07	2.43	1.01	0.07

Next, by simple calculation, we can get that $\frac{3}{2}\eta_1^2 = 0.79935$, $\frac{3}{2}\eta_2^2 = 0.82140$, $\frac{3}{2}\eta_3^2 = 0.88935$, $\frac{3}{2}\eta_4^2 = 0.93615$, $\frac{3}{2}\eta_5^2 = 1.03335$. According to the data and results obtained in Table 2, $\min_{1\leq k\leq n}\{\tilde{\mu}_k\} = 1.13495$, and it is clear that condition (12) and

$\frac{3}{2}\eta_k^2 \leq \min_{1\leq k\leq n}\{\tilde{\mu}_k\}$ in Theorem 2 are satisfied. In summary, we can derive that SLCCSN (10) is IOS and the simulation results are shown in Figures 1–6 by using the Euler method with the step size of 0.1.

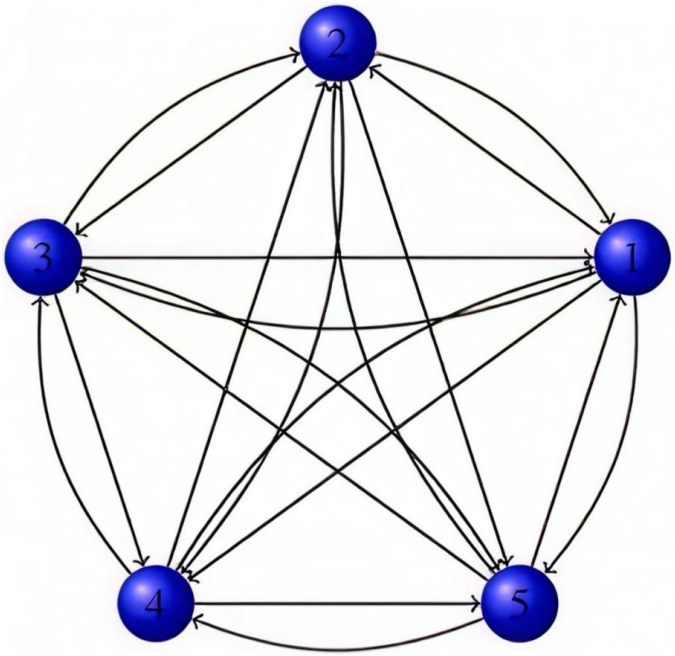


Figure 1. The struction diagram of SLCCSN (10).

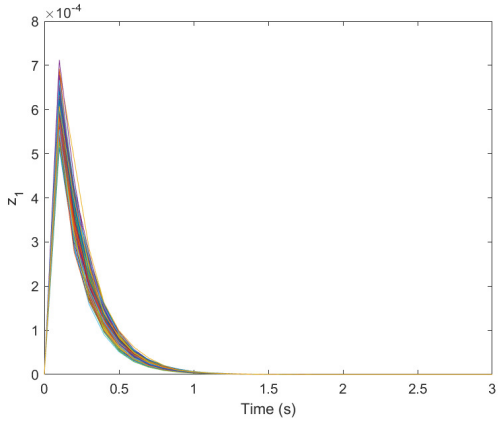


Figure 2. The sample path of SLCCSN (10) with control or input $\tilde{u}_1 = 0.73x_1$.

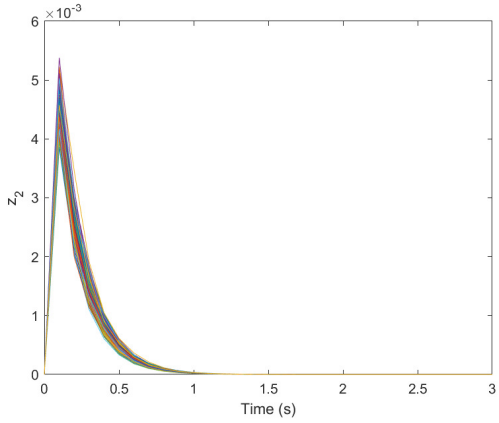


Figure 3. The sample path of SLCCSN (10) with control or input $\tilde{u}_2 = 0.74x_2$.

