

Article

On the Algebraic Independence of the Values of Functions That Are Certain Integrals Involving the ${}_1F_1(1; \lambda + 1; z)$ Hypergeometric Function [†]

Vasily Gorelov ^{1,*} and Gennady Voronov ^{2,‡}

¹ Moscow Power Engineering Institute, National Research University, 111250 Moscow, Russia

² Department of Higher Mathematics, MIREA, Russian Technological University, 119454 Moscow, Russia; voronov_g@mirea.ru

* Correspondence: gorelov.va@mail.ru; Tel.: +7-495-362-77-74

[†] The results of the work are obtained in the framework of the state contract of the Ministry of Education and Science of the Russian Federation (project no. FSWF-2023-0012).

[‡] These authors contributed equally to this work.

Abstract

Indefinite integrals of products of exponential functions, power functions and generalized hypergeometric functions of some types are considered. Necessary and sufficient conditions are established for the algebraic independence of large sets of such functions (for various parameters) and their derivatives, as well as their values. All the algebraic relations between these functions are written out explicitly.

Keywords: generalized hypergeometric functions; algebraic independence; Siegel's method; E-functions

MSC: 11J91



Academic Editor: Gradimir V. Milovanović

Received: 11 June 2025

Revised: 11 July 2025

Accepted: 18 July 2025

Published: 25 July 2025

Citation: Gorelov, V.; Voronov, G. On the Algebraic Independence of the Values of Functions That Are Certain Integrals Involving the ${}_1F_1(1; \lambda + 1; z)$ Hypergeometric Function. *Axioms* **2025**, *14*, 572. <https://doi.org/10.3390/axioms14080572>

Copyright: © 2025 by the authors.

Licensee MDPI, Basel, Switzerland.

This article is an open access article distributed under the terms and

conditions of the Creative Commons Attribution (CC BY) license

(<https://creativecommons.org/licenses/by/4.0/>).

1. Introduction

This work is devoted to the development and generalization of the results of references [1,2].

Let $\mathbb{A}$ be the set of all algebraic numbers, $\mathbb{L}\{\omega_1, \dots, \omega_n\}$ be the linear space over $\mathbb{Q}$ generated by numbers $\omega_1, \dots, \omega_n \in \mathbb{C}$, $\mathbb{C}[z^{\pm 1}]$ be the ring $\mathbb{C}[z, z^{-1}]$, and $\mathbb{C}\langle v_1, \dots, v_n \rangle$ be the differential field obtained by joining to $\mathbb{C}$ analytical functions $v_1, \dots, v_n$.

The Siegel–Shidlovsky method (see [3–5]) remains to this day one of the main transcendental number theory methods. This method allows us to prove the transcendence and algebraic independence of the values of the so-called E-functions (see [3–5]). The set of E-functions is a ring, which is closed with respect to differentiation, integration and substitutions of the argument z by αz for $\alpha \in \mathbb{A}$. To apply the method, it is necessary to know the algebraic properties of the functions under consideration.

A well-known example of E-functions is the general hypergeometric E-functions ${}_l\varphi_q(\vec{v}; \vec{\lambda}; \alpha z^{q-l})$, where

$${}_l\varphi_q(\vec{v}; \vec{\lambda}; z) = {}_{l+1}F_q \left(\begin{matrix} 1, v_1, \dots, v_l \\ \lambda_1, \dots, \lambda_q \end{matrix} \middle| z \right) = \sum_{n=0}^{\infty} \frac{(v_1)_n \dots (v_l)_n}{(\lambda_1)_n \dots (\lambda_q)_n} z^n,$$

$(v)_0 = 1$, $(v)_n = v(v+1) \dots (v+n-1)$, $0 \leq l < q$, $\vec{v} = (v_1, \dots, v_l) \in \mathbb{Q}^l$, $\vec{\lambda} \in (\mathbb{Q} \setminus \mathbb{Z}_{\leq 0})^q$, $\alpha \in \mathbb{A}$.

K. Siegel ([3], § 2) formulated the conjecture that every E-function that is the solution of the linear differential equation with coefficients from $\mathbb{C}(z)$ coincides with a polynomial P in z and a finite set of hypergeometric E-functions.

In reference [6], J. Fresan and P. Jossen proved that in the general case Siegel's conjecture is incorrect (for a more detailed history of the issue, see, for example, [1,6–8]). Consequently, there is a need for methods that allow one to establish algebraic independence over $\mathbb{C}(z)$ of E-functions that are not expressed through hypergeometric E-functions using arithmetic operations.

Reference [1] considered the following functions:

$$V_{\lambda,\alpha}(z) = e^{\alpha z} \int_0^z e^{-\alpha t} \varphi_\lambda(t) dt = z + \left(\alpha + \frac{1}{\lambda+1} \right) \frac{z^2}{2} + \dots,$$

where $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, $\alpha \in \mathbb{C}$, $\varphi_\lambda(z)$ is a function introduced by A.B. Shidlovsky,

$$\varphi_\lambda(z) = {}_1F_1 \left(\begin{matrix} 1 \\ \lambda+1 \end{matrix} \middle| z \right) = 1 + \sum_{n=1}^{\infty} \frac{z^n}{(\lambda+1) \dots (\lambda+n)} = \lambda z^{-\lambda} e^z \int_0^z t^{\lambda-1} e^{-t} dt.$$

In the last equality, we use the notation of A.B. Shidlovsky (see [5], Ch. 5, § 2): if $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function, then

$$\lambda z^{-\lambda} \int_0^z t^{\lambda-1} f(t) dt = a_0 + \lambda \sum_{n=1}^{\infty} \frac{a_n}{\lambda+n} z^n, \quad \lambda \in \mathbb{C} \setminus \mathbb{Z}_{<0}.$$

“The Shidlovsky integral” $\lambda \int_0^z t^{\lambda-1} f(t) dt$ for $\lambda = \lambda_0 \notin \mathbb{Z}_{<0}$ can be found as integral $\lambda \int_0^z t^{\lambda-1} f(t) dt$ with parameter λ , $\operatorname{Re} \lambda > 0$, if the parameter λ is replaced with λ_0 in the final result.

Reference [2] considered more general functions

$$V_{\lambda,\nu,\alpha}(z) = \nu z^{-\nu} e^{\alpha z} \int_0^z t^{\nu-1} e^{-\alpha t} \varphi_\lambda(t) dt = 1 + \left(\alpha + \frac{\nu}{\lambda+1} \right) \frac{z}{\nu+1} + \dots, \quad (1)$$

where $\lambda, \nu \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, $\alpha \in \mathbb{C}$, satisfying the equations

$$y'' + \left(-\alpha - 1 + \frac{\lambda + \nu + 1}{z} \right) y' + \left(\alpha - \frac{\nu + (\lambda + 1)\alpha}{z} + \frac{\lambda \nu}{z^2} \right) y = \frac{\lambda \nu}{z^2}.$$

In this case, $V_{\lambda,\alpha}(z) = z V_{\lambda,1,\alpha}(z)$, $V_{\lambda,0,\alpha}(z) = e^{\alpha z}$, and $V_{\lambda,\nu,\alpha}(0) = 1$. Some functions $V_{\lambda,\nu,\alpha}(z)$ may coincide with polynomials in exponential functions, power functions, and functions $\varphi_\mu(\gamma z)$ (see Lemma 2 of this paper). The following theorem is also valid.

