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Functional Differential Equations with Non-Canonical Operator: Oscillatory Features of Solutions

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Abstract

This study focuses on investigating the asymptotic and oscillatory behavior of a new class of fourth-order nonlinear neutral differential equations. This research aims to achieve a qualitative advancement in the analysis and understanding of the relationships between the corresponding function and its derivatives. By utilizing various techniques, innovative criteria have been developed to ensure the oscillation of all solutions of the studied equations without resorting to additional constraints. Effective analytical tools are provided, contributing to a deeper theoretical understanding and expanding their application scope. The paper concludes by presenting examples that illustrate the practical impact of the results, highlighting the theoretical value of the research in the field of functional differential equations.

Keywords: noncanonical operator; fourth-order differential equations; oscillation

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1. Introduction

In this paper, we are concerned with the asymptotic and oscillatory behavior of solutions of fourth-order delay differential equations of the form

$$D_4 y(\top) + \int_a^b g(\top, z) y(l(\top, z)) \mathrm{d}z = 0, \tag{1}$$

where $0 \leq a < b$, and

$$D_4 y(\top) = \left(\rho_3(\top) \left(\rho_2(\top) \left(\rho_1(\top) y'(\top) \right)' \right)' \right)'.$$

In the following, we will assume that:

(H1) $\rho_i \in C^{(4-i)}(\top_0, \infty), i = 1, 2, 3, g(\top, z) \in C([\top_0, \infty) \times [a, b], (0, \infty)), l(\top, z) \in C([\top_0, \infty) \times [a, b], (0, \infty))$;

(H2) $\rho_i(\top) > 0, g(\top, z) > 0, l(\top, z) < \top$ for $z \in [a, b]$, $\lim_{\top \to \infty} l(\top, z) = \infty, l_1(\top) = l(\top, b)$ and $l_1'(\top) > 0$.

Definition 1. *Under a solution of (1) we mean a nontrivial function* $y(\top) \in C^1\left(\left[\top_y, \infty\right)\right)$, *which has the property*

$$D_4 y(\top) \in C\left(\left[\top_y, \infty\right)\right), \quad \top_y \geq \top_0,$$

and satisfies (1) on $\left[\top_y, \infty\right)$. *We only consider those solutions of (1) which exist on some half-line* $\left[\top_y, \infty\right)$ *and satisfy*

$$\sup\{|y(\top)| : t \leq \top < \infty\} > 0$$

for any $t \geq \top_y$. *A solution of (1) is said to be oscillatory if it has arbitrarily large zeros on* $\left[T_y, \infty\right)$. *Otherwise, it is said to be nonoscillatory. Equation (1) is called oscillatory if all its solutions are oscillatory.*

For the sake of simplicity, we define the operators:

$$\begin{aligned} D_1 y(\top) &= \rho_1(\top) y'(\top), \\ D_i y(\top) &= \rho_i(\top)(D_{i-1} y(\top))', \quad i = 2, 3. \end{aligned}$$

Differential equations play a crucial role in understanding and solving complex problems in various fields of modern technology. As systems in engineering, physics, biology, economics, and computer science become increasingly intricate, differential equations provide the mathematical framework to model dynamic behavior and predict outcomes over time. Whether analyzing the flow of electricity through circuits, the behavior of materials under stress, or the evolution of populations in ecological models, differential equations offer a precise method to describe how different variables interact and change. With advancements in computational power, the ability to solve and simulate these equations has accelerated innovation in fields such as robotics, artificial intelligence, medical technology, and environmental science. The study of differential equations is, therefore, not only fundamental to theoretical research but also indispensable for the continued progress and application of modern technological solutions [1–5].

The study of oscillation theory, a field rooted in mathematical analysis and physics, has profound implications across various disciplines, particularly in medicine and engineering. Oscillatory phenomena are observed in numerous systems, from the rhythmic beating of the human heart to the vibrations of mechanical structures. In medical science, understanding the principles of oscillation has led to advancements in diagnostic tools, such as heart rate monitoring, brainwave analysis, and the modelling of complex biological processes. Engineering, on the other hand, leverages oscillation theory to design systems that can endure or exploit oscillatory behavior, such as in control systems, mechanical resonators, and signal processing technologies [6–11].

Mathematical models that study the oscillation of solutions to differential equations are used in various areas of the medical field, especially in modeling biological systems and physiological processes. For example, the following models

$$\frac{\mathrm{d}^2}{\mathrm{d}\top^2} y + g^2(\top) y = 0,$$

where g is the angular frequency, are used to study systems that exhibit periodic behavior, such as the oscillations of the heart or respiratory rhythms. As another important example of the use of differential equation, we mention the famous Van der Pol oscillator equation

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\top^2} - m(1 - y)^2 \frac{\mathrm{d}y}{\mathrm{d}\top} + y = 0, \tag{2}$$

where m is a damping term and y is the displacement. The work in (2) used it to model biological rhythms and oscillatory behaviors like heartbeats, neural firing, and other cyclic phenomena in physiology.

Mathematical models based on fourth-order differential equations are used to study oscillatory phenomena in various fields, including the medical field. These models are particularly useful when analyzing systems with higher-order behavior or those requiring more complex representations. For example, differential equations

$$\frac{\mathrm{d}^4 y}{\mathrm{d}\top^4} - m^4 y = 0$$

are used in mechanical systems; these types of equations describe the vibrations of beams or plates and also the mechanical aspects of the heart valve's motion.

The study of the oscillation of solutions to fourth-order differential equations is a key area in mathematical analysis with significant applications in various scientific fields, including physics, engineering, and applied mathematics. This study not only enhances our understanding of the inherent properties of such equations but also contributes to the development of methods for controlling and optimizing the behavior of real-world systems. The authors in [12–14] established new oscillation criteria for fourth-order delay differential equations, particularly for equations of the form

$$\Big(\rho(\top)(x(\top) + g(\top)x(\sigma(\top)))''\Big)'' + g(\top)y(l(\top)) = 0,$$

where

$$\int_0^\infty \frac{\top}{\rho(\top)}\mathrm{d}\top < \infty.$$

Grace et al. [15] explored more efficient criteria for analyzing the asymptotic and oscillatory properties of solutions of equation

$$\Big(\rho_3\Big(\rho_2(\rho_1 y'(\top))'\Big)'\Big)' + g(\top)y(l(\top)) = 0,$$

where

$$\int_0^\infty \frac{1}{\rho_i(\top)}\mathrm{d}\top < \infty, \ \ i = 1, 2, 3.$$

The biggest challenge facing researchers lies in the difficulty of providing criteria that ensure the exclusion of all possible cases for the existence of non-oscillatory solutions in order to guarantee the oscillatory behavior of the equation's solutions. Therefore, Trench [16] resorted to benefiting from the transformation for Equation (2) to be in the following canonical form:

$$\Big(v_4(\top)\Big(v_3(\top)\Big(v_2(\top)(v_1(\top)y(\top))'\Big)'\Big)'\Big)' + g(\top)y(l(\top)) = 0,$$

where $v_i(\top) \in C([\top_0, \infty), \mathbb{R})$, and

$$\int_0^\infty \frac{1}{v_i(\top)}\mathrm{d}\top < \infty, \ \ i = 1, 2, 3.$$

The researchers have also addressed special cases of Equation (2) in several research papers, including, for example, [15,17–27] and the references mentioned therein.

