

Article

Critical Sets and Unavoidable Sets of Strictly Concentric Magic Squares of Odd Order and Their Application to Prime Strictly Concentric Magic Squares of Order 5 [†]

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Abstract

There has been much interest in the mathematical investigation of critical sets and unavoidable sets in Latin Squares, Sudoku, and their applications to practical problems in areas such as agriculture and cryptology. This paper considers the associated structures of Strictly Concentric Magic Squares (SCMSs) and Prime Strictly Concentric Magic Squares (PSCMSs). A framework of formal definitions is given that leads to the definitions of critical sets and unavoidable sets. Minimal critical sets are of interest in Latin Squares, and in this article, the cardinality of minimal critical sets of SCMS is given for all n , n odd. Two families of unavoidable sets are established for SCMS, leading to a complete classification of unavoidable sets of minimum PSCMS of order 5.

Keywords: magic squares; prime; concentric; completability; critical sets; unavoidable sets

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1. Introduction and Preliminaries

The study of completability of partial grids and their applications to areas such as experimental design, error-correcting codes, and computer science [1,2] is evident in the literature for Latin Squares and Sudoku. Keedwell [3] briefly considered partial magic squares along with Latin Squares and Sudoku grids, investigating **critical sets** (sets comprising sufficient clues/information to ensure the unique completability) of normal magic squares of orders 3 and 4. Keedwell postulated that concentric magic squares may possess sufficient structure to enable the determination of critical sets within grids of a larger order. While a partial grid containing all values specified by a critical set ensures unique completion, **unavoidable sets** of a grid specify cell values that must be present in any critical set [4,5].

The current authors provided the first formal treatment of Strictly Concentric Magic Squares (SCMSs) in the literature [6], presenting foundational material on, and terminology for, this structure, following from the literature on Latin Squares [7]. That paper presents a full enumeration of minimum Prime Strictly Concentric Magic Squares (PSCMSs) of order 5, which are squares composed entirely of prime numbers and having a minimum centre cell value. This paper extends this foundational material on SCMSs to critical sets and unavoidable sets, providing definitions and results on SCMSs in general, before applying

these results to PSCMSs of order 5. A complete classification for unavoidable sets of minimum PSCMSs of order 5 is given.

The basic properties of magic squares that are needed in this paper are now given. A **magic square** of order n is an n -by- n grid containing n^2 distinct integers positioned such that all rows, columns, and main diagonals sum to the same value known as the **magic constant**. A **normal magic square**, NMS, contains the integers 1 to n^2 . A **prime magic square**, PMS, contains n^2 distinct primes. A cell of a grid of order n at row i , $i = 1, \dots, n$, and column j , $j = 1, \dots, n$, is denoted (i, j) and has value a_{ij} . A **triple** (i, j, a_{ij}) specifies a value a_{ij} in row i and column j of a grid. Denote by A_n the cells of a grid of order n , i.e., a set of (i, j) tuples, and denote by $\mathbf{A}_n$ a set of triples (i, j, a_{ij}) . A **centre subsquare** of order m , m odd, of a magic square of order n , n odd, comprises the centre m by m cells of the magic square, for $m = n - 2i$, $i = 1, \dots, \frac{n-1}{2}$. The smallest such centre subsquare is of order 1. A **magic centre subsquare** of order m of a magic square of order n is a centre subsquare which is itself a magic square. The centre subsquare of order 1 is here considered a trivial magic centre subsquare, but otherwise the concept of concentric magic squares given here follows Andrews [8]. A magic square of order n , $n \geq 3$, for which its order $(n - 2)$ centre subsquare is a magic centre subsquare is termed a **concentric magic square**, CMS.

This paper focuses on SCMS of order n , n odd, but where the definitions apply to both odd and even orders they are given in general. Throughout this paper it is taken that the centre cell value of the grid is always known and is denoted M .

Definition 1. A magic square of order n , $n \geq 5$ and odd, is **strictly concentric**, denoted an SCMS, if each of its order $n - 2i$ centre subsquares, $i = 1, \dots, \frac{n-3}{2}$, is a CMS. A centre subsquare of order 3 is here considered a trivial SCMS. A SCMS containing n^2 distinct primes is denoted a PSCMS.

In the exploration of completability of SCMS later in this paper, the concepts of borders and paired cells will be of importance due to constraints on the values in such cells. Definitions 2–6, and Table 1 below are taken from [6].

Definition 2. For an SCMS of order n , n odd, a cell (i, j) has a **paired cell** $(\bar{i}, \bar{j})$, such that

$$(\bar{i}, \bar{j}) = \begin{cases} (n - i + 1, n - i + 1) & i = 1, \dots, n, i \neq \frac{n+1}{2}, j = i & (1) \\ (n - i + 1, i) & i = 1, \dots, n, i \neq \frac{n+1}{2}, j = n - i + 1 & (2) \\ (i, n - j + 1) & i = 2, \dots, n - 1, j \neq i, i + j \leq n \text{ when } i > j \\ & \text{and } i + j \geq n + 2 \text{ when } i < j & (3) \\ (n - i + 1, j) & j = 2, \dots, n - 1, j \neq i, i + j \leq n \text{ when } j > i \\ & \text{and } i + j \geq n + 2 \text{ when } j < i & (4) \end{cases}$$

Definition 2 is illustrated in Figure 1, in which the number in a cell relates to an equation number from Definition 2, and in combination with the following Greek letter indicates a pairing with another cell.

1 α	4 α	4 β	4 γ	2 α
3 α	1 β	4 δ	2 β	3 α
3 β	3 γ		3 γ	3 β
3 δ	2 β	4 δ	1 β	3 δ
2 α	4 α	4 β	4 γ	1 α

Figure 1. Illustration of paired cells for an SCMS of order 5; the cell numbers relate to the equation numbers given in Definition 2, followed by a Greek letter denoting pairings.

Definition 3. An SCMS of order n , n odd, and each of its centre subsquares has a border which comprises those cells which are adjacent to its respective outer edge. Let B_n be the set of **border cells** of the SCMS of order n , and let B_{n-2i} be the set of border cells of its centre subsquares of order $n - 2i$, $i = 1, \dots, \frac{n-3}{2}$. Let $B^n = \cup B_{n-2i}$, $i = 0, \dots, \frac{n-3}{2}$.

That is, B^n denotes the set of all the cells of an SCMS of order n , n odd, apart from the trivial centre subsquare.

Definition 4. A **border pair** $(a_{ij}, a_{\bar{i}\bar{j}})$ is a pair of values placed in cells in B^n for an SCMS of order n , n odd, where (i, j) and $(\bar{i}, \bar{j})$ are paired cells.

An SCMS of order n , n odd, with centre cell value M has magic constant $S_n = nM$. It follows that the elements of each border pair of an SCMS of order n , n odd, sum to $2M$ [6].

Definition 5. Denote two values summing to $2M$, where M is the centre cell value of an SCMS, as a **pair of complement values**. Hence, all border pairs of an SCMS are pairs of complement values. Denote two primes summing to $2M$, where M is the centre cell value of a PSCMS, as a **pair of complement primes**. Hence, all border pairs of a PSCMS are pairs of complement values (primes).

Definition 6. An SCMS of order n , n odd, that has minimum centre cell value, M , is termed a **minimum SCMS**. A PSCMS of order n , n odd, that has minimum centre cell prime, M , is termed a **minimum PSCMS**.

The main focus of this paper is PSCMSs of order 5. All examples of SCMSs of order 5 given in this paper are minimum PSCMSs, and all examples of SCMS of order 3 are of valid centre subsquares of minimum PSCMSs of order 5. The magic constant of a minimum PSCMS of order 5, which has $M = 251$, is given in [9] and the number of minimum PSCMSs of order 5 is enumerated in [6].

The specific permutation operations in Table 1 (first given by the authors in [6]) may be performed on the outer border of the grid of order n and on the borders of its centre subsquares. Let s denote the order of the grid or its centre subsquares, $s = 3, 5, \dots, n$, s odd. When applying the permutation operations to the border of the grid or a centre subsquare of order s , consider the rows and columns of the border to be numbered $1, \dots, s$. Note that the first two permutation operations may not be performed on the centre subsquare of order 3, since there is only a single non-corner border pair, in each case [6].

Table 1. Permutation operations on the border pairs in cells of an SCMS of order n , n odd, where the permutation order given is the smallest number of such permutations required to return to the initial state [6].

Permutation	Order
Permute the border pairs in columns i , $i \neq 1$ and $i \neq s$, noting that the values remain in their original row	$(s - 2)!$
Permute the border pairs in rows i , $i \neq 1$ and $i \neq s$, noting that the values remain in their original column	$(s - 2)!$
Permute the value in $(1, j)$ with the value in (s, j) for all $j = 1, \dots, s$	2
Permute the value in $(i, 1)$ with the value in (i, s) for all $i = 1, \dots, s$	2
Permute the value in $(1, j)$ with the value in (i, s) and the value in (s, j) with the value in $(i, 1)$ for all $i = 1, \dots, s$ where $i = s - j + 1$	2

Definition 7. Two Strictly Concentric Magic Squares, $\mathbf{A}_n^1$ and $\mathbf{A}_n^2$, of order n with magic constant nM , n odd, are **equivalent** if one can be obtained from the other by undergoing specific permutations, given in Table 1. Otherwise the magic squares are **non-equivalent**.