Theorem 1 (see [2], Theorem 2). *E-functions $V_{\lambda,\nu,\alpha}(z)$ and $V'_{\lambda,\nu,\alpha}(z)$ are algebraically dependent over $\mathbb{C}(z)$ if and only if $\nu = 0$, or $\alpha \in \mathbb{Q}$, $\lambda \in \mathbb{Z}$, $\nu \in \mathbb{Z}_{>\lambda}$, or $\alpha = 1$, $\nu - \lambda \in \mathbb{Z} \setminus \{0\}$, or $\alpha = 1$, $\lambda = 0$, or $\alpha = 2$, $\nu = 2\lambda$, or $\alpha \neq 1$, $\nu - \lambda = k \in \mathbb{Z}_{\geq 2}$,*

$$\lambda \sum_{n=0}^{k-1} \frac{(1-k)_n \alpha^{n+1}}{(2-\lambda-k)_n (\alpha-1)^n} = 0, \quad (2)$$

or $\nu = 2\lambda - k$, $k \in \mathbb{Z}_{\geq 3}$, $\alpha = 2$,

$$\sum_{n=0}^{k-1} \frac{(2\lambda-k)_n}{2^n (\lambda-k)_{n+1}} = 0, \quad (3)$$

or $v = 2\lambda + k$, $k \in \mathbb{Z}_{\geq 3}$, $\alpha = 2$,

$$\sum_{n=0}^{k-1} \frac{2^n(1-\lambda-k)_n}{(1-2\lambda-k)_{n+1}} = 0. \quad (4)$$

In this paper, we find necessary and sufficient conditions for algebraic independence and dependence of functions $V_{\lambda,\nu,\alpha}(\xi z)$, $V'_{\lambda,\nu,\alpha}(\xi z)$ for various parameters together with functions $\varphi_\mu(\gamma z)$, and power and exponential functions. From this, we can draw conclusions about the algebraic properties of the values of these functions. The main result is the following theorem.

Theorem 2. Let $\lambda_i, \nu_i \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, $\xi_i \in \mathbb{C} \setminus \{0\}$, $\alpha_i \in \mathbb{C}$, $i = 1, \dots, m$, $\mu_k \in \mathbb{C} \setminus \mathbb{Z}$, $\gamma_k \in \mathbb{C} \setminus \{0\}$, $k = 1, \dots, n$, $\mu_k - \mu_l \notin \mathbb{Z}$ for $\gamma_k = \gamma_l$, $k \neq l$; the numbers $\beta_1, \dots, \beta_p \in \mathbb{C}$, as well as $\zeta_1, \dots, \zeta_q \in \mathbb{C}$, are linearly independent over $\mathbb{Q}$, $\zeta_1 \in \mathbb{Q}$, $n, p \in \mathbb{Z}_{\geq 0}$, $m, q \in \mathbb{N}$. Then, $2m + n + p + q$ functions

$$V_{\lambda_1, \nu_1, \alpha_1}(\xi_1 z), V'_{\lambda_1, \nu_1, \alpha_1}(\xi_1 z), \dots, V_{\lambda_m, \nu_m, \alpha_m}(\xi_m z), V'_{\lambda_m, \nu_m, \alpha_m}(\xi_m z), \\ \varphi_{\mu_1}(\gamma_1 z), \dots, \varphi_{\mu_n}(\gamma_n z), e^{\beta_1 z}, \dots, e^{\beta_p z}, z^{\zeta_1}, \dots, z^{\zeta_q} \quad (5)$$

are algebraically dependent over $\mathbb{C}$ if and only if at least one of the following conditions is met:

1. $\nu_i = 0$, or $\lambda_i \in \mathbb{Z}$, $\xi_i \in \mathbb{L}\{\beta_1, \dots, \beta_p\}$, or $\lambda_i \in \mathbb{Z}$, $\nu_i \in \mathbb{Z}_{>\lambda}$, $\xi_i \in \mathbb{L}\{\alpha_i \xi_i, \beta_1, \dots, \beta_p\}$, or $\lambda_i \in \mathbb{Z}$, $\nu_i - \mu_j \in \mathbb{Z}$, $\nu_i - \mu_k \in \mathbb{Z}$, $\alpha_i \xi_i = \gamma_j$, $(\alpha_i - 1)\xi_i = \gamma_k$, or $\lambda_i - \mu_j \in \mathbb{Z}$, $\xi_i = \gamma_j$ for some $1 \leq i \leq m$, $j, k \in \{1, \dots, n\}$;
2. $\lambda_i = 0$, $\alpha_i = 1$, or $\lambda_i - \nu_i \in \mathbb{Z} \setminus \{0\}$, $\alpha_i = 1$, or $\lambda_i \neq \nu_i$, $\nu_i \in \mathbb{Z}$, $\alpha_i = 1$, $\xi_i \in \mathbb{L}\{\beta_1, \dots, \beta_p\}$, or $\lambda_i \neq \nu_i$, $\alpha_i = 1$, $\nu_i - \mu_k \in \mathbb{Z}$, $\xi_i = \gamma_k$, for some $1 \leq i \leq m$, $1 \leq k \leq n$;
3. $\nu_i - \lambda_i = k \in \mathbb{N}$, $\alpha_i \neq 1$ and either equality (2) is true for $\lambda = \lambda_i$, $k \geq 2$, or $\lambda_i - \mu_j \in \mathbb{Z}$, $\alpha_i \xi_i = \gamma_j$ for some $1 \leq i \leq m$, $1 \leq j \leq n$;
4. $\lambda_i - \lambda_l \in \mathbb{Z}$, $\xi_i = \xi_l$, or $\lambda_i, \lambda_l \in \mathbb{Z}$, $\xi_l \in \mathbb{L}\{\xi_i, \beta_1, \dots, \beta_p\}$ for some $1 \leq i < l \leq m$;
5. $\lambda_i \neq \nu_i$, $\lambda_l \neq \nu_l$, $\alpha_i = \alpha_l = 1$ and either $\nu_i, \nu_l \in \mathbb{Z}_{\geq 0}$, $\xi_l \in \mathbb{L}\{\xi_i, \beta_1, \dots, \beta_p\}$, or $\nu_i - \nu_l \in \mathbb{Z}$, $\xi_i = \xi_l$ for some $1 \leq i < l \leq m$;
6. $\lambda_i + \lambda_l = \nu_i = \nu_l$, $\xi_i + \xi_l = \xi_i \alpha_i = \xi_l \alpha_l$ for some $i, l \in \{1, \dots, m\}$;
7. $\lambda_i + \lambda_l - \nu_i = r \in \mathbb{Z} \setminus \{0\}$, $\nu_l - \nu_i = s \in \mathbb{Z}_{\geq 0}$, $\xi_i + \xi_l = \xi_i \alpha_i = \xi_l \alpha_l$ and either $\nu_i \in \mathbb{Z}$, $\xi_i + \xi_l \in \mathbb{L}\{\beta_1, \dots, \beta_p\}$, or $\nu_i - \mu_k \in \mathbb{Z}$, $\xi_i + \xi_l = \gamma_k$, $1 \leq k \leq n$, or for $r > 0$

