

## Article

# Structured Distance to Normality of Dirichlet–Neumann Tridiagonal Toeplitz Matrices

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## Abstract

This paper conducts a rigorous study on the spectral properties and operator-space distances of perturbed Dirichlet–Neumann tridiagonal (PDNT) Toeplitz matrices, with emphasis on their asymptotic behaviors. We establish explicit closed-form solutions for the eigenvalues and associated eigenvectors, highlighting their fundamental importance for characterizing matrix stability in the presence of perturbations. By exploiting the structural characteristics of PDNT Toeplitz matrices, we obtain closed-form expressions quantifying the distance to normality, the deviation from normality.

**Keywords:** Toeplitz matrix; eigenvalue; normal matrix; distance; Frobenius norm

**MSC:** 15A18; 15A60

## 1. Introduction

When solving the wave equations using the method of separation of variables in Cartesian coordinates, one often encounters eigenvalue problems of the following form. It is necessary to find  $\mu$  and  $N(x)$  that satisfy [1]

$$\frac{d^2 N}{dx^2} + \mu N = 0, \quad (1)$$

and the function must satisfy the homogeneous Dirichlet and Neumann boundary conditions (BCs) at  $m$  and  $n$ , i.e.,

$$N(m) = 0, \quad N'(n) = 0, \quad x \in [m, n]. \quad (2)$$

Discretize Equation (1) by letting  $x = m + hl$  and  $N_h = N(m + hl)$ , where  $l$  is the step size and  $h$  is an integer index ( $h = 1, \dots, z$ ). We then carry out second-order finite difference discretization incorporating operator symmetry. BCs are fulfilled through odd-even extensions, and the scheme is formulated in matrix form. Substituting symmetric constraints remove boundary variables to reduce the number of problem variables. Hence, the matrix eigenvalue problem



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$$\begin{pmatrix} -1 & 1 & & 0 \\ 1 & 0 & 1 & \\ & \ddots & \ddots & \ddots \\ & & 1 & 0 & 1 \\ 0 & & & 2 & 0 \end{pmatrix}_{z \times z} \begin{pmatrix} N_1 \\ N_2 \\ \vdots \\ N_{z-1} \\ N_z \end{pmatrix} = \lambda \begin{pmatrix} N_1 \\ N_2 \\ \vdots \\ N_{z-1} \\ N_z \end{pmatrix} \quad (3)$$

is obtained.

The structure of (3) shows the following: matching left–right vectors and a corresponding square matrix. It is evident that the aforementioned coefficient matrix can be diagonalized via the discrete sine transform of type VIII (DST8). The core of this research lies in exploring a broader family of matrices diagonalizable by DST8.

In domains like resistance network modeling [2,3], quantum anomalous Hall effect analysis [4], and molecular orbital theory [5]. Fields such as partial differential equations [6–9], time series decomposition [10], and regularization of discrete ill-posed problems via Tikhonov methods [11,12] also rely on the application of these matrices. It is essential to study the fundamental computational characteristics of tridiagonal Toeplitz matrices.

Numerous studies [13–17] have rigorously analyzed the determinant properties and eigenvalue systems of bordered periodic tridiagonal matrices. Comprehensive discussions on determinants, inverses, and norm equalities/inequalities for perturbed tridiagonal Toeplitz matrices are available in references [18–22]. Systematic analysis approaches for two regular matrix pairs are available in [23–25]. Matrix nearness problems have drawn considerable research attention, with significant contributions found in [26–32] and subsequent studies. Works [33–35] investigate  $\varepsilon$ -pseudospectral separations in banded Toeplitz matrices, revealing characteristic spectral behaviors. The study of tridiagonal Toeplitz matrix operators is motivated by their dual strengths: efficient computation of key metrics and multidisciplinary applicability. Biswa Datta pioneered the investigation of multiple themes, including the inverse eigenvalue problems examined here, in foundational works [36–39].

This paper is organized as follows: We begin by clarifying the symbolic representations and associated definitions utilized in this research. Section 2 provides explicit analytic representations for eigenvalues and eigenvectors of PDNT Toeplitz matrices. Section 4 examines spectral distance and matrix normalization. Sections 3 and 5 explore the structured distances between PDNT Toeplitz matrices and two algebraic varieties. Section 6 presents numerical examples.

### Concept and Symbol

This work employs the Euclidean vector norm  $\|\cdot\|_2$  and its induced matrix counterpart. Additionally, the Frobenius norm  $\|\cdot\|_F$ , with its distinct applications, is employed to quantify matrices and vectors. The following provides explanations of the symbols used in this paper:

- (1)  $\tilde{Q}$ : The subspace of  $\mathbf{C}^{z \times z}$  formed by PDNT Toeplitz matrices.
- (2)  $\mathcal{U}$ : The algebraic variety formed by normal matrices within  $\mathbf{C}^{z \times z}$ .
- (3)  $\mathcal{U}_{\tilde{Q}}$ :  $\mathcal{U} \cap \tilde{Q}$
- (4)  $\mathcal{V}$ : The collection of matrices with multiple eigenvalues constitutes an algebraic set in  $\mathbf{C}^{z \times z}$ .
- (5)  $\mathcal{V}_{\tilde{Q}}$ :  $\mathcal{V} \cap \tilde{Q}$
- (6)  $(\cdot)^T$ : This symbol represents the transpose operation.
- (7)  $(\cdot)^H$ : This symbol signifies Hermitian adjoint.

### Definition 1.

$$\mathbb{E} = \begin{bmatrix} \gamma_0 - \sqrt{\gamma_1 \gamma_2} & \gamma_1 & & & & 0 \\ & \gamma_2 & \gamma_0 & \gamma_1 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & \ddots \\ & & & & \ddots & \ddots \\ & & & & & \gamma_1 \\ & & & & & \gamma_2 & \gamma_0 & \sqrt{2}\gamma_1 \\ 0 & & & & & \sqrt{2}\gamma_2 & \gamma_0 \end{bmatrix} \in \mathbb{C}^{z \times z}. \quad (4)$$

Standardized as PDNT Toeplitz matrix  $\mathbb{E} = (z; -\sqrt{\gamma_1 \gamma_2}, \gamma_2, \gamma_0, \gamma_1, \sqrt{2}\gamma_2, \sqrt{2}\gamma_1)$ , a  $z$ -order square matrix conforming to Equation (4). Phase parameters are further defined as follows:

$$\omega_1 = \arg \gamma_2, \omega_2 = \arg \gamma_1, \omega_3 = \arg \gamma_0. \quad (5)$$

When  $\gamma_0 = 0$ , the matrix is given by  $\mathbb{E}_0 = (z; -\sqrt{\gamma_1 \gamma_2}, \gamma_2, 0, \gamma_1, \sqrt{2}\gamma_2, \sqrt{2}\gamma_1)$ .