All of the operations in Table 1 are feasible and each forms a group; further details can be found in [6].

Let $\mathbb{N}'$ be a subset of the set of natural numbers $\mathbb{N}$. There exist pairs of complement values $a_{ij}, \bar{a}_{ij} \in \mathbb{N}$ such that $a_{ij} + \bar{a}_{ij} = 2M$, where M is the centre cell value of an SCMS of order n , n odd, and $a_{ij}, \bar{a}_{ij}$ are the values in paired cells. Likewise, this property holds for pairs of complement primes $a_{ij}, \bar{a}_{ij} \in \mathbb{P}$, where $\mathbb{P}'$ is a subset of the set of prime numbers $\mathbb{P}$ and their sum is the centre cell value of a PSCMS of order n , n odd.

Firstly, in this paper, definitions are given for SCMSs, and secondly for PSCMSs, in relation to the completability of partial grids. Recall that the centre cell value, M , of the SCMS is known for a given grid; the pairs of values summing to $2M$ form the corresponding set $\mathbb{N}'$. The larger the chosen value of M , the more combinations of values summing to $2M$ there will be, and hence, the number of combinations of values summing to $2M$ grows, and thus the cardinality of the set of values in $\mathbb{N}'$ is larger. It is noted that as $\mathbb{N}'$ is used to fill the remaining cells in the SCMS, then M must be chosen to be suitably large in order for there to be sufficient values for the given grid. Likewise, this holds for $\mathbb{P}'$ and the filling of cells in PSCMS. Given the value M , the set $\mathbb{N}'$ (or $\mathbb{P}'$) is easily determined.

These definitions are used in Section 2 to explore properties of partial SCMSs relating to their completability, and for the identification of critical sets in Section 3. Section 4 provides definitions for unavoidable sets of SCMSs, and Section 5 establishes two families of unavoidable sets for SCMSs, and provides a complete classification for unavoidable sets of minimum PSCMSs of order 5. The paper concludes by providing open problems in this area.

2. Completability of Partial SCMS with Non-Empty Centre Cell

In this paper, the term partial SCMS denotes a partial grid with a non-empty centre cell, as given in Definition 8.

Definition 8. A grid of order n is termed **partial** if between 1 and $n^2 - 1$ cells are non-empty, including the centre cell. The set of tuples (i, j) of the non-empty cells is denoted H_n . When the non-empty cells have values such that the partial grid is **completable** to an SCMS then the set of triples (i, j, a_{ij}) is denoted $\mathbf{H}_n$ and termed a **partial SCMS**.

An example of a partial PSCMS of order 3 is given in Figure 2a.

Definition 9. Let $\mathbb{A}_n^{\mathbf{H}}$ be the set of all SCMSs to which $\mathbf{H}_n$ can be completed. Hence, if $|\mathbb{A}_n^{\mathbf{H}}| = 1$ then $\mathbf{H}_n$ is **uniquely completable**, and if $|\mathbb{A}_n^{\mathbf{H}}| > 1$ then $\mathbf{H}_n$ is **multiply completable**.

Only partial grids which are partial SCMSs, $\mathbf{H}_n$, are considered in this paper; i.e., the partial grid is completable to an SCMS. Operations can be defined for assigning values, a_{ij} , to empty cells. The terms strongly and weakly completable are used in a manner similar to the established definitions in the literature for related combinatorial structures [10].

Definition 10. A cell of a partial SCMS, $\mathbf{H}_n$, is **row-completable** (or **column-completable**) if $n - 1$ cells are non-empty in the given row (or column). A cell (i, j) is **pairwise-completable** if its paired cell $(\bar{i}, \bar{j})$ is non-empty.

Definition 11. An empty cell (i, j) in a partial SCMS, $\mathbf{H}_n$, that can be filled uniquely using a row/column/pairwise completable operation is referred to as **forced**.

Definition 12. A completable partial SCMS, $\mathbf{H}_n$, is termed a **strong partial SCMS**, $\mathbf{H}_n^s$, if every empty cell becomes forced at some iteration of completion; otherwise, it is a **weak partial SCMS**, $\mathbf{H}_n^w$.

Lemma 1. A strong partial SCMS, $\mathbf{H}_n^s$, is uniquely completable. A weak partial SCMS, $\mathbf{H}_n^w$, is either uniquely or multiply completable.

Proof. From Definition 12, all empty cells in a strong partial SCMS, $\mathbf{H}_n^s$, are forced; hence, there is only one way to complete the grid. A weak partial SCMS, $\mathbf{H}_n^w$, has at least one cell which is not forced. For specific partial SCMSs, one or more given cells that are not forced can be completed in more than one way (one such example is given in Figure 3c). For other specific partial SCMSs with one or more given cells that are not forced, all such cells are completable in only one way (one such example is give in Figure 3a). $\square$

Recall that grids of order 3 are here considered trivial SCMSs. Attention will be given first to SCMSs of order 3 before considering SCMSs of order 5 and of odd order $n > 5$.

Lemma 2. A partial SCMS of order 3, $\mathbf{H}_3$, with two non-empty, non-paired cells in the border, B_3 , including at least one corner cell, is strongly completable and hence uniquely completable.

Proof. Without loss of generality, assume the partial SCMS includes $(1, 1, a_{11})$ and one other non-paired triple, (i, j, a_{ij}) , in addition to the centre cell triple $(2, 2, a_{22})$.

If $i = 1$ (or $j = 1$), then the third cell in the row (or column) is forced completable and the row (or column) is completed, and hence $(3, k, a_{3,k})$ for $k = 1, 2, 3$ are pairwise completable (or likewise for $(l, 3, a_{l,3})$ for $l = 1, 2, 3$). The remaining two cells are forced.

If $i \neq 1$ and $j \neq 1$, then the paired cell of a_{ij} is in either row 1 or column 1 and forced. Hence, the above argument applies. Hence, the grid is strongly completable and from Lemma 1 is uniquely completable. $\square$

Corollary 1. A partial SCMS of order 3, $\mathbf{H}_3$, with two non-empty non-paired cells in the border, B_n , neither of which is a corner cell, is weakly and uniquely completable.

Proof. The paired cells of the two non-empty cells in the border are forced, leaving just the corner cells empty. When all corner cells are empty, no cell is forced, so the grid is not strongly completable. However, there is only one way of completing the four corners to satisfy the magic constant constraint of a magic square. $\square$

Lemma 3. A partial SCMS of order 3, $\mathbf{H}_3$, with fewer than two non-empty cells in the border, B_3 , is weakly completable and multiply completable.

Proof. Consider the case of a single non-empty corner cell, in addition to the centre cell. Without loss of generality, assume the partial SCMS includes $(1, 1, a_{11})$; hence, $(3, 3, a_{33})$ is pairwise completable and no other cell is immediately forced. There exist $a_{12}, a_{13} \in \mathbb{N}'$ such that $\sum_{j=1}^3 a_{1j} = 3M$ and a_{12}, a_{13} are not paired, and $a_{21}, a_{31} \in \mathbb{N}'$ such that $\sum_{i=1}^3 a_{i1} = 3M$, a_{21}, a_{31} are not paired and $a_{31} = \bar{a}_{13}$. As $a_{12} + a_{13} = a_{21} + a_{31}$, then, another completion of the partial SCMS exists in which both the paired values in positions $(1, 3)$ and $(3, 1)$ are permuted, and the non-paired values in positions $(1, 2)$ and $(2, 1)$ are permuted. Hence there are at least two completions, and therefore the SCMS is multiply completable.

Consider the case of a single non-empty, non-corner cell, in addition to the centre cell. Without loss of generality, assume the partial SCMS includes $(1, 2, a_{12})$; hence, $(3, 2, a_{32})$ is pairwise completable and no other cell is immediately forced. There exist three values in $\mathbb{N}'$ summing to $3M$, with their complement values also summing to $3M$, in order to form the first column and the third column, such that the column and row sums are satisfied. Each of these can be placed in either the first or the third column, and hence there are always at least two completions and therefore the SCMS is multiply completable.

With all cells empty, other than the centre cell, there are no fewer completions than in the cases above. $\square$

Lemma 3 is illustrated for $M = 251$ in Figure 2, with a partial PSCMS of order 3, which is multiply completable, as shown in Figure 2a and its two equivalent completions shown in Figure 2b,c.

461		
	251	

(a)

461	23	269
59	251	443
233	479	41

(b)

461	59	233
23	251	479
269	443	41

(c)

Figure 2. A partial PSCMS of order 3 which is multiply completable and its two equivalent completions. (a) Two non-empty cells in a grid of order 3; (b) first completion of (a); (c) second completion of (a).

Corollary 2. For a partial SCMS of order 3 to be strongly completable, it must contain at least two non-empty, non-paired cells, including at least one corner cell in addition to the centre cell.

Proof. Follows directly from Lemmas 2 and 3 and Corollary 1. $\square$

From Lemma 1, a weak partial SCMS, $\mathbf{H}_n^w$, is either uniquely or multiply completable, and an example of each case is given in Example 1 for $n = 5$ with $M = 251$.