$$\alpha_i \lambda_i \lambda_l \sum_{k=1}^r \frac{(1-\lambda_i)_{r-k} (1-\lambda_l)_{k-1}}{(\alpha_i - 1)^k} - \\ - \lambda_i \lambda_l \sum_{k=0}^{\omega-1} \frac{(v_i)_k (1-\lambda_i)_{r-k-1}}{\alpha_l^k (1-\alpha_l)^{-k}} + (-1)^r \lambda_l \sum_{k=\omega}^{s-1} \frac{(v_i)_k (\alpha_l - 1)^k}{(\lambda_i + 1)_{k-r} \alpha_l^k} = 0, \quad (6)$$

$\omega = \min(r, s)$, and for $r < 0$

$$\alpha_i \sum_{k=0}^{-r-1} \frac{(r-\lambda_i+1)_k}{(\lambda_l+1)_k (1-\alpha_i)^{-k}} + \lambda_l \sum_{k=0}^{s-1} \frac{(v_i)_k (\alpha_l - 1)^k}{(\lambda_i - r)_{k+1} \alpha_l^k} = 0, \quad (7)$$

for some $i, l \in \{1, \dots, m\}$.

Empty sums are set to zero everywhere.

It is easy to see that, under condition 7, $\alpha_i \neq 1$, $\alpha_l \neq 1$, $(1 - \alpha_l)^{-1} = 1 - \alpha_i$; hence, for $s = 0$, it follows that it is symmetric with respect to indices i and l . Note that conditions 6 and 7 can be combined. Note also that the constraints on the parameters λ for specific values of k , following from equalities (3) and (4), can be obtained with a smaller amount of computations from, respectively, (6) and (7), setting $s = 0$, $i = l$.

2. Auxiliary Statements

The function $\varphi_\lambda(z)$ is a solution of the following equation:

$$y' = \left(1 - \frac{\lambda}{z}\right)y + \frac{\lambda}{z} \quad (8)$$

(see [5], Ch. 5, § 2). The following identities hold:

$$\varphi_0(z) = e^z, \quad (9)$$

$$\varphi_\lambda(z) = \frac{z^l}{(\lambda+1) \dots (\lambda+l)} \varphi_{\lambda+l}(z) + 1 + \sum_{n=1}^{l-1} \frac{z^n}{(\lambda+1) \dots (\lambda+n)}, \quad (10)$$

where $l \in \mathbb{N}$, and

$$V'_{\lambda,\nu,\alpha}(z) = \left(\alpha - \frac{\nu}{z}\right)V_{\lambda,\nu,\alpha}(z) + \frac{\nu}{z}\varphi_\lambda(z). \quad (11)$$

Without reducing generality, we set $\zeta_1 = 1/b$, $b \in \mathbb{N}$.

Lemma 1 (see [5], Ch. 5, § 3; [9], Lemma 4; [10], Lemma 7). *Under the conditions of Theorem 2, functions (5) are algebraically independent over $\mathbb{C}$.*

Taking into account identities (9) and (10), the conditions of Lemma 1 are necessary and sufficient.

Lemma 2 (see [2], Theorem 1). *Let $\lambda, \nu \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, $\alpha \in \mathbb{C}$, $k \in \mathbb{N}$. Then,*

$$V_{\lambda,0,\alpha}(z) = e^{\alpha z}; \quad (12)$$

$$V_{0,\nu,\alpha}(z) = e^z \varphi_\nu((\alpha-1)z); \quad (13)$$

$$V_{k,\nu,\alpha}(z) = \frac{k!(1-\alpha)^k}{(1-\nu)_k} e^z \varphi_\nu((\alpha-1)z) + c_k \varphi_\nu(\alpha z) + P_k \varphi_k(z) + \tilde{P}_k, \quad (14)$$

$$\nu \notin \{1, \dots, k\}, \quad c_k \in \mathbb{Q}(\nu, \alpha), \quad P_k, \tilde{P}_k \in \mathbb{Q}(\nu, \alpha)[z], \quad P_k \neq 0;$$

$$V_{\lambda,\nu,1}(z) = \frac{\nu}{\nu-\lambda} \varphi_\lambda(z) - \frac{\lambda}{\nu-\lambda} \varphi_\nu(z), \quad \nu \neq \lambda; \quad (15)$$

$$V_{\lambda,\lambda+1,\alpha}(z) = \frac{1}{1-\alpha} (\varphi_{\lambda+1}(z) - \alpha \varphi_{\lambda+1}(\alpha z)), \quad \alpha \neq 1;$$

$$V_{\lambda,\lambda+k,\alpha}(z) = c_k \varphi_{\lambda+k}(\alpha z) + \frac{1}{z} P_k \varphi_\lambda(z) + \frac{1}{z} \tilde{P}_k, \quad \alpha \neq 1, \quad (16)$$

$$c_k \in \mathbb{Q}(\lambda, \alpha), \quad P_k, \tilde{P}_k \in \mathbb{Q}(\lambda, \alpha)[z^{-1}], \quad P_k \neq 0;$$

$$V_{\lambda,2\lambda,2}(z) = \varphi_\lambda^2(z); \quad (17)$$

$$V_{\lambda,2\lambda-k,2}(z) = \frac{(k-2\lambda)(-z)^k}{2(-\lambda)_{k+1}} \varphi_\lambda^2(z) + P_k \varphi_\lambda(z) + \tilde{P}_k \varphi_{2\lambda-k}(2z) + \tilde{P}_k, \quad (18)$$

$$\begin{aligned} \lambda &\notin \{0, 1, \dots, k\}, P_k, \tilde{P}_k \in \mathbb{Q}(\lambda)[z], P_k \neq 0, \hat{P}_k \in \mathbb{Q}(\lambda); \\ V_{\lambda, 2\lambda+k, 2}(z) &= \frac{(\lambda+1)_{k-1}(2\lambda+k)}{2z^k} \varphi_{\lambda}^2(z) + \frac{1}{z} P_k \varphi_{\lambda}(z) + \hat{P}_k \varphi_{2\lambda+k}(2z) + \frac{1}{z} \tilde{P}_k, \\ \lambda &\neq 0, P_k, \tilde{P}_k \in \mathbb{Q}(\lambda)[z^{-1}], P_k \neq 0, \hat{P}_k \in \mathbb{Q}(\lambda). \end{aligned} \quad (19)$$