**Definition 2.** For a matrix  $\mathbb{E} \in \mathbb{C}^{z \times z}$ , the Frobenius norm distance to the set of normal matrices is defined as the minimal norm of the difference between  $\mathbb{E}$  and  $\mathbb{E}_{\mathcal{U}} \in \mathcal{U}$ , expressed as follows:

$$d_F(\mathbb{E}, \mathcal{U}) = \min_{\mathbb{E}_{\mathcal{U}} \in \mathcal{U}} \|\mathbb{E} - \mathbb{E}_{\mathcal{U}}\|_F. \quad (6)$$

See [26,28,40–43] for detailed discussions.

**Definition 3.** The Frobenius norm measures the distance between PDNT Toeplitz matrix  $\mathbb{E}$  and the set  $\mathcal{U}_{\tilde{Q}}$ , formally defined as follows:

$$d_F(\mathbb{E}, \mathcal{U}_{\tilde{Q}}) = \min_{\mathbb{E}_{\mathcal{U}} \in \mathcal{U}_{\tilde{Q}}} \|\mathbb{E} - \mathbb{E}_{\mathcal{U}}\|_F. \quad (7)$$

**Definition 4.** The Frobenius distance between PDNT Toeplitz matrix  $\mathbb{E}$  and the family  $\mathcal{V}_{\tilde{Q}}$  can be expressed as follows:

$$d_F(\mathbb{E}, \mathcal{V}_{\tilde{Q}}) = \min_{\mathbb{E}_{\mathcal{V}} \in \mathcal{V}_{\tilde{Q}}} \|\mathbb{E} - \mathbb{E}_{\mathcal{V}}\|_F. \quad (8)$$

## 2. Eigenvalues and Associated Eigenvectors

This section utilizes similarity transformations to construct explicit representations of eigenvalues and eigenvectors for the PDNT Toeplitz matrix  $\mathbb{E}$  defined in (4). Let

$$\Lambda_z = \text{diag}(1, \sqrt{\frac{\gamma_1}{\gamma_2}}, \dots, (\sqrt{\frac{\gamma_1}{\gamma_2}})^{z-2}, (\sqrt{\frac{\gamma_1}{\gamma_2}})^{z-1}).$$

From  $|\gamma_1| = |\gamma_2|$  it follows that  $\Lambda_z$  is unitary. This property implies  $\Lambda_z \Lambda_z^H = I_z$ , equivalently  $\Lambda_z^{-1} = \Lambda_z^H$ . Here,  $I_z$  is the  $z$ -dimensional identity matrix,  $\Lambda_z^{-1}$  denotes the inverse of the matrix  $\Lambda_z$  and  $\Lambda_z^H$  represents the Hermitian adjoint of  $\Lambda_z$ .

Define the matrix

$$\mathbb{S}_z^{VIII} = \left( \frac{2}{\sqrt{2z-1}} d_k d_j \sin \frac{(2k-1)(2j-1)\pi}{2(2z-1)} \right)_{k,j=1}^z,$$

$$\text{where } d_q = \begin{cases} 1 & q \neq z, \\ \frac{\sqrt{2}}{2} & q = z. \end{cases}$$

The eighth discrete sine transform matrix  $\mathbb{S}_z^{VIII}$ , defined by its orthogonality property [44], automatically obeys the relation

$$(\mathbb{S}_z^{VIII})^{-1} = (\mathbb{S}_z^{VIII})^T = \mathbb{S}_z^{VIII}.$$

The following equation is derived by diagonalizing  $\mathbb{E}$  and performing a series of matrix multiplications with sophisticated algebraic manipulations:

$$(\mathbb{S}_z^{VIII})^{-1} \Lambda_z \mathbb{E} \Lambda_z^{-1} (\mathbb{S}_z^{VIII}) = \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_z). \quad (9)$$

Equation (9) enables the expression of  $\mathbb{E}$  as

$$\mathbb{E} = \Lambda_z^{-1} (\mathbb{S}_z^{VIII}) \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_z) (\mathbb{S}_z^{VIII})^{-1} \Lambda_z,$$

where  $\lambda_j$  is defined by

$$\lambda_j = \gamma_0 + 2\sqrt{\gamma_1 \gamma_2} \cos \frac{(2j-1)\pi}{2z-1}, j = 1, \dots, z. \quad (10)$$

Equation (9) explicitly presents the diagonalization  $\text{diag}(\lambda_1, \dots, \lambda_z)$  of PDNT Toeplitz matrix  $\mathbb{E}$ , with  $\lambda_j$  as its eigenvalues.

Left-multiplication of Equation (9) by  $\Lambda_z^{-1} \mathbb{S}_z^{VIII}$  constructs the eigensystem:

$$\mathbb{E} \Lambda_z^{-1} \mathbb{S}_z^{VIII} = \Lambda_z^{-1} \mathbb{S}_z^{VIII} \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_z),$$

rewritten as

$$\mathbb{E}(v^{(1)}, v^{(2)}, \dots, v^{(z)}) = (v^{(1)}, v^{(2)}, \dots, v^{(z)}) \text{diag}(\lambda_1, \lambda_2, \dots, \lambda_z), \quad (11)$$

where the vectors  $v^{(j)} = (v_1^{(j)}, \dots, v_z^{(j)})^T$  satisfy

$$v_k^{(j)} = \frac{2}{\sqrt{2z-1}} d_k d_j \left( \sqrt{\frac{\gamma_2}{\gamma_1}} \right)^{k-1} \sin \frac{(2k-1)(2j-1)\pi}{2(2z-1)}, k = 1, \dots, z, \quad j = 1, \dots, z,$$

$$\text{where } d_q = \begin{cases} 1 & q \neq z, \\ \frac{\sqrt{2}}{2} & q = z. \end{cases}$$

Equation (11) is equivalent to

$$\mathbb{E} v^{(j)} = \lambda_j v^{(j)}, j = 1, 2, \dots, z. \quad (12)$$

The right eigenvector  $v^{(j)} = (v_1^{(j)}, \dots, v_z^{(j)})^T$  is derived directly from Equation (12). The eigenvalues of  $\mathbb{E}$  derived from Equations (5) and (10) is

$$\lambda_j(\mathbb{E}) = \gamma_0 + 2\sqrt{|\gamma_1 \gamma_2|} e^{\frac{(\omega_1 + \omega_2)i}{2}} \cos \frac{(2j-1)\pi}{2z-1}, j = 1, \dots, z. \quad (13)$$

If  $\gamma_1 \gamma_2 \neq 0$ ,  $\mathbb{E}$  possesses  $z$  distinct eigenvalues. The eigenvalues lie on a closed segment in the complex plane

$$\mathcal{S}_\lambda(\mathbb{E}) = \left\{ \gamma_0 + b e^{\frac{(\omega_1 + \omega_2)i}{2}} : b \in \mathbf{R}, |b| \leq 2\sqrt{|\gamma_1 \gamma_2|} \cos \frac{\pi}{2z-1} \right\} \subset \mathbf{C}. \quad (14)$$

For matrix  $\mathbb{E}$ , its spectral radius admits the following expression

$$\rho(\mathbb{E}) = \max \left\{ \left| \gamma_0 + 2\sqrt{|\gamma_1 \gamma_2|} e^{\frac{(\omega_1 + \omega_2)i}{2}} \cos \frac{\pi}{2z-1} \right|, \left| \gamma_0 - 2\sqrt{|\gamma_1 \gamma_2|} e^{\frac{(\omega_1 + \omega_2)i}{2}} \right| \right\}.$$