Example 1. Figure 3a,c each show a partial PSCMS, $\mathbf{H}_5$, with $M = 251$ and a non-empty centre subsquare of order 3. In Figure 3a the triples $(1, 1, 419)$, $(1, 2, 71)$, $(1, 3, 311)$, $(2, 1, 11)$, $(3, 1, 113)$ and their complement pairs are non-empty and highlighted. The empty cells A, B, C and their complements have only one completion, which is given in Figure 3b. In Figure 3c the triples $(1, 1, 419)$, $(1, 2, 71)$, $(2, 1, 11)$, $(3, 1, 113)$ and their complement pairs are non-empty and highlighted. The empty cells A, B, C, D and their complements have two possible completions, given in Figure 3b,d.

Theorem 1. For any partial SCMS, $\mathbf{H}_n$, of order n , n odd,

1. If $\mathbf{H}_n$ is a strong partial SCMS, then it is uniquely completable.
2. If $\mathbf{H}_n$ is multiply completable, then it is a weak partial SCMS.
3. If $\mathbf{H}_n$ is a weak partial SCMS, then it is either uniquely completable or multiply completable.
4. If $\mathbf{H}_n$ is uniquely completable, then it is either a strong or a weak partial SCMS.

Proof. 1. This follows immediately from Lemma 1.

2. If a grid is multiply completable, then it cannot be a strong partial SCMS, so it is a weak partial SCMS.

3. This follows immediately from Lemma 1.

4. From conditions 1 and 3, both strong and weak partial SCMSs can be uniquely completable.

$\square$

419	71	311	A	B
11	461	23	269	491
113	59	251	443	389
C	233	479	41	$\bar{C}$
$\bar{B}$	431	191	$\bar{A}$	83

(a)

419	71	311	401	53
11	461	23	269	491
113	59	251	443	389
263	233	479	41	239
449	431	191	101	83

(b)

419	71	D	A	B
11	461	23	269	491
113	59	251	443	389
C	233	479	41	$\bar{C}$
$\bar{B}$	431	$\bar{D}$	$\bar{A}$	83

(c)

419	71	401	311	53
11	461	23	269	491
113	59	251	443	389
263	233	479	41	239
449	431	101	191	83

(d)

Figure 3. Two weak partial PSCMS, H_5^w , with non-empty border cells highlighted, and two completions. (a) A weak partial PSCMS, H_5^w , which is uniquely completable; (b) unique completion of the partial PSCMS given in (a) and one completion of the partial PSCMS given in (c); (c) a weak partial PSCMS, H_5^w , which is multiply completable; (d) second completion of the partial PSCMS given in (c), equivalent to (b).

3. Strong Partial SCMSs and Critical Sets of Partial SCMSs

Recall that H_n^s is the set of cells of a strong partial SCMS and that H_n^s is the set of triples of that strong partial SCMS.

Definition 13. A *minimal strong partial SCMS* of order n , denoted $H_n^{s,min}$, is a strong partial SCMS of minimum cardinality.

There may be more than one minimal strong partial SCMS that completes to a given grid A_n . Two examples of minimal strong partial SCMSs are given in Figure 4a,b which are both uniquely completable to the grid in Figure 4c. The cardinality of the minimal strong partial SCMS, $H_n^{s,min}$, is shown in Theorem 3.

311	113		401	
	461	23		239
		251		
				53
		491		

(a)

		11		419
			269	
		251		353
449			41	
	389			191

(b)

311	113	11	401	419
263	461	23	269	239
149	59	251	443	353
449	233	479	41	53
83	389	491	101	191

(c)

Figure 4. Two minimal strong partial PSCMSs, $H_5^{s,min}$, and their unique completion, A_5 . (a) First minimal strong partial PSCMS $H_5^{s,min}$; (b) second minimal strong partial PSCMS $H_5^{s,min}$; (c) unique completion of the minimal strong partial PSCMS in (a,b).

The definition of a critical set in a Latin Square is given in [11] and a similar definition is given here for critical sets of SCMSs.

Definition 14. A critical set, $\mathbf{V}_n$, of an SCMS, $\mathbf{A}_n$, of order n , is a set $\mathbf{V}_n = \{(i, j, a_{ij}) \mid i, j \in \{1, \dots, n\}, a_{ij} \in \mathbb{N}'\}$ such that

1. $\mathbf{A}_n$ is the only SCMS of order n which has entry a_{ij} in position (i, j) for all $(i, j, a_{ij}) \in \mathbf{V}_n$;
2. No proper subset of $\mathbf{V}_n$ satisfies (1).

Definition 14 can be re-expressed as Definition 15.

Definition 15. A critical set $\mathbf{V}_n$ is a partial SCMS that is uniquely completable to an SCMS $\mathbf{A}_n$, i.e., $|\mathbb{A}_n^{\mathbf{V}}| = 1$, and is such that if any triple is removed from $\mathbf{V}_n$, the resulting partial SCMS is multiply completable; i.e., for any $\mathbf{H}_n \subset \mathbf{V}_n$, $|\mathbb{A}_n^{\mathbf{H}}| > 1$.

Definition 16. A critical set, $\mathbf{V}_n$, is termed a **strong critical set** if it is also a strong partial SCMS, $\mathbf{H}_n^s$; else it is a **weak critical set**.

Definition 17. A **minimal critical set** of an SCMS of order n , $\mathbf{A}_n$, denoted as $\mathbf{V}_n^{\min}$, is a critical set of minimum cardinality.

Let V_n denote the set of cells of a critical set, $\mathbf{V}_n$, and let $\mathbb{V}_n^{\mathbf{A}}$ be the set of all critical sets of $\mathbf{A}_n$. Consider $\exists \mathbf{V}_n^{\min} \in \mathbb{V}_n^{\mathbf{A}}$ such that $\forall \mathbf{V}_n \in \mathbb{V}_n^{\mathbf{A}}, |\mathbf{V}_n^{\min}| \leq |\mathbf{V}_n|$, and hence $\mathbf{V}_n^{\min}$ is a minimal critical set of $\mathbf{A}_n$. There may be more than one minimal critical set of a given SCMS.

Remark 1 follows immediately from Definition 14, using strong completable or weak completable.

Remark 1. A critical set, $\mathbf{V}_n$, describes a partial SCMS, $\mathbf{H}_n$, which is uniquely completable.

A comparison of the cardinality of the minimal strong partial SCMS, $\mathbf{H}_n^{s, \min}$, and the cardinality of the minimal critical set, $\mathbf{V}_n^{\min}$, of an SCMS of order n , n odd, are now given, and the cardinality of the minimal strong partial SCMS of order n , n odd, is then calculated. Bounds are given for the cardinality of a minimal critical set of an SCMS of order n , n odd.

Lemma 4. $|\mathbf{V}_n^{\min}| \leq |\mathbf{H}_n^{s, \min}|$.

Proof. Every strong partial SCMS is either a critical set or contains a critical set. $\square$

Theorem 2. A minimal strong partial SCMS of order n has $2j - 4$ non-empty cells in the border of each square of order j , $j = 3, 5, \dots, n$, and in each of these borders, there is at most one cell of any non-empty border pair and at least one non-empty corner cell.

Proof. Recall that the centre cell is non-empty.

Suppose that for each border of a partial SCMS, all corner cells are empty, and suppose that all non-corner cells in one row and one column of each border are non-empty. Then, each border of order j contains $2j - 4$ non-empty cells. All empty non-corner cells are pairwise completable, but the corner cells are not forced. Such a grid is not a strong partial SCMS.

Suppose that for each border of a partial SCMS, one corner cell is non-empty, and suppose that $2j - 6$ non-corner, non-paired cells are non-empty. By Definition 11, $2j - 5$ empty cells of each border are forced, but two pairs are not forced, and these are non-corner cells. From the permutation operations, Table 1, these are not uniquely completable.

Lastly, suppose that for each border of a partial SCMS, one corner cell is non-empty, and suppose that $2j - 5$ non-corner, non-paired cells are non-empty. By Definition 11, all non-empty cells of each border are forced. $\square$

Corollary 3. *If an SCMS of order n , n odd, exists, then $|\mathbf{H}_n^{s,min}| = 2n - 4 + |\mathbf{H}_{n-2}^{s,min}|$ where $|\mathbf{H}_1^{s,min}| = 1$.*

Proof. The proof follows immediately from Theorem 2. $\square$

It is noted that $|\mathbf{H}_n^{s,min}|$, n odd, follows the integer sequence A058331 [12], $a(x) = 2x^2 + 1$.

Theorem 3. *If an SCMS of order n , n odd, exists, then $|\mathbf{H}_n^{s,min}| = \frac{1}{2}(n^2 - 2n + 3)$.*

Proof. From Theorem 2, for j odd, $j = (2i + 1)$ where $i = 1, \dots, \frac{n-1}{2}$, the size of the set of non-empty cells in the outer border of the order j grid, $\mathbf{Z}_j$, is $|\mathbf{Z}_j| = 2j - 4 = 2(2i + 1) - 4 = 4i - 2$. Hence, $\sum_{j=1}^{\frac{n-1}{2}} |\mathbf{Z}_j| = 4 \sum_{i=1}^{\frac{n-1}{2}} i - \sum_{i=1}^{\frac{n-1}{2}} 2 = \frac{4}{2}(\frac{n-1}{2})(\frac{n-1}{2} + 1) - (n - 1) = \frac{1}{2}(n^2 - 2n + 1)$. Since the centre cell is not included in a border but is always non-empty, then $|\mathbf{H}_n^{s,min}| = \sum_{j=3}^n |\mathbf{Z}_j| + 1$ and hence $|\mathbf{H}_n^{s,min}| = \frac{1}{2}(n^2 - 2n + 3)$. $\square$

Theorem 4. *Given an SCMS of order n , n odd, $n \geq 5$, the size of the minimal critical set, $|\mathbf{V}_n^{min}|$, satisfies $\frac{1}{2}(n^2 - 4n + 9) \leq |\mathbf{V}_n^{min}| \leq \frac{1}{2}(n^2 - 2n + 3)$.*

Proof. First consider the upper bound of the size of the minimal critical set, from Lemma 4 $|\mathbf{V}_n^{min}| \leq |\mathbf{H}_n^{s,min}|$ and from Theorem 3 $|\mathbf{H}_n^{s,min}| = \frac{1}{2}(n^2 - 2n + 3)$.