The main objective of this work is to advance fundamental studies by introducing new criteria for the oscillation of a special class of fourth-order differential equations.

This is achieved by reducing the number of possible cases for nonoscillatory solutions, thereby simplifying and expanding the scope of results derived from previous research—particularly those cited in [15,27]. Theorems 8 and 10 reduce the possible nonoscillatory solutions of Equation (1) to only (P_7) or (P_8), fundamentally facilitating the examination of the oscillation of solutions to Equation (1). Therefore, this work provides substantial contributions to the oscillation theory of differential equations.

Notation and Definitions

For the sake of clarity, we will use the following notation:

$$\begin{aligned}\Pi_i(\top) &= \int_\top^\infty \frac{1}{\rho_i(s)}\mathrm{d}s,\\ \Pi_{ij}(\top) &= \int_\top^\infty \frac{1}{\rho_i(s)}\Pi_j(s)\mathrm{d}s\end{aligned}$$

and

$$\Pi_{ijk}(\top) = \int_\top^\infty \frac{1}{\rho_i(s)}\Pi_{jk}(s)\mathrm{d}s,$$

where $i, j, k \in \{1, 2, 3\}$.

Remark 1. *In this paper, we take into account the following remarks:*

(1) *All functional inequalities are supposed to hold eventually; this means all functional inequalities are satisfied for all $\top$ large enough.*
(2) *If y is a solution of (1), then $-y$ is also a solution of (1). Thus, when analyzing nonoscillatory solutions of Equation (1), it is sufficient to consider only the eventually positive solutions.*

Definition 2. *We say that (1) has property M if any nonoscillatory solution y of (1) satisfies*

$$\lim_{\top\to\infty} \rho_1(\top)y'(\top) = \lim_{\top\to\infty} y(\top) = 0. \tag{3}$$

2. Preliminary Results

We start with some properties, which are useful when proving the main results.

Lemma 1 ([28]). *Let $y(\top) > 0$ be a solution of (1). Then*

$$D_4 y(\top) < 0,$$

and there are the following eight possible cases:

$$\begin{aligned}
\text{Case } (P_1) &: D_1 y(\top) > 0, D_2 y(\top) > 0, D_3 y(\top) > 0,\\
\text{Case } (P_2) &: D_1 y(\top) > 0, D_2 y(\top) > 0, D_3 y(\top) < 0,\\
\text{Case } (P_3) &: D_1 y(\top) > 0, D_2 y(\top) < 0, D_3 y(\top) < 0,\\
\text{Case } (P_4) &: D_1 y(\top) > 0, D_2 y(\top) < 0, D_3 y(\top) > 0,\\
\text{Case } (P_5) &: D_1 y(\top) < 0, D_2 y(\top) > 0, D_3 y(\top) > 0,\\
\text{Case } (P_6) &: D_1 y(\top) < 0, D_2 y(\top) < 0, D_3 y(\top) > 0,\\
\text{Case } (P_7) &: D_1 y(\top) < 0, D_2 y(\top) > 0, D_3 y(\top) < 0,\\
\text{Case } (P_8) &: D_1 y(\top) < 0, D_2 y(\top) < 0, D_3 y(\top) < 0.
\end{aligned}$$

Lemma 2. *Assume that (1) holds. Then*

(*I*) $\Pi_i(\top)\Pi_j(\top) = \Pi_{ij}(\top) + \Pi_{ji}(\top)$,

(*II*) $\Pi_{321}(\top) = \Pi_{123}(\top) + \Pi_{32}(\top)\Pi_1(\top) - \Pi_3(\top)\Pi_{12}(\top)$.

Proof. Note that

$$-\left(\frac{1}{\rho_i(\top)}\Pi_j(\top) + \frac{1}{\rho_j(\top)}\Pi_i(\top)\right) = \big(\Pi_i(\top)\Pi_j(\top)\big)'.$$

Integrating the above equation from $\top$ to ∞, we have

$$\begin{aligned}\Pi_i(\top)\Pi_j(\top) &= \int_\top^\infty \frac{1}{\rho_i(s)}\Pi_j(s)\mathrm{d}s + \int_\top^\infty \frac{1}{\rho_j(s)}\Pi_i(s)\mathrm{d}s \\ &= \Pi_{ij}(\top) + \Pi_{ji}(\top).\end{aligned}$$

Now, we aim to prove that

$$\Pi_{321}(\top) - \Pi_{123}(\top) = \Pi_{32}(\top)\Pi_1(\top) - \Pi_3(\top)\Pi_{12}(\top).$$

It is easy to see that

$$\begin{aligned}(\Pi_{32}(\top)\Pi_1(\top) - \Pi_3(\top)\Pi_{12}(\top))' &= \frac{1}{\rho_3(s)}\Pi_1(\top)\Pi_2(s) + \frac{1}{\rho_1(s)}\Pi_{32}(\top) \\ &\quad -\frac{1}{\rho_1(s)}\Pi_3(\top)\Pi_2(s) - \frac{1}{\rho_3(s)}\Pi_{12}(\top)\end{aligned}$$

Using (I), we obtain

$$\begin{aligned}(\Pi_{32}(\top)\Pi_1(\top) - \Pi_3(\top)\Pi_{12}(\top))' &= \frac{1}{\rho_3(s)}(\Pi_{12}(\top) + \Pi_{21}(\top)) \\ &\quad +\frac{1}{\rho_1(s)}\Pi_{32}(\top) \\ &\quad -\frac{1}{\rho_1(s)}(\Pi_{32}(\top) + \Pi_{23}(s)) \\ &\quad -\frac{1}{\rho_3(s)}\Pi_{12}(\top).\end{aligned}$$

That is

$$\begin{aligned}(\Pi_{32}(\top)\Pi_1(\top) - \Pi_3(\top)\Pi_{12}(\top))' &= \frac{1}{\rho_3(s)}\Pi_{21}(\top) + \frac{1}{\rho_1(s)}\Pi_{32}(\top) \\ &\quad -\frac{1}{\rho_1(s)}\Pi_{32}(\top) - \frac{1}{\rho_1(s)}\Pi_{23}(s).\end{aligned}$$

Integrating the above equation from $\top$ to ∞, we have

$$\begin{aligned}\Pi_{32}(\top)\Pi_1(\top) - \Pi_3(\top)\Pi_{12}(\top) &= \int_\top^\infty \frac{1}{\rho_3(s)}\Pi_{21}(\top)\mathrm{d}s - \int_\top^\infty \frac{1}{\rho_1(s)}\Pi_{23}(s)\mathrm{d}s \\ &= \Pi_{321}(\top) - \Pi_{123}(\top).\end{aligned}$$

The proof is complete. □

Now, we present important properties for nonoscillatory solutions of (1) when cases (P_7) or (P_8) is satisfied.