Under the nonsingularity of  $\mathbb{E}$ , the spectral radius of  $\mathbb{E}^{-1}$  can be computable by Equation (13)

$$\rho(\mathbb{E}^{-1}) = \max_{j=1,\dots,z} \left| \gamma_0 + 2\sqrt{|\gamma_1\gamma_2|} e^{\frac{(\omega_1+\omega_2)i}{2}} \cos \frac{(2j-1)\pi}{2z-1} \right|^{-1}.$$

For  $\gamma_1\gamma_2 \neq 0$ , the elements of the right eigenvector  $v^{(j)} = [v_1^{(j)}, \dots, v_z^{(j)}]^T$  belonging to  $\lambda_j(\mathbb{E})$  are determined by

$$v_k^{(j)} = \frac{2}{\sqrt{2z-1}} d_k d_j \left( \sqrt{\frac{\gamma_2}{\gamma_1}} \right)^{k-1} \sin \frac{(2k-1)(2j-1)\pi}{2(2z-1)}, k = 1, \dots, z, j = 1, \dots, z, \quad (15)$$

Correspondingly, the left eigenvector  $u^{(j)} = [u_1^{(j)}, \dots, u_z^{(j)}]^T$  is expressed as follows:

$$u_k^{(j)} = \frac{2}{\sqrt{2z-1}} d_k d_j \left( \sqrt{\frac{\gamma_1}{\gamma_2}} \right)^{k-1} \sin \frac{(2k-1)(2j-1)\pi}{2(2z-1)}, k = 1, \dots, z, j = 1, \dots, z, \quad (16)$$

where  $d_q = \begin{cases} 1 & q \neq z, \\ \frac{\sqrt{2}}{2} & q = z. \end{cases}$  The overline represents complex conjugation.

As shown, specifying the dimension and  $\frac{\gamma_2}{\gamma_1}$  ratio of the PDNT Toeplitz matrix  $\mathbb{E}$  uniquely determines its left and right eigenvectors.

### 3. The Structured Distance Between PDNT Toeplitz and the Family of Normal Matrices

This section focuses on a PDNT Toeplitz matrix, examining both its distance from and deviation to the family of normal matrices.

**Theorem 1.** The matrix  $\mathbb{E}$  defined by (4) is normal if and only if the equality

$$|\gamma_1| = |\gamma_2| \quad (17)$$

holds.

**Proof.**  $\mathbb{E}^H \mathbb{E} = \mathbb{E} \mathbb{E}^H$  can be proved, an equality equivalent to (17).  $\square$

The above theorem indicates that a normal PDNT Toeplitz matrix admits the following representation.

$$\mathbb{E}' = \begin{bmatrix} \gamma_0 - \Psi e^{\frac{(\omega_1' + \omega_2')i}{2}} & \Psi e^{i\omega_2'} & & & O \\ \Psi e^{i\omega_1'} & \gamma_0 & \Psi e^{i\omega_2'} & & \\ & \Psi e^{i\omega_1'} & . & . & \\ & & . & . & . \\ & & & . & . \\ O & & & \sqrt{2}\Psi e^{i\omega_1'} & \sqrt{2}\Psi e^{i\omega_2'} \\ & & & \sqrt{2}\Psi e^{i\omega_1'} & \gamma_0 \end{bmatrix}, \quad (18)$$

where  $\mathbb{E}' = (z; -\Psi e^{\frac{(\omega_1' + \omega_2')i}{2}}, \Psi e^{i\omega_1'}, \gamma_0, \Psi e^{i\omega_2'}, \sqrt{2}\Psi e^{i\omega_1'}, \sqrt{2}\Psi e^{i\omega_2'})$ , with parameters satisfying  $\gamma_0 \in \mathbf{C}$ ,  $\Psi \geq 0$ , and  $\omega_1', \omega_2' \in \mathbf{R}$ .

Based on Equation (13), we obtain explicit solutions for the eigenvalues of matrix  $\mathbb{E}'$ .

$$\lambda_j(\mathbb{E}') = \gamma_0 + 2\Psi e^{\frac{(\omega_1' + \omega_2')}{2}i} \cos \frac{(2j-1)\pi}{2z-1}, j = 1, \dots, z.$$

In the complex plane, the eigenvalues of  $\mathbb{E}'$  are distributed on a closed line segment that can be parameterized as follows:

$$\mathcal{S}_{\lambda(\mathbb{E}')} = \left\{ \gamma_0 + be^{\frac{(\omega_1' + \omega_2')}{2}i} : b \in \mathbf{R}, |b| \leq 2\Psi \cos \frac{\pi}{2z-1} \right\} \subset \mathbf{C}.$$

**Theorem 2.** For the PDNT Toeplitz matrix  $\mathbb{E}$ , the Frobenius norm minimization problem  $\min \|\mathbb{E} - \mathbb{E}_{\mathcal{U}}\|_F$  has a unique optimal solution  $\mathbb{E}^* = (z; -\sqrt{\gamma_1^* \gamma_2^*}, \gamma_2^*, \gamma_0^*, \gamma_1^*, \sqrt{2}\gamma_2^*, \sqrt{2}\gamma_1^*) \in \mathcal{U}_{\tilde{Q}}$ , with parameters  $\gamma_0^* = \gamma_0$ ,  $\gamma_1^* = \Psi^* e^{i\omega_2}$ ,  $\gamma_2^* = \Psi^* e^{i\omega_1}$ , where  $\omega_1$  and  $\omega_2$  are given by (5) and  $\Psi^*$  is computed as  $\frac{\sqrt{|\gamma_1 \gamma_2|} + z(|\gamma_1| + |\gamma_2|)}{2z+1}$ .

**Proof.** According to Theorem 1, the Frobenius norm minimum  $\min \|\mathbb{E} - \mathbb{E}_{\mathcal{U}}\|_F$  is attained by the matrix  $\mathbb{E}^* \in \mathcal{U}_{\tilde{Q}}$ , with the condition  $|\gamma_1^*| = |\gamma_2^*|$ . The parameter assignments are necessarily chosen as follows:

$$\gamma_0^* = \gamma_0, \gamma_1^* = \Psi^* e^{i\omega_2}, \gamma_2^* = \Psi^* e^{i\omega_1}.$$

To achieve the global minimum of the objective function  $g(\Psi) = (\Psi - \sqrt{|\gamma_1 \gamma_2|})^2 + z((\Psi - \gamma_1)^2 + (\Psi - \gamma_2)^2)$ , the parameter  $\Psi$  must be determined. The unique global minimizer is given by  $\Psi^* = \frac{\sqrt{|\gamma_1 \gamma_2|} + z(|\gamma_1| + |\gamma_2|)}{2z+1}$ , thereby completing the proof of the theorem.  $\square$

**Corollary 1.** Let  $\mathbb{E}^* = (z; -\sqrt{\gamma_1^* \gamma_2^*}, \gamma_2^*, \gamma_0^*, \gamma_1^*, \sqrt{2}\gamma_2^*, \sqrt{2}\gamma_1^*) \in \mathcal{U}_{\tilde{Q}}$  be the optimal normal approximation to  $\mathbb{E}$ , with eigenvalues

$$\lambda_j(\mathbb{E}^*) = \gamma_0 + 2\Psi^* e^{\frac{(\omega_1 + \omega_2)}{2}i} \cos \frac{(2j-1)\pi}{2z-1}, j = 1, \dots, z, \quad (19)$$

where  $\omega_1$  and  $\omega_2$  are given by Equation (5).

