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Article

New Class of Specific Functions with Fractional Derivatives

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Abstract

Let $\mathcal{A}_n$ be the class of specific analytic functions $f(z) = z + \sum_{k=1}^{\infty} a_{1+\frac{k}{n}} z^{1+\frac{k}{n}}$ $(n \in \mathbb{N} = \{1, 2, 3, \cdots\})$ in the open unit disk $\mathbb{U}$. For $f(z) \in \mathcal{A}_n$, fractional derivatives $D_z^{\lambda} f(z)$ and $D_z^{j+\lambda} f(z)$ $(0 \leq \lambda < 1, j \in \mathbb{N})$ are defined by using Gamma functions. Applying such fractional derivatives, we introduce a new subclass $\mathcal{A}_n(j, \lambda, \alpha, \beta)$ of $\mathcal{A}_n$. In this paper, we establish sufficient conditions for $f(z)$ for $\mathcal{A}_n(j, \lambda, \alpha, \beta)$, coefficient inequalities for $|a_{1+\frac{1}{n}}|$ and $|a_{1+\frac{k}{n}}|$ $(k = 2, 3, 4, \cdots)$ of $f(z) \in \mathcal{A}_n(j, \lambda, \alpha, \beta)$, and some interesting argument properties of fractional derivatives for $f(z) \in \mathcal{A}_n$ through an example.

Keywords: specific function; fractional integral; fractional derivative; subordination; coefficient inequality; argument property

MSC: 30C45; 30C80

1. Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ that are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0$ and $f'(0) = 1$. For this class, we introduce a subclass $\mathcal{A}_n$ of $\mathcal{A}$ consisting of specific functions $f(z)$ given by

$$f(z) = z + \sum_{k=1}^{\infty} a_{1+\frac{k}{n}} z^{1+\frac{k}{n}} \quad (n \in \mathbb{N} = \{1, 2, 3, \cdots\}).$$

This class was introduced by Güney, Breaz and Owa [1]. They considered $f(z) \in \mathcal{A}_n$ given by

$$f(z) = \frac{z}{(1-\sqrt[n]{z})^{2n(1-\alpha)}} = z + \sum_{k=1}^{\infty} \frac{\prod_{\ell=1}^{k}(\ell + 2n(1-\alpha) - 1)}{k!} z^{1+\frac{k}{n}} \quad (n \in \mathbb{N} = \{1, 2, 3, \cdots\})$$

for $0 \leq \alpha < 1$ and showed that

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} = Re\left\{\frac{1 + (1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}}\right\} > \alpha \quad (z \in \mathbb{U}).$$

For $f(z) \in \mathcal{A}_n$, we define the fractional integral of order λ as follows:

$$D_z^{-\lambda} f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(t)}{(z-t)^{1-\lambda}} dt \quad (\lambda > 0),$$

where $\Gamma(\lambda)$ is the Gamma function.

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The fractional derivative of order λ $(0 \leq \lambda < 1)$ for $f(z) \in \mathcal{A}_n$ is defined by

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(t)}{(z-t)^{\lambda}} dt.$$

Further, $D_z^{j+\lambda} f(z)$ $(j \in \mathbb{N})$ is defined by

$$D_z^{j+\lambda} f(z) = \frac{d^j}{dz^j} \left(D_z^{\lambda} f(z) \right).$$

The definitions $D_z^{-\lambda} f(z)$, $D_z^{\lambda} f(z)$ and $D_z^{j+\lambda} f(z)$ were given by Owa [2,3]. With the above definitions, we know that

$$D_z^0 f(z) = f(z),$$
$$D_z^j f(z) = f^{(j)}(z)$$

and

$$D_z^{j+\lambda} f(z) = D_z^{\lambda} \left(f^{(j)}(z) \right).$$

Thus, we have

$$D_z^{\lambda} f(z) = \frac{1}{\Gamma(2-\lambda)} z^{1-\lambda} + \sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)} a_{1+\frac{k}{n}} z^{1+\frac{k}{n}-\lambda}$$

and

$$D_z^{j+\lambda} f(z) = \frac{\prod_{\ell=1}^{j}(2-\lambda-\ell)}{\Gamma(2-\lambda)} z^{1-\lambda-j} + \sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)} \left(\prod_{\ell=1}^{j} \left(\frac{k}{n} + 2 - \lambda - \ell \right) \right) a_{1+\frac{k}{n}} z^{\frac{k}{n}+1-\lambda-j}$$

for $j \in \mathbb{N}$. With the above the fractional derivative, we introduce the subclass $\mathcal{A}_n(j, \lambda, \alpha, \beta)$ of $\mathcal{A}_n$ as follows:

$$Re\left(e^{i\alpha} \frac{z D_z^{j+\lambda} f(z)}{(2-\lambda-j) D_z^{j-1+\lambda} f(z)} \right) > \beta \cos\alpha \quad (z \in \mathbb{U}) \tag{1}$$

with some real α $(|\alpha| < \frac{\pi}{2})$ and β $(0 \leq \beta < 1)$, where $j \in \mathbb{N}$ and $0 \leq \lambda < 1$.

If $j = 1$, $\lambda = 0$ and $\alpha = 0$ in (1), then we have

$$Re\left(\frac{z f'(z)}{f(z)} \right) > \beta \quad (z \in \mathbb{U})$$

for the class $\mathcal{A}_n(1, 0, 0, \beta)$. Functions $f(z) \in \mathcal{A}_n$ in the class $\mathcal{A}_n(1, 0, 0, \beta)$ were discussed by Güney, Breaz and Owa [1]. Let us consider a function $f(z) \in \mathcal{A}_n$ given by

$$D_z^{j-1+\lambda} f(z) = \frac{z^{2-\lambda-j}}{\Gamma(2-\lambda)(1-\sqrt[n]{z})^t} \tag{2}$$

with

$$t = 2n(2-\lambda-j)(1-\beta)e^{-i\alpha}\cos\alpha \quad (j \in \mathbb{N}).$$

Then, $f(z)$ satisfies

$$\frac{z D_z^{j+\lambda} f(z)}{D_z^{j-1+\lambda} f(z)} = (2-\lambda-j) + \frac{t}{n} \left(\frac{\sqrt[n]{z}}{1-\sqrt[n]{z}} \right)$$

and

$$Re\left(e^{i\alpha}\frac{zD_z^{j+\lambda}f(z)}{(2-\lambda-j)D_z^{j-1+\lambda}f(z)}\right) = \cos\alpha + 2(1-\beta)\cos\alpha Re\left(\frac{\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)$$
$$> \cos\alpha - (1-\beta)\cos\alpha$$
$$= \beta\cos\alpha \quad (z \in \mathbb{U}).$$

Therefore, $f(z) \in \mathcal{A}_n$ given by (2) belongs to the class $\mathcal{A}_n(j,\lambda,\alpha,\beta)$. Accordingly, it is very important to provide example functions $f(z)$ for the new class $\mathcal{A}_n(j,\lambda,\alpha,\beta)$.

2. Conditions of $f(z)$ for the Class $\mathcal{A}_n(j,\lambda,\alpha,\beta)$

We first consider the following condition on $f(z)$ for the class $\mathcal{A}_n(j,\lambda,\alpha,\beta)$.

Theorem 1. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\left|\frac{zD_z^{j+\lambda}f(z)}{(2-\lambda-j)D_z^{j-1+\lambda}f(z)} - 1\right| < (1-\beta)\cos\alpha \quad (z \in \mathbb{U}) \tag{3}$$

for some real α ($|\alpha| < \frac{\pi}{2}$), $0 \le \beta < 1$, $0 \le \lambda < 1$ and $j \in \mathbb{N}$, then $f(z) \in \mathcal{A}_n(j,\lambda,\alpha,\beta)$.