Lemma 3. *Let $y(\top) > 0$ be a solution of (1) satisfying (P_7). If*

$$\int_{\top_0}^{\infty} \left(\int_a^b g(\top, z) \mathrm{d}z \right) \Pi_3(s) \Pi_1(l(s,z)) \mathrm{d}s = \infty, \tag{4}$$

then property M is verified.

Proof. Assume that $y(\top) > 0$ is a solution of (1) satisfying (P_7) of Lemma 1, for $\top \geq \top_1 \geq \top_0$. Using the fact that $y(\top) > 0$ and $y'(\top) < 0$, there exists a finite nonnegative

$$\lim_{\top \to \infty} y(\top) = \ell \geq 0.$$

Assume on the contrary that $\ell > 0$. Then there exists a $\top \geq \top_1$ such that $y(l(\top, z)) \geq \ell > 0$. Integrating (1) from $\top_1$ to $\top$, we obtain

$$\begin{aligned} -(D_2 y(\top))' &\geq \frac{1}{\rho_3(\top)} \int_{\top_1}^{\top} \int_a^b g(\top, z) y(l(\top, z)) \mathrm{d}z \mathrm{d}s \\ &\geq \frac{\ell}{\rho_3(\top)} \int_{\top_1}^{\top} g(s) \mathrm{d}s. \end{aligned} \tag{5}$$

Integrating from $\top_1$ to ∞, we obtain

$$\begin{aligned} \left(\rho_2(\rho_1 y')\right)'(\top_1) &\geq \ell \int_{\top_1}^{\infty} \frac{1}{\rho_3(u)} \int_{\top_1}^{u} \int_a^b g(\top, z) \mathrm{d}z \mathrm{d}s \mathrm{d}u \\ &= \ell \int_{\top_1}^{\infty} \left(\int_a^b g(\top, z) \mathrm{d}z \right) \Pi_3(s) \mathrm{d}s, \end{aligned}$$

which contradicts (4), and thus, $\lim_{\top \to \infty} y(\top) = \ell = 0$.

Since $-\rho_1 y' > 0$ is decreasing, there exists $\lim_{\top \to \infty} -\rho_1 y'(\top) = \ell$ such that $\ell \geq 0$. Let $\ell > 0$. Then

$$-\rho_1 y'(\top) > \ell, \ \top \geq \top_1.$$

Integrating this inequality from $\top$ to ∞, we find

$$\frac{y(\top)}{\ell} \geq \Pi_1(\top). \tag{6}$$

Combining (6) and (5), we have

$$-\rho_3(\top)(D_2 y(\top))' \geq \ell \int_{\top_1}^{\top} \left(\int_a^b g(\top, z) \mathrm{d}z \right) \Pi_1(l(s,z)) \mathrm{d}s.$$

An integration from $\top_1$ to ∞ yields

$$\left(\rho_2(\rho_1 y')\right)'(\top_1) \geq \ell \int_{\top_1}^{\infty} \frac{1}{\rho_3(u)} \int_{\top_1}^{u} \left(\int_a^b g(\top, z) \mathrm{d}z \right) \Pi_1(l(s,z)) \mathrm{d}s \mathrm{d}u.$$

So switching the order of integration between $u \in [\top_1, \infty)$ and $s \in [\top_1, u)$ gives

$$\left(\rho_2(\rho_1 y')\right)'(\top_1) \geq \ell \int_{\top_1}^{\infty} \left(\int_a^b g(\top, z) \mathrm{d}z \right) \left(\int_s^{\infty} \frac{1}{\rho_3(u)} \mathrm{d}u \right) \Pi_1(l(s,z)) \mathrm{d}s.$$

That is

$$\left(\rho_2(\rho_1 y')\right)'(\top_1) \geq \ell \int_{\top_1}^{\infty} \left(\int_a^b g(\top, z) \mathrm{d}z \right) \Pi_3(s) \Pi_1(l(s,z)) \mathrm{d}s,$$

which clearly contradicts (4). The proof is complete. □

Theorem 1. *Assume that $y(\top) > 0$ is a solution of (1) satisfying (P_7), and that (4) holds. Then*

(i) $\left(\frac{y(\top)}{\Pi_{12}(\top)}\right)' \leq 0$;

(ii) $\left(\frac{y(\top)}{\Pi_{123}(\top)}\right)' \geq 0$.

Proof. Assume that $y(\top) > 0$ is a positive solution of (1) satisfying (P_7) of Lemma 1, for $\top \geq \top_1 \geq \top_0$.

(i) Using (3) and the fact that $D_2 y(\top) > 0$ is decreasing, we get

$$-\rho_1(\top)y'(\top) = \int_{\top}^{\infty} \frac{1}{\rho_2(s)} D_2 y(s) \mathrm{d}s \leq D_2 y(\top)\Pi_2(\top),$$

which yields

$$\left(\frac{\rho_1(\top)}{\Pi_2(\top)} y'(\top)\right)' = \frac{1}{\rho_2(\top)\Pi_2^2(\top)} \left(D_2 y(\top)\Pi_2(\top) + \rho_1(\top)y'(\top)\right) \geq 0.$$

Thus, $(\rho_1(\top)y'(\top)/\Pi_2(\top))' \geq 0$. From (3), it follows that

$$-y(\top) = \int_{\top}^{\infty} \frac{\rho_1(s)\Pi_2(s)y'(s)}{\rho_1(s)\Pi_2(s)} \mathrm{d}s \geq \frac{\rho_1(\top)\Pi_{12}(\top)}{\Pi_2(\top)} y'(\top),$$

and we find that

$$\left(\frac{1}{\Pi_{12}(\top)} y(\top)\right)' = \frac{\rho_1(\top)\Pi_{12}(\top)y'(\top) + \Pi_2(\top)y(\top)}{\rho_1(\top)\Pi_{12}^2(\top)} \leq 0.$$

That is $\left(\frac{y(\top)}{\Pi_{12}(\top)}\right)' \leq 0$.