All eigenvalues lie within the closed line segment in the complex plane

$$\mathcal{S}_{\lambda(\mathbb{E}^*)} = \left\{ \gamma_0 + be^{\frac{(\omega_1 + \omega_2)}{2}i} : b \in \mathbf{R}, |b| \leq 2\Psi^* \cos \frac{\pi}{2z-1} \right\}.$$

$\mathbb{E} \notin \mathcal{U}_{\tilde{Q}}$  holds if and only if the line segment defined in Equation (14) is contained within this segment. The spectral radius of  $\mathbb{E}^*$  admits the following characterization.

$$\rho(\mathbb{E}^*) = \max \left\{ \left| \gamma_0 + 2\Psi^* e^{\frac{(\omega_1 + \omega_2)}{2}i} \cos \frac{\pi}{2z-1} \right|, \left| \gamma_0 - 2\Psi^* e^{\frac{(\omega_1 + \omega_2)}{2}i} \cos \frac{\pi}{2z-1} \right| \right\},$$

where  $\Psi^* = \frac{\sqrt{|\gamma_1 \gamma_2|} + z(|\gamma_1| + |\gamma_2|)}{2z+1}$ .

Regarding the distance to normality of the PDNT Toeplitz matrix, the following result presents a succinct representation.

**Theorem 3.** Consider the PDNT Toeplitz matrix  $\mathbb{E} = (z; -\sqrt{\gamma_1\gamma_2}, \gamma_2, \gamma_0, \gamma_1, \sqrt{2}\gamma_2, \sqrt{2}\gamma_1)$ . The Frobenius distance between  $\mathbb{E}$  and the normal matrix variety  $\mathcal{U}_{\tilde{Q}}$  is

$$d_F(\mathbb{E}, \mathcal{U}_{\tilde{Q}}) = \sqrt{\frac{z^2}{2z+1}(|\gamma_1| - |\gamma_2|)^2 + \frac{z(|\gamma_1| + |\gamma_2|)}{2z+1}(\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|})^2}, \quad (20)$$

with  $\mathcal{U}_{\tilde{Q}}$  representing the algebraic variety of normal PDNT Toeplitz matrices.

**Proof.** The proof is established by substituting Theorem 2 into Equation (7), with the following derivation

$$\begin{aligned} \|\mathbb{E} - \mathbb{E}^*\|_F^2 &= z(|\gamma_1 - \gamma_1^*|^2 + |\gamma_2 - \gamma_2^*|^2) + |\sqrt{\gamma_1\gamma_2} - \sqrt{\gamma_1^*\gamma_2^*}|^2 \\ &= (2z+1)^2\Psi^* - 2\Psi^* \left[ z(|\gamma_1| + |\gamma_2|) + \sqrt{|\gamma_1\gamma_2|} \right] + z(|\gamma_1|^2 + |\gamma_2|^2) + |\gamma_1\gamma_2| \\ &= -(2z+1)\Psi^{*2} + z(|\gamma_1|^2 + |\gamma_2|^2) + |\gamma_1\gamma_2| \\ &= \frac{z^2}{2z+1}(|\gamma_1| - |\gamma_2|)^2 + \frac{z(|\gamma_1| + |\gamma_2|)}{2z+1}(\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|})^2. \end{aligned}$$

This establishes the desired result.  $\square$

**Example 1.** For a PDNT Toeplitz matrix

$$A = \begin{pmatrix} -1 & i & 0 & 0 & 0 \\ -4i & 1 & i & 0 & 0 \\ 0 & -4i & 1 & i & 0 \\ 0 & 0 & -4i & 1 & \sqrt{2}i \\ 0 & 0 & 0 & -4\sqrt{2}i & 1 \end{pmatrix}.$$

By virtue of Equation (20), we can show that

$$d_F(A, \mathcal{U}_{\tilde{Q}}) = \frac{5\sqrt{110}}{11},$$

meanwhile, direct computation shows that the unstructured F-distance

$$d_{Fu} = 3\sqrt{10}.$$

These two results compute the distance from structured and unstructured perspectives, respectively.

**Remark 1.** Theorem 3 establishes that the Frobenius distance  $d_F(\mathbb{E}, \mathcal{U}_{\tilde{Q}})$  from PDNT Toeplitz matrix  $\mathbb{E}$  to  $\mathcal{U}_{\tilde{Q}}$  shows no dependence on  $\gamma_0$ . However, the normal PDNT Toeplitz matrix  $\mathbb{E}^*$  nearest to  $\mathbb{E}$  varies with changes in  $\gamma_0$ . Briefly put, while variations in  $\gamma_0$  leave the distance to  $\mathcal{U}_{\tilde{Q}}$  invariant, they alter the optimal approximating matrices within this algebraic variety. In particular, the matrix pair  $\mathbb{E}_1 = (z; -\sqrt{\gamma_1\gamma_2}, \gamma_2, \gamma_0^1, \gamma_1, \sqrt{2}\gamma_2, \sqrt{2}\gamma_1)$  and  $\mathbb{E}_2 = (z; -\sqrt{\gamma_1\gamma_2}, \gamma_2, \gamma_0^2, \gamma_1, \sqrt{2}\gamma_2, \sqrt{2}\gamma_1)$  satisfies the following equality:

$$\|\mathbb{E}_1^* - \mathbb{E}_2^*\|_F = \|\mathbb{E}_1 - \mathbb{E}_2\|_F = \sqrt{z}|\gamma_0^1 - \gamma_0^2|.$$

The Distance from Matrix  $\mathbb{E}_0$  to Matrix Family  $\mathcal{U}_{\tilde{Q}}$

The deviation of matrix  $\mathbb{E}$  from normality is measured by the Frobenius norm, with the specific expression

$$\Delta_F(\mathbb{E}) = (\|\mathbb{E}\|_F^2 - \sum_{j=1}^z |\lambda_j|^2)^{\frac{1}{2}}, \mathbb{E} \in \mathbb{C}^{z \times z}, \quad (21)$$

as defined in [41].