Proof. We define a function $w(z)$ as follows:

$$\frac{zD_z^{j+\lambda}f(z)}{(2-\lambda-j)D_z^{j-1+\lambda}f(z)} - 1 = (1-\beta)\cos\alpha w(z) \quad (z \in \mathbb{U})$$

for $f(z) \in \mathcal{A}_n$ which satisfies (3). Then, we see that $w(z)$ is analytic in $\mathbb{U}$, that $w(0) = 0$ and that $|w(z)| < 1$ $(z \in \mathbb{U})$. For such a function $w(z)$, we have

$$Re\left(e^{i\alpha}\frac{zD_z^{j+\lambda}f(z)}{(2-\lambda-j)D_z^{j-1+\lambda}f(z)}\right) = Re(e^{i\alpha}(1+(1-\beta)\cos\alpha w(z)))$$
$$= \cos\alpha + (1-\beta)\cos\alpha Re(e^{i\alpha}w(z))$$
$$\ge \cos\alpha - (1-\beta)\cos\alpha|e^{i\alpha}w(z)|$$
$$> \beta\cos\alpha \quad (z \in \mathbb{U}).$$

This shows us that $f(z) \in \mathcal{A}_n(j,\lambda,\alpha,\beta)$. □

If we consider $j = 1$ and $\lambda = 0$ in Theorem 1, then we have the following corollary.

Corollary 1. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < (1-\beta)\cos\alpha \quad (z \in \mathbb{U})$$

for some real α ($|\alpha| < \frac{\pi}{2}$) and $0 \le \beta < 1$, then

$$Re\left(e^{i\alpha}\frac{zf'(z)}{f(z)}\right) > \beta\cos\alpha \quad (z \in \mathbb{U}).$$

Next, we derive the following theorem.

Theorem 2. *If $f(z) \in \mathcal{A}_n$ satisfies*

$$\Gamma(1-\lambda)\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)}\left(1-\lambda+\frac{k}{n(1-\beta)}\sec\alpha\right)\left|a_{1+\frac{k}{n}}\right| \leq 1 \tag{4}$$

for some real α $(|\alpha| < \frac{\pi}{2})$, $0 \leq \beta < 1$ and $0 \leq \lambda < 1$, then $f(z) \in \mathcal{A}_n(1,\lambda,\alpha,\beta)$. If $f(z) \in \mathcal{A}_n$ satisfies

$$\begin{aligned}\frac{\Gamma(2-\lambda)}{\prod_{\ell=1}^{j}(2-\lambda-\ell)}\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)}&\left(2-\lambda-j+\frac{k}{n(1-\beta)}\sec\alpha\right)\\ &\times\left(\prod_{\ell=1}^{j-1}\left(\frac{k}{n}+2-\lambda-\ell\right)\right)\left|a_{1+\frac{k}{n}}\right| \leq 1\end{aligned} \tag{5}$$

for some real α $(|\alpha| < \frac{\pi}{2})$, $0 \leq \beta < 1$, $0 \leq \lambda < 1$ and $j = 2,3,\ldots$, then $f(z) \in \mathcal{A}_n(1,\lambda,\alpha,\beta)$.

Proof. For $f(z) \in \mathcal{A}_n$ and $j \geq 2$, we have that

$$\begin{aligned} zD_z^{j+\lambda}f(z) - (2-\lambda-j)D_z^{j-1+\lambda}f(z) &= \frac{\prod_{\ell=1}^{j}(2-\lambda-\ell)}{\Gamma(2-\lambda)}z^{2-\lambda-j}\\ &+\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)}\left(\prod_{\ell=1}^{j}\left(\frac{k}{n}+2-\lambda-\ell\right)\right)a_{1+\frac{k}{n}}z^{\frac{k}{n}+2-\lambda-j}\\ &-(2-\lambda-j)\left\{\frac{\prod_{\ell=1}^{j-1}(2-\lambda-\ell)}{\Gamma(2-\lambda)}z^{2-\lambda-j}\right.\\ &\left.+\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)}\left(\prod_{\ell=1}^{j-1}\left(\frac{k}{n}+2-\lambda-\ell\right)\right)a_{1+\frac{k}{n}}z^{\frac{k}{n}+2-\lambda-j}\right\}\\ &=\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)}\left(\frac{k}{n}\right)\left(\prod_{\ell=1}^{j-1}\left(\frac{k}{n}+2-\lambda-\ell\right)\right)a_{1+\frac{k}{n}}z^{\frac{k}{n}+2-\lambda-j}. \end{aligned}$$

It follows from the above that

$$\begin{aligned} &\left|\frac{zD_z^{j+\lambda}f(z)}{(2-\lambda-j)D_z^{j-1+\lambda}f(z)}-1\right|\\ &=\left|\frac{\frac{\Gamma(2-\lambda)}{\prod_{\ell=1}^{j}(2-\lambda-\ell)}\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)}\left(\frac{k}{n}\right)\left(\prod_{\ell=1}^{j-1}\left(\frac{k}{n}+2-\lambda-\ell\right)\right)a_{1+\frac{k}{n}}z^{\frac{k}{n}}}{1+\frac{\Gamma(2-\lambda)}{\prod_{\ell=1}^{j}(2-\lambda-\ell)}\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)}\left(\prod_{\ell=1}^{j-1}\left(\frac{k}{n}+2-\lambda-\ell\right)\right)a_{1+\frac{k}{n}}z^{\frac{k}{n}}}\right|\\ &<\frac{\frac{\Gamma(2-\lambda)}{\prod_{\ell=1}^{j}(2-\lambda-\ell)}\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)}\left(\frac{k}{n}\right)\left(\prod_{\ell=1}^{j-1}\left(\frac{k}{n}+2-\lambda-\ell\right)\right)\left|a_{1+\frac{k}{n}}\right|}{1-\frac{\Gamma(2-\lambda)}{\prod_{\ell=1}^{j-1}(2-\lambda-\ell)}\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)}\left(\prod_{\ell=1}^{j-1}\left(\frac{k}{n}+2-\lambda-\ell\right)\right)\left|a_{1+\frac{k}{n}}\right|} \end{aligned}$$

for $z \in \mathbb{U}$. Thus, we know by Theorem 1 that if $f(z)$ satisfies

$$
\begin{aligned}
&\frac{\Gamma(2-\lambda)}{\prod_{\ell=1}^{j}(2-\lambda-\ell)} \sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)} \left(\frac{k}{n}\right) \left(\prod_{\ell=1}^{j-1}\left(\frac{k}{n}+2-\lambda-\ell\right)\right) \left|a_{1+\frac{k}{n}}\right| \\
&\leq (1-\beta)\cos\alpha \left\{1 - \frac{\Gamma(2-\lambda)}{\prod_{\ell=1}^{j-1}(2-\lambda-\ell)} \sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)} \left(\prod_{\ell=1}^{j-1}\left(\frac{k}{n}+2-\lambda-\ell\right)\right) \left|a_{1+\frac{k}{n}}\right| \right\},
\end{aligned} \tag{6}
$$

then $f(z) \in \mathcal{A}_n(j, \lambda, \alpha, \beta)$. Finally, we see that the inequality (6) implies (5).

If $j = 1$, we also have that

$$
\left|\frac{zD_z^{1+\lambda} f(z)}{(1-\lambda)D_z^{\lambda} f(z)} - 1\right| < \frac{\Gamma(1-\lambda)\sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)}\left(\frac{k}{n}\right)\left|a_{1+\frac{k}{n}}\right|}{1-\Gamma(2-\lambda)\sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)}\left|a_{1+\frac{k}{n}}\right|}.
$$

Noting that if $f(z)$ satisfies

$$
\begin{aligned}
&\Gamma(1-\lambda) \sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)} \left(\frac{k}{n}\right) \left|a_{1+\frac{k}{n}}\right| \\
&\leq (1-\beta)\cos\alpha \left(1 - \Gamma(2-\lambda) \sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)} \left|a_{1+\frac{k}{n}}\right|\right),
\end{aligned}
$$

we obtain the inequality in (4). Further, if we consider a function $f(z) \in \mathcal{A}_n$ given by

$$
f(z) = z + \sum_{k=1}^{\infty} c_{1+\frac{k}{n}} z^{1+\frac{k}{n}}
$$

with

$$
c_{1+\frac{k}{n}} = \frac{\Gamma\left(\frac{k}{n}+2-\lambda\right)e^{i\theta}}{\Gamma(1-\lambda)\Gamma\left(\frac{k}{n}+2\right)\left(1-\lambda+\frac{k}{n(1-\beta)}\sec\alpha\right)k(k+1)} \quad (0 \leq \theta < 2\pi),
$$

then $f(z)$ satisfies

$$
\begin{aligned}
&\Gamma(1-\lambda) \sum_{k=1}^{\infty} \frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)} \left(1-\lambda+\frac{k}{n(1-\beta)}\sec\alpha\right) \left|c_{1+\frac{k}{n}}\right| \\
&= \sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \sum_{k=1}^{\infty} \left(\frac{1}{k} - \frac{1}{k+1}\right) = 1.
\end{aligned}
$$

