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Oscillation Criteria for Hybrid Second-Order Neutral Delay Difference Equations with Mixed Coefficients

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Abstract

This paper explores the oscillatory behavior of a class of second-order hybrid-type neutral delay difference equations. A novel approach is introduced to transform these complex trinomial equations into a simpler binomial form by utilizing solutions from an associated linear difference equation. By employing comparison techniques and summation-averaging methods, we establish new oscillation criteria which guarantee that all solutions exhibit oscillatory behavior. Our findings extend to an existing oscillation theory and are applicable even to non-neutral second-order equations. A couple of examples are presented to highlight the impact and novelty of the obtained results.

Keywords: trinomial difference equation; neutral; second-order; binomial form; oscillation

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1. Introduction

Neutral difference equations form a significant class of functional equations characterized by the presence of highest-order difference terms, both with and without delays or advances. These equations generalize classical difference equations by incorporating neutral terms that involve the unknown sequence or its differences at shifted arguments. Owing to their broad applicability, neutral difference equations have been extensively studied across various fields, including economics, population dynamics, control systems, circuit theory, and signal processing (see, for example, the monographs [1–3] and the paper [4]).

In recent decades, considerable attention has been devoted to both linear and non-linear neutral difference equations. As discrete analogs of differential equations, they offer a robust framework for modeling processes that evolve in discrete time or are represented by discrete data. A central focus has been the investigation of qualitative properties of solutions—particularly stability, boundedness, and oscillation—which are essential for understanding the dynamics of the modeled systems (see, for instance, the monographs [2,3,5]).

Oscillation theory, in particular, seeks to identify conditions under which solutions of difference equations exhibit periodic or oscillatory behavior. While extensive research has

addressed neutral difference equations with positive coefficients, relatively little attention has been paid to those involving both positive and negative coefficients. Nevertheless, such equations arise in various contexts, including population dynamics with delayed and resource-dependent regimes, economic systems with delayed investment or consumption, circuit theory with delayed feedback, and signal processing and control systems with memory effects. The limited attention is largely due to the analytical challenges posed by mixed coefficients, which complicate the derivation of oscillation criteria.

This paper addresses this gap by investigating the oscillatory behavior of a broad class of second-order hybrid-type neutral difference equations featuring both positive and negative coefficients. Specifically, we consider the equation of the form

$$\Delta(\mu(\varphi)\Delta\theta(\varphi)) - \phi_1(\varphi)\delta(\varphi + 1) + \phi_2(\varphi)\delta(\sigma(\varphi)) = 0, \quad \varphi \geq \varphi_0 > 0, \quad (1)$$

where Δ denotes the forward difference operator defined by $\Delta\theta(\varphi) = \theta(\varphi + 1) - \theta(\varphi)$ and $\theta(\varphi) = \delta(\varphi) + b(\varphi)\delta(\tau(\varphi))$. We assume the following conditions:

- (A1) $\{\mu(\varphi)\}$, $\{\phi_1(\varphi)\}$ and $\{\phi_2(\varphi)\}$ are sequences of positive real numbers with $\sum_{\varphi=\varphi_0}^{\infty} \frac{1}{\mu(\varphi)} = \infty$;
- (A2) $\{b(\varphi)\}$ is a non-negative real sequence with $0 \leq b(\varphi) \leq d < 1$;
- (A3) $\{\sigma(\varphi)\}$, $\{\tau(\varphi)\}$ are sequences of integers such that $\tau(\varphi) \leq \varphi - 1$ and $\sigma(\varphi) \leq \varphi - 1$ with $\lim_{\varphi \rightarrow \infty} \sigma(\varphi) = \lim_{\varphi \rightarrow \infty} \tau(\varphi) = \infty$.

A solution of Equation (1) is a real sequence $\delta(\varphi)$ that satisfies the equation for all sufficiently large φ and for which $\sup |\delta(\varphi)| : \varphi \geq \Theta > 0$ for all $\Theta > \varphi_0$.

We adopt the standard definition of oscillation: a non-trivial solution is said to be oscillatory if it is neither eventually positive nor eventually negative. Equation (1) is said to be oscillatory if all of its solutions are oscillatory.

Equation (1) encompasses both the second-order neutral delay difference equation

$$\Delta(\mu(\varphi)\Delta\theta(\varphi)) + \phi_2(\varphi)\delta(\sigma(\varphi)) = 0, \quad (2)$$

and the unstable-type neutral difference equation

$$\Delta(\mu(\varphi)\Delta\theta(\varphi)) - \phi_1(\varphi)\delta(\varphi + 1) = 0. \quad (3)$$

Accordingly, we classify Equation (1) as a *hybrid-type* neutral difference equation. The oscillatory properties of Equations (2) and (3) have been extensively studied in the literature: see [6–13] for linear neutral terms, refs. [5,14,15] for sublinear terms, and [16–20] for superlinear terms. It is important to note that the solution spaces of Equations (2) and (3) are fundamentally different. Let S denote the set of all non-oscillatory solutions. For Equation (2), we define:

$$S = S_1,$$

where, for a positive solution $\delta(\varphi)$, the sequence satisfies:

$$\theta(\varphi) \in S_1 \Leftrightarrow \theta(\varphi) > 0, \quad \mu(\varphi)\Delta\theta(\varphi) > 0, \quad \Delta(\mu(\varphi)\Delta\theta(\varphi)) \leq 0.$$