Lemma 3. Let $\lambda, \nu \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, $\alpha \in \mathbb{C}$, $k \in \mathbb{Z}_{\geq 0}$. Then,

$$\begin{aligned} V_{\lambda+k, \nu, \alpha}(z) &= \frac{(\lambda+1)_k(1-\alpha)^k}{(\lambda-\nu+1)_k} V_{\lambda, \nu, \alpha}(z) + \nu \sum_{n=0}^{k-1} \frac{(-\lambda-k)_n(1-\alpha)^n}{(\nu-\lambda-k)_{n+1}} \varphi_{\lambda+k-n}(z) + \\ &+ \alpha \left(\sum_{n=0}^{k-1} \frac{(-\lambda-k)_{n+1}(1-\alpha)^n}{(\nu-\lambda-k)_{n+1}} \right) \varphi_{\nu}(\alpha z), \end{aligned} \quad (20)$$

if $\nu \notin \{\lambda+1, \dots, \lambda+k\}$;

$$\begin{aligned} V_{\lambda, \nu, \alpha}(z) &= \frac{(\lambda-\nu+1)_k}{(\lambda+1)_k(1-\alpha)^k} V_{\lambda+k, \nu, \alpha}(z) + \nu \sum_{n=1}^k \frac{(\lambda-\nu+1)_{n-1}}{(\lambda+1)_n(1-\alpha)^n} \varphi_{\lambda+n}(z) - \\ &- \alpha \left(\sum_{n=0}^{k-1} \frac{(\lambda-\nu+1)_n}{(\lambda+1)_n(1-\alpha)^{n+1}} \right) \varphi_{\nu}(\alpha z), \end{aligned} \quad (21)$$

if $\alpha \neq 1$;

$$\begin{aligned} V_{\lambda, \nu-k, \alpha}(z) &= \frac{(\nu-k)(1-\alpha)^k z^k}{(\lambda-\nu+1)_k \nu} V_{\lambda, \nu, \alpha}(z) + (\nu-k) \sum_{n=0}^{k-1} \frac{(-\lambda)_n(1-\alpha)^n}{(\nu-\lambda-k)_{n+1}} \varphi_{\lambda-n}(z) - \\ &- \lambda \left(\sum_{n=0}^{k-1} \frac{(\nu-k)_n(\alpha-1)^n}{(\nu-\lambda-k)_{n+1} \alpha^n} \right) \varphi_{\nu-k}(\alpha z) + R, \quad R \in \mathbb{Q}(\lambda, \alpha)[z], \end{aligned} \quad (22)$$

if $\nu \notin \{0, \lambda+1, \dots, \lambda+k\}$.

Proof of Lemma 3. Identities (20) and (22) are, respectively, identities (25) and (26) from the article ref-Gorelov2. Identity (21) is obtained similarly from ([2] equality (20)) by induction on k . $\square$

Lemma 4. Let $\lambda_1, \lambda_2, \nu \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, $\lambda_1 + \lambda_2 = \nu$, $\alpha_1, \alpha_2 \in \mathbb{C}$, $\xi_1, \xi_2 \in \mathbb{C} \setminus \{0\}$, $\xi_1 + \xi_2 = \xi_1 \alpha_1 = \xi_2 \alpha_2$. Then,

$$\lambda_2 V_{\lambda_1, \nu, \alpha_1}(\xi_1 z) + \lambda_1 V_{\lambda_2, \nu, \alpha_2}(\xi_2 z) = \nu \varphi_{\lambda_1}(\xi_1 z) \varphi_{\lambda_2}(\xi_2 z). \quad (23)$$

Proof of Lemma 4. If we divide both sides of equality (23) by $e^{(\xi_1+\xi_2)z} = e^{\xi_1 \alpha_1 z} = e^{\xi_2 \alpha_2 z}$ and $z^{-\nu}$, differentiate with respect to z and divide by $z^{\nu-1} e^{-(\xi_1+\xi_2)z}$, then, taking into account (1) and (8), we get the correct identity $\lambda_2 \varphi_{\lambda_1}(\xi_1 z) + \lambda_1 \varphi_{\lambda_2}(\xi_2 z) = \lambda_1 \varphi_{\lambda_2}(\xi_2 z) + \lambda_2 \varphi_{\lambda_1}(\xi_1 z)$. Having performed these operations in reverse order, in view of $V_{\lambda, \nu, \alpha}(0) = 1$, we arrive at (23). $\square$

Identity (23) with $\lambda_1 = \lambda_2 = \nu/2$, $\xi_1 = \xi_2 = 1$, $\alpha_1 = \alpha_2 = 2$ goes over to (17).

Lemma 5. Let $\lambda_1, \lambda_2, \nu \in \mathbb{C} \setminus \mathbb{Z}_{<0}$, $\lambda_1 + \lambda_2 - \nu = r \in \mathbb{Z} \setminus \{0\}$, $\alpha_1, \alpha_2 \in \mathbb{C}$, $\xi_1, \xi_2 \in \mathbb{C} \setminus \{0\}$, $\xi_1 + \xi_2 = \xi_1 \alpha_1 = \xi_2 \alpha_2$ (it follows from this that $\alpha_i \neq 1$). Then, if $r > 0$, then

$$\begin{aligned} & \frac{(-\lambda_2)_{r+1}}{(\alpha_1 - 1)^r} V_{\lambda_1, \nu, \alpha_1}(\xi_1 z) + (-\lambda_1)_{r+1} V_{\lambda_2, \nu, \alpha_2}(\xi_2 z) = \\ & = -\nu(-\xi_1 z)^r \varphi_{\lambda_1}(\xi_1 z) \varphi_{\lambda_2}(\xi_2 z) + \nu \lambda_2 \left(\sum_{n=1}^r \frac{(1 - \lambda_2)_{n-1} (-\xi_1 z)^{r-n}}{(\alpha_1 - 1)^n} \right) \varphi_{\lambda_1}(\xi_1 z) + \\ & \quad + \nu \lambda_1 \left(\sum_{n=1}^r (1 - \lambda_1)_{r-n} (-\xi_1 z)^{n-1} \right) \varphi_{\lambda_2}(\xi_2 z) - \\ & \quad - \nu \lambda_1 \lambda_2 \sum_{n=1}^r \frac{(1 - \lambda_2)_{n-1}}{(\alpha_1 - 1)^n} \sum_{m=1}^{r-n} (1 - \lambda_1)_{r-n-m} (-\xi_1 z)^{m-1} - \\ & \quad - \alpha_1 \lambda_1 \lambda_2 \left(\sum_{n=1}^r \frac{(1 - \lambda_1)_{r-n} (1 - \lambda_2)_{n-1}}{(\alpha_1 - 1)^n} \right) \varphi_{\nu}((\xi_1 + \xi_2)z). \quad (24) \end{aligned}$$