If we take a function $f(z) \in \mathcal{A}_n$ given by

$$
f(z) = z + \sum_{k=1}^{\infty} c_{1+\frac{k}{n}} z^{1+\frac{k}{n}}
$$

with

$$c_{1+\frac{k}{n}} = \frac{\left(\prod_{\ell=1}^{j}(2-\lambda-\ell)\right)\Gamma\left(\frac{k}{n}+2-\lambda\right)e^{i\theta}}{\Gamma(2-\lambda)\Gamma\left(\frac{k}{n}+2\right)\left(2-\lambda-j+\frac{k}{n(1-\beta)}\sec\alpha\right)} \times \frac{1}{\left(\prod_{\ell=1}^{j-1}\left(\frac{k}{n}+2-\lambda-\ell\right)\right)k(k+1)} \quad (0 \leq \theta < 2\pi),$$

then $f(z)$ satisfies the equality in (5). □

3. Coefficient Inequalities of $f(z)$ for $\mathcal{A}_n(j,\lambda,\alpha,\beta)$

We would like to discuss coefficient inequalities of $f(z)$ for the class $\mathcal{A}_n(j,\lambda,\alpha,\beta)$. We introduce the lemma established by Carathéodory [4].

Lemma 1. *If a function $p(z)$ given by*

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$$

is analytic in $\mathbb{U}$ and $\text{Re}\,p(z) > \alpha$ $(z \in \mathbb{U})$ with $0 \leq \alpha < 1$, then

$$|c_k| \leq 2(1-\alpha) \quad (z \in \mathbb{U}). \tag{7}$$

The equality in (7) is satisfied by $p(z)$ such that

$$p(z) = \frac{1+(1-2\alpha)z}{1-z} = 1 + 2(1-\alpha)\sum_{k=1}^{\infty} z^k.$$

With the above lemma, we see the following lemma.

Lemma 2. *If a function $p(z)$ given by*

$$p(z) = 1 + \sum_{k=1}^{\infty} c_{\frac{k}{n}} z^{\frac{k}{n}} \quad (n \in \mathbb{N})$$

is analytic in $\mathbb{U}$ and $\text{Re}\,p(z) > \alpha$ $(z \in \mathbb{U})$ with $0 \leq \alpha < 1$, then

$$|c_{\frac{k}{n}}| \leq 2(1-\alpha) \quad (k \in \mathbb{N}). \tag{8}$$

The equality in (8) is satisfied by $p(z)$ such that

$$p(z) = \frac{1+(1-2\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}} = 1 + 2(1-\alpha)\sum_{k=1}^{\infty} z^{\frac{k}{n}}.$$

Applying Lemma 2, we prove the following theorem.

Theorem 3. *If $f(z) \in \mathcal{A}_n(j,\lambda,\alpha,\beta)$, then*

$$\left|a_{1+\frac{1}{n}}\right| \leq \frac{2n\left|e^{i\alpha}-\beta\cos\alpha\right|\Gamma\left(\frac{1}{n}+2-\lambda\right)\prod_{\ell=1}^{j}|2-\lambda-\ell|}{\Gamma(2-\lambda)\Gamma\left(\frac{1}{n}+2\right)\prod_{\ell=1}^{j-1}\left|\frac{1}{n}+2-\lambda-\ell\right|} \tag{9}$$

and

$$\left|a_{1+\frac{k}{n}}\right| \leq \frac{2n\left|e^{i\alpha}-\beta\cos\alpha\right|\Gamma\left(\frac{k}{n}+2-\lambda\right)\prod_{\ell=1}^{j}|2-\lambda-\ell|}{k\Gamma(2-\lambda)\Gamma\left(\frac{k}{n}+2\right)\prod_{\ell=1}^{j-1}\left|\frac{k}{n}+2-\lambda-\ell\right|} \times\prod_{m=1}^{k-1}\left(1+\frac{2n}{m}\left|e^{i\alpha}-\beta\cos\alpha\right||2-\lambda-j|\right) \tag{10}$$

for $k=2,3,4,\ldots$, *where* $j=2,3,4,\ldots$. *The equalities in* (9) *and* (10) *are satisfied by* $f(z)\in\mathcal{A}_n$ *such that*

$$D_z^{j-1+\lambda}f(z)=\left(z\left(1-\sqrt[n]{z}\right)^t\right)^{2-\lambda-j}$$

with

$$t=2n\left(\beta\cos\alpha e^{-i\alpha}-1\right).$$

Proof. Let us consider a function $p(z)$ as follows:

$$p(z)=\frac{e^{i\alpha}\frac{zD_z^{j+\lambda}f(z)}{(2-\lambda-j)D_z^{j-1+\lambda}f(z)}-\beta\cos\alpha}{e^{i\alpha}-\beta\cos\alpha}=1+\sum_{k=1}^{\infty}c_{\frac{k}{n}}z^{\frac{k}{n}} \tag{11}$$

for $f(z)\in\mathcal{A}_n(j,\lambda,\alpha,\beta)$. Then, we know that $p(z)$ is analytic in $\mathbb{U}$, $p(0)=1$ and that $Rep(z)>0$ $(z\in\mathbb{U})$. Thus, Lemma 2 implies that

$$\left|c_{\frac{k}{n}}\right|\leq 2\quad(k\in\mathbb{N}) \tag{12}$$

and the equality in (12) is satisfied by

$$p(z)=\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}.$$

By (11), we see

$$\begin{aligned}&e^{i\alpha}zD_z^{j+\lambda}f(z)-\beta(2-\lambda-j)\cos\alpha D_z^{j-1+\lambda}f(z)\\&\quad=(2-\lambda-j)\left(e^{i\alpha}-\beta\cos\alpha\right)p(z)D_z^{j-1+\lambda}f(z).\end{aligned}$$

It is clear that

$$\begin{aligned}&e^{i\alpha}zD_z^{j+\lambda}f(z)-\beta(2-\lambda-j)\cos\alpha D_z^{j-1+\lambda}f(z)\\&\quad=\frac{\left(e^{i\alpha}-\beta\cos\alpha\right)}{\Gamma(2-\lambda)}\prod_{\ell=1}^{j}(2-\lambda-\ell)z^{2-\lambda-j}\\&\quad+\sum_{k=1}^{\infty}\left\{\left(e^{i\alpha}\left(\frac{k}{n}+2-\lambda-\ell\right)-\beta(2-\lambda-j)\right)\frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)}\right.\\&\quad\left.\times\left(\prod_{\ell=1}^{j-1}\left(\frac{k}{n}+2-\lambda-\ell\right)\right)\right\}a_{1+\frac{k}{n}}z^{\frac{k}{n}+2-\lambda-j}.\end{aligned} \tag{13}$$