In order to present the above formulas, we first give the following expression.

$$\sum_{j=1}^z \cos^2 \frac{(2j-1)\pi}{2z-1} = \begin{cases} 1, & z=1, \\ \frac{2z+1}{4}, & z>1. \end{cases} \quad (22)$$

**Proof.** To derive the formula, we first transform the original expression using the double-angle formula, obtaining

$$\sum_{j=1}^z \cos^2 \frac{(2j-1)\pi}{2z-1} = \frac{1}{2} \sum_{j=1}^z (1 + \cos \frac{2(2j-1)\pi}{2z-1}).$$

With  $t = \frac{2\pi}{2z-1}$ , when  $z > 1$ , by applying the product-to-sum formula, we simplify

$$\begin{aligned} 2 \sin t \sum_{j=1}^z \cos(2j-1)t &= 2 \sin t (\cos t + \cos 3t + \cdots + \cos(2z-3)t + \cos(2z-1)t) \\ &= \sin 2t + \sin 4t - \sin 2t + \cdots + \sin(2z-2)t - \sin(2z-4)t + \sin 2z - \sin(2z-2)t \\ &= \sin 2zt \end{aligned}$$

from which we can deduce that

$$\sum_{j=1}^z \cos(2j-1)t = \frac{\sin 2zt}{2 \sin t} = \frac{1}{2}.$$

Consequently,

$$\sum_{j=1}^z \cos^2 \frac{(2j-1)\pi}{2z-1} = \frac{z}{2} + \frac{1}{2} \sum_{j=1}^z \cos \frac{4j\pi}{2v+1} = \frac{z}{2} + \frac{1}{4} = \frac{2z+1}{4}.$$

When  $z = 1$ ,

$$\cos^2 \frac{(2j-1)\pi}{2z-1} = 1.$$

Thus, the proof is completed.  $\square$

**Theorem 4.** Given  $z > 1$  and the family  $\mathcal{U}_{\tilde{Q}}$  consisting of all normal PDNT Toeplitz matrices, the deviation  $\Delta_F(\mathbb{E}_0)$  of the PDNT Toeplitz matrix  $\mathbb{E}_0$  is related to the distance  $d_F(\mathbb{E}_0, \mathcal{U}_{\tilde{Q}})$  as

$$\Delta_F(\mathbb{E}_0) = \left( \sqrt{\frac{(2z+1) [z(|\gamma_1| - |\gamma_2|)^2 + 2|\gamma_1\gamma_2|]}{z^2(|\gamma_1| - |\gamma_2|)^2 + z(|\gamma_1| + |\gamma_2|)(\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|})^2}} \right) d_F(\mathbb{E}_0, \mathcal{U}_{\tilde{Q}}). \quad (23)$$

**Proof.** By applying Equations (21) and (22), the Frobenius deviation of the PDNT Toeplitz matrix  $\mathbb{E}_0$  from  $\mathcal{U}_{\tilde{Q}}$  is computed as follows:

for  $z > 1$ ,

$$\begin{aligned} \Delta_F(\mathbb{E}_0) &= \sqrt{|\gamma_1\gamma_2| + (z-2)(|\gamma_1|^2 + |\gamma_2|^2) + 2|\gamma_1|^2 + 2|\gamma_2|^2 - \sum_{j=1}^z \left| 2\sqrt{|\gamma_1\gamma_2|} \cos \frac{(2j-1)\pi}{2z-1} \right|^2} \\ &= \sqrt{|\gamma_1\gamma_2| + z(|\gamma_1|^2 + |\gamma_2|^2) - 4|\gamma_1\gamma_2| \sum_{j=1}^z \left| \cos \frac{(2j-1)\pi}{2z-1} \right|^2} \\ &= \sqrt{|\gamma_1\gamma_2| + z(|\gamma_1|^2 + |\gamma_2|^2) - (2z-1)|\gamma_1\gamma_2|}. \end{aligned} \quad (24)$$

And  $\Delta_F(\mathbb{E}_0) = 0$ , when  $z = 1$ .

Thus, the deviation is given by the piecewise function

$$\Delta_F(\mathbb{E}_0) = \begin{cases} 0, & z = 1, \\ \sqrt{z(|\gamma_1| - |\gamma_2|)^2 + 2|\gamma_1\gamma_2|}, & z > 1. \end{cases}$$

Moreover, using Equation (20), the deviation for  $z > 1$  can be compactly expressed as follows:

$$\Delta_F(\mathbb{E}_0) = \sqrt{\frac{(2z+1)[z(|\gamma_1| - |\gamma_2|)^2 + 2|\gamma_1\gamma_2|]}{z^2(|\gamma_1| - |\gamma_2|)^2 + z(|\gamma_1| + |\gamma_2|)(\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|})^2}} d_F(\mathbb{E}_0, \mathcal{U}_{\tilde{Q}}).$$

□

**Lemma 1** ([42]). *The Frobenius distance  $d_F(\mathbb{E}, \mathcal{U})$  between a matrix  $\mathbb{E} \in \mathbb{C}^{z \times z}$  and the normal matrix family  $\mathcal{U}$  is bounded by:*

$$\frac{\Delta_F(\mathbb{E})}{\sqrt{z}} \leq d_F(\mathbb{E}, \mathcal{U}) \leq \Delta_F(\mathbb{E}). \quad (25)$$

which Equation (25) explicitly links the deviation  $\Delta_F(\mathbb{E}_0)$  to the distance.

**Corollary 2.** *For  $z > 1$ , define  $\theta$  as*

$$\theta = \sqrt{\frac{(2z+1)[z(|\gamma_1| - |\gamma_2|)^2 + 2|\gamma_1\gamma_2|]}{z^2(|\gamma_1| - |\gamma_2|)^2 + z(|\gamma_1| + |\gamma_2|)(\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|})^2}}, \quad (26)$$

such that Equation (23) reduces to

$$\Delta_F(\mathbb{E}_0) = \theta d_F(\mathbb{E}_0, \mathcal{U}_{\tilde{Q}}). \quad (27)$$

Substituting this into Equation (25) yields

$$\frac{\theta}{\sqrt{z}} d_F(\mathbb{E}_0, \mathcal{U}_{\tilde{Q}}) \leq d_F(\mathbb{E}_0, \mathcal{U}) \leq \theta d_F(\mathbb{E}_0, \mathcal{U}_{\tilde{Q}}). \quad (28)$$

## 4. Spectral Distance and Normalization Analysis

### 4.1. The Distance of the Spectra of $\mathbb{E}$ and $\mathbb{E}^*$

**Theorem 5.** *Let  $\mathbb{E}^*$  be the nearest normal PDNT Toeplitz matrix to  $\mathbb{E}$ . Define the eigenvalue vectors of  $\mathbb{E}$  and  $\mathbb{E}^*$  as*

$$\begin{aligned} \vec{A} &= [\lambda_1(\mathbb{E}), \lambda_2(\mathbb{E}), \dots, \lambda_z(\mathbb{E})], \\ \vec{A}^* &= [\lambda_1(\mathbb{E}^*), \lambda_2(\mathbb{E}^*), \dots, \lambda_z(\mathbb{E}^*)], \end{aligned}$$

whose Euclidean distance is given by the piecewise formula

$$\|\vec{A} - \vec{A}^*\|_2 = \begin{cases} \frac{z}{\sqrt{2z+1}} (\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|})^2, & z > 1, \\ \frac{2z}{2z+1} (\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|})^2, & z = 1. \end{cases} \quad (29)$$

Moreover, the limit

$$\lim_{\mathbb{E} \rightarrow \mathbb{E}^*} \frac{\|\vec{A} - \vec{A}^*\|_2}{d_F(\mathbb{E}, \mathcal{U}_{\tilde{Q}})} = 0 \quad (30)$$

holds, where  $d_F(\mathbb{E}, \mathcal{U}_{\tilde{Q}})$  is specified in (20).