If $r < 0$, then

$$\begin{aligned} & \frac{(\alpha_1 - 1)^{-r}}{(\lambda_2 + 1)_{-r-1}} V_{\lambda_1, \nu, \alpha_1}(\xi_1 z) + \frac{1}{(\lambda_1 + 1)_{-r-1}} V_{\lambda_2, \nu, \alpha_2}(\xi_2 z) = \nu(\xi_1 z)^r \varphi_{\lambda_1}(\xi_1 z) \varphi_{\lambda_2}(\xi_2 z) - \\ & \quad - \nu \left(\sum_{n=0}^{-r-1} \frac{(\alpha_1 - 1)^n (\xi_1 z)^{r+n}}{(\lambda_2 + 1)_n} \right) \varphi_{\lambda_1}(\xi_1 z) - \nu \left(\sum_{n=0}^{-r-1} \frac{(\xi_1 z)^{r+n}}{(\lambda_1 + 1)_n} \right) \varphi_{\lambda_2}(\xi_2 z) + \\ & \quad + \nu \sum_{n=0}^{-r-1} \frac{(\alpha_1 - 1)^n}{(\lambda_2 + 1)_n} \sum_{m=0}^{-r-n-1} \frac{(\xi_1 z)^{r+n+m}}{(\lambda_1 + 1)_m} + \\ & \quad + \alpha_1 \left(\sum_{n=0}^{-r-1} \frac{(\alpha_1 - 1)^n}{(\lambda_1 + 1)_{-r-n-1} (\lambda_2 + 1)_n} \right) \varphi_{\nu}((\xi_1 + \xi_2)z). \quad (25) \end{aligned}$$

Proof of Lemma 5. Let $\lambda_0 = \nu - \lambda_2 = \lambda_1 - r$. If $\lambda_0 \notin \mathbb{Z}_{<0}$, then according to (23)

$$\lambda_2 V_{\lambda_0, \nu, \alpha_1}(\xi_1 z) + \lambda_0 V_{\lambda_2, \nu, \alpha_2}(\xi_2 z) = \nu \varphi_{\lambda_0}(\xi_1 z) \varphi_{\lambda_2}(\xi_2 z). \quad (26)$$

Let $\lambda_1 + \lambda_2 - \nu = r > 0$. Then, $\lambda_1 - \lambda_0 = r \in \mathbb{N}$ and from identity (21) follows

$$\begin{aligned} V_{\lambda_0, \nu, \alpha_1}(\xi_1 z) &= \frac{(1 - \lambda_2)_r}{(\lambda_1 - r + 1)_r (1 - \alpha_1)^r} V_{\lambda_1, \nu, \alpha_1}(\xi_1 z) + \\ & \quad + \nu \sum_{n=1}^r \frac{(1 - \lambda_2)_{n-1}}{(\lambda_1 - r + 1)_n (1 - \alpha_1)^n} \varphi_{\lambda_1 - r + n}(\xi_1 z) - \\ & \quad - \alpha_1 \left(\sum_{n=1}^r \frac{(1 - \lambda_2)_{n-1}}{(\lambda_1 - r + 1)_{n-1} (1 - \alpha_1)^n} \right) \varphi_{\nu}(\xi_1 \alpha_1 z). \end{aligned}$$

Substituting this expression into (26), multiplying the resulting equality by $(-1)^r (\lambda_1 - r + 1)_r = (-\lambda_1)_r$ and using (10), after simple transformations, we obtain (24). Identity (24) is also valid in the case of $\lambda_1 - r \in \mathbb{Z}_{<0}$, since for fixed r, ξ_1, ξ_2 all terms included in it are continuous in λ_1 and λ_2 .

Let $\lambda_1 + \lambda_2 - \nu = r < 0$. Then, $\lambda_0 - \lambda_1 = -r \in \mathbb{N}$, $\nu - \lambda_1 + r = \lambda_2$ and according to (20)

$$\begin{aligned} V_{\lambda_0, \nu, \alpha_1}(\xi_1 z) &= \frac{(\lambda_1 + 1)_{-r} (1 - \alpha_1)^{-r}}{(-1)^{-r} (\lambda_2)_{-r}} V_{\lambda_1, \nu, \alpha_1}(\xi_1 z) + \\ &+ \nu \sum_{n=0}^{-r-1} \frac{(r - \lambda_1)_n (1 - \alpha_1)^n}{(\lambda_2)_{n+1}} \varphi_{\lambda_1 - r - n}(\xi_1 z) + \\ &+ \alpha_1 \left(\sum_{n=0}^{-r-1} \frac{(r - \lambda_1)_{n+1} (1 - \alpha_1)^n}{(\lambda_2)_{n+1}} \right) \varphi_{\nu}(\xi_1 \alpha_1 z). \end{aligned}$$

Substituting this expression into (26), dividing the resulting equality by $(\lambda_1 + 1)_{-r}$ and using (10), we obtain (25). Lemma 5 is proved. $\square$

Identities (24) and (25) generalize, respectively, (18) and (19).

If functions (5) are algebraically independent over $\mathbb{C}$, then it is convenient to carry out algebraic operations with them formally, as with the variables

$$u_1, \dots, u_n, x_1, \dots, x_p, z_1, \dots, z_q \quad (27)$$

corresponding to them. In addition, the operation of differentiation of rational functions depending, respectively, on

$$e^{\beta_1 z}, \dots, e^{\beta_p z}, z^{\zeta_1}, \dots, z^{\zeta_q} \quad (28)$$

and (5), with respect to z , corresponds to the operators

$$\begin{aligned} D &= \sum_{j=1}^p \beta_j x_j \frac{\partial}{\partial x_j} + \sum_{j=1}^q \zeta_j \frac{z_j}{z_1^b} \frac{\partial}{\partial z_j}, \\ D_1 &= D + \sum_{i=1}^n \left(\left(\gamma_i - \frac{\mu_i}{z_1^b} \right) u_i + \frac{\mu_i}{z_1^b} \right) \frac{\partial}{\partial u_i}. \end{aligned}$$

Let P be a polynomial in variables (27). Then, $z_1^b D_1 P$ is also a polynomial in variables (27).