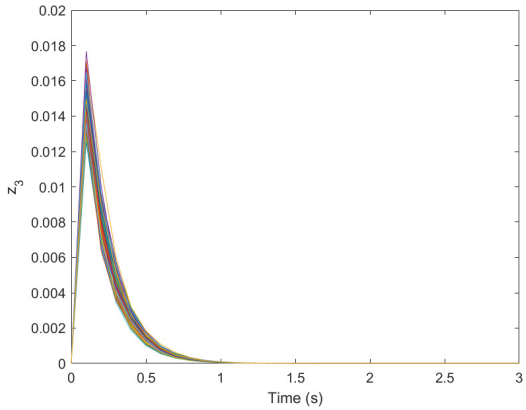


Figure 4. The sample path of SLCCSN (10) with control or input $\tilde{u}_3 = 0.77x_3$.

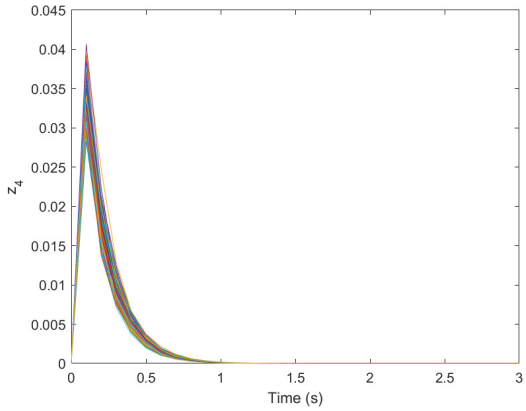


Figure 5. The sample path of SLCCSN (10) with control or input $\tilde{u}_4 = 0.79x_4$.

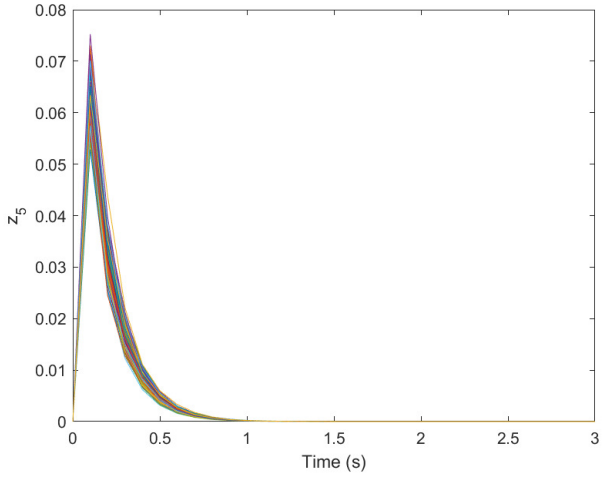


Figure 6. The sample path of SLCCSN (10) with control or input $\tilde{u}_5 = 0.83x_5$.

Table 2. The calculation results of the parameters in Table 1.

$\frac{1}{2}\zeta_k^2 + \frac{1}{2}q_k + \frac{1}{2}r_k - p_k$	$\frac{1}{2}\omega_k^2 + \frac{1}{2}q_k + \frac{1}{2}r_k + 1 - s_k$	$\tilde{\mu}_k$
−1.65480	−1.13495	1.13495
−1.65455	−1.15455	1.15455
−1.64375	−1.17420	1.17420
−1.65255	−1.25320	1.25320
−1.74495	−1.35255	1.35255

6. Conclusions

In this article, we have studied IOS for SCNCS (1) under the interference of white noise. And based on Kirchhoff's matrix tree theorem in graph theory, Lyapunov method and stochastic analysis skills, we have proposed some sufficient criteria to ensure SCNCS (1) to satisfy IOS. Moreover, we have applied the results to SLCCSN (10) and performed numerical test to illustrate the effectiveness of our theories. However, in real-world applications, there are not only other noise interference, but also some time delay phenomena in practical problems. The existence of time delay will have a non-negligible impact on the system, especially on the stability. Therefore, the influence of other noise interference and time delay on SCNCS (1) satisfying IOS is the direction of our future research.