Further, we calculate

$$
\begin{aligned}
&(2-\lambda-j)\Big(e^{i\alpha}-\beta\cos\alpha\Big)p(z)D_z^{j-1+\lambda}f(z)\\
&=(2-\lambda-j)\Big(e^{i\alpha}-\beta\cos\alpha\Big)\left(1+\sum_{k=1}^{\infty}c_{\frac{k}{n}}z^{\frac{k}{n}}\right)\times\left(\frac{\prod_{\ell=1}^{j-1}(2-\lambda-\ell)}{\Gamma(2-\lambda)}z^{2-\lambda-j}\right.\\
&\left.+\sum_{k=1}^{\infty}\frac{\Gamma\Big(\frac{k}{n}+2\Big)}{\Gamma\Big(\frac{k}{n}+2-\lambda\Big)}\left(\prod_{\ell=1}^{j-1}\Big(\frac{k}{n}+2-\lambda-\ell\Big)a_{1+\frac{k}{n}}z^{\frac{k}{n}+1-\lambda-j}\right)\right)\\
&=(2-\lambda-j)(e^{i\alpha}-\beta\cos\alpha)\left\{\frac{\prod_{\ell=1}^{j-1}(2-\lambda-\ell)}{\Gamma(2-\lambda)}z^{2-\lambda-j}\right.\\
&+\left(c_{\frac{1}{n}}\frac{\prod_{\ell=1}^{j-1}(2-\lambda-\ell)}{\Gamma(2-\lambda)}+\frac{\Gamma\Big(\frac{1}{n}+2\Big)}{\Gamma\Big(\frac{1}{n}+2-\lambda\Big)}\prod_{\ell=1}^{j-1}\Big(\frac{1}{n}+2-\lambda-\ell\Big)a_{1+\frac{1}{n}}\right)z^{\frac{1}{n}+2-\lambda-j}\\
&+\left(c_{\frac{2}{n}}\frac{\prod_{\ell=1}^{j-1}(2-\lambda-\ell)}{\Gamma(2-\lambda)}+c_{\frac{1}{n}}\frac{\Gamma\big(\frac{2}{n}+2\big)}{\Gamma\big(\frac{2}{n}+2-\lambda\big)}\prod_{\ell=1}^{j-1}\Big(\frac{1}{n}+2-\lambda-\ell\Big)a_{1+\frac{1}{n}}\right.\\
&\left.+\frac{\Gamma\big(\frac{2}{n}+2\big)}{\Gamma\big(\frac{2}{n}+2-\lambda\big)}\prod_{\ell=1}^{j-1}\Big(\frac{2}{n}+2-\lambda-\ell\Big)a_{1+\frac{2}{n}}\right)z^{\frac{2}{n}+2-\lambda-j}\\
&+\left(c_{\frac{3}{n}}\frac{\prod_{\ell=1}^{j-1}(2-\lambda-\ell)}{\Gamma(2-\lambda)}+c_{\frac{2}{n}}\frac{\Gamma\Big(\frac{1}{n}+2\Big)}{\Gamma\Big(\frac{1}{n}+2-\lambda\Big)}\prod_{\ell=1}^{j-1}\Big(\frac{2}{n}+2-\lambda-\ell\Big)a_{1+\frac{1}{n}}\right.\\
&+c_{\frac{1}{n}}\frac{\Gamma\big(\frac{2}{n}+2\big)}{\Gamma\big(\frac{2}{n}+2-\lambda\big)}\prod_{\ell=1}^{j-1}\Big(\frac{2}{n}+2-\lambda-\ell\Big)a_{1+\frac{2}{n}}\\
&\left.+\frac{\Gamma\big(\frac{3}{n}+2\big)}{\Gamma\big(\frac{3}{n}+2-\lambda\big)}\prod_{\ell=1}^{j-1}\Big(\frac{3}{n}+2-\lambda-\ell\Big)a_{1+\frac{3}{n}}\right)z^{\frac{3}{n}+2-\lambda-j}\\
&+\ldots+\left(c_{\frac{k}{n}}\frac{\prod_{\ell=1}^{j-1}(2-\lambda-\ell)}{\Gamma(2-\lambda)}\right.\\
&+c_{\frac{k-1}{n}}\frac{\Gamma\Big(\frac{1}{n}+2\Big)}{\Gamma\Big(\frac{1}{n}+2-\lambda\Big)}\prod_{\ell=1}^{j-1}\Big(\frac{1}{n}+2-\lambda-\ell\Big)a_{1+\frac{1}{n}}\\
&+c_{\frac{k-2}{n}}\frac{\Gamma\big(\frac{2}{n}+2\big)}{\Gamma\big(\frac{2}{n}+2-\lambda\big)}\prod_{\ell=1}^{j-1}\Big(\frac{2}{n}+2-\lambda-\ell\Big)a_{1+\frac{2}{n}}\\
&+\ldots+c_{\frac{2}{n}}\frac{\Gamma\Big(\frac{k-2}{n}+2\Big)}{\Gamma\Big(\frac{k-2}{n}+2-\lambda\Big)}\prod_{\ell=1}^{j-1}\Big(\frac{k-2}{n}+2-\lambda-\ell\Big)a_{1+\frac{k-2}{n}}\\
&+c_{\frac{1}{n}}\frac{\Gamma\Big(\frac{k-1}{n}+2\Big)}{\Gamma\Big(\frac{k-1}{n}+2-\lambda\Big)}\prod_{\ell=1}^{j-1}\Big(\frac{k-1}{n}+2-\lambda-\ell\Big)a_{1+\frac{k-1}{n}}\\
&\left.\left.+\frac{\Gamma\Big(\frac{k}{n}+2\Big)}{\Gamma\Big(\frac{k}{n}+2-\lambda\Big)}\prod_{\ell=1}^{j-1}\Big(\frac{k}{n}+2-\lambda-\ell\Big)a_{1+\frac{k}{n}}\right)z^{\frac{k}{n}+2-\lambda-j}+\ldots\right\}.
\end{aligned}
\tag{14}
$$

Therefore, for $k=1$, we have

$$\frac{e^{i\alpha}\Gamma\left(\frac{1}{n}+2\right)}{n\Gamma\left(\frac{1}{n}+2-\lambda\right)}\prod_{\ell=1}^{j-1}\left(\frac{1}{n}+2-\lambda-\ell\right)a_{1+\frac{1}{n}}$$
$$=c_{\frac{1}{n}}(2-\lambda-j)\left(e^{i\alpha}-\beta\cos\alpha\right)\frac{\prod_{\ell=1}^{j-1}(2-\lambda-\ell)}{\Gamma(2-\lambda)}.$$

Using $|c_{\frac{1}{n}}|\leq 2$ for $p(z)$, we have

$$\left|a_{1+\frac{1}{n}}\right|\leq\frac{2n|e^{i\alpha}-\beta\cos\alpha|\Gamma\left(\frac{1}{n}+2-\lambda\right)\prod_{\ell=1}^{j-1}|2-\lambda-\ell|}{\Gamma(2-\lambda)\Gamma\left(\frac{1}{n}+2\right)\prod_{\ell=1}^{j-1}\left|\frac{1}{n}+2-\lambda-\ell\right|}. \tag{15}$$

This proves the coefficient inequality (9) for $a_{1+\frac{1}{n}}$. □

Considering the coefficient for $z^{\frac{2}{n}+2-\lambda-j}$ in (13) and (14), we prove

$$\frac{2e^{i\alpha}\Gamma\left(\frac{2}{n}+2\right)}{n\Gamma\left(\frac{2}{n}+2-\lambda\right)}\prod_{\ell=1}^{j-1}\left(\frac{2}{n}+2-\lambda-\ell\right)a_{1+\frac{2}{n}}$$
$$=(2-\lambda-j)\left(e^{i\alpha}-\beta\cos\alpha\right)\left(c_{\frac{2}{n}}\frac{\prod_{\ell=1}^{j-1}(2-\lambda-\ell)}{\Gamma(2-\lambda)}\right.$$
$$\left.+c_{\frac{1}{n}}\frac{\Gamma\left(\frac{1}{n}+2\right)}{\Gamma\left(\frac{1}{n}+2-\lambda\right)}\prod_{\ell=1}^{j-1}\left(\frac{1}{n}+2-\lambda-\ell\right)a_{1+\frac{1}{n}}\right).$$

Therefore, using (8) and (15), it follows that

$$\left|a_{1+\frac{2}{n}}\right|\leq\frac{2n|e^{i\alpha}-\beta\cos\alpha|\Gamma\left(\frac{2}{n}+2-\lambda\right)\prod_{\ell=1}^{j-1}|2-\lambda-\ell|}{2\Gamma(2-\lambda)\Gamma\left(\frac{2}{n}+2\right)\prod_{\ell=1}^{j-1}\left|\frac{2}{n}+2-\lambda-\ell\right|}$$
$$\times\left(1+2n|2-\lambda-j|\left|e^{i\alpha}-\beta\cos\alpha\right|\right).$$

This shows the coefficient inequality for $a_{1+\frac{2}{n}}$ in (10).