**Proof.** By combining the eigenvalue expressions in Equations (10) and (19) with the trigonometric summation identity in Equation (22), we derive the spectral distance as follows:

$$\begin{aligned} \|\vec{A} - \vec{A}^*\|_2 &= \sqrt{\sum_{j=1}^z |\lambda_j(\mathbb{E}) - \lambda_j(\mathbb{E}^*)|^2} \\ &= \frac{2z}{2z+1} (\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|})^2 \sqrt{\sum_{j=1}^z \left| \cos \frac{(2j-1)\pi}{2z-1} \right|^2}. \\ \|\vec{A} - \vec{A}^*\|_2 &= \begin{cases} \frac{z}{\sqrt{2z+1}} (\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|})^2, z > 1, \\ \frac{2z}{2z+1} (\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|})^2, z = 1. \end{cases} \end{aligned}$$

Equation (30) is established by substituting Equations (20) and (29) into the limit expression followed by algebraic simplifications.  $\square$

**Theorem 6.** For the case where  $\mathbb{E} \notin \mathcal{U}_{\tilde{Q}}$ , it logically follows that

$$\frac{\|\vec{A} - \vec{A}^*\|_2}{d_F(\mathbb{E}, \mathcal{U}_{\tilde{Q}})} = \begin{cases} \frac{\sqrt{z} |\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|}|}{\sqrt{z (\sqrt{|\gamma_1|} + \sqrt{|\gamma_2|})^2 + \|\gamma_1 - \gamma_2\|}}, & z > 1, \\ \frac{2 |\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|}|}{\sqrt{3 (\sqrt{|\gamma_1|} + \sqrt{|\gamma_2|})^2 + \|\gamma_1 - \gamma_2\|}}, & z = 1. \end{cases} \quad (31)$$

**Proof.** The combined application of Equations (20) and (29) establishes the quantitative relation between the eigenvalue vector distance and the Frobenius distance expressed as follows:

when  $z = 1$ ,

$$\frac{\|\vec{A} - \vec{A}^*\|_2}{d_F(\mathbb{E}, \mathcal{U}_{\tilde{Q}})} = \frac{2\sqrt{z} |\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|}|}{\sqrt{(2z+1) \left( z (\sqrt{|\gamma_1|} + \sqrt{|\gamma_2|})^2 + \|\gamma_1 - \gamma_2\| \right)}},$$

when  $z > 1$ ,

$$\frac{\|\vec{A} - \vec{A}^*\|_2}{d_F(\mathbb{E}, \mathcal{U}_{\tilde{Q}})} = \frac{\sqrt{z} |\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|}|}{\sqrt{z (\sqrt{|\gamma_1|} + \sqrt{|\gamma_2|})^2 + \|\gamma_1 - \gamma_2\|}}.$$

$\square$

#### 4.2. Normalized Distance of Matrix $\mathbb{E}_0$ to the $\mathcal{U}_{\tilde{Q}}$ Family

**Theorem 7.** For a PDNT Toeplitz matrix  $\mathbb{E}_0$  with  $(\gamma_1, \gamma_2) \neq (0, 0)$ , the normalized Frobenius distance to the normal matrix family  $\mathcal{U}_{\tilde{Q}}$  is given

$$\frac{d_F(\mathbb{E}_0, \mathcal{U}_{\tilde{Q}})}{\|\mathbb{E}_0\|_F} = \sqrt{\frac{\frac{z(z+1)}{2z+1} (1 - \frac{|\gamma_2|}{|\gamma_1|})^2 - \frac{2z}{2z+1} \sqrt{\frac{|\gamma_2|}{|\gamma_1|}} (1 - \sqrt{\frac{|\gamma_2|}{|\gamma_1|}})^2}{\frac{|\gamma_2|}{|\gamma_1|} + z(1 + (\frac{|\gamma_2|}{|\gamma_1|})^2)}}, \quad (32)$$

with its value bounded in

$$0 \leq \frac{d_F(\mathbb{E}_0, \mathcal{U}_{\tilde{Q}})}{\|\mathbb{E}_0\|_F} < \sqrt{\frac{z+1}{2z+1}}.$$

**Proof.** For  $\gamma_1\gamma_2 \neq 0$ , Equation (20) yields the normalized structured distance

$$\begin{aligned} \frac{d_F(\mathbb{E}_0, \mathcal{U}_{\tilde{Q}})}{\|\mathbb{E}_0\|_F} &= \sqrt{\frac{\frac{z^2}{2z+1}(|\gamma_1| - |\gamma_2|)^2 + \frac{z(|\gamma_1| + |\gamma_2|)}{2z+1}(\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|})^2}{|\gamma_1\gamma_2| + z(|\gamma_1|^2 + |\gamma_2|^2)}} \\ &= \sqrt{\frac{\frac{z^2}{2z+1}(1 - \frac{|\gamma_2|}{|\gamma_1|})^2 + \frac{z}{2z+1}(1 + \frac{|\gamma_2|}{|\gamma_1|})(1 - \sqrt{\frac{|\gamma_2|}{|\gamma_1|}})^2}{|\frac{\gamma_2}{\gamma_1}| + z(1 + |\frac{\gamma_2}{\gamma_1}|^2)}} \\ &= \sqrt{\frac{\frac{z^2}{2z+1}(1 - \frac{|\gamma_1|}{|\gamma_2|})^2 + \frac{z}{2z+1}(1 + \frac{|\gamma_1|}{|\gamma_2|})(1 - \sqrt{\frac{|\gamma_1|}{|\gamma_2|}})^2}{|\frac{\gamma_1}{\gamma_2}| + z(1 + |\frac{\gamma_1}{\gamma_2}|^2)}}. \end{aligned}$$

The normalized distance decreases from  $\sqrt{\frac{z+1}{2z+1}}$  to 0 as either  $\frac{|\gamma_1|}{|\gamma_2|}$  or  $\frac{|\gamma_2|}{|\gamma_1|}$  increases from 0 to 1. So the bound is

$$0 \leq \frac{d_F(\mathbb{E}_0, \mathcal{U}_{\tilde{Q}})}{\|\mathbb{E}_0\|_F} < \sqrt{\frac{z+1}{2z+1}}.$$

□

**Remark 2.** From Equation (32), the normalized distance from matrix  $\mathbb{E}_0$  to  $\mathcal{U}_{\tilde{Q}}$  is

$$\frac{d_F(\mathbb{E}_0, \mathcal{U}_{\tilde{Q}})}{\|\mathbb{E}_0\|_F} = \begin{cases} 0 & |\gamma_1| = |\gamma_2|, \\ \sqrt{\frac{z+1}{2z+1}} & \text{exactly one of } \gamma_1 \text{ and } \gamma_2 \text{ is 0.} \end{cases} \quad (33)$$