Lemma 6 (see [10], Lemma 5). *Let function (28) be algebraically independent over $\mathbb{C}$, and $P \neq 0$ be a polynomial in $p + q$ variables $x_1, \dots, x_p, z_1, \dots, z_q$ with coefficients from $\mathbb{C}$. Then, the polynomial $z_1^b D_1 P$ is divisible by P if and only if*

$$P = \sigma x_1^{k_1} \dots x_p^{k_p} z_1^{b_1} \dots z_q^{b_q}, \quad \sigma \in \mathbb{C} \setminus \{0\}, \quad k_i, b_i \in \mathbb{Z}_{\geq 0}. \quad (29)$$

Lemma 7. *Let function (5) be algebraically independent over $\mathbb{C}$, and let $P \neq 0$ be a polynomial in $n + p + q$ variables (27) with coefficients from $\mathbb{C}$. Then, the polynomial $z_1^{t_1} \dots z_q^{t_q} x_1^{s_1} \dots x_p^{s_p} D_1 P$, where $t_i, s_i \in \mathbb{Z}_{\geq 0}$, $t_1 \geq b$, is divisible by P if and only if P has the form (29).*

Proof of Lemma 7. Let the polynomial P contain at least one variable u_i , $1 \leq i \leq n$, i.e.,

$$P = P_s u_i^s + \dots + P_1 u_i + P_0, \quad P_s \neq 0, \quad s \geq 1,$$

where $P_s, \dots, P_0$ are polynomials in all variables (27) except u_i . Then,

$$D_1 P = \left(D_1 P_s + s \left(\gamma_i - \frac{\mu_i}{z} \right) P_s \right) u_i^s + \dots$$

can be treated as a polynomial in u_i with coefficients from the field of rational functions over $\mathbb{C}$ in all variables (27) except u_i . Because the degrees in u_i of the polynomials $D_1 P$ and

P are equal, their quotient is equal to the quotient of dividing the coefficients of u_i^s and does not depend on u_i . Therefore,

$$\frac{D_1 P}{P} = \frac{D_1 P_s}{P_s} + s \left(\gamma_i - \frac{\mu_i}{z} \right),$$

whence $P = \sigma P_s e^{s \gamma_i z} z^{-s \mu_i}$, $\sigma \in \mathbb{C} \setminus \{0\}$. Hence, P does not contain the variables

$$u_1, \dots, u_n \quad (30)$$

(both in the case of $\gamma_i \in \mathbb{L}\{\beta_1, \dots, \beta_p\}$, $\mu_i \in \mathbb{L}\{\zeta_1, \dots, \zeta_q\}$, and otherwise) and in view of Lemma 6 has the form (29). $\square$

Lemma 8 (see [11,12]). *Let $\mathbb{F}$ be a differential field of meromorphic functions containing the field $\mathbb{C}$, and $\psi_1, \dots, \psi_m$ be functions whose derivatives belong to $\mathbb{F}$. Then, $\psi_1, \dots, \psi_m$ are algebraically dependent over $\mathbb{F}$ if and only if $\alpha_1 \psi_1 + \dots + \alpha_m \psi_m \in \mathbb{F}$ for some $\alpha_1, \dots, \alpha_m \in \mathbb{C}$, $|\alpha_1| + \dots + |\alpha_m| \neq 0$.*

3. Proof of Theorem 2

The sufficiency of condition 1 of Theorem 2 follows from identities (9)–(14), condition 2—from (9)–(11), (13), (15), condition 3—from (10), (11), (16), condition 4—from (9)–(11), condition 5—from (9)–(11), (15), and conditions 6 and 7—from Lemmas 4 and 5 and identities (10), (11), (22).

To prove the necessity of the conditions of the theorem in view of (1) and (11), it suffices to find out when m integrals

$$\int_0^{\zeta_i z} t^{v_i-1} e^{-\alpha_i t} \varphi_{\lambda_i}(t) dt, \quad i = 1, \dots, m \quad (31)$$

are algebraically independent over the field $\mathbb{F} = \mathbb{C} \langle \varphi_{\mu_1}(\gamma_1 z), \dots, \varphi_{\mu_n}(\gamma_n z), e^{\beta_1 z}, \dots, e^{\beta_p z}, z^{\zeta_1}, \dots, z^{\zeta_q} \rangle$, where, changing (if necessary) the numbers $\mu_k, \gamma_k, \beta_k, \zeta_k, p, q, n$ and the numbering of the functions, we now set $\lambda_i \notin \mathbb{Z}$ for $i \leq \kappa$, $\lambda_i \in \mathbb{Z}$ for $i > \kappa$, $\kappa \leq m$, $\mu_i = \lambda_i$, $\gamma_i = \zeta_i$, $i = 1, \dots, \kappa$, $-\zeta_i \alpha_i = \sum_k c_{i,k} \beta_k$, $\zeta_i = \sum_k \tilde{c}_{i,k} \beta_k$, $v_i - 1 = \sum_k b_{i,k} \zeta_k$, $\mu_j = \sum_k \tilde{b}_{j,k} \zeta_k$, $c_{i,k}, \tilde{c}_{i,k}, b_{i,k}, \tilde{b}_{j,k} \in \mathbb{Z}$.

If we assume that the integrals (31) are algebraically dependent over $\mathbb{F}$, then according to Lemma 8

$$\sum_{i=1}^m a_i \int_0^{\zeta_i z} t^{v_i-1} e^{-\alpha_i t} \varphi_{\lambda_i}(t) dt = \frac{P}{Q}, \quad (32)$$

where $a_i \in \mathbb{C}$, $\sum_i |a_i| \neq 0$, P, Q are polynomials over $\mathbb{C}$ in functions (5). Differentiating equality (32), we obtain

$$\sum_{i=1}^m a_i \zeta_i^{v_i} z^{v_i-1} e^{-\zeta_i \alpha_i z} \varphi_{\lambda_i}(\zeta_i z) = (P/Q)'.$$

Replacing functions (5) in this equality with variables (27), we have

$$a_1 \zeta_1^{v_1} z_1^{b_{1,1}} \dots z_q^{b_{1,q}} x_1^{c_{1,1}} \dots x_p^{c_{1,p}} u_1 + \dots + a_\kappa \zeta_\kappa^{v_\kappa} z_1^{b_{\kappa,1}} \dots z_q^{b_{\kappa,q}} x_1^{c_{\kappa,1}} \dots x_p^{c_{\kappa,p}} u_\kappa + \\ + R = \frac{QD_1 P - PD_1 Q}{Q^2}, \quad R \in \mathbb{C}[x_1, \dots, x_p, z_1, \dots, z_q]. \quad (33)$$

Set $(P, Q) = 1$. Then, from (33) it follows that the polynomial $z_1^{t_1} \dots z_q^{t_q} x_1^{s_1} \dots x_p^{s_p} D_1 Q$ for some $t_i, s_i \in \mathbb{Z}_{\geq 0}$, $t_1 \geq b$ is divisible by Q . Hence, according to Lemmas 7 and 1, Q has the form (29) and does not contain the variables (30).