Furthermore, we have from (13) and (14) that

$$\frac{ke^{i\alpha}\Gamma\left(\frac{k}{n}+2\right)}{n\Gamma\left(\frac{k}{n}+2-\lambda\right)}\prod_{\ell=1}^{j-1}\left(\frac{k}{n}+2-\lambda-\ell\right)a_{1+\frac{k}{n}}$$
$$=(2-\lambda-j)\left(e^{i\alpha}-\beta\cos\alpha\right)\left(c_{\frac{k}{n}}\frac{\prod_{\ell=1}^{j-1}(2-\lambda-\ell)}{\Gamma(2-\lambda)}\right.$$
$$+c_{\frac{k-1}{n}}\frac{\Gamma\left(\frac{1}{n}+2\right)}{\Gamma\left(\frac{1}{n}+2-\lambda\right)}\prod_{\ell=1}^{j-1}\left(\frac{1}{n}+2-\lambda-\ell\right)a_{1+\frac{1}{n}}$$
$$+c_{\frac{k-2}{n}}\frac{\Gamma\left(\frac{2}{n}+2\right)}{\Gamma\left(\frac{2}{n}+2-\lambda\right)}\prod_{\ell=1}^{j-1}\left(\frac{2}{n}+2-\lambda-\ell\right)a_{1+\frac{2}{n}}$$
$$\left.+\ldots+c_{\frac{1}{n}}\frac{\Gamma\left(\frac{k-1}{n}+2\right)}{\Gamma\left(\frac{k-1}{n}+2-\lambda\right)}\prod_{\ell=1}^{j-1}\left(\frac{k-1}{n}+2-\lambda-\ell\right)a_{1+\frac{k-1}{n}}\right).$$

Thus, by mathematical induction, we also prove that

$$\left|a_{1+\frac{k}{n}}\right| \leq \frac{2n|e^{i\alpha}-\beta\cos\alpha|\Gamma\left(\frac{k}{n}+2-\lambda\right)\prod_{\ell=1}^{j}|2-\lambda-\ell|}{k\Gamma(2-\lambda)\Gamma\left(\frac{k}{n}+2\right)\prod_{\ell=1}^{j-1}\left|\frac{k}{n}+2-\lambda-\ell\right|} \times\prod_{m=1}^{k-1}\left(1+\frac{2n}{m}\left|e^{i\alpha}-\beta\cos\alpha\right||2-\lambda-\ell|\right)$$

for $k=2,3,4,\ldots$.

Finally, considering a function $p(z)$ given by

$$p(z)=\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}=1+2\sum_{k=1}^{\infty}z^{\frac{k}{n}},$$

Equation (11) implies that

$$\begin{aligned}\frac{zD_z^{j+\lambda}f(z)}{D_z^{j-1+\lambda}f(z)}&=(2-\lambda-j)\left(e^{-i\alpha}\beta\cos\alpha+(1-e^{-i\alpha}\beta\cos\alpha)\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)\\&=(2-\lambda-j)\frac{1+(1-2e^{-i\alpha}\beta\cos\alpha)\sqrt[n]{z}}{1-\sqrt[n]{z}}\end{aligned}$$

and that

$$\frac{D_z^{j+\lambda}f(z)}{D_z^{j-1+\lambda}f(z)}=(2-\lambda-j)\left(\frac{1}{z}+2(1-e^{-i\alpha}\beta\cos\alpha)\frac{\sqrt[n]{z}}{z(1-\sqrt[n]{z})}\right).$$

This gives us

$$\left(\log D_z^{j-1+\lambda}f(z)\right)'=(2-\lambda-j)\left((\log z)'+2n(\beta\cos\alpha e^{-i\alpha}-1)\left(\log(1-\sqrt[n]{z})\right)'\right).$$

Thus, we obtain the following:

$$D_z^{j-1+\lambda}f(z)=\left(z(1-\sqrt[n]{z})^t\right)^{2-\lambda-j}$$

with

$$t=2n(\beta\cos\alpha e^{-i\alpha}-1).$$

Taking $\lambda=0$ in Theorem 3, we see the following corollary.

Corollary 2. *If $f(z)\in\mathcal{A}_n(j,0,\alpha,\beta)$, then*

$$\left|a_{1+\frac{1}{n}}\right| \leq \frac{2n\left|e^{i\alpha}-\beta\cos\alpha\right|\prod_{\ell=1}^{j}|2-\ell|}{\prod_{\ell=1}^{j-1}\left|\frac{1}{n}+2-\ell\right|}$$

and

$$\left|a_{1+\frac{k}{n}}\right| \leq \frac{2n\left|e^{i\alpha}-\beta\cos\alpha\right|\prod_{\ell=1}^{j}|2-\ell|}{k\prod_{\ell=1}^{j-1}\left|\frac{k}{n}+2-\ell\right|}\times\prod_{m=1}^{k-1}\left(1+\frac{2n}{m}\left|e^{i\alpha}-\beta\cos\alpha\right||2-j|\right)$$

for $k=2,3,4,\ldots$, where $j=2,3,4,\ldots$.

4. Argument Properties of Fractional Derivatives

For analytic functions $f(z)$ and $g(z)$ in $\mathbb{U}$, $f(z)$ is said to be subordinate to $g(z)$ in $\mathbb{U}$, written $f(z) \prec g(z)$ $(z \in \mathbb{U})$, if there exists a function $w(z)$ analytic in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ $(z \in \mathbb{U})$, such that $f(z) = g(w(z))$ $(z \in \mathbb{U})$. If $g(z)$ is univalent in $\mathbb{U}$, then $f(z) \prec g(z)$ $(z \in \mathbb{U})$ if and only if $f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$ [5]. For subordinations, Miller and Mocanu [6] gave the following lemma.

Lemma 3. *Let $\beta_0 = 1.21872\ldots$ be the solution of $\beta\pi = \frac{3}{2}\pi - Tan^{-1}\beta$ and $\alpha = \alpha(\beta) = \beta + 2Tan^{-1}\left(\frac{\beta}{\pi}\right)$ for $0 < \beta < \beta_0$. If $g(z)$ is analytic in $\mathbb{U}$ with $g(0) = 1$, then*

$$g(z) + zg'(z) \prec \left(\frac{1+z}{1-z}\right)^{\alpha} \quad (z \in \mathbb{U}) \tag{16}$$

gives

$$g(z) \prec \left(\frac{1+z}{1-z}\right)^{\beta} \quad (z \in \mathbb{U}). \tag{17}$$

It follows that the subordination given in (16) implies that

$$|\arg(g(z) + zg'(z))| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U})$$

and the subordination (17) implies that

$$|\arg g(z)| < \frac{\pi}{2}\beta \quad (z \in \mathbb{U}).$$

We also introduce the next lemma, which was established by Nunokawa [7].

Lemma 4. *Let $g(z)$ be analytic in $\mathbb{U}$ with $g(0) = 1$ and $g(z) \neq 0$ $(z \in \mathbb{U})$. If there exists a point $z_0 \in \mathbb{U}$ such that*

$$|\arg g(z)| < \frac{\pi}{2}\alpha \quad (|z| < |z_0|)$$

and

$$|\arg g(z_0)| = \frac{\pi}{2}\alpha$$

for some real $\alpha > 0$, then we have

$$\frac{z_0 g'(z_0)}{g(z_0)} = ik\alpha,$$

where

$$k \geq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad (for \arg g(z_0) = \frac{\pi}{2}\alpha), \tag{18}$$

$$k \leq \frac{1}{2}\left(a + \frac{1}{a}\right) \quad (for \arg g(z_0) = -\frac{\pi}{2}\alpha), \tag{19}$$

and

$$g(z_0)^{\frac{1}{\alpha}} = \pm ia \quad (a > 0).$$

Now we have the following theorem.