**Remark 3.** The equality  $d_F(\mathbb{E}, \mathcal{U}_{\tilde{Q}}) = d_F(\mathbb{E}_0, \mathcal{U}_{\tilde{Q}})$  implies the normalized distance bound for the PDNT Toeplitz matrix  $\mathbb{E}$

$$\begin{aligned} 0 &\leq \frac{d_F(\mathbb{E}, \mathcal{U}_{\tilde{Q}})}{\|\mathbb{E}\|_F} = \frac{d_F(\mathbb{E}_0, \mathcal{U}_{\tilde{Q}})}{\|\mathbb{E}_0\|_F} \frac{\|\mathbb{E}_0\|_F}{\|\mathbb{E}\|_F} \\ &= \frac{d_F(\mathbb{E}_0, \mathcal{U}_{\tilde{Q}})}{\|\mathbb{E}_0\|_F} \sqrt{\frac{|\gamma_1\gamma_2| + z(|\gamma_1|^2 + |\gamma_2|^2)}{|\gamma_1\gamma_2| + z(|\gamma_1|^2 + |\gamma_2|^2) + z|\gamma_0|^2 - 2|\gamma_0|\sqrt{|\gamma_1\gamma_2|}}} \\ &\leq \frac{d_F(\mathbb{E}_0, \mathcal{U}_{\tilde{Q}})}{\|\mathbb{E}_0\|_F} \leq \sqrt{\frac{z+1}{2z+1}}, \end{aligned}$$

with the upper bound attained if and only if  $\gamma_0 = 0$ ,  $\sqrt{|\gamma_1\gamma_2|} = 0$  and  $\mathbb{E}$  is bidiagonal.

## 5. The Distance Between PDNT Toeplitz Matrix $\mathbb{E}$ and Matrix Family $\mathcal{V}_{\tilde{Q}}$

**Remark 4.** For the PDNT Toeplitz matrix  $\mathbb{E} = (z; -\sqrt{\gamma_1\gamma_2}, \gamma_2, \gamma_0, \gamma_1, \sqrt{2}\gamma_2, \sqrt{2}\gamma_1)$ , the minimal distance matrix  $\mathbb{E}^+$  in  $\mathcal{V}_{\tilde{Q}}$  (with respect to Frobenius norm) is given by

$$\mathbb{E}^+ = \begin{cases} (z; 0, \gamma_2, \gamma_0, 0, \sqrt{2}\gamma_2, 0) & |\gamma_1| = \min\{|\gamma_1|, |\gamma_2|\}, \\ (z; 0, 0, \gamma_0, \gamma_1, 0, \sqrt{2}\gamma_1) & |\gamma_2| = \min\{|\gamma_1|, |\gamma_2|\}. \end{cases} \quad (34)$$

**Corollary 3.** For the PDNT Toeplitz matrix  $\mathbb{E}$ , the Frobenius distance to the family  $\mathcal{V}_{\tilde{Q}}$  is calculated as follows:

- when  $\mathbb{E} \in \tilde{Q}$ ,

$$d_F(\mathbb{E}, \mathcal{V}_{\tilde{Q}}) = d_F(\mathbb{E}, \mathbb{E}^+) = \sqrt{z \min\{|\gamma_1|, |\gamma_2|\}^2 + |\gamma_1\gamma_2|}, \quad (35)$$

- when  $\mathbb{E} \in \mathcal{U}_{\tilde{Q}}$ ,

$$d_F(\mathbb{E}, \mathcal{V}_{\tilde{Q}}) = d_F(\mathbb{E}, \mathbb{E}^+) = \sqrt{z}|\gamma_1| = \sqrt{z}|\gamma_2|,$$

- when  $\mathbb{E} \notin \mathcal{U}_{\tilde{Q}}$ ,

$$d_F(\mathbb{E}^*, \mathcal{V}_{\tilde{Q}}) = d_F(\mathbb{E}^*, \mathbb{E}^+) = \sqrt{z} \left( \frac{\sqrt{|\gamma_1 \gamma_2|} + z(|\gamma_1| + |\gamma_2|)}{2z + 1} \right), \quad (36)$$

where  $\mathbb{E}^*$  is the matrix in  $\mathcal{U}_{\tilde{Q}}$  closest to  $\mathbb{E}$ , and  $\mathbb{E}^+$  is defined by Equation (34).

From Equations (35) and (36), we deduce

$$\begin{aligned} d_F(\mathbb{E}^*, \mathcal{V}_{\tilde{Q}}) - d_F(\mathbb{E}, \mathcal{V}_{\tilde{Q}}) &= d_F(\mathbb{E}^*, \mathbb{E}^+) - d_F(\mathbb{E}, \mathbb{E}^+) \\ &= \sqrt{z} \left( \frac{\sqrt{|\gamma_1 \gamma_2|} + z(|\gamma_1| + |\gamma_2|)}{2z + 1} \right) - \sqrt{z \min\{|\gamma_1|, |\gamma_2|\}^2 + |\gamma_1 \gamma_2|} \\ &= U d_F(\mathbb{E}, \mathcal{U}_{\tilde{Q}}), \end{aligned}$$

$$\text{where } U = \frac{\sqrt{z} \left( \frac{\sqrt{|\gamma_1 \gamma_2|} + z(|\gamma_1| + |\gamma_2|)}{2z + 1} \right) - \sqrt{z \min\{|\gamma_1|, |\gamma_2|\}^2 + |\gamma_1 \gamma_2|}}{\sqrt{\frac{z^2}{2z+1} (|\gamma_1| - |\gamma_2|)^2 + \frac{z(|\gamma_1| + |\gamma_2|)}{2z+1} (\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|})^2}}.$$

We introduce the ratio  $g$  as

$$g = \frac{\min\{|\gamma_1|, |\gamma_2|\}}{\max\{|\gamma_1|, |\gamma_2|\}}, \quad (37)$$

which will be utilized to estimate the bounds of the normalized distance between  $\mathbb{E}_0$  and  $\mathcal{V}_{\tilde{Q}}$  in the following proof.

**Theorem 8.** When  $\gamma_0 = 0$ , the normalized Frobenius distance from a PDNT Toeplitz matrix  $\mathbb{E}_0$  to the matrix family  $\mathcal{V}_{\tilde{Q}}$  satisfies

$$\frac{d_F(\mathbb{E}_0, \mathcal{V}_{\tilde{Q}})}{\|\mathbb{E}_0\|_F} \leq \sqrt{\frac{z+1}{2z+1}},$$

with equality holding when  $\mathbb{E}_0$  is normal.