(ii) Using the fact that $D_3 y(\top) < 0$ is decreasing function, we see that

$$-D_2 y(\top) \leq \int_{\top}^{\infty} \frac{1}{\rho_3(s)} D_3 y(s) \mathrm{d}s \leq D_3 y(\top)\Pi_3(\top),$$

which implies that

$$\left(\frac{1}{\Pi_3(\top)} D_2 y(\top)\right)' = \frac{1}{\rho_3(\top)\Pi_3^2(\top)} \left(D_3 y(\top)\Pi_3(\top) + D_2 y(\top)\right) \geq 0,$$

and thus, $(D_2 y(\top)/\Pi_3(\top))' \geq 0$. Therefore,

$$\begin{aligned} -\rho_1(\top)y'(\top) &= \int_{\top}^{\infty} \frac{\Pi_3(s) D_2 y(s)}{\rho_2(s)\Pi_3(s)} \mathrm{d}s \\ &\geq \frac{\Pi_{23}(\top)}{\Pi_3(\top)} D_2 y(\top). \end{aligned}$$

It follows that $(\rho_1(\top)y'(\top)/\Pi_{23}(\top))' \geq 0$. Now, it is easy to see that

$$\begin{aligned} -y(\top) &= \int_{\top}^{\infty} \frac{\rho_1(s)\Pi_{23}(s)y'(s)}{\rho_1(s)\Pi_{23}(s)} \mathrm{d}s \\ &\leq \rho_1(\top) \frac{\Pi_{123}(\top)}{\Pi_{23}(\top)} y'(\top). \end{aligned}$$

That is,

$$\left(\frac{1}{\Pi_{123}(\top)} y(\top)\right)' = \frac{1}{\rho_1(\top)\Pi_{123}^2(\top)} \left(\rho_1(\top)y'(\top)\Pi_{123}(\top) + y(\top)\Pi_{23}(\top)\right) \geq 0,$$

which leads to $\left(\frac{y(\top)}{\Pi_{123}(\top)}\right)' \geq 0$. □

Theorem 2. *Assume that $y(\top) > 0$ is a solution of (1) satisfying (P_8). Then*

$$\left(\frac{y(\top)}{\Pi_1(\top)}\right) \geq 0. \tag{7}$$

Proof. Let $y(\top) > 0$ be a solution of (1) satisfying (P_8) in Lemma 1, for $\top \geq \top_1 \geq \top_0$. Using monotonic property of $\rho_1(\top)y'(\top)$, we obtain

$$-y(\top) = \int_{\top}^{\infty} \frac{\rho_1(s)}{\rho_1(s)} y'(s)\mathrm{d}s \leq \rho_1(\top)y'(\top)\Pi_1(\top),$$

which implies that

$$\left(\frac{1}{\Pi_1(\top)}y(\top)\right)' = \frac{1}{\rho_1(\top)\Pi_1^2(\top)}\rho_1(\top)y'(\top)\Pi_1(\top) + y(\top) \geq 0.$$

The proof is complete. □

3. Main Results

In the following subsection, we present the conditions ensuring the nonexistence of solutions of types (P_1)–(P_8).

3.1. Absence of Solutions in the Classes (P_1)–(P_4)

Theorem 3. *Let $y(\top) > 0$ be a solution of (1). If*

$$\int_{\top_1}^{\infty} \Pi_{32}(s)\left(\int_a^b g(\top, z)\mathrm{d}z\right)\mathrm{d}s = \infty, \tag{8}$$

then cases (P_1)–(P_4) in Lemma 1 are impossible.

Proof. Let $y(\top) > 0$ be a solution of (1) satisfying (P_1) or (P_4) in Lemma 1, for $\top \geq \top_1 \geq \top_0$. Using the fact that $y' \geq 0$, there exists a constant $k > 0$ such that $y(\top) \geq k$ for $\top \geq \top_1$. Integrating (1) from $\top_1$ to ∞, we get

$$\begin{aligned} D_3 y(\top_1) &\geq \int_{\top_1}^{\infty}\int_a^b g(\top, z)y(l(\top, z))\mathrm{d}z\mathrm{d}s \\ &\geq k\int_{\top_1}^{\infty}\int_a^b g(\top, z)\mathrm{d}z\mathrm{d}s, \end{aligned}$$

which contradicts (8). Now, let $y(\top) > 0$ be a solution of (1) satisfying (P_2) in Lemma 1, for $\top \geq \top_1$. Integrating (1) from $\top_1$ to $\top$, since $y'(\top) \geq 0$, we have

$$-\frac{1}{k}D_3 y(\top) \geq \int_{\top_1}^{\top}\int_a^b g(\top, z)\mathrm{d}z\mathrm{d}s. \tag{9}$$

Integrating (9) from $\top_1$ to ∞, we obtain

$$\begin{aligned} \frac{1}{k}D_2 y(\top_1) &\geq \int_{\top_1}^{\infty}\frac{1}{\rho_3(u)}\int_{\top_1}^{u}\int_a^b g(\top, z)\mathrm{d}z\mathrm{d}s\mathrm{d}u \\ &= \int_{\top_1}^{\infty}\Pi_3(u)\left(\int_a^b g(\top, z)\mathrm{d}z\right)\mathrm{d}u, \end{aligned}$$

which contradicts (8).

Now, let $y(\top) > 0$ be a solution of (1) satisfying (P_3) in Lemma 1, for $\top \geq \top_1$. As we see before, we get (9). Integrating (9) from $\top_1$ to $\top$, we find

$$-D_2 y(\top) \geq k \int_{\top_1}^{\top} \frac{1}{\rho_3(u)} \int_{\top_1}^{u} \int_a^b g(\top, z) \mathrm{d}z \mathrm{d}s \mathrm{d}u.$$

An integration from $\top_1$ to ∞ yields

$$\begin{aligned} \frac{1}{k} D_1 y(\top_1) &\geq \int_{\top_1}^{\infty} \frac{1}{\rho_2(v)} \int_{\top_1}^{v} \frac{1}{\rho_3(u)} \int_{\top_1}^{v} \int_a^b g(\top, z) \mathrm{d}z \mathrm{d}s \mathrm{d}u \mathrm{d}v \\ &= \int_{\top_1}^{\infty} \frac{1}{\rho_3(u)} \int_{\top_1}^{u} \left(\int_a^b g(\top, z) \mathrm{d}z \right) \mathrm{d}s \int_u^{\infty} \frac{1}{\rho_2(v)} \mathrm{d}v \mathrm{d}u \\ &= \int_{\top_1}^{\infty} \frac{1}{\rho_3(u)} \Pi_2(u) \int_{\top_1}^{u} \left(\int_a^b g(\top, z) \mathrm{d}z \right) \mathrm{d}s \mathrm{d}u \\ &= \int_{\top_1}^{\infty} \Pi_{32}(s) g(s) \mathrm{d}s, \end{aligned}$$

which is a contradiction to our initial assumption. The proof is complete. □

3.2. Absence of Solutions in the Classes (P_5), (P_6)

Theorem 4. *Let $y(\top) > 0$ be a solution of (1). If*

$$\int_{\top_1}^{\infty} \Pi_{12}(l(s)) \left(\int_a^b g(\top, z) \mathrm{d}z \right) \mathrm{d}s = \infty, \tag{10}$$

then cases (P_5) and (P_6) in Lemma 1 are impossible.