Now consider an SCMS of order 3; from Lemma 2 and Corollary 1 there are two non-empty cells in the border for the grid to be uniquely completable. Hence, a critical set contains no fewer than two triples, and with the inclusion of the centre cell value contains three triples. When $n = 3$, $|\mathbf{H}_3^{s,min}| = 3$ and hence there are exactly three non-empty cells for the minimal critical set, $|\mathbf{V}_3^{min}| = 3$.

For j odd, $j = (2i + 3)$ where $i = 1, \dots, \frac{n-3}{2}$, the borders of order j of the centre subsquares and the outer border ($j = n$) of the SCMS of order n each have $4j - 4$ cells. For an SCMS of order n , $n > 3$, $2j - 6$ non-empty cells are required in each border for the SCMS to be uniquely completable; else it is multiply completable due to the permutation operations in Table 1. Hence, $|\mathbf{Z}_j| = 2j - 6 = 2(2i + 3) - 6 = 4i$. Hence, $\sum_{j=1}^{\frac{n-1}{2}} |\mathbf{Z}_j| = 4 \sum_{i=1}^{\frac{n-3}{2}} i = \frac{4}{2}(\frac{n-3}{2})(\frac{n-3}{2} + 1)$, giving $\sum_{j=1}^{\frac{n-1}{2}} |\mathbf{Z}_j| = \frac{1}{2}(n^2 - 4n + 3)$.

The centre subsquare of order 3 has three non-empty cells, and hence $|\mathbf{V}_n^{min}| = \sum_{j=5}^n |\mathbf{Z}_j| + 3$, and thus $|\mathbf{V}_n^{min}| = \frac{1}{2}(n^2 - 4n + 9)$. $\square$

Construction 1. *Take an empty grid of order 3 and fill in the centre cell to form a valid partial SCMS. Secondly, place one value in a corner to form a valid partial SCMS. Thirdly, place a further value in any cell that is not paired with that of the previously placed value to form a valid partial SCMS.*

Theorem 5. *A strong partial SCMS of order 3 that is also a minimal critical set is specified by a partial grid obtained by Construction 1.*

Proof. Using Construction 1, the number of non-empty cells is 3, which satisfies $\frac{1}{2}(n^2 - 2n + 3)$, which is both the size of the minimal strong partial SCMS and the upper bound of the size of the minimal critical set. If either of the values in the border are

removed, then the resulting grid is multiply completable, and hence the remaining triples do not form a critical set nor a strong partial SCMS. $\square$

Without loss of generality, let the two non-empty cells of a partial SCMS of order 3, specified in Construction 1, be the cells (1,1) and (1,2) illustrated using a PSCMS in Figure 5a with its completion given in Figure 5b. Hence, $|\mathbf{V}_3^{min}| = 3$, which satisfies the upper bound of Theorem 4.

461	23	
	251	

(a)

461	23	269
59	251	443
233	479	41

(b)

Figure 5. A partial PSCMS, $\mathbf{H}_3$, specified by a minimal critical set, $\mathbf{V}_3^{min}$ and its unique completion, $\mathbf{A}_3$. (a) Three non-empty cells in a grid of order 3; (b) unique completion of (a).

All critical sets of different sizes in Latin Squares of order at most 6, with their cardinality, are given in [13]. The cardinality of all minimal critical sets, $|\mathbf{V}_n^{min}|$, of an SCMS of order n , n odd, with known M and hence known $\mathbb{P}'$, can be determined. For illustration, the SCMSs of order 5 (which are also minimum PSCMSs) have been taken from [6], and the cardinality of their minimal critical sets is given.

Theorem 6. The cardinality of all minimal critical sets of an SCMS of order 5 is $|\mathbf{V}_5^{min}| = 7$, which satisfies $\frac{1}{2}(n^2 - 4n + 9)$, the lower bound of Theorem 4.

Proof. Suppose for contradiction there are six non-empty cells, including the centre cell. Given any SCMS, suppose there are six non-empty cells, including the centre cell. It can be clearly seen using the permutation operations in Table 1 that, irrespective of which cells are non-empty, the grid is multiply completable, and hence the non-empty cells do not form a critical set. An example of a critical set of size 7 of an SCMS of order 5 (which is also a minimum PSCMS) is given in Figure 6, and hence the cardinality of the minimal critical set of an SCMS of order 5 is 7. $\square$

	71	311		
11	461	23		
113		251		

(a)

419	71	311	401	53
11	461	23	269	491
113	59	251	443	389
263	233	479	41	239
449	431	191	101	83

(b)

Figure 6. A partial SCMS (which is also a minimum PSCMS), $\mathbf{H}_5$, specified by a minimal critical set, $\mathbf{V}_5^{min}$, of an SCMS and its unique completion, $\mathbf{A}_5$. (a) A grid specified by a minimal critical set of an SCMS of order 5; (b) unique completion of the partial SCMS in (a).

Theorem 6 will be useful for the classification of unavoidable sets of PSCMS of order 5 in Section 5.

While the concepts of strong and weak critical sets are extended above from the literature on Latin Squares to SCMS, a further concept of ‘totally weak’ critical sets [10] (where no cell is initially forced) does not apply. Consider an SCMS with a valid centre cell

value, the completion of any single cell results in a forced cell completion by a pairwise-completion operation. A set containing only the centre cell is not a critical set as the partial grid so defined is not uniquely completable. If both values in paired cells are non-empty and the grid is uniquely completable, then one of the values can be removed and the grid is still uniquely completable, and so the filled cells do not form a critical set. Hence, the concept of a totally weak critical set does not exist for any SCMS.

4. Unavoidable Sets of SCMSs

Consider the two SCMS given in Figure 7. Now consider the shaded cells to be empty, forming partial grids; each of the grids so formed is multiply completable. Consider the partial SCMS corresponding to Figure 7a, and assume now that exactly one of the shaded cells is non-empty; then, the resulting partial grid is uniquely completable. Similarly, consider the partial SCMS corresponding to Figure 7b, and assume now that exactly one of the shaded cells is non-empty. In this case, the resulting partial grid is either uniquely completable or multiply completable to SCMSs that are equivalent. Sets of cells that have these properties are referred to as **unavoidable sets**. Figure 7a provides an example of a proper unavoidable set, since the addition of one non-empty cell results in the partial SCMS being uniquely completable. Figure 7b provides an example of an improper unavoidable set since the addition of one non-empty cell results in a partial SCMS which is either uniquely completable or multiply completable to SCMSs that are equivalent. Formal definitions for these are given in Definition 18.

353	11	41	401	449
191	431	83	239	311
269	59	251	443	233
389	263	419	71	113
53	491	461	101	149

(a)

353	11	41	401	449
191	431	83	239	311
269	59	251	443	233
389	263	419	71	113
53	491	461	101	149

(b)

Figure 7. Unavoidable sets of SCMSs. (a) A proper unavoidable set; (b) an improper unavoidable set.

Recall that H_n denotes the non-empty cells of the partial SCMS, $\mathbf{H}_n$. Correspondingly, denote by H'_n the set of empty cells. Unavoidable sets are considered in the literature on Latin Squares, Sudoku, and magic squares (e.g., [14]). Similar to Sudoku [5], we formally define an unavoidable set of an SCMS.

Definition 18. An unavoidable set, D_n , of an SCMS, $\mathbf{A}_n$, is such that the partial SCMS, $\mathbf{H}_n$, for which $H_n = \mathbf{A}_n \setminus D_n$, is multiply completable, i.e., $|\mathbb{A}_n^{\mathbf{H}_n}| > 1$ and either

1. The addition to $\mathbf{H}_n$ of any triple $(i, j, a_{i,j}) \in \mathbf{A}_n$, where $(i, j) \in D_n$, produces a partial SCMS, $\mathbf{G}_n$, which is uniquely completable; or
2. The addition to $\mathbf{H}_n$ of any triple $(i, j, a_{i,j}) \in \mathbf{A}_n$ where $(i, j) \in D_n$, produces a partial SCMS, $\mathbf{G}_n$, which is either multiply completable to SCMS that are equivalent or possibly, for certain choices of triple $(i, j, a_{i,j})$, uniquely completable.

It follows that $D_n \subset \mathbf{A}_n$, where $\mathbf{A}_n$ is the set of cells corresponding to $\mathbf{A}_n$. Note that there is no set of triples, $\mathbf{D}_n$, corresponding to the set of cells, D_n , as the cells are empty and hence the values in the cells of D_n are not defined.