Theorem 4. *If $f(z) \in A_n$ satisfies*

$$\left|\arg\left(\frac{D_z^{j+\lambda} f(z)}{z^{1-j-\lambda}}\left(\frac{zD_z^{j+1+\lambda} f(z)}{D_z^{j+\lambda}} + j + \lambda\right)\right)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}) \tag{20}$$

for some $\alpha > 0$, $0 \leq \lambda < 1$ and $j \in \mathbb{N}$, then

$$\left|\arg\left(\frac{D_z^{j+\lambda} f(z)}{z^{1-j-\lambda}}\right)\right| < \frac{\pi}{2}\alpha\beta \quad (z \in \mathbb{U}),$$

where

$$\beta = \frac{2}{\pi}\int_0^1 Sin^{-1}\left(\frac{2\rho}{1+\rho^2}\right)d\rho = 0.55872876\ldots. \tag{21}$$

Proof. For $f(z) \in A_n$ satisfying (20), we define the function $g(z)$ as follows:

$$\begin{aligned} g(z) &= \frac{\Gamma(2-\lambda)D_z^{j+\lambda} f(z)}{\prod_{\ell=1}^{j}(2-\lambda-\ell)z^{1-j-\lambda}} \\ &= 1 + \frac{\Gamma(2-\lambda)}{\prod_{\ell=1}^{j}(2-\lambda-\ell)}\sum_{k=1}^{\infty}\frac{\Gamma\left(\frac{k}{n}+2\right)}{\Gamma\left(\frac{k}{n}+2-\lambda\right)}\prod_{\ell=1}^{j}\left(\frac{k}{n}+2-\lambda-\ell\right)a_{1+\frac{k}{n}}z^{\frac{k}{n}}. \end{aligned} \tag{22}$$

Then, $g(z)$ is analytic in $\mathbb{U}$ with $g(0) = 1$ and $g(z) \neq 0$ $(z \in \mathbb{U})$, as follows from (20). For the above-defined $g(z)$, we see that

$$\begin{aligned} \arg g(z) &= \arg\left\{\frac{1}{z}(zg(z))\right\} \\ &= \arg\left\{\frac{1}{z}\int_0^z (tg(t))'dt\right\} \\ &= \arg\left\{\frac{1}{z}\int_0^r \frac{d}{d\rho}(\rho e^{i\theta}g(\rho e^{i\theta}))e^{i\theta}d\rho\right\} \\ &= \arg\left\{\int_0^r\left(g(\rho e^{i\theta}) + \rho e^{i\theta}\frac{d}{d\rho}g(\rho e^{i\theta})\right)d\rho\right\}, \end{aligned}$$

where $z = re^{i\theta}$ and $t = \rho e^{i\theta}$. Let

$$0 = \rho_0 < \rho_1 < \rho_2 < \ldots < \rho_{n-1} < \rho_n = r\ ,\ \sum_{\ell=1}^{n}\rho_\ell = \rho$$

and

$$\rho_\ell - \rho_{\ell-1} = \delta \quad (\ell = 1, 2, 3, \ldots, n).$$

Then, we have

$$\begin{aligned} |\arg g(z)| &= \left|\arg\left\{\lim_{n\to\infty}\sum_{\ell=1}^{n}\delta(g(\rho_\ell e^{i\theta}) + \rho_\ell e^{i\theta}g'(\rho_\ell e^{i\theta}))\right\}\right| \\ &\leq \lim_{n\to\infty}\sum_{\ell=1}^{n}\delta\left|\arg(g(\rho_\ell e^{i\theta}) + \rho_\ell e^{i\theta}g'(\rho_\ell e^{i\theta}))\right|. \end{aligned}$$

Since

$$|\arg g(z)+zg'(z)|=\left|\arg\left(\frac{D_z^{j+\lambda}f(z)}{z^{1-j-\lambda}}\left(\frac{zD_z^{j+1+\lambda}f(z)}{D_z^{j+\lambda}f(z)}+j+\lambda\right)\right)\right|<\frac{\pi}{2}\alpha$$

by (20), using Lemma 4, we obtain the following:

$$\begin{aligned}|\arg g(z)| &= \left|\arg\left(\frac{D_z^{j+\lambda}f(z)}{z^{1-j-\lambda}}\right)\right| \\ &\leq \lim_{n\to\infty}\sum_{\ell=1}^{n}\delta\left|\arg\left(\frac{1+\rho_\ell e^{i\theta}}{1-\rho_\ell e^{i\theta}}\right)^{\alpha}\right| \\ &= \alpha\int_0^r Sin^{-1}\left(\frac{2\rho}{1+\rho^2}\right)d\rho \\ &< \alpha\int_0^1 Sin^{-1}\left(\frac{2\rho}{1+\rho^2}\right)d\rho=\frac{\pi}{2}\alpha\beta.\end{aligned}$$

This completes the proof of the theorem. □

Taking $j=1$ and $\lambda=0$ in Theorem 4, we have the following corollary.

Corollary 3. *If $f(z)\in A_n$ satisfies*

$$\left|\arg(f'(z)+zf''(z))\right|<\frac{\pi}{2}\alpha \quad (z\in\mathbb{U}),$$

for some $\alpha>0$, then

$$\left|\arg(f'(z)\right|<\frac{\pi}{2}\alpha\beta \quad (z\in\mathbb{U}),$$

where β is given by (21).

Remark 1. *If we take $\alpha=\frac{3}{2}$, then we have*

$$\alpha\beta=\frac{3}{\pi}\left(1-\frac{2}{\pi}\log 2\right)=0.533546701\ldots.$$

Next, applying Lemma 4, as developed by Nunokawa [7], we prove the following theorem.

Theorem 5. *If $f(z)\in A_n$ satisfies*

$$\left|\arg\left(\frac{zD_z^{j+1+\lambda}f(z)}{D_z^{j+\lambda}f(z)}+\frac{\Gamma(2-\lambda)D_z^{j+\lambda}f(z)}{\prod_{\ell=1}^{j}(2-\lambda-\ell)z^{1-j-\lambda}}+j+\lambda-1\right)\right|<Tan^{-1}\beta(\alpha) \quad (z\in\mathbb{U}) \tag{23}$$

for some α $(0<\alpha<1)$, $0\leq\lambda<1$ and $j\in\mathbb{N}$, then

$$\left|\arg\left(\frac{D_z^{j+\lambda}f(z)}{z^{1-j-\lambda}}\right)\right|<\frac{\pi}{2}\alpha \quad (z\in\mathbb{U}),$$

where

$$\beta(\alpha)=\frac{\alpha}{2}\left\{\left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{1-\alpha}{2}}+\left(\frac{1-\alpha}{1+\alpha}\right)^{\frac{1+\alpha}{2}}\right\}\sec\left(\frac{\pi}{2}\alpha\right)+\tan\left(\frac{\pi}{2}\alpha\right). \tag{24}$$

Proof. We consider a function $g(z)$ given by (22) for $f(z) \in A_n$. Then, $g(z)$ is analytic in $\mathbb{U}$ with $g(0) = 1$. For the above-defined $g(z)$, we see that

$$\frac{zg'(z)}{g(z)} + g(z) = \frac{zD_z^{j+1+\lambda} f(z)}{D_z^{j+\lambda} f(z)} + \frac{\Gamma(2-\lambda) D_z^{j+\lambda} f(z)}{\prod_{\ell=1}^{j} (2-\lambda-\ell) z^{1-j-\lambda}} + j + \lambda - 1.$$

We suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$|\arg g(z)| = \left| \arg\left(\frac{D_z^{j+\lambda} f(z)}{z^{1-j-\lambda}} \right) \right| < \frac{\pi}{2}\alpha \quad (|z| < |z_0|)$$

and

$$|\arg g(z_0)| = \left| \arg\left(\frac{D_z^{j+\lambda} f(z_0)}{z_0^{1-j-\lambda}} \right) \right| = \frac{\pi}{2}\alpha.$$

Then, Lemma 4 gives us the following:

$$\frac{z_0 g'(z_0)}{g(z_0)} = \frac{z_0 D_z^{j+1+\lambda} f(z_0)}{D_z^{j+\lambda} f(z_0)} + j + \lambda - 1 = ik\alpha,$$

where k is given by (18) and (19). If $\arg g(z_0) = \frac{\pi}{2}\alpha$, then $g(z_0) = a^\alpha e^{i\frac{\pi}{2}\alpha}$ and

$$\begin{aligned}
\arg\left(\frac{z_0 g'(z_0)}{g(z_0)} + g(z_0) \right) &= \arg\left(\frac{z_0 D_z^{j+1+\lambda} f(z_0)}{D_z^{j+\lambda} f(z_0)} + \frac{\Gamma(2-\lambda) D_z^{j+\lambda} f(z_0)}{\prod_{\ell=1}^{j} (2-\lambda-\ell) z_0^{1-j-\lambda}} + j + \lambda - 1 \right) \\
&= \arg\left(ik\alpha + a^\alpha e^{i\frac{\pi}{2}\alpha} \right) \\
&= \arg\left\{ a^\alpha \cos\left(\frac{\pi}{2}\alpha \right) + i\left(k\alpha + a^\alpha \sin\left(\frac{\pi}{2}\alpha \right) \right) \right\} \\
&\geq \arg\left\{ a^\alpha \cos\left(\frac{\pi}{2}\alpha \right) + i\left(\frac{\alpha}{2}\left(a + \frac{1}{a} \right) + a^\alpha \sin\left(\frac{\pi}{2}\alpha \right) \right) \right\} \\
&= Tan^{-1}\left\{ \frac{\alpha}{2}\left(a^{1-\alpha} + a^{-1-\alpha} \right) \sec\left(\frac{\pi}{2}\alpha \right) + \tan\left(\frac{\pi}{2}\alpha \right) \right\} \\
&\geq Tan^{-1}\left\{ \frac{\alpha}{2}\left(\left(\frac{1+\alpha}{1-\alpha} \right)^{\frac{1-\alpha}{2}} + \left(\frac{1-\alpha}{1+\alpha} \right)^{\frac{1+\alpha}{2}} \right) \sec\left(\frac{\pi}{2}\alpha \right) + \tan\left(\frac{\pi}{2}\alpha \right) \right\} \\
&= Tan^{-1}\beta(\alpha).
\end{aligned}$$