Let us ask ourselves whether the polynomial P is a linear function of the variables (30). Suppose that

$$P/Q = P_s u_k^s + P_{s-1} u_k^{s-1} + \dots + P_0, \quad P_s \neq 0, \quad s \geq 2, \quad 1 \leq k \leq n,$$

where $P_s, P_{s-1}, \dots, P_0$ are polynomials in the variables

$$x_1^{\pm 1}, \dots, x_p^{\pm 1}, z_1^{\pm 1}, \dots, z_q^{\pm 1} \quad (34)$$

and (30), except for u_k . Because

$$D_1(P/Q) = \left(D_1 P_s + s \left(\gamma_k - \frac{\mu_k}{z} \right) P_s \right) u_k^s + \dots,$$

Then, from (33)

$$D_1 P_s + s \left(\gamma_k - \frac{\mu_k}{z} \right) P_s = 0, \quad P_s = c_1 z^{s\mu_k} e^{-s\gamma_k z}, \quad c_1 \in \mathbb{C} \setminus \{0\},$$

$$D_1 P_{s-1} + (s-1) \left(\gamma_k - \frac{\mu_k}{z} \right) P_{s-1} + c_1 s \mu_k z^{s\mu_k-1} e^{-s\gamma_k z} = \delta a_i \xi_i^{\nu_i} z^{\nu_i-1} e^{-\xi_i \alpha_i z}, \quad (35)$$

where $1 \leq i \leq \kappa$, and $\delta \neq 0$ only in the case of $i = k \leq \kappa$, $s = 2$, $\gamma_k = \xi_k$, $\mu_k = \lambda_k$. Substituting functions (5) instead of variables (27) into (35), we get a differential equation, the general solution of which has the following form:

$$P_{s-1} = cz^{(s-1)\mu_k} e^{-(s-1)\gamma_k z} - c_1 s \mu_k z^{(s-1)\mu_k} e^{-(s-1)\gamma_k z} \int z^{\mu_k-1} e^{-\gamma_k z} dz + \\ + \delta a_k \xi_k^{\nu_k} z^{\lambda_k} e^{-\gamma_k z} \int z^{\nu_k-\lambda_k-1} e^{\gamma_k(1-\alpha_k)z} dz, \quad c \in \mathbb{C}.$$

Because $c_1 s \mu_k \neq 0$, the obtained equality contradicts Lemma 1 (as well as Lemma 7 from [10]), with the exception of the case $\mu_k \in \mathbb{N}$, which is impossible, and the case $s = 2$, $\gamma_k = \xi_k$, $\alpha_k = 2$, $2\lambda_k - \nu_k \in \mathbb{Z}$, specified in conditions 6 and 7 of Theorem 2.

Therefore, we can limit ourselves to the case when the degree of the polynomial P in (32) with respect to each of the variables (30) is less than or equal to 1.

Let us assume that, with respect to the set of variables (30), the polynomial P has degree $s \geq 2$. Without reducing generality, we can assume that P/Q includes a monomial

$$A_1 = P_1 u_1 \dots u_s, \quad P_1 \neq 0,$$

where P_1 is a polynomial in variables (34). Then,

$$D_1 A_1 = \left(DP_1 + \left(\gamma_1 - \frac{\mu_1}{z} + \dots + \gamma_s - \frac{\mu_s}{z} \right) P_1 \right) u_1 \dots u_s + \dots,$$

$$DP_1 + \left(\gamma_1 - \frac{\mu_1}{z} + \dots + \gamma_s - \frac{\mu_s}{z} \right) P_1 = 0,$$

$$P_1 = c_1 z^{\mu_1 + \dots + \mu_s} e^{-(\gamma_1 + \dots + \gamma_s)z}, \quad c_1 \in \mathbb{C} \setminus \{0\}.$$

If P/Q includes other monomials of degree s , for example,

$$A_{s+1} = P_{s+1} u_2 \dots u_s u_{s+1}, \dots, A_n = P_n u_2 \dots u_s u_n,$$

then, similarly,

$$P_{s+1} = c_{s+1}z^{\mu_2+\dots+\mu_s+\mu_{s+1}}e^{-(\gamma_2+\dots+\gamma_s+\gamma_{s+1})z}, \dots, P_n = c_nz^{\mu_2+\dots+\mu_s+\mu_n}e^{-(\gamma_2+\dots+\gamma_s+\gamma_n)z},$$

where $c_{s+1}, \dots, c_n \in \mathbb{C}$. Consider also the monomial

$$A_0 = P_0u_2 \dots u_s.$$

By equating the coefficients of $u_2 \dots u_s$ in (33), in the case $s > 2$ we obtain

$$\begin{aligned} DP_0 + \left(\gamma_2 - \frac{\mu_2}{z} + \dots + \gamma_s - \frac{\mu_s}{z}\right)P_0 + \\ + c_1\mu_1z^{\mu_1+\dots+\mu_s-1}e^{-(\gamma_1+\dots+\gamma_s)z} + c_{s+1}\mu_{s+1}z^{\mu_2+\dots+\mu_s+\mu_{s+1}-1}e^{-(\gamma_2+\dots+\gamma_s+\gamma_{s+1})z} + \\ + \dots + c_n\mu_nz^{\mu_2+\dots+\mu_s+\mu_n-1}e^{-(\gamma_2+\dots+\gamma_s+\gamma_n)z} = 0. \end{aligned} \quad (36)$$

Equation (36) has the general solution

$$\begin{aligned} P_0 = cz^{\mu_2+\dots+\mu_s}e^{-(\gamma_2+\dots+\gamma_s)z} - c_1\mu_1z^{\mu_2+\dots+\mu_s}e^{-(\gamma_2+\dots+\gamma_s)z} \int z^{\mu_1-1}e^{-\gamma_1z}dz - \\ - c_{s+1}\mu_{s+1}z^{\mu_2+\dots+\mu_s}e^{-(\gamma_2+\dots+\gamma_s)z} \int z^{\mu_{s+1}-1}e^{-\gamma_{s+1}z}dz - \\ - \dots - c_n\mu_nz^{\mu_2+\dots+\mu_s}e^{-(\gamma_2+\dots+\gamma_s)z} \int z^{\mu_n-1}e^{-\gamma_nz}dz, \quad c \in \mathbb{C}. \end{aligned}$$

Since $c_1\mu_1 \neq 0$, this equality contradicts Lemma 1 (as well as Lemma 7 from [10]), except for the case $\mu_1 \in \mathbb{N}$ and the case $\mu_1 - \mu_i \in \mathbb{Z}$, $\gamma_1 = \gamma_i$, which are impossible. If $s = 2$, then equating the coefficients of u_2 in (33), instead of (36), we get

$$\begin{aligned} DP_0 + \left(\gamma_2 - \frac{\mu_2}{z}\right)P_0 + c_1\mu_1z^{\mu_1+\mu_2-1}e^{-(\gamma_1+\gamma_2)z} + \\ + c_3\mu_3z^{\mu_2+\mu_3-1}e^{-(\gamma_2+\gamma_3)z} + \dots + c_n\mu_nz^{\mu_2+\mu_n-1}e^{-(\gamma_2+\gamma_n)z} = \\ = \delta a_i \tilde{\alpha}_i^{\nu_i} z^{\nu_i-1} e^{-\xi_i \alpha_i z}, \quad 1 \leq i \leq m, \end{aligned} \quad (37)$$

and $\delta \neq 0$ only in the case $\gamma_2 = \xi_i$, $\mu_2 = \lambda_i$. Equation (37) has the general solution