**Proof.** Let  $g = \frac{|\gamma_1|}{|\gamma_2|}$ . The following relation can be derived from Equations (7) and (34),

$$\frac{d_F(\mathbb{E}_0, \mathcal{V}_{\tilde{Q}})}{\|\mathbb{E}_0\|_F} = \frac{d_F(\mathbb{E}_0, \mathbb{E}^+)}{\|\mathbb{E}_0\|_F} = \frac{\sqrt{z|\gamma_1|^2 + |\gamma_1 \gamma_2|}}{\sqrt{z(|\gamma_1|^2 + |\gamma_2|^2) + |\gamma_1 \gamma_2|}} = \frac{\sqrt{zg^2 + g}}{\sqrt{z(g^2 + 1) + g}} \leq \sqrt{\frac{z+1}{2z+1}}.$$

With  $g$  gradually changing from 1 to 0, the normalized structured distance transitions from  $\sqrt{\frac{z+1}{2z+1}}$  to 0. For  $g = \frac{|\gamma_2|}{|\gamma_1|}$ , an entirely analogous derivation can be made using symmetry.  $\square$

By considering what has been discussed previously, we can derive the subsequent conclusions:

- (1) When  $|\gamma_1| = |\gamma_2|$ ,  $\frac{d_F(\mathbb{E}_0, \mathcal{V}_{\tilde{Q}})}{\|\mathbb{E}_0\|_F} = \sqrt{\frac{z+1}{2z+1}}$ ;
- (2) When  $\gamma_2 \neq 0$ ,  $\lim_{|\gamma_1| \rightarrow 0} \frac{d_F(\mathbb{E}_0, \mathcal{V}_{\tilde{Q}})}{\|\mathbb{E}_0\|_F} = 0$ ;
- (3) When  $\gamma_1 \neq 0$ ,  $\lim_{|\gamma_2| \rightarrow 0} \frac{d_F(\mathbb{E}_0, \mathcal{V}_{\tilde{Q}})}{\|\mathbb{E}_0\|_F} = 1$ .

## 6. Examples of Parameter

This section numerically verifies the properties of PDNT Toeplitz matrices and their eigenvalues as analyzed previously. The spectral distance between  $\mathbb{E}$  and its closest normal matrix  $\mathbb{E}^*$  is obtained through Equations (20) and (31).

$$\|\vec{A}(\mathbb{E}) - \vec{A}(\mathbb{E}^*)\|_2 = \begin{cases} \frac{\sqrt{z}|\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|}|}{\sqrt{z(\sqrt{|\gamma_1|} + \sqrt{|\gamma_2|})^2 + ||\gamma_1| - |\gamma_2||}} d_F(\mathbb{E}, \mathcal{U}_{\tilde{Q}}), & z > 1, \\ \frac{2|\sqrt{|\gamma_1|} - \sqrt{|\gamma_2|}|}{\sqrt{3[(\sqrt{|\gamma_1|} + \sqrt{|\gamma_2|})^2 + ||\gamma_1| - |\gamma_2||]}} d_F(\mathbb{E}, \mathcal{U}_{\tilde{Q}}), & z = 1. \end{cases} \quad (38)$$

For the matrix  $\mathbb{E}_{(g)}$  defined in

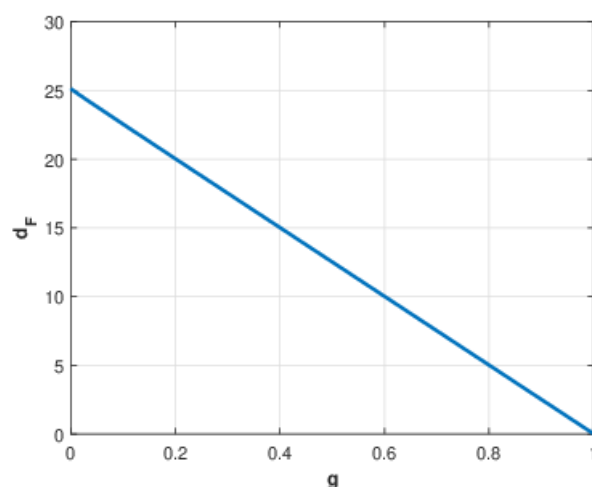
$$\mathbb{E}_{(g)} = (50; -\sqrt{-5g(4+3i)}, (4+3i)g, 16-3i, -5, \sqrt{2}(4+3i)g, -5\sqrt{2}), \quad (39)$$

with  $0 < g < 1$ , where  $g$  is the ratio in (37).

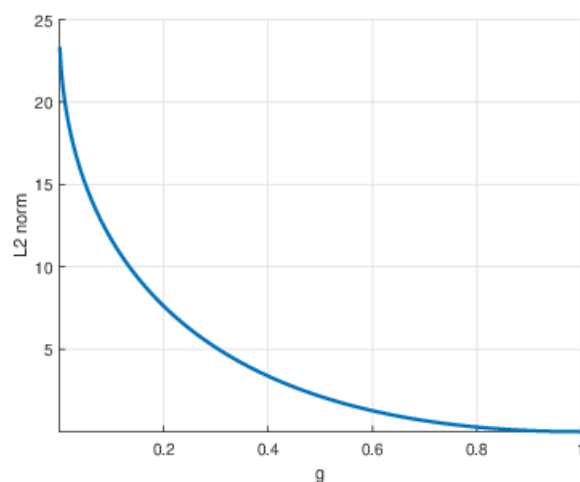
Figure 1 illustrates the linear relationship between the variable  $d_F(\mathbb{E}_{(g)}, \mathcal{U}_{\tilde{Q}})$  and  $g$ . It can be seen that as  $g$  gradually increases from 0 to 1,  $d_F(\mathbb{E}_{(g)}, \mathcal{U}_{\tilde{Q}})$  shows a linear decreasing trend. Figure 2 depicts the variation law of the  $\|\vec{A}(\mathbb{E}_{(g)}) - \vec{A}(\mathbb{E}_{(g)}^*)\|_2$  with respect to the parameter  $g$ . When  $g$  starts to increase from 0, the spectral norm decays rapidly, showing a nonlinear decreasing trend; as  $g$  approaches 1, the spectral norm gradually becomes gentle. Table 1 presents numerical results of matrix  $\mathbb{E}_{(g)}$ . Based on the example related to parameter  $g$  established by Equation (39), the above results can be calculated by combining Equations (20) and (38). By combining these formulas, what is shown in the figure can be strictly proven.

**Table 1.** Partial parameter examples.

| $g$ | $d_F(\mathbb{E}_{(g)}, \mathcal{U}_{\tilde{Q}})$ | $\ \vec{A}(\mathbb{E}_{(g)}) - \vec{A}(\mathbb{E}_{(g)}^*)\ _2$ |
|-----|--------------------------------------------------|-----------------------------------------------------------------|
| 0.1 | 22.5300                                          | 11.6439                                                         |
| 0.3 | 17.5074                                          | 5.1011                                                          |
| 0.5 | 12.5018                                          | 2.1413                                                          |
| 0.9 | 2.5000                                           | 0.0658                                                          |



**Figure 1.** The relationship between  $d_F(\mathbb{E}_{(g)}, \mathcal{U}_{\tilde{Q}})$  and  $g$  (horizontal axis:  $g$ ; vertical axis:  $d_F(\mathbb{E}_{(g)}, \mathcal{U}_{\tilde{Q}})$ , denoted as  $d_F$ ).



**Figure 2.** The relationship between  $\|\vec{A}(\mathbb{E}_{(g)}) - \vec{A}(\mathbb{E}_{(g)}^*)\|_2$  and  $g$  (horizontal axis:  $g$ ; vertical axis:  $\|\vec{A}(\mathbb{E}_{(g)}) - \vec{A}(\mathbb{E}_{(g)}^*)\|_2$ , denoted as L2 norm).

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