Proof. Assume on the contrary that $y(\top) > 0$ is a solution of (1) satisfying (P_5) in Lemma 1, for $\top \geq \top_1 \geq \top_0$. Since $(D_2 y(\top))' > 0$, there exists a constant $k > 0$ such that

$$\frac{1}{k} D_2 y(\top) \geq 0 \text{ for } \top \geq \top_1. \tag{11}$$

Integrating the (11) from $\top$ to ∞, we get

$$-\rho_1(\top) y'(\top) \geq k \int_{\top}^{\infty} \frac{1}{\rho_2(s)} \mathrm{d}s. \tag{12}$$

Integration (12) from $l(\top, z)$ to ∞, we have

$$\frac{1}{k} y(l_1(\top)) \geq \int_{l_1(\top)}^{\infty} \frac{1}{\rho_1(u)} \int_u^{\infty} \frac{1}{\rho_2(s)} \mathrm{d}s \mathrm{d}u = k \Pi_{12}(l_1(\top)). \tag{13}$$

An integration of (1) from $\top_1$ to ∞ and according to (13), we obtain

$$\begin{aligned} D_3 y(\top_1) &\geq \int_{\top_1}^{\infty} \int_a^b g(\top, z) y(l(\top, z)) \mathrm{d}z \mathrm{d}s \\ &\geq k \int_{\top_1}^{\infty} \left(\int_a^b g(\top, z) \mathrm{d}z \right) \Pi_{12}(l(s, z)) \mathrm{d}s, \end{aligned}$$

which is a contradiction to (10).

Now, let $y(\top) > 0$ be a solution of (1) satisfying (P_6) in Lemma 1, for $\top \geq \top_1 \geq \top_0$. Using $D_1 y < 0$ and $(D_1 y)' < 0$, there exists a constant $k > 0$ such that

$$D_1 y(\top) = \rho_1(\top) y'(\top) \leq -k \text{ for } \top \geq \top_1. \tag{14}$$

Integrating (14) from $l_1(\top)$ to ∞, we have

$$\frac{1}{k}y(l_1(\top)) \geq \int_{l_1(\top)}^{\infty} \frac{1}{\rho_1(s)}\mathrm{d}s. \tag{15}$$

Integrating (1) from $\top_1$ to ∞ and using (15), we arrive at

$$\begin{aligned} D_3y(\top_1) &\geq \int_{\top_1}^{\infty}\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\mathrm{d}s \\ &\geq k\int_{\top_1}^{\infty}\left(\int_a^b g(\top,z)\mathrm{d}z\right)\int_{l_1(s)}^{\infty}\frac{1}{\rho_1(u)}\mathrm{d}u\mathrm{d}s \\ &= \int_{\top_1}^{\infty}\left(\int_a^b g(\top,z)\mathrm{d}z\right)\Pi_1(l(s,z))\mathrm{d}s, \end{aligned}$$

and we are led to a contradiction. The proof is complete. □

3.3. Absence of Solutions in the Class (P_7)

Theorem 5. *Assume that $y(\top) > 0$ is a solution of (1), and that (4) holds. If*

$$\limsup_{\top\to\infty}\left\{\frac{\Phi_1(s,\top)}{\Pi_{12}(l_1(\top))} + \frac{\int_{\top}^{\infty}\left(\int_a^b g(\top,z)\mathrm{d}z\right)F(s,l_1(\top))\Pi_{123}(l_1(s))\mathrm{d}s}{\Pi_{123}(l_1(\top))}\right\} > 1, \tag{16}$$

where

$$\begin{aligned} \Phi_1(s,\top) &= \Pi_{123}(l_1(\top))\int_{\top_1}^{l_1(\top)}\left(\int_a^b g(\top,z)\mathrm{d}z\right)\Pi_{12}(l(s,z))\mathrm{d}s \\ &+ \int_{l_1(\top)}^{\top}\left(\int_a^b g(\top,z)\mathrm{d}z\right)F(s,l_1(\top))\mathrm{d}s, \end{aligned}$$

and

$$F(s,\top) = \Pi_{12}(l_1(s))[\Pi_{321}(s) + \Pi_{12}(\top)\Pi_3(s) - \Pi_1(\top)\Pi_{32}(s)],$$

then case (P_7) in Lemma 1 is impossible.

Proof. Assume on the contrary that $y(\top) > 0$ is a solution of (1) satisfying (P_7) of Lemma 1, for $\top \geq \top_1 \geq \top_0$. Integrating (1) from $\top_1$ to $\top$, then integrating the result from $\top$ to ∞, it follows that

$$D_2y(\top) \geq \int_{\top}^{\infty}\frac{1}{\rho_3(u)}\int_{\top_1}^{u}\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\mathrm{d}s\mathrm{d}u.$$

That is,

$$\begin{aligned} \left(\rho_1(\top)y'(\top)\right)' &\geq \frac{\Pi_3(\top)\int_{\top_1}^{\top}\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\mathrm{d}s}{\rho_2(\top)} \\ &+ \frac{\int_{\top}^{\infty}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\Pi_3(s)\mathrm{d}s}{\rho_2(\top)}. \end{aligned} \tag{17}$$

Integrating (17) from $\top$ to ∞, we obtain

$$\begin{aligned}
-\rho_1(\top)y'(\top) &\geq \int_{\top}^{\infty}\frac{\Pi_3(u)}{\rho_2(u)}\int_{\top_1}^{u}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s\mathrm{d}u\\
&\quad+\int_{\top}^{\infty}\frac{1}{\rho_2(u)}\int_{u}^{\infty}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\Pi_3(s)\mathrm{d}s\mathrm{d}u\\
&= \Pi_{23}(\top)\int_{\top_1}^{\top}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s\\
&\quad+\int_{\top}^{\infty}\Pi_{23}(s)\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s\\
&\quad+\int_{\top}^{\infty}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\Pi_3(s)[\Pi_2(\top)-\Pi_2(s)]\mathrm{d}s.
\end{aligned}$$

It follows from Lemma 2 that $\Pi_{23}(s)+\Pi_{32}(s)=\Pi_2(s)\Pi_3(s)$, and so

$$\begin{aligned}
-y'(\top) &= \frac{\Pi_{23}(\top)}{\rho_1(\top)}\int_{\top_1}^{\top}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s\\
&\quad+\frac{\Pi_2(\top)}{\rho_1(\top)}\int_{\top}^{\infty}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\Pi_3(s)\mathrm{d}s\\
&\quad-\frac{1}{\rho_1(\top)}\int_{\top}^{\infty}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\Pi_{32}(s)\mathrm{d}s.
\end{aligned}$$