A given SCMS can have more than one unavoidable set, and these sets can overlap. Strictly, unavoidable sets are sub-structures that are present in all of the SCMSs to which a given partial grid completes.

For the purpose of classifying unavoidable sets, this paper will explore these sets in relation to completability of partial grids.

Definition 19. An unavoidable set is here considered a *proper unavoidable set* if it conforms to condition 1 of Definition 18. An unavoidable set is here considered an *improper unavoidable set* if it conforms to condition 2 of Definition 18.

In cases of proper unavoidable sets, $\mathbb{A}_n^H$ contains non-equivalent and equivalent completions of the partial grid H_n , and the addition of a further triple constrains the completion either to a unique grid or to only multiple equivalent grids in $\mathbb{A}_n^H$. In cases of improper unavoidable sets, $\mathbb{A}_n^H$ contains multiple completions of the partial grid, H_n , which are non-equivalent, equivalent or both, and the addition of a further triple constrains the completion to a unique grid in $\mathbb{A}_n^H$.

Definition 20. A partial SCMS, H_n , completable to any $A_n \in \mathbb{A}_n^H$, may contain cells that are forced and completable and a number of unavoidable sets. Let the set of unavoidable sets of a partial SCMS, H_n , be $\mathbb{D}_n^H$, and so $\bigcup_{D_n \in \mathbb{D}_n^H} D_n \subseteq H'_n$.

This paper does not address overlapping unavoidable sets as it is not beneficial in classifying the forms of unavoidable sets. Only partial SCMSs with a set of empty cells that correspond to a single unavoidable set of all grids in $\mathbb{A}_n^H$ are considered.

Recall from critical sets and strong partial SCMSs that whenever one half of a complement pair is included in a partial SCMS, its partner is immediately forced. For the purpose of the following analysis of unavoidable sets, it is assumed that for any partial SCMS considered, either both triples of a given complement pair are included or neither are. Hence any unavoidable set will be a collection of paired cells.

Corollary 4. Any two empty, non-corner cells in any single row or column of a border of an SCMS of order n , $n \geq 5$, (with their paired cells also empty) form a proper unavoidable set.

Proof. From Table 1, the values in the paired cells in rows (columns) $2, \dots, n-1$ of the grid can be permuted to form another valid completion, and hence the cells form an unavoidable set. By fixing any one valid value in an empty cell, the grid is uniquely completable, so the cells form a proper unavoidable set. $\square$

To illustrate two separate proper unavoidable sets, an SCMS of order larger than 5 is required. A normal SCMS of order 7 from [15] is used in Figure 8, the first proper unavoidable set is in the border of order 7 (highlighted blue) and the second is in the border of order 5 (highlighted grey).

46	1	2	3	42	41	40
45	35	13	14	32	31	5
44	34	28	21	26	16	6
7	17	23	25	27	33	43
11	20	24	29	22	30	39
12	19	37	36	18	15	38
10	49	48	47	8	9	4

Figure 8. Normal SCMS of order 7 containing two unavoidable sets.

5. Classification of Unavoidable Sets of Order 5

Different patterns of empty cells are now explored on partial PSCMSs of order 5, completable to minimum PSCMSs, where the interest is in the minimum number of empty cells in a given form of unavoidable set (defined below). There are two forms of proper unavoidable sets and an additional three forms of improper unavoidable sets that are identified for SCMSs of order 5. These are identified within the minimum PSCMSs of order 5 given in [6], and a complete classification for these structures is provided in Sections 5.1 and 5.2.

Forms 1 and 2 are proper unavoidable sets (Definitions 21 and 22) and define patterns of empty cells that are mutually exclusive. Likewise, Forms 3, 4 and 5 are improper unavoidable sets (Definitions 23–25) and define patterns of empty cells that are mutually exclusive to each other and to those patterns of proper unavoidable sets. It is possible that, for multiple patterns of empty cells, the corresponding partial grids complete to the same PSCMSs. That is, two partial PSCMSs with different patterns of empty cells could have one or more of their completions in common. Given a PSCMS with fixed prime M , $\mathbb{P}'$ denotes the set of all primes that can be paired to sum to $2M$.

Every minimum PSCMS of order 5 has $M = 251$. There are thirteen pairs of primes summing to $2M$, $|\mathbb{P}'| = 26$, with twelve needed for a grid. There are two non-equivalent centre subsquares, and for the first centre subsquare (denoted centre subsquare 1, Figure 9a), there are six different lists of twelve pairs of primes that form a minimum PSCMS of order 5, and for the second centre subsquare (denoted centre subsquare 2, Figure 9b), there are nine different lists of twelve pairs of primes that form a minimum PSCMS of order 5.

Each grid with either centre subsquare 1 or 2 and its corresponding list of primes is referred to as a **type**, and hence, there are six types with centre subsquare 1 and nine types with centre subsquare 2. These types are given in Table 2. There are 35 non-equivalent grids and 80,640 grids in total; a full enumeration can be found at [6].

461	23	269	431	83	239
59	251	443	59	251	443
233	479	41	263	419	71

(a) (b)

Figure 9. The two non-equivalent centre subsquares of order 3 for a minimum PSCMS of order 5. (a) Centre subsquare 1; (b) centre subsquare 2.

Table 2. Primes used in B_5 of the minimum PSCMS of order 5 with magic centre subsquares 1 and 2 (taken from [6]).

Type	List of Primes in the Border of Order 5
1A	11, 53, 71, 83, 101, 113, 191, 239, 263, 311, 389, 401, 419, 431, 449, 491
1B	11, 53, 83, 101, 113, 149, 191, 239, 263, 311, 353, 389, 401, 419, 449, 491
1C	11, 71, 83, 101, 113, 149, 191, 239, 263, 311, 353, 389, 401, 419, 431, 491
1D	53, 71, 83, 101, 113, 149, 191, 239, 263, 311, 353, 389, 401, 419, 431, 449
1E	11, 53, 71, 83, 101, 113, 149, 239, 263, 353, 389, 401, 419, 431, 449, 491
1F	11, 53, 71, 83, 101, 149, 191, 239, 263, 311, 353, 401, 419, 431, 449, 491
2A	11, 23, 41, 53, 101, 113, 149, 191, 311, 353, 389, 401, 449, 461, 479, 491
2B	11, 23, 41, 53, 101, 113, 149, 233, 269, 353, 389, 401, 449, 461, 479, 491
2C	11, 23, 41, 53, 101, 113, 191, 233, 269, 311, 389, 401, 449, 461, 479, 491
2D	11, 23, 41, 101, 113, 149, 191, 233, 269, 311, 353, 389, 401, 461, 479, 491
2E	11, 23, 41, 53, 101, 149, 191, 233, 269, 311, 353, 401, 449, 461, 479, 491
2F	11, 23, 41, 53, 113, 149, 191, 233, 269, 311, 353, 389, 449, 461, 479, 491
2G	11, 23, 53, 101, 113, 149, 191, 233, 269, 311, 353, 389, 401, 449, 479, 491
2H	11, 41, 53, 101, 113, 149, 191, 233, 269, 311, 353, 389, 401, 449, 461, 491
2I	23, 41, 53, 101, 113, 149, 191, 233, 269, 311, 353, 389, 401, 449, 461, 479

5.1. Proper Unavoidable Sets of Minimum PSCMSs of Order 5

Definition 21. Consider a partial PSCMS of order 5, H_5 , such that all empty cell tuples form a single unavoidable set and are located within the outer border. If all the multiple completions of H_5 , in A_5^H , use a single subset of $\mathbb{P}'$ and are equivalent using the permutations of paired cells in columns 2, 3, 4 of the grid or the permutations of paired cells in rows 2, 3, 4 of the grid (from Table 1), then the unavoidable set is defined to be of Form 1 denoted $D_5^{H,1}$.

An unavoidable set of Form 1 is given in Figure 10 in a minimum PSCMS of order 5.

Example 2. Figure 10 shows a partial PSCMS, multiply completable to minimum PSCMSs of order 5, with a Form 1 unavoidable set of size 4 in cells (1, 2), (1, 3), (5, 2), (5, 3) and its two equivalent completions.

419			401	53	419	71	311	401	53	419	311	71	401	53
11	461	23	269	491	11	461	23	269	491	11	461	23	269	491
113	59	251	443	389	113	59	251	443	389	113	59	251	443	389
263	233	479	41	239	263	233	479	41	239	263	233	479	41	239
449			101	83	449	431	191	101	83	449	191	431	101	83
(a)					(b)					(c)				

Figure 10. Form 1 unavoidable set of size 4 in a minimum PSCMS of order 5 with its two completions. (a) Form 1 unavoidable set of a minimum PSCMS of order 5; (b) first completion of (a), with highlighted paired cells given the same colour; (c) second completion of (a).

The completions shown in Figure 10b,c are equivalent, using the same 24 border pairs. The addition of any triple to the partial PSCMS in Figure 10a determines a unique completion, and hence the empty cells describe a proper unavoidable set.

Lemma 5. An unavoidable set of Form 1 is a proper unavoidable set.

Proof. A Form 1 unavoidable set satisfies Definition 19 and hence condition 1 of Definition 18. $\square$

Theorem 7. If the empty cells of a partial PSCMS, H_5 , completable to a minimum PSCMS, are an unavoidable set of Form 1, $D_5^{H,1}$, then $|D_5^{H,1}| = 4$, and partial PSCMSs of order 5 exist that contain such unavoidable sets of size 4.