This contradicts the condition (23) of the theorem. If $\arg g(z_0) = -\frac{\pi}{2}\alpha$, then we also have that

$$\begin{aligned}
\arg\left(\frac{z_0 g'(z_0)}{g(z_0)} + g(z_0) \right) &= \arg\left(\frac{z_0 D_z^{j+1+\lambda} f(z_0)}{D_z^{j+\lambda} f(z_0)} + \frac{\Gamma(2-\lambda) D_z^{j+\lambda} f(z_0)}{\prod_{\ell=1}^{j} (2-\lambda-\ell) z_0^{1-j-\lambda}} + j + \lambda - 1 \right) \\
&\leq -Tan^{-1}\left\{ \frac{\alpha}{2}\left(\left(\frac{1+\alpha}{1-\alpha} \right)^{\frac{1-\alpha}{2}} + \left(\frac{1-\alpha}{1+\alpha} \right)^{\frac{1+\alpha}{2}} \right) \sec\left(\frac{\pi}{2}\alpha \right) + \tan\left(\frac{\pi}{2}\alpha \right) \right\} \\
&= -Tan^{-1}\beta(\alpha).
\end{aligned}$$

This also contradicts the condition (23) of the theorem for $g(z_0) = -\frac{\pi}{2}\alpha$. Therefore, we say that there is no $z_0 \in \mathbb{U}$ such that

$$|\arg g(z)| = \left| \arg\left(\frac{D_z^{j+\lambda} f(z)}{z^{1-j-\lambda}} \right) \right| < \frac{\pi}{2}\alpha \quad (|z| < |z_0|)$$

and

$$|\arg g(z_0)| = \left|\arg\left(\frac{D_z^{j+\lambda} f(z_0)}{z_0^{1-j-\lambda}}\right)\right| = \frac{\pi}{2}\alpha.$$

This implies that

$$|\arg g(z)| = \left|\arg\left(\frac{D_z^{j+\lambda} f(z)}{z^{1-j-\lambda}}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).$$

□

Taking $j = 1$ and $\lambda = 0$ in Theorem 5, we have the following corollary.

Corollary 4. *If $f(z) \in A_n$ satisfies*

$$\left|\arg\left(\frac{zf''(z)}{f'(z)} + f'(z)\right)\right| < Tan^{-1}\beta(\alpha) \quad (z \in \mathbb{U}),$$

for some real α $(0 < \alpha < 1)$, then

$$|\arg f'(z)| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}),$$

where $\beta(\alpha)$ is given by (24).

Next, we consider $f(z) \in A_n$ given by

$$\frac{\Gamma(2-\lambda) z^{j+\lambda} D_z^{j+\lambda} f(z)}{\prod_{\ell=1}^{j}(2-\lambda-\ell)} = \int_0^z \left(\frac{1+\sqrt[n]{t}}{1-\sqrt[n]{t}}\right)^{\alpha} dt \quad (z \in \mathbb{U})$$

with $0 \leq \alpha < 1$, $0 \leq \lambda < 1$ and $j \in \mathbb{N}$. Then $f(z)$ satisfies

$$\begin{aligned}
&\left|\arg\left(\frac{\Gamma(2-\lambda) z^{j+\lambda} D_z^{j+\lambda} f(z)}{\prod_{\ell=1}^{j}(2-\lambda-\ell)}\right)'\right| \\
&= \left|\arg\left(z^{j+\lambda}\left((j+\lambda)\frac{D_z^{j+\lambda} f(z)}{z} + D_z^{j+1+\lambda} f(z)\right)\right)\right| \\
&= \alpha\left|\arg\left(\frac{1+\sqrt[n]{z}}{1-\sqrt[n]{z}}\right)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).
\end{aligned}$$

To discuss such an argument problem, we need the result devised by Fejér and Riesz [8] (or by Tsuji [9]).

Lemma 5. *Let a function $f(z)$ be analytic in $|z| \leq 1$. Then, $f(z)$ satisfies*

$$\int_{-1}^{1} |f(z)|^{\alpha} |dz| \leq \frac{1}{2}\int_{|z|=1} |f(z)|^{\alpha} |dz| \quad (\alpha > 0), \tag{25}$$

where the above integral on the left-hand side is considered along the real axis.

Remark 2. *When we make the change of variables in Lemma 5, Equation (25) becomes*

$$\int_{-r}^{r} |f(\rho e^{i\theta})|^{\alpha} d\rho \leq \frac{r}{2}\int_0^{2\pi} |f(re^{i\theta})|^{\alpha} d\theta. \tag{26}$$

Further, Gwynme [10] gave the following lemma.

Lemma 6. *Let $f(z)$ be a complex valued harmonic function defined on a neighborhood of a closed disc of radius one and center the origin in the complex plane. Then*

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\rho})\frac{1-r^2}{1+r^2-2r\cos(\theta-\rho)}d\rho$$

and

$$\frac{1}{2\pi}\int_0^{2\pi} f(e^{i\rho})\frac{1-r^2}{1+r^2-2r\cos\rho}d\rho = 1.$$

Now, we derive the following theorem.

Theorem 6. *If $f(z) \in A_n$ satisfies*

$$\left|\frac{zD_z^{j+1+\lambda}f(z)}{D_z^{j+\lambda}f(z)}\right| \leq \frac{\alpha}{2} \quad (z \in \mathbb{U}) \tag{27}$$

for some α $(0 \leq \alpha < 1)$, then

$$\left|\arg\left(z^{j-1+\lambda}D_z^{j+\lambda}f(z)\right)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}),$$

where $j \in \mathbb{N}$ and $0 \leq \lambda < 1$.

Proof. We note that

$$\log\left(\frac{\Gamma(2-\lambda)z^{j-1+\lambda}D_z^{j+\lambda}f(z)}{\prod_{\ell=1}^{j}(2-\lambda-\ell)}\right) = \log\left|\frac{\Gamma(2-\lambda)z^{j-1+\lambda}D_z^{j+\lambda}f(z)}{\prod_{\ell=1}^{j}(2-\lambda-\ell)}\right| + i\arg(z^{j-1+\lambda}D_z^{j+\lambda}f(z)) \tag{28}$$

and

$$\begin{aligned}\log\left(\frac{\Gamma(2-\lambda)z^{j-1+\lambda}D_z^{j+\lambda}f(z)}{\prod_{\ell=1}^{j}(2-\lambda-\ell)}\right) &= \int_0^z\left(\log\left(\frac{\Gamma(2-\lambda)t^{j-1+\lambda}D_z^{j+\lambda}f(t)}{\prod_{\ell=1}^{j}(2-\lambda-\ell)}\right)'\right)dt \\ &= \int_0^z\left(\frac{j-1+\lambda}{t}+\frac{D_z^{j+1+\lambda}f(t)}{D_z^{j+\lambda}f(t)}\right)dt.\end{aligned} \tag{29}$$