Notations: In this paper, $\mathbb{R}^m$ represents the m-dimensional real vector space and $\mathbb{R}^{m\times n}$ represents the set of $m \times n$-dimensional real matrices and $\mathbb{R}^+ = [0, +\infty)$. The superscript "T" represents the transpose of the vector or matrix. The Euclidean norm is defined as $|x| = (\sum_{k=1}^{m} x_k^2)^{\frac{1}{2}}$ for vector $x = (x_1, \ldots, x_m)^{\mathrm{T}} \in \mathbb{R}^m$ and $|y| = (\sum_{k=1}^{n} y_k^2)^{\frac{1}{2}}$ for vector $y = (y_1, \ldots, y_n)^{\mathrm{T}} \in \mathbb{R}^n$. In the main text, without causing singularities, we also represent vectors with x_k, y_k. $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ denotes a complete probability space, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration satisfying the usual conditions, $\mathbb{P}$ is a probability measure, and $\mathbb{E}$ is the expectation of $\mathbb{P}$. On the complete probability space, $B(t)$ represents a one-dimensional Brownian motion. Define $\mathbb{N} = \{1, 2, \ldots, n\}$, a function $\alpha(\cdot) : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{K}_\infty$ if it is strictly increasing, continuous and unbounded, and satisfies $\alpha(0) = 0$. A function $\beta(\cdot, \cdot) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is of class $\mathcal{KL}$ if the first argument is of class $\mathcal{K}$ and the second argument decreases to zero. $C^{2,1}(\mathbb{R}^n \times \mathbb{R}^+; \mathbb{R}^+)$ represents the family of all nonnegative functions $V(x, t)$ on $\mathbb{R}^n \times \mathbb{R}^+$ which are continuously twice differentiable in x and once in t.

We shall introduce some basic concepts of directed graphs (digraphs). A directed graph $\mathcal{G} = (V, E)$ can be regarded as a composite of the vertex set $V(\mathcal{G})$ and the arc set $E(\mathcal{G})$. Provided that a subgraph $\mathcal{H}$ of $\mathcal{G}$ shares the same vertex set with $\mathcal{G}$, $\mathcal{H}$ is spanning. Where there exists an arc pointing from initial vertex k to terminal vertex h in $\mathcal{G}$, there is $(k, h) \in E(\mathcal{G})$. Therefore, we can assign a positive weight a_{kh} to the arc connecting initial vertex h and terminal vertex k. Obviously, $a_{kh} = 0$ is equivalent to $(h, k) \notin E(\mathcal{G})$. After introducing weights into a digraph, we denote the weighted digraph by $(\mathcal{G}, A)$. When it comes to the weight $W(\mathcal{H})$ of a subdigraph $\mathcal{H}$, we refer to the product of the weights on all its arcs. A vertex is said to be isolated if it has no arcs that are either directed towards or away from it. A self-loop is an arc that connects a vertex to itself. A walk is a finite alternating sequence of vertices and the connecting arcs between them, in addition, it begins and ends with a vertex. Particularly, when the sequence starts and ends at the same vertex, a walk is said to be a closed walk. What is more, a path is a walk that has no repeated vertices. A cycle is a closed walk without repeated vertices. If for every pair of distinct vertices k and h, there is a path from vertex k to vertex h and a path from vertex h to vertex k as well, a directed graph is said to be strongly connected (see Figure 7 as an example). Other notations will be explained where they first appear.

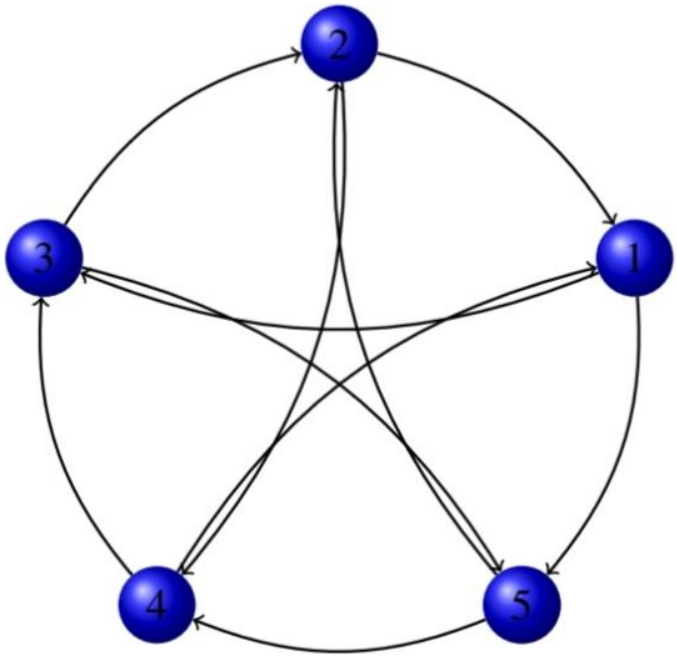


Figure 7. A strongly connected digraph $\mathcal{G}$ with 5 vertices.

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