Proof. Consider first a partial PSCMS of order 5 with fewer than four empty cells; it is uniquely completable. Next consider a partial PSCMS of order 5 with exactly four empty cells: the addition of any triple determines a unique completion. Finally, consider a partial PSCMS of order 5 with more than four empty cells; the addition of a triple does not guarantee a uniquely completable grid. An example of an unavoidable set, $D_5^{H,1}$, for which $|D_5^{H,1}| = 4$ is given in Figure 10. $\square$

Definition 22. Consider a partial PSCMS of order 5, H_5 , such that all empty cell tuples form a single unavoidable set and are located within the outer border. If all the multiple completions of H_5 , in A_5^H , are non-equivalent and each completion uses a different subset of $\mathbb{P}'$, then the empty cell tuples form an unavoidable set of Form 2 denoted $D_5^{H,2}$.

An unavoidable set of Form 2 is given in Example 3 in a minimum PSCMS of order 5.

Example 3. Figure 11 shows a partial PSCMS, multiply completable to a minimum PSCMS of order 5, with a Form 2 unavoidable set of size 6 in cells (1,4), (1,5), (4,1), (4,5), (5,1), (5,4) and its two non-equivalent completions.

311	11	113		
149	461	23	269	353
263	59	251	443	239
	233	479	41	
	491	389		191

(a)

311	11	113	419	401
149	461	23	269	353
263	59	251	443	239
431	233	479	41	71
101	491	389	83	191

(b)

311	11	113	401	419
149	461	23	269	353
263	59	251	443	239
449	233	479	41	53
83	491	389	101	191

(c)

Figure 11. Minimal Form 2 unavoidable set of size 6 in a minimum PSCMS of order 5 with its two completions. (a) Minimal Form 2 unavoidable set of a minimum PSCMS of order 5; (b) first completion of (a), with highlighted paired cells given the same colour; (c) second completion of (a).

The completions shown in Figure 11b,c are non-equivalent and use different subsets of $\mathbb{P}'$. The addition of any triple to the partial PSCMS in Figure 11a determines a unique completion, and hence the empty cells describe a proper unavoidable set.

Lemma 6. An unavoidable set of Form 2 is a proper unavoidable set.

Proof. A Form 2 unavoidable set satisfies Definition 19 and hence condition 1 of Definition 18. $\square$

Theorem 8. If the empty cells of a partial PSCMS, $\mathbf{H}_5$, completable to a minimum PSCMS, are an unavoidable set of Form 2, $D_5^{\mathbf{H},2}$, then $|D_5^{\mathbf{H},2}| \geq 6$, and partial PSCMSs of order 5 exist that contain such unavoidable sets of size 6.

Proof. Consider first a partial PSCMS, completable to minimum PSCMS of order 5, with fewer than six empty cells. Assume for contradiction that the grid has four empty cells; then, either it is uniquely completable and hence the cells do not form an unavoidable set, or it is a Form 1 unavoidable set. Now assume that the grid has any fewer than four empty cells; then, it does not contain an unavoidable set as the grid is uniquely completable. An example of an unavoidable set, $D_5^{\mathbf{H},2}$, for which $|D_5^{\mathbf{H},2}| = 6$ is given in Figure 11. $\square$

5.2. Improper Unavoidable Sets of Minimum PSCMSs of Order 5

Definition 23. Consider a partial PSCMS of order 5, $\mathbf{H}_5$, such that all empty cell tuples form a single unavoidable set and are located in both the outer border and the centre subsquare of order 3. For the empty cell tuples to form an unavoidable set of Form 3, denoted $D_5^{\mathbf{H},3}$, then all completions of $\mathbf{H}_5$, in $\mathbb{A}_5^{\mathbf{H}}$, that use the same subset of $\mathbb{P}'$ are equivalent. There are also completions of $\mathbf{H}_5$, in $\mathbb{A}_5^{\mathbf{H}}$, that use different subsets of $\mathbb{P}'$, and hence these are non-equivalent.

It follows that a partial PSCMS, completable to a minimum PSCMS of order 5, having a single unavoidable set of Form 3 will have multiple equivalent and non-equivalent completions and trivially all completions that have different centre subsquares of order 3 are non-equivalent. Figure 12a shows a minimal unavoidable set of Form 3 in a minimum PSCMS of order 5. If a minimum PSCMS of order 5 has a Form 3 unavoidable set, then there are empty cells in both the border of order 3 and the border of order 5. The fixing of any empty cell either gives a uniquely completable partial PSCMS or it gives a multiply completable partial PSCMS where all completions are equivalent.

Example 4. Figure 12 shows a partial PSCMS, multiply completable to a minimum PSCMS of order 5, with a Form 3 unavoidable set of size 12 in cells (1, 3), (1, 5), (2, 2), (2, 3), (2, 4), (3, 1), (3, 5), (4, 2), (4, 3), (4, 4), (5, 1), (5, 3) and its two non-equivalent completions. The multiple equivalent completions are not given.

353	11		401	
191				311
	59	251	443	
389				113
	491		101	149

(a)

353	11	41	401	449
191	431	83	239	311
269	59	251	443	233
389	263	419	71	113
53	491	461	101	149

(b)

353	11	71	401	419
191	461	23	269	311
239	59	251	443	263
389	233	479	41	113
83	491	431	101	149

(c)

Figure 12. Minimal Form 3 unavoidable set of size 12 in a minimum PSCMS of order 5 with its two non-equivalent completions. (a) Minimal Form 3 unavoidable set of a minimum PSCMS of order 5; (b) first completion of (a), with highlighted paired cells given the same colour; (c) second completion of (a).

The completions shown in Figure 12b,c are non-equivalent and use different subsets of $\mathbb{P}'$. The addition of any triple to the partial PSCMS in Figure 12a determines either a unique completion or multiple equivalent completions, and hence the empty cells describe an improper unavoidable set.

Lemma 7. An unavoidable set of Form 3 is an improper unavoidable set.

Proof. A Form 3 unavoidable set satisfies Definition 19 and hence condition 2 of Definition 18. $\square$

Example 5. Figure 13a shows a partial PSCMS, multiply completable to a minimum PSCMS of order 5, with a Form 3 unavoidable set of size 12. Figure 13b shows, in yellow, the triple (1, 3, 41) added to the grid in (a), and its two equivalent completions are given in Figure 13c,d. Figure 13e shows, in yellow, the triple (2, 2, 431) added to the grid in (a) and its unique completion is given in Figure 13f. Hence the unavoidable set is improper.

Theorem 9. If the empty cells of a partial PSCMS, $\mathbf{H}_5$, completable to a minimum PSCMS, are an unavoidable set of Form 3, $D_5^{\mathbf{H}_3}$, then $|D_5^{\mathbf{H}_3}| \geq 12$, and a partial PSCMS of order 5 exist that contain such unavoidable sets of size 12.

Proof. There are exactly two possible centre subsquares for the minimum PSCMS of order 5, and these differ by three pairs. Hence, in order for the centre subsquare to have more than one unique completion, there must be six empty cells in the centre subsquare of order 3. If there are fewer than four empty cells (two pairs) in the border, then all empty cells are forced, and hence the border is uniquely completable, and therefore the grid is uniquely completable.

For the minimum PSCMS of order 5, $|\mathbb{P}'| = 26$, so each grid uses twelve pairs out of a possible thirteen, and hence each subset can only differ by one pair [6]. If there are exactly four empty cells in the border, then either the grid is uniquely completable or all multiple completions of the grid are equivalent. An example of a Form 3 unavoidable set of size 12 is given in Figure 12, and hence, the smallest cardinality of a Form 3 unavoidable set is 12. $\square$

353	11		401	
191				311
	59	251	443	
389				113
	491		101	149
(a)				
353	11	41	401	
191				311
	59	251	443	
389				113
	491		101	149
(b)				
353	11	41	401	449
191	431	83	239	311
269	59	251	443	233
389	263	419	71	113
53	491	461	101	149
(c)				
353	11	41	401	449
191	263	419	71	311
269	59	251	443	233
389	431	83	239	113
53	491	461	101	149
(d)				
353	11		401	
191	431			311
	59	251	443	
389				113
	491		101	149
(e)				
353	11		401	449
191	431	83	239	311
269	59	251	443	233
389	263	419	71	113
53	491	461	101	149
(f)				

Figure 13. Demonstration of placing two different triples in a Form 3 unavoidable set. (a) Multiply completable partial PSCMS with a Form 3 improper unavoidable set; (b) placing the triple (1, 3, 41), highlighted in yellow, into the grid in (a); (c) first completion of the grid in (b); (d) second completion of the grid in (b) equivalent to the grid in (c); (e) placing the triple (2, 2, 431), highlighted in yellow, into the grid in (a); (f) unique completion of (e) (which is also equal to first completion of (b)).

Definition 24. Consider a partial PSCMS of order 5, H_5 , such that all empty cell tuples form a single unavoidable set and are located within the outer border. For the empty cell tuples to form an unavoidable set of Form 4, denoted $D_5^{H_4}$, then all completions use the same subset of $\mathbb{P}'$ for each of the multiple equivalent and non-equivalent completions in $\mathbb{A}_5^H$.