Integrating from $\top$ to ∞ and using Lemma 2, we see that

$$\begin{aligned}
y(\top) &\geq \Pi_{123}(\top)\int_{\top_1}^{\top}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s\\
&\quad+\int_{\top}^{\infty}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\Pi_{123}(s)\mathrm{d}s\\
&\quad+\int_{\top}^{\infty}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\Pi_3(s)[\Pi_{12}(\top)-\Pi_{12}(s)]\mathrm{d}s\\
&\quad-\int_{\top}^{\infty}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\Pi_{32}(s)[\Pi_1(\top)-\Pi_1(s)]\mathrm{d}s\\
&= \Pi_{123}(\top)\int_{\top_1}^{\top}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s\\
&\quad+\int_{\top}^{\infty}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\Pi_{321}(s)\mathrm{d}s\\
&\quad+\Pi_{12}(\top)\int_{\top}^{\infty}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\Pi_3(s)\mathrm{d}s\\
&\quad-\Pi_1(\top)\int_{\top}^{\infty}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\Pi_{32}(s)\mathrm{d}s\\
&= \Pi_{123}(\top)\int_{\top_1}^{\top}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s\\
&\quad+\int_{\top}^{\infty}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)F(s,\top)\mathrm{d}s.
\end{aligned}$$

Then

$$\begin{aligned} y(l_1(\top)) &\geq \Pi_{123}(l_1(\top))\int_{\top_1}^{l_1(\top)}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s \\ &\quad+\int_{l_1(\top)}^{\infty}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)F(s,l_1(\top))\mathrm{d}s \\ &= \Pi_{123}(l_1(\top))\int_{\top_1}^{l_1(\top)}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s \\ &\quad+\int_{l_1(\top)}^{\top}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)F(s,l_1(\top))\mathrm{d}s \\ &\quad+\int_{l_1(\top)}^{\infty}\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)F(s,l_1(\top))\mathrm{d}s. \end{aligned}$$

Using the fact that $(y(\top)/\Pi_{12}(\top))$ is decreasing and $(y(\top)/\Pi_{123}(\top))$ is increasing, the last inequality yields

$$\begin{aligned} y(l_1(\top)) &\geq \Pi_{123}(l_1(\top))\frac{y(l_1(\top))}{\Pi_{12}(l_1(\top))}\int_{\top_1}^{l_1(\top)}\left(\int_a^b g(\top,z)\mathrm{d}z\right)\Pi_{12}(l(s,z))\mathrm{d}s \\ &\quad+\frac{y(l_1(\top))}{\Pi_{12}(l_1(\top))}\int_{l_1(\top)}^{\top}\left(\int_a^b g(\top,z)\mathrm{d}z\right)F(s,l_1(\top))\Pi_{12}(l(s,z))\mathrm{d}s \\ &\quad+\frac{y(l_1(\top))}{\Pi_{123}(l_1(\top))}\int_{\top}^{\infty}\left(\int_a^b g(\top,z)\mathrm{d}z\right)F(s,l_1(\top))\Pi_{123}(l(s,z))\mathrm{d}s. \end{aligned}$$

It follows that

$$\begin{aligned} &\frac{\Pi_{123}(l_1(\top))}{\Pi_{12}(l_1(\top))}\int_{\top_1}^{l_1(\top)}\left(\int_a^b g(\top,z)\mathrm{d}z\right)\Pi_{12}(l(\top,z))\mathrm{d}s \\ &+\frac{1}{\Pi_{12}(l(\top))}\int_{l_1(\top)}^{\top}\left(\int_a^b g(\top,z)\mathrm{d}z\right)F(s,l_1(\top))\Pi_{12}(l(s,z))\mathrm{d}s \\ &+\frac{1}{\Pi_{123}(l_1(\top))}\int_{\top}^{\infty}\left(\int_a^b g(\top,z)\mathrm{d}z\right)F(s,l_1(\top))\Pi_{123}(l_1(s))\mathrm{d}s \leq 1, \end{aligned}$$

a contradiction. The proof is complete. □

3.4. Absence of Solutions in the Class (P_8)

Theorem 6. *Let* $y(\top) > 0$ *be a solution of (1). If*

$$\limsup_{\top\to\infty}\left\{\Phi_2(s,\top)+\frac{\int_{\top}^{\infty}\left(\int_a^b g(\top,z)\mathrm{d}z\right)\Pi_{321}(s)\Pi_1(l(s,z))\mathrm{d}s}{\Pi_1(l_1(\top))}\right\} > 1, \tag{18}$$

where

$$\Phi_2(s,\top)=\int_{\top_1}^{l_1(\top)}\left(\int_a^b g(\top,z)\mathrm{d}z\right)H(s,l_1(\top))\mathrm{d}s+\int_{l_1(\top)}^{\top}\left(\int_a^b g(\top,z)\mathrm{d}z\right)\Pi_{321}(s)\mathrm{d}s,$$

and

$$H(s,\top)=\Pi_{32}(s)\Pi_1(\top)+\Pi_3(s)\Pi_{12}(\top)+\Pi_{123}(\top),$$

then case (P_8) *in Lemma 1 is impossible.*

Proof. Assume on the contrary that $y(\top) > 0$ is a solution of (1) satisfying (P_8) in Lemma 1, for $\top \geq \top_1 \geq \top_0$. Integrating (1) twice from $\top_1$ to $\top$, we arrive at

$$-\rho_2(\top)\big(\rho_1(\top)y'(\top)\big)' \geq \int_{\top_1}^{\top} (\Pi_3(s) - \Pi_3(\top))\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s. \tag{19}$$

Integrating (19) from $\top_1$ to $\top$, we obtain

$$\begin{aligned} -\rho_1(\top)y'(\top) &\geq \int_{\top_1}^{\top} \frac{1}{\rho_2(u)} \int_{\top_1}^{u} (\Pi_3(s) - \Pi_3(\top))\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s\mathrm{d}u \\ &= \int_{\top_1}^{\top} \Pi_3(s)(\Pi_2(s) - \Pi_2(\top))\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s \\ &\quad - \int_{\top_1}^{\top} \left((\Pi_{23}(s) - \Pi_{23}(\top)) \int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s. \end{aligned}$$

Using Lemma 2, we get

$$-\rho_1(\top)y'(\top) \geq \int_{\top_1}^{\top} (\Pi_{32}(s) - \Pi_3(s)\Pi_2(\top) + \Pi_{23}(s))\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s. \tag{20}$$