It follows from (28) and (29) that

$$\begin{aligned}\left|\arg\left(z^{j-1+\lambda}D_z^{j+\lambda}f(z)\right)\right| &= \left|Im\left(\int_0^z\left(\frac{j-1+\lambda}{t}+\frac{D_z^{j+1+\lambda}f(t)}{D_z^{j+\lambda}f(t)}\right)dt\right)\right| \\ &\leq \left|Im\left(\int_0^r\left(\frac{j-1+\lambda}{\rho e^{i\theta}}+\frac{D_z^{j+1+\lambda}f(\rho e^{i\theta})}{D_z^{j+\lambda}f(\rho e^{i\theta})}\right)e^{i\theta}d\rho\right)\right| \\ &= \int_0^r\left|Im\left(\frac{j-1+\lambda}{\rho}+\frac{e^{i\theta}D_z^{j+1+\lambda}f(\rho e^{i\theta})}{D_z^{j+\lambda}f(\rho e^{i\theta})}\right)\right|d\rho \\ &\leq \int_{-r}^r\left|\frac{D_z^{j+1+\lambda}f(\rho e^{i\theta})}{D_z^{j+\lambda}f(\rho e^{i\theta})}\right|d\rho,\end{aligned}$$

where $z = re^{i\theta}$ $(0 \le \theta < 2\pi)$, $0 \le r < 1$ and $0 \le \rho \le r$. Using (26) for $\alpha = 1$, we see that

$$\begin{aligned}\left|\arg\left(z^{j-1+\lambda} D_z^{j+\lambda} f(z)\right)\right| &\le \frac{r}{2}\int_0^{2\pi}\left|\frac{D_z^{j+1+\lambda} f(re^{i\theta})}{D_z^{j+\lambda} f(re^{i\theta})}\right| d\theta \\ &= \frac{1}{2}\int_0^{2\pi}\left|\frac{re^{i\theta} D_z^{j+1+\lambda} f(re^{i\theta})}{D_z^{j+\lambda} f(re^{i\theta})}\right| d\theta.\end{aligned} \tag{30}$$

Therefore, if $f(z)$ satisfies (27), then $f(z)$ satisfies

$$\left|\arg\left(z^{j-1+\lambda} D_z^{j+\lambda} f(z)\right)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).$$

□

Taking $j = 1$ and $\lambda = 0$ in Theorem 6, we have the following corollary.

Corollary 5. *If $f(z) \in A_n$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| \le \frac{\alpha}{2} \quad (z \in \mathbb{U})$$

for some α $(0 \le \alpha < 1)$, then

$$\left|\arg f'(z)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).$$

Further, on applying Lemma 6, we arrive at the following theorem.

Theorem 7. *If $f(z) \in A_n$ satisfies*

$$\left|\frac{zD_z^{j+1+\lambda} f(z)}{D_z^{j+\lambda} f(z)}\right| < \frac{\alpha}{2} Re\left(\frac{1+\beta\sqrt[n]{z}}{1-\sqrt[n]{z}}\right) \quad (z \in \mathbb{U}) \tag{31}$$

for some α $(0 \le \alpha < 1)$ and β $(\beta \neq -1)$, then

$$\left|\arg\left(z^{j-1+\lambda} D_z^{j+\lambda} f(z)\right)\right| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}),$$

where $j \in \mathbb{N}$ and $0 \le \lambda < 1$.

Proof. From (30), we see that

$$\left|\arg\left(z^{j-1+\lambda} D_z^{j+\lambda} f(z)\right)\right| < \frac{1}{2}\int_0^{2\pi}\left|\frac{re^{i\theta} D_z^{j+1+\lambda} f(re^{i\theta})}{D_z^{j+\lambda} f(re^{i\theta})}\right| d\theta$$

for $f(z) \in A_n$. It follows that (31) gives

$$\begin{aligned}\left|\arg\left(z^{j-1+\lambda} D_z^{j+\lambda} f(z)\right)\right| &< \frac{\alpha}{4}\int_0^{2\pi} Re\left(\frac{1+\beta\sqrt[n]{r}e^{i\frac{\theta}{n}}}{1-\sqrt[n]{r}e^{i\frac{\theta}{n}}}\right) d\theta \\ &= \frac{\alpha}{4}\int_0^{2\pi}\left(\frac{1+(\beta-1)\sqrt[n]{r}\cos\left(\frac{\theta}{n}\right)-\beta(\sqrt[n]{r})^2}{1-2\sqrt[n]{r}\cos\left(\frac{\theta}{n}\right)-(\sqrt[n]{r})^2}\right) d\theta \\ &= \frac{\alpha}{4}\int_0^{2\pi}\left\{\frac{1-\beta}{2}+\left(\frac{1+\beta}{2}\right)\frac{1-(\sqrt[n]{r})^2}{1+(\sqrt[n]{r})^2-2\sqrt[n]{r}\cos\left(\frac{\theta}{n}\right)}\right\} d\theta.\end{aligned} \tag{32}$$

Applying Lemma 6, we obtain

$$\int_0^{2\pi} \frac{1-(\sqrt[n]{r})^2}{1+(\sqrt[n]{r})^2 - 2\sqrt[n]{r}\cos\left(\frac{\theta}{n}\right)} d\theta = n\int_0^{\frac{2\pi}{n}} \frac{1-(\sqrt[n]{r})^2}{1+(\sqrt[n]{r})^2 - 2\sqrt[n]{r}\cos\rho} d\rho \leq 2\pi.$$

Therefore, (32) becomes

$$\left|\arg\left(z^{j-1+\lambda} D_z^{j+\lambda} f(z)\right)\right| < \frac{\pi}{2}\alpha \ \ (z \in \mathbb{U}).$$

□

Setting $j = 1$ and $\lambda = 0$ in Theorem 7, we obtain the following corollary.

Corollary 6. *If $f(z) \in A_n$ satisfies*

$$\left|\frac{zf''(z)}{f'(z)}\right| < \frac{\alpha}{2} Re\left(\frac{1+\beta\sqrt[n]{z}}{1-\sqrt[n]{z}}\right) \quad (z \in \mathbb{U})$$

for some α $(0 \leq \alpha < 1)$ and β $(\beta \neq -1)$, then

$$|\arg f'(z)| < \frac{\pi}{2}\alpha \quad (z \in \mathbb{U}).$$

Example 1. *We consider a function $f(z) \in A_n$ given by*

$$\frac{\Gamma(2-\lambda) z^{j-1+\lambda} D_z^{j+\lambda} f(z)}{\prod_{\ell=1}^{j}(2-\lambda-\ell)} = \left(\frac{2}{2-\sqrt[n]{z}}\right)^{3\alpha} \quad (z \in \mathbb{U})$$

with $0 < \alpha < 1$. If we write that

$$w(z) = \frac{2}{2-\sqrt[n]{z}} \quad (z \in \mathbb{U}),$$

$w(z)$ satisfies

$$\left|w(z) - \frac{4}{3}\right| < \frac{2}{3} \quad (z \in \mathbb{U})$$

and

$$|\arg w(z)| = \left|\arg\left(\frac{2}{2-\sqrt[n]{z}}\right)\right| < \frac{\pi}{6} \quad (z \in \mathbb{U}).$$

Therefore, we have

$$\left|\arg\left(z^{j-1+\lambda} D_z^{j+\lambda} f(z)\right)\right| = 3\alpha\left|\arg\left(\frac{2}{2-\sqrt[n]{z}}\right)\right| < \frac{\pi}{2}\alpha \ \ (z \in \mathbb{U}).$$

For such a function $f(z)$, we obtain

$$\begin{aligned}
\left|\frac{zD_z^{j+1+\lambda} f(z)}{D_z^{j+\lambda} f(z)}\right| &= \left|\frac{3\alpha}{n}\left(\frac{\sqrt[n]{z}}{2-\sqrt[n]{z}}\right) - (j-1+\lambda)\right| \\
&< \frac{3\alpha}{n}\left|\frac{\sqrt[n]{z}}{2-\sqrt[n]{z}}\right| + (j-1+\lambda) \\
&< \frac{3\alpha}{n} + (j-1+\lambda) \ \ (z \in \mathbb{U}).
\end{aligned}$$

If we consider some β *such that*

$$\beta \leq \frac{n-12}{n} - \frac{4(j-1+\lambda)}{\alpha},$$

then

$$\begin{aligned}\left|\frac{zD_z^{j+1+\lambda}f(z)}{D_z^{j+\lambda}f(z)}\right| &< \frac{3\alpha}{n} + (j-1+\lambda) \\ &\leq \frac{\alpha(1-\beta)}{4} \\ &< \frac{\alpha}{2}Re\left(\frac{1+\beta\sqrt[n]{z}}{1-\sqrt[n]{z}}\right) \quad (z \in \mathbb{U}).\end{aligned}$$

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