From Definition 24, the multiple completions in $\mathbb{A}_5^H$ are always a mix of equivalent and non-equivalent completions, but the addition of any triple always results in a partial grid that has multiple equivalent completions.

Example 6 shows a minimal unavoidable set of Form 4 in a minimum PSCMS of order 5. Like Form 1 and Form 2, Form 4 unavoidable sets comprise cells entirely in the of order 5 and contain the minimum number of empty cells in a border for there to be multiple completions such that at least two are non-equivalent and the list of primes is fixed.

Example 6. Figure 14 shows a partial PSCMS, multiply completable to minimum PSCMS of order 5, with a Form 4 unavoidable set of size 8 in cells (1, 2), (1, 3), (2, 1), (2, 5), (4, 1), (4, 5), (5, 2), (5, 3) and its two non-equivalent completions. The multiple equivalent completions are not given.

The completions shown in Figure 14b,c are non-equivalent and use the same subset of $\mathbb{P}'$. The addition of any triple to the partial PSCMS in Figure 14a determines multiple equivalent completions, and hence the empty cells describe an improper unavoidable set.

Lemma 8. An unavoidable set of Form 4 is an improper unavoidable set.

Proof. A Form 4 unavoidable set satisfies Definition 19 and hence condition 2 of Definition 18. $\square$

101			449	431
	461	23	269	
353	59	251	443	149
	233	479	41	
71			53	401

(a)

101	11	263	449	431
311	461	23	269	191
353	59	251	443	149
419	233	479	41	83
71	491	239	53	401

(b)

101	83	191	449	431
239	461	23	269	263
353	59	251	443	149
491	233	479	41	11
71	419	311	53	401

(c)

Figure 14. Minimal Form 4 unavoidable set of size 8 in a minimum PSCMS of order 5 with its two non-equivalent completions. (a) Minimal Form 4 unavoidable set of a minimum PSCMS of order 5; (b) first completion of the grid in (a), with highlighted paired cells given the same colour; (c) second completion of the grid in (a).

Example 7. Figure 15a shows a partial PSCMS, multiply completable to a minimum PSCMS of order 5, with a Form 4 unavoidable set of size 8. Figure 15b shows, in yellow, the triple (1, 2, 11) added to the grid in (a), and its two equivalent completions are given in Figure 15c,d. Hence the unavoidable set is improper.

Theorem 10. If the empty cells of a partial PSCMS, H_5 , completable to a minimum PSCMS, are an unavoidable set of Form 4, $D_5^{H,4}$, then $|D_5^{H,4}| \geq 8$, and partial PSCMSs of order 5 exist that contain such unavoidable sets of size 8.

Proof. Consider first a partial PSCMS, completable to minimum PSCMS of order 5, with fewer than six empty cells. If the grid has four empty cells, then either it is uniquely completable, and hence the cells do not form an unavoidable set, or it has a Form 1 unavoidable set. Any fewer than four cells can never be an unavoidable set as the grid is always uniquely completable. If the partial PSCMS had exactly six empty cells, then it is immediately uniquely completable, or it is completable only to equivalent grids, as the subset of $\mathbb{P}'$ is fixed. An example of an unavoidable set, $D_5^{H,4}$, for which $|D_5^{H,4}| = 8$ is given in Figure 14. $\square$

Definition 25. Consider a partial PSCMS of order 5, H_5 , such that all empty cell tuples form a single unavoidable set and are located in both the outer border and the centre subsquare of order 3. For the empty cell tuples to form an unavoidable set of Form 5, denoted $D_5^{H,5}$, then all completions use the same subset of $\mathbb{P}'$. PSCMS may be formed using both centre subsquares. All PSCMSs with the same centre subsquare are equivalent.

All grids with Form 5 unavoidable sets have completions using both centre subsquares, and trivially, all pairs of completions using different centre subsquares are non-equivalent. As with Form 4, the multiple completions in $\mathbb{A}_5^H$ will always be a mix of equivalent and non-equivalent completions, but the addition of any triple always results in a partial grid that has multiple equivalent completions.

Example 8. Figure 16 shows a partial PSCMS, multiply completable to a minimum PSCMS of order 5, with a Form 5 unavoidable set of size 14 in cells (1, 2), (1, 4), (2, 2), (2, 3), (2, 4), (3, 1), (3, 5), (4, 1), (4, 2), (4, 3), (4, 4), (4, 5), (5, 2), (5, 4) and its two non-equivalent completions. The multiple equivalent completions are not given.

101			449	431
	461	23	269	
353	59	251	443	149
	233	479	41	
71			53	401

(a)

101	11		449	431
	461	23	269	
353	59	251	443	149
	233	479	41	
71			53	401

(b)

101	11	263	449	431
311	461	23	269	191
353	59	251	443	149
419	233	479	41	83
71	491	239	53	401

(c)

101	11	263	449	431
419	461	23	269	83
353	59	251	443	149
311	233	479	41	191
71	491	239	53	401

(d)

Figure 15. Demonstration of placing a triple, highlighted in yellow, in a Form 4 unavoidable set. (a) Multiply completable partial PSCMS with a Form 4 improper unavoidable set; (b) placing the triple (1, 2, 11) into the grid in (a); (c) first completion of the grid in (b); (d) second completion of the grid in (b) equivalent to the grid in (c).

The completions shown in Figure 16b,c are non-equivalent and use the same subset of $\mathbb{P}'$. The addition of any triple to the partial PSCMS in Figure 16a determines multiple equivalent completions, and hence the empty cells describe an improper unavoidable set. A further, detailed analysis of minimal Form 5 unavoidable sets in the minimum PSCMS of order 5 is now given.

Lemma 9. *An unavoidable set of Form 5 is an improper unavoidable set.*

Proof. A Form 5 unavoidable set satisfies Definition 19 and hence condition 2 of Definition 18. $\square$

311		53		401
113				389
	59	251	443	
101		449		191

(a)

311	71	53	419	401
113	461	23	269	389
239	59	251	443	263
491	233	479	41	11
101	431	449	83	191

(b)

311	11	53	479	401
113	431	83	239	389
269	59	251	443	233
461	263	419	71	41
101	491	449	23	191

(c)

Figure 16. Minimal Form 5 unavoidable set of size 14 in a minimum PSCMS of order 5 with its two non-equivalent completions. (a) Minimal Form 5 unavoidable set of a minimum PSCMS of order 5; (b) first completion of the grid in (a), with highlighted paired cells given the same colour; (c) second completion of the grid in (a).

Example 9. Figure 17a shows a partial PSCMS, completable to a minimum PSCMS of order 5, with a Form 5 unavoidable set of size 14. Figure 17b shows, in yellow, the triple (1, 2, 71) added to the grid in (a), and its four equivalent completions are given in Figure 17c–f. Hence the unavoidable set is improper.

Lemma 10. *If a partial PSCMS, H_5 , completable to a minimum PSCMS, contains an unavoidable set of Form 5, $D_5^{H,5}$, then $|D_5^{H,5}| \geq 12$.*

311		53		401
113				389
	59	251	443	
101		449		191

(a)

311	71	53		401
113				389
	59	251	443	
101		449		191

(b)

311	71	53	419	401
113	461	23	269	389
239	59	251	443	263
491	233	479	41	11
101	431	449	83	191

(c)

311	71	53	419	401
113	461	23	269	389
491	59	251	443	11
239	233	479	41	263
101	431	449	83	191

(d)

311	71	53	419	401
113	233	479	41	389
239	59	251	443	263
491	461	23	269	11
101	431	449	83	191

(e)

311	71	53	419	401
113	233	479	41	389
491	59	251	443	11
239	461	23	269	263
101	431	449	83	191

(f)

Figure 17. Demonstration of placing a triple, highlighted in yellow, in a Form 5 unavoidable set. (a) Multiply completable partial PSCMS with a Form 5 improper unavoidable set; (b) placing the triple (1, 2, 71) into the grid in (a); (c) first completion of the grid in (b); (d) second completion of the grid in (b) equivalent to the grid in (c); (e) third completion of the grid in (b) equivalent to the grids in (c,d); (f) fourth completion of the grid in (b) equivalent to the grids in (c–e).

Proof. There exist two unique centre subsquares of order 3 for the minimum PSCMS of order 5, and these centre subsquares consist of the centre cell value M and four pairs of complement primes, one of which is common to both centre subsquares. Hence, the existence of completions using both centre subsquares requires that three paired cells (six cells) are empty in the centre subsquare and three paired cells (six cells) in the border are empty. Therefore, at least twelve cells are empty. If fewer cells are empty in the border, since all completions use the same subset of $\mathbb{P}'$ then the partial grid is completable only to equivalent grids. $\square$

If three pairs of cells are empty in the centre subsquare of order 3 and three pairs are empty in the border, there are five distinctly different patterns in which these empty cells could be arranged, shown in Figure 18, where the grey cells have values and the white cells are empty. It will now be shown that none of these patterns of twelve empty cells can form a Form 5 unavoidable set.