Integrating (20) from $\top$ to ∞, we find

$$\begin{aligned} y(\top) &\geq \int_{\top}^{\infty} \frac{(\Pi_{32}(s) - \Pi_3(s)\Pi_2(u) + \Pi_{23}(u)) \int_{\top_1}^{u} \left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)}{\rho_1(u)}\mathrm{d}s\mathrm{d}u \\ &= \int_{\top_1}^{\top} \left(\Pi_{32}(s)\Pi_1(\top) \int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s \\ &\quad - \int_{\top_1}^{\top} \left(\Pi_3(s)\Pi_{12}(\top) \int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s \\ &\quad + \int_{\top_1}^{\top} \Pi_{123}(\top)\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s \\ &\quad + \int_{\top}^{\infty} \Pi_{32}(s)\Pi_1(s)\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s \\ &\quad - \int_{\top}^{\infty} \Pi_3(s)\Pi_{12}(s)\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s \\ &\quad + \int_{\top}^{\infty} \left(\Pi_{123}(s) \int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s \end{aligned}$$

From

$$\Pi_{321}(\top) = \Pi_{123}(\top) + \Pi_{32}(\top)\Pi_1(\top) - \Pi_3(\top)\Pi_{12}(\top),$$

we conclude that

$$\begin{aligned} y(\top) &\geq \int_{\top_1}^{\top} (\Pi_{32}(s)\Pi_1(\top) - \Pi_3(s)\Pi_{12}(\top) + \Pi_{123}(\top))\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s \\ &\quad + \int_{\top}^{\infty} \Pi_{321}(s)\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s. \end{aligned}$$

Thus,

$$\begin{aligned} y(\top) &\geq \int_{\top_1}^{\top} H(s,\top)\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s \\ &\quad + \int_{\top}^{\infty} \Pi_{321}(s)\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s, \end{aligned} \tag{21}$$

and

$$\begin{aligned} y(l_1(\top)) &\geq \int_{\top_1}^{l_1(\top)} H(s,l_1(\top))\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s \\ &\quad + \int_{l_1(\top)}^{\infty} \Pi_{321}(s)\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s \\ &= \int_{\top_1}^{l_1(\top)} H(s,l_1(\top))\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s \\ &\quad + \int_{l_1(\top)}^{\top} \Pi_{321}(s)\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s \\ &\quad + \int_{\top}^{\infty} \Pi_{321}(s)\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s. \end{aligned}$$

From $y'(\top) < 0$ and $(y(\top)/\Pi_1(\top))' > 0$, we obtain

$$\begin{aligned} y(l_1(\top)) &\geq y(l_1(\top)) \int_{\top_1}^{l_1(\top)} H(s,l_1(\top))\left(\int_a^b g(\top,z)\mathrm{d}z\right)\mathrm{d}s \\ &\quad + y(l_1(\top)) \int_{l_1(\top)}^{\top} \Pi_{321}(s)\left(\int_a^b g(\top,z)\mathrm{d}z\right)\mathrm{d}s \\ &\quad + \frac{1}{\Pi_1(l_1(\top))} y(l_1(\top)) \int_{l_1(\top)}^{\infty} \Pi_1(l(s,z))\Pi_{321}(s)\left(\int_a^b g(\top,z)\mathrm{d}z\right)\mathrm{d}s, \end{aligned}$$

we find that

$$\begin{aligned} 1 &\geq \int_{\top_1}^{l_1(\top)} H(s,l_1(\top))\left(\int_a^b g(\top,z)\mathrm{d}z\right)\mathrm{d}s + \int_{l_1(\top)}^{\top} \Pi_{321}(s)\left(\int_a^b g(\top,z)\mathrm{d}z\right)\mathrm{d}s \\ &\quad + \frac{1}{\Pi_1(l_1(\top))} \int_{\top}^{\infty} \Pi_1(l(s,z))\Pi_{321}(s)\left(\int_a^b g(\top,z)\mathrm{d}z\right)\mathrm{d}s. \end{aligned}$$

We arrive at contradiction to (18). The proof is complete. □

Corollary 1. *Assume that $y(\top) > 0$ is a solution of (1). If*

$$\limsup_{\top\to\infty}\left\{\Phi_3(s,\top) + \frac{1}{\Pi_1(l_1(\top))}\int_{\top}^{\infty}\left(\int_a^b g(\top,z)\mathrm{d}z\right)\Pi_{321}(s)\Pi_1(l(\top,z))\mathrm{d}s\right\} > 1, \quad (22)$$

where

$$\Phi_3(s,\top) = \Pi_{321}(l_1(\top))\int_{\top_1}^{l_1(\top)}\left(\int_a^b g(\top,z)\mathrm{d}z\right)\mathrm{d}s + \int_{l_1(\top)}^{\top}\left(\int_a^b g(\top,z)\mathrm{d}z\right)\Pi_{321}(s)\mathrm{d}s,$$

then case (P_8) in Lemma 1 is impossible.

Proof. Assume on the contrary that $y(\top) > 0$ is a solution of (1) satisfying (P_8) in Lemma 1 for $\top \geq \top_1 \geq \top_0$. As in the proof of Theorem 6, we have (21). Using properties of $H(s,\top)$ and by Lemma 2, we see that

$$H(\top,\top) \leq H(s,\top) = \Pi_{321}(s) \quad (23)$$

for $s \in (\top_1,\top)$. Substituting (23) into (21), we obtain

$$\begin{aligned} y(\top) &\geq \Pi_{321}(\top)\int_{\top_1}^{\top}\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\mathrm{d}s \\ &\quad + \int_{\top}^{\infty}\Pi_{321}(s)\left(\int_a^b g(\top,z)y(l(\top,z))\mathrm{d}z\right)\mathrm{d}s. \end{aligned}$$

That is,

$$\begin{aligned} y(l_1(\top)) \geq\ & \Pi_{321}(l(\top)) \int_{\top_1}^{l_1(\top)} \left(\int_a^b g(\top,z) y(l(\top,z)) \mathrm{d}z \right) \mathrm{d}s \\ & + \int_{l_1(\top)}^{\top} \Pi_{321}(s) \left(\int_a^b g(\top,z) y(l(\top,z)) \mathrm{d}z \right) \mathrm{d}s \\ & + \int_{\top}^{\infty} \Pi_{321}(s) \left(\int_a^b g(\top,z) y(l(\top,z)) \mathrm{d}z \right) \mathrm{d}s. \end{aligned}$$

It follows from properties $y'(\top) < 0$ and $(y(\top)/\Pi_1(\top))' > 0$, we obtain

$$\begin{aligned} 1 \geq\ & \Pi_{321}(l_1(\top)) \int_{\top_1}^{l_1(\top)} \left(\int_a^b g(\top,z) \mathrm{d}z \right) \mathrm{d}s + \int_{l_1(\top)}^{\top} \left(\int_a^b g(\top,z) \mathrm{d}z \right) \Pi_{321}(s) \mathrm{d}s \\ & + \frac{1}{\Pi_1(l_1(\top))} \int_{\top}^{\infty} \left(\int_a^b g(\top,z) \mathrm{d}z \right) \Pi_{321}(s) \Pi_1(l(s,z)) \mathrm{d}s, \end{aligned}$$

which contradicts (22). The proof is complete. □

3.5. Oscillatory Criteria

Combining Theorems 3–6, one can easily provide fundamentally new oscillatory criteria of (1).