$$\begin{aligned} P_0 = cz^{\mu_2}e^{-\gamma_2z} - c_1\mu_1z^{\mu_2}e^{-\gamma_2z} \int z^{\mu_1-1}e^{-\gamma_1z}dz - \\ - c_3\mu_3z^{\mu_2}e^{-\gamma_2z} \int z^{\mu_3-1}e^{-\gamma_3z}dz - \dots - c_n\mu_nz^{\mu_2}e^{-\gamma_2z} \int z^{\mu_n-1}e^{-\gamma_nz}dz + \\ + \delta a_i \tilde{\alpha}_i^{\nu_i} z^{\mu_2}e^{-\gamma_2z} \int z^{\nu_i-\mu_2-1}e^{(\gamma_2-\xi_i \alpha_i)z}dz, \quad c \in \mathbb{C}, \end{aligned}$$

where $c_1\mu_1 \neq 0$, which is impossible due to Lemma 1 except for the already considered cases $\mu_1 \in \mathbb{N}$ and $\mu_j - \mu_k \in \mathbb{Z}$, $\gamma_j = \gamma_k$, as well as the case $\gamma_2 = \xi_i$, $\mu_2 = \lambda_i$, $-\gamma_1 = \xi_i(1 - \alpha_i)$, $\mu_1 + \mu_2 - \nu_i \in \mathbb{Z}$. By swapping u_1 and u_2 , we obtain in the last case $\gamma_1 = \xi_l$, $\mu_1 = \lambda_l$, $-\gamma_2 = \xi_l(1 - \alpha_l)$, $\mu_1 + \mu_2 - \nu_l \in \mathbb{Z}$. Hence $\lambda_i + \lambda_l - \nu_i \in \mathbb{Z}$, $\nu_i - \nu_l \in \mathbb{Z}$, $\xi_i + \xi_l = \xi_i \alpha_i = \xi_l \alpha_l$, which corresponds to conditions 6 and 7 of Theorem 2.

Therefore, if conditions 6 and 7 of Theorem 2 are not satisfied, then the polynomial P has degree 1 with respect to the set of the variables (30), and

$$P/Q = B_1u_1 + \dots + B_nu_n + B,$$

where $B_1, \dots, B_n, B$ are polynomials in the variables (34). Hence, the right side of equality (33) has the following form:

$$\left(B'_1 + \left(\gamma_1 - \frac{\mu_1}{z}\right)B_1\right)u_1 + \dots + \left(B'_n + \left(\gamma_n - \frac{\mu_n}{z}\right)B_n\right)u_n + \frac{\mu_1}{z}B_1 + \dots + \frac{\mu_n}{z}B_n + B'.$$

If $\lambda_i \notin \mathbb{Z}$ for some $1 \leq i \leq m$ and condition 4 of Theorem 2 are not satisfied, then $B_i \neq 0$, $\mu_i = \lambda_i$, $\gamma_i = \xi_i$. Then,

$$B'_i + \left(\gamma_i - \frac{\mu_i}{z}\right)B_i = a_i \xi_i^{\nu_i} z^{\nu_i-1} e^{-\xi_i \alpha_i z}.$$

This equation has a solution

$$B_i = cz^{\lambda_i} e^{-\xi_i z} + a_1 \xi_i z^{\lambda_i} e^{-\xi_i z} \int z^{\nu_i - \lambda_i - 1} e^{\xi_i (1 - \alpha_i) z} dz, \quad c \in \mathbb{C},$$

which is impossible due to Lemma 1, except for the cases $\nu_i - \lambda_i \in \mathbb{N}$ and $\alpha_i = 1$. Theorem 2 is proved.

4. Conclusions

1. Theorem 2 has a functional–theoretic formulation, but from it, using A.B. Shidlovsky’s theorem [5], one can easily derive conditions that are necessary and sufficient for the algebraic independence of the values of the functions under consideration at algebraic points. This yields new examples of transcendental numbers and algebraically independent sets of numbers.
2. Due to the cumbersomeness of Formulas (13)–(19) and (23)–(25), it was appropriate to check them with a computer for typos. For this purpose, the Wolfram Mathematica computer algebra system was used, which allows one, in particular, to perform arithmetic operations with rational numbers absolutely accurately. In recent decades, this system (as well as MathCAD, Maple, and others) has been widely used both in theoretical and purely applied research (see, for example, refs. [13–15]). To check any equality for typos, it is enough to select the parameters of all the functions included in it as rational, and then compare the coefficients of the Taylor expansions of the left and right sides of this equality in powers of z up to N , where N is a sufficiently large natural number.

Author Contributions: Conceptualization and writing—review and editing by V.G. and G.V. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Gorelov, V.A. On the Algebraic Independence of the Values of Functions Associated with Hypergeometric Functions. *Axioms* **2023**, *12*, 36. [CrossRef]
2. Gorelov, V.A. On algebraic properties of functions related to hypergeometric functions. *Math. Notes* **2025**, *117*, 383–391. [CrossRef]
3. Siegel, C.L. Über einige Anwendungen Diophantischer Approximationen. *Abh. Preuss. Acad. Wiss. Phys.-Math. Kl.* **1929**, *1*, 1–70.
4. Siegel, C.L. *Transcendental Numbers*; Princeton University Press: Princeton, NJ, USA, 1949.
5. Shidlovsky, A.B. *Transcendental Numbers*; Walter de Gruyter: Berlin, Germany; New York, NY, USA, 1989.
6. Fresan, J.; Jossen, P. A non-hypergeometric E-function. *Ann. of Math.* **2021**, *194*, 903–942. [CrossRef]
7. Fischler, S.; Rivoal, T. On Siegel’s Problem for E-Functions. Available online: <https://arxiv.org/pdf/1910.06817.pdf> (accessed on 4 June 2020).

8. Andre, Y. Séries Gevrey de type arithmétique. I. Théoremes de pureté et de dualité *Ann. of Math.* **2000**, *151*, 705–740.
9. Shidlovsky, A.B. On transcendence and algebraic independence of the values of some functions. *Trans. Moscow Math. Soc.* **1959**, *11*, 283–320.
10. Gorelov, V.A. On the algebraic properties of solutions of inhomogeneous hypergeometric equations. *Math. Notes* **2016**, *99*, 663–675. [\[CrossRef\]](#)
11. Ostrowski, A. Sur les relations algébriques entre les intégrales indéfinies. *Acta Math.* **1946**, *78*, 315–318. [\[CrossRef\]](#)
12. Kolchin, E.R. Algebraic groups and algebraic dependence. *Amer. J. Math.* **1968**, *90*, 1151–1164. [\[CrossRef\]](#)
13. Wolfram, S. *The Mathematica Book*, 5th ed.; Addison-Wesley: White Plains, NY, USA, 2002.
14. Dzhemesyuk, I.A.; Gorbunov, S.G. Mathematical modeling of the influence of relaxation processes on the temperature field in an elastic half-space. *Rus. Technol. J.* **2017**, *5*, 40–50.
15. Riddle, A. *Applied Electronic Engineering with Mathematica*; Wesley: New York, NY, USA, 1994.

Disclaimer/Publisher’s Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.