Lemma 11. *There is no unavoidable set of Form 5, $D_5^{H,5}$, such that $|D_5^{H,5}| = 12$.*

Proof. It can easily be seen that the pattern relating to Case 1 in Figure 18 is always completable only to equivalent grids since it contains a forced completable cell in the border. Likewise for Case 2. Now considering Cases 3, 4 and 5, for every pattern of empty cells, where $M = 251$, either the partial grid is completable only to equivalent grids or the multiple completions use different lists of primes. Hence there are no Form 5 unavoidable sets in any minimum PSCMS of order 5 having only twelve empty cells. $\square$

Theorem 11. *If the empty cells of a partial PSCMS, $\mathbf{H}_5$, completable to a minimum PSCMS, are an unavoidable set of Form 5, $D_5^{\mathbf{H},5}$, then $|D_5^{\mathbf{H},5}| \geq 14$, and partial PSCMSs of order 5 exist that contain such unavoidable sets of size 14.*

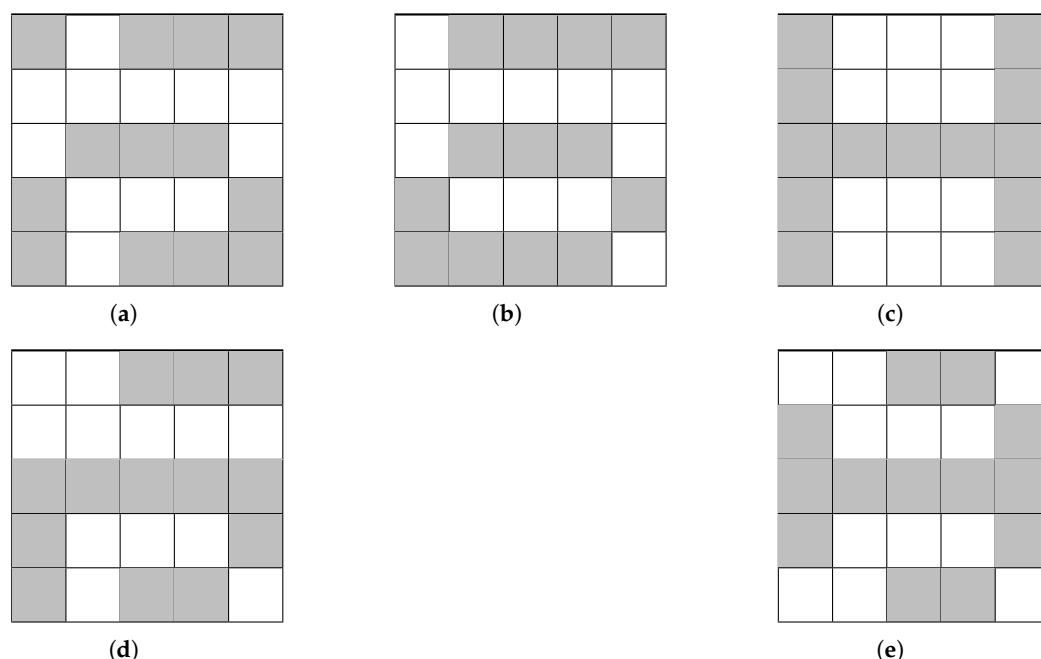


Figure 18. Patterns of twelve empty cell, shown in white, in grids of order 5 where six of the empty cells are in the order 3 centre subsquare. (a) Case 1; (b) case 2; (c) case 3; (d) case 4; (e) case 5.

Proof. From Lemmas 10 and 11, $|D_5^{\mathbf{H},5}| > 12$. Since an unavoidable set consists of paired cells, then $|D_5^{\mathbf{H},5}| \geq 14$, and an example of an unavoidable set of Form 5 consisting of 14 cells is given in Example 8. $\square$

Theorem 12. *If a partial PSCMS, $\mathbf{H}_5$, completable to a minimum PSCMS, contains an unavoidable set of:*

1. *Form 1, the minimum cardinality of the unavoidable set is 4.*
2. *Form 2, the minimum cardinality of the unavoidable set is 6.*
3. *Form 3, the minimum cardinality of the unavoidable set is 12.*
4. *Form 4, the minimum cardinality of the unavoidable set is 8.*
5. *Form 5, the minimum cardinality of the unavoidable set is 14.*

Proof. This follows from Theorems 7–11. $\square$

It can be noted that, from Definition 12, a strong partial SCMS, $\mathbf{H}_n^s$, is a set of triples such that the partial SCMS, $\mathbf{H}_n$, so defined is strongly completable. A strong partial SCMS ensures that there are no possible unavoidable sets of any form in the partial completable grid. If this is not the case, then the grid is not strongly completable. Hence, any strong partial SCMS includes at least one triple from every possible unavoidable set.

If a partial PSCMS, completable to minimum PSCMS of order 5, contains a single Form 5 unavoidable set, then it has completions including both centre subsquares of order 3, shown in Figure 9. The requirement to use the same subset of $\mathbb{P}'$ to complete the grid restricts the completions to two non-equivalent grids, one of which has centre subsquare 1 (Figure 9a) and the other has centre subsquare 2 (Figure 9b). This establishes a relationship between pairs of specific types, described in this section. The following Lemma establishes which types cannot be in such a pairing, and Theorem 13 specifies all valid pairings.

Recall from the start of this section that for centre subsquare 1, there are six different lists of primes, referred to here as types of centre subsquare 1, and for centre subsquare 2, there are nine different lists of primes, referred to here as types of centre subsquare 2, that can be used to form minimum PSCMS of order 5. In order for a partial PSCMS, completable to minimum PSCMS of order 5, to contain a single Form 5 unavoidable set, the subset $\mathbb{P}'$ used to complete the grid must be a valid list of primes for both a grid with centre subsquare 1 and a grid with centre subsquare 2.

Lemma 12. *No unavoidable set of Form 5 exists in five types of minimum PSCMSs of order 5.*

Proof. For all minimum PSCMSs of order 5, $\mathbb{P}' = \{11, 23, 41, 53, 59, 71, 83, 101, 113, 149, 191, 233, 239, 263, 269, 311, 353, 389, 401, 419, 431, 443, 449, 461, 479, 491\}$.

For one type of centre subsquare 1, the omitted pair is (71, 431), which occurs in centre subsquare 2. For three types of centre subsquare 2, the omitted pairs are (233, 269), (41, 461) and (23, 479), respectively, which all occur in centre subsquare 1. For one type of centre subsquare 2, the omitted pair is (101, 401), which is not found in centre subsquare 1, but it is in every subset of primes for each grid with centre subsquare 1. Hence an unavoidable set of Form 5 does not exist in these types. $\square$

Theorem 13. *Form 5 unavoidable sets exist in ten types of minimum PSCMSs of order 5.*

Proof. For all minimum PSCMSs of order 5, $\mathbb{P}' = \{11, 23, 41, 53, 59, 71, 83, 101, 113, 149, 191, 233, 239, 263, 269, 311, 353, 389, 401, 419, 431, 443, 449, 461, 479, 491\}$.

Consider a partial PSCMS, $\mathbf{H}_5$, completable to minimum PSCMS, with an unavoidable set of Form 5. If $\mathbf{H}_5$ is completable using $\mathbb{P}'$, and if the omitted pair from $\mathbb{P}'$ is any of the following five pairs, (149, 353), (53, 449), (11, 491), (191, 311), (113, 389), then $|\mathbb{A}_5^{\mathbf{H}}| > 2$ with exactly two non-equivalent completions, one of which is a minimum PSCMS of order 5 with centre subsquare 1, and the other a minimum PSCMS of order 5 with centre subsquare 2. $\square$

6. Conclusions

This paper provides foundational and theoretical underpinnings for SCMSs and PSCMSs. The concepts of critical sets and unavoidable sets previously defined for Latin Squares and Sudoku are defined for SCMSs, and two families of unavoidable sets of SCMSs are established. A full classification is given for unavoidable sets of minimum PSCMSs of order 5.

The definitions and results offer much scope for further work in this area. It remains an open problem whether there exist SCMSs with minimal critical set cardinality of 7 which are not also minimum PSCMSs. Another open problem is to determine how many unavoidable sets of different sizes exist on PSCMSs of higher orders. One trivial result is provided by Lemma 13.

Lemma 13. *If the empty cells of a partial PSCMS, $\mathbf{H}_n$, are an unavoidable set of Form 1, $D_n^{\mathbf{H},1}$, then $|D_n^{\mathbf{H},1}| = 4$.*

Proof. Consider first a partial PSCMS with fewer than four empty cells; it is uniquely completable. Next consider a partial PSCMS with exactly four empty cells: the addition of any triple determines a unique completion. Finally, consider a partial PSCMS with more than four empty cells; the addition of a triple does not guarantee a uniquely completable grid. $\square$

Similar to puzzles being constructed from Sudoku and Latin Square grids, puzzles may be constructed from SCMSs and PSCMSs, utilising their strong structural features and knowledge of their unavoidable sets to minimise the number of clues (filled cells) provided [16]. Applications of completable partial grids in Latin Squares and Sudoku to codes, cryptography and experimental design are well established in the literature [17–19]. Recently, applications of magic squares to cryptography, have also been identified as being of interest [20], and the current authors intend that the foundational material provided in this paper both enable and encourage further work in applications of SCMSs and PSCMSs.

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Abbreviations

The following abbreviations are used in this manuscript:

NMS	Normal Magic Square
CMS	Concentric Magic Square
SCMS	Strictly Concentric Magic Square
PSCMS	Prime Strictly Concentric Magic Square

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