Theorem 7. *Assume that*

$$\int_{\top_0}^{\infty} \left(\int_a^b g(\top,z) \mathrm{d}z \right) \Pi_3(s) \Pi_1(l(s,z)) \mathrm{d}s = \infty, \tag{24}$$

$$\int_{\top_1}^{\infty} \Pi_{32}(s) \left(\int_a^b g(\top,z) \mathrm{d}z \right) \mathrm{d}s = \infty, \tag{25}$$

$$\limsup_{\top \to \infty} \left\{ \frac{\Phi_1(s,\top)}{\Pi_{12}(l_1(\top))} + \frac{\int_{\top}^{\infty} \left(\int_a^b g(\top,z) \mathrm{d}z \right) F(s, l_1(\top)) \Pi_{123}(l_1(s)) \mathrm{d}s}{\Pi_{123}(l_1(\top))} \right\} > 1 \tag{26}$$

and if

$$\limsup_{\top \to \infty} \left\{ \Phi_2(s,\top) + \frac{\int_{\top}^{\infty} \left(\int_a^b g(\top,z) \mathrm{d}z \right) \Pi_{321}(s) \Pi_1(l(s,z)) \mathrm{d}s}{\Pi_1(l_1(\top))} \right\} > 1, \tag{27}$$

or

$$\limsup_{\top \to \infty} \left\{ \Phi_3(s,\top) + \frac{1}{\Pi_1(l_1(\top))} \int_{\top}^{\infty} \left(\int_a^b g(\top,z) \mathrm{d}z \right) \Pi_{321}(s) \Pi_1(l(\top,z)) \mathrm{d}s \right\} > 1, \tag{28}$$

where Φ_1, F and Φ_2, Φ_3 are defined as in Theorems 5, 6 and Corollary 1, respectively, hold, then (1) is oscillatory.

4. Examples

Example 1. *Consider the following equation:*

$$\frac{\mathrm{d}}{\mathrm{d}\top} \left(\top^2 \frac{\mathrm{d}}{\mathrm{d}\top} \left(\top^2 \frac{\mathrm{d}}{\mathrm{d}\top} \left(\top^2 \frac{\mathrm{d}}{\mathrm{d}\top} y(\top) \right) \right) \right) + a\top^2 y(\lambda\top) = 0, \top \geq \top_0 > 0, \tag{29}$$

where $a \in (0,\infty), \lambda \in (0,1)$. It is easy to note that

$$\Pi_i(\top) = \frac{1}{\top}, \Pi_{ij}(\top) = \frac{1}{2\top^2}, \Pi_{123}(\top) = \Pi_{321}(\top) = \frac{1}{6\top^3}.$$

Clearly one an see that conditions (25) and (24) hold. Based on Condition (26), we have

$$\lambda^2\left(\frac{36}{a}+1\right)-9\lambda-18\ln\lambda^{-1}<9. \tag{30}$$

Also, Condition (27) implies that

$$\frac{1}{6}17a+a\ln\lambda^{-1}>6. \tag{31}$$

Now, by applying Theorem 7, we conclude that (29) is oscillatory if (30) and (31) hold.

Furthermore, according to Theorem 7, we see that conditions (25) and (24) hold, and condition (26) implies (30). By (28), we obtain

$$\frac{a\lambda^3}{18}-\frac{a}{6}\ln\lambda>1. \tag{32}$$

We conclude that (29) is oscillatory if (30) and (32) hold.

Example 2. *Consider the following equations:*

$$\frac{\mathrm{d}}{\mathrm{d}\top}\left(\top^2\frac{\mathrm{d}}{\mathrm{d}\top}\left(\top^2\frac{\mathrm{d}}{\mathrm{d}\top}\left(\top^2\frac{\mathrm{d}}{\mathrm{d}\top}y(\top)\right)\right)\right)+a\top^2y(0.8\top)=0, \tag{33}$$

$\top\geq\top_0>0$, *where* $a\in(0,\infty)$. *According to Theorem 7, similarly as in Example (29), we see that (33) is oscillatory if* $a>1.963$, *while, in [27], (33) is oscillatory if* $a>7.913$.

Furthermore, by Theorem 7, we see that (34) is oscillatory if $a>15.24$.

Remark 2. *Considering Example 2, we find that Theorem (27) yields better results than (28) and also outperforms the results mentioned in [27]. So our results are more efficient than the previous ones.*

Example 3. *Consider the following equation:*

$$\frac{\mathrm{d}}{\mathrm{d}\top}\left(\top^2\frac{\mathrm{d}}{\mathrm{d}\top}\left(\top^2\frac{\mathrm{d}}{\mathrm{d}\top}\left(\top^2\frac{\mathrm{d}}{\mathrm{d}\top}y(\top)\right)\right)\right)+a\top^2y(0.9\top)=0, \tag{34}$$

$\top\geq\top_0>0$, *where* $a\in(0,\infty)$. *By Theorem 7, similarly as in Example (29), we conclude that (34) is oscillatory if* $a>1.77$.

According to the best condition in [15], (34) is oscillatory if $a>3.50$.

5. Conclusions

This study thoroughly investigates the oscillatory properties of noncanonical fourth-order differential equations. By establishing several theorems, we have formulated criteria that ensure the oscillation of all solutions to the studied equation. In contrast to the approaches commonly adopted in most of the previous literature, our work introduces precise and simplified conditions that systematically eliminate the possibility of non-oscillatory solutions; see, for example, [12–15,27]. Based on these findings, we establish strict oscillation criteria that guarantee all solutions of the considered equations exhibit oscillatory behavior. Our results demonstrate significant superiority (see Examples 1 and 2), while also offering ease of application and effectiveness in addressing key challenges associated with non-standard differential equations.

Our results not only extend and refine existing theoretical frameworks but also open new directions for further research, fostering deeper exploration in this field. We suggest

that future studies explore the application of the approach adopted in this paper to higher-order equations, as well as its applicability to more general forms, for example

$$D_4 y(\top) + \sum_{i=1}^{k} g(\top, z) y(l(\top, z)) \mathrm{d}z = 0.$$

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