In contrast, for Equation (3), the set S is given by:

$$S = S_0 \cup S_2,$$

where, for a positive solution $\delta(\varphi)$, the sequence satisfies:

$$\theta(\varphi) \in S_0 \Leftrightarrow \theta(\varphi) > 0, \quad \mu(\varphi)\Delta\theta(\varphi) < 0, \quad \Delta(\mu(\varphi)\Delta\theta(\varphi)) > 0,$$

$$\theta(\varphi) \in S_2 \Leftrightarrow \theta(\varphi) > 0, \quad \mu(\varphi)\Delta\theta(\varphi) > 0, \quad \Delta(\mu(\varphi)\Delta\theta(\varphi)) > 0.$$

Thus, for Equation (1), which includes both positive and negative terms, the structure of the non-oscillatory solution space is not immediately evident.

A common strategy in oscillation theory for trinomial difference equations is to omit one of the terms (see [21–23]). Omitting the negative term in Equation (1) yields the inequality:

$$\Delta(\mu(\varphi)\Delta\theta(\varphi)) + \phi_2(\varphi)\delta(\sigma(\varphi)) \operatorname{sgn} \delta(\varphi) \geq 0, \quad (4)$$

which contradicts the typical inequality associated with Equation (2). Similarly, omitting the positive term leads to:

$$\Delta(\mu(\varphi)\Delta\theta(\varphi)) - \phi_1(\varphi)\delta(\varphi + 1) \operatorname{sgn} \delta(\varphi) \leq 0, \quad (5)$$

which again conflicts with standard analytical conditions. Consequently, relatively few studies have addressed Equation (1), which includes both positive and negative components.

In [1,5,6,24], the authors examined the oscillation and non-oscillation of the first-order neutral difference equation:

$$\Delta(\delta(\varphi) - b(\varphi)\delta(\varphi - \tau)) + \phi_1(\varphi)\delta(\varphi - \tau_1) - \phi_2(\varphi)\delta(\varphi - \tau_2) = 0, \quad (6)$$

using fixed-point methods and summation-averaging techniques.

Furthermore, second-order neutral difference equations of the form:

$$\Delta^2(\delta(\varphi) + \lambda b(\varphi)\delta(\varphi - \tau)) + \phi_1(\varphi)\delta(\varphi - \tau_1) - \phi_2(\varphi)\delta(\varphi - \tau_2) = 0, \quad (7)$$

where $\lambda = \pm 1$, were studied in [25,26]. Notably, the theory of non-oscillation appears more developed than that of oscillation. For example, in [26], under the condition:

$$\sum_{\varphi=\varphi_2}^{\infty} \sum_{s=\tau-\tau_1+\tau_2}^{\varphi-1} \phi_2(s) < \infty,$$

the authors used comparison and summation-averaging methods to establish criteria ensuring that every solution is either oscillatory or tends to zero asymptotically.

In [27–31], the following second-order neutral difference equation, or its special cases, was studied:

$$\Delta(\mu(\varphi)\Delta(\theta(\varphi) + \lambda b(\varphi)\theta(\varphi - \tau))) + \phi_1(\varphi)f(\delta(\varphi - \tau_1)) - \phi_2(\varphi)f(\delta(\varphi - \tau_2)) = 0, \quad (8)$$

where $\lambda = \pm 1$. By assuming the inequality:

$$\phi_1(\varphi) - \phi_2(\varphi - \tau_2 + \tau_1) \geq M > 0, \quad (9)$$

the authors derived several conditions under which every solution is either oscillatory or asymptotically tends to zero.

Observe that the equations considered in [25–31] involve deviating arguments in both the positive and negative terms. However, Equation (1) contains a positive term with delay and a negative term without delay. Therefore, Equation (1) studied in this paper is fundamentally different from those previously examined in the literature.

Furthermore, we employ a novel method to transform the trinomial equation into a binomial form, which allows for a clear identification of the structure of non-oscillatory solutions of the transformed equation. Hence, this transformation is crucial for establishing oscillation criteria for Equation (1).

By applying comparison techniques and the summation averaging method, we derive several sufficient conditions ensuring the oscillation of all solutions of the transformed binomial-type equation, which in turn guarantees the oscillation of all solutions of Equation (1). Two illustrative examples are provided to demonstrate the applicability and robustness of the main results.

2. Preliminary Results

This section presents several preliminary results that serve as foundational tools for proving our main theorems. The key findings, detailed in Section 3, establish connections between the solution properties of the second-order neutral delay difference equation given by (1) and those of an associated auxiliary second-order linear difference equation of the form:

$$\Delta(\mu(\varphi)\Delta\zeta(\varphi)) - \phi_1(\varphi)\zeta(\varphi + 1) = 0, \quad \varphi \geq \varphi_0. \quad (10)$$

The initial result is derived from an alternative formulation of the linear difference operator:

$$L(\theta(\varphi)) = \Delta(\mu(\varphi)\Delta\theta(\varphi)) - \phi_1(\varphi)\theta(\varphi + 1), \quad (11)$$

in terms of a positive solution $\zeta(\varphi)$ of (10). It is well-known that if $\mu(\varphi)$ and $\phi_1(\varphi)$ are positive real sequences, then Equation (10) is non-oscillatory. Note that Equation (1) can be written in an equivalent form as:

$$\Delta(\mu(\varphi)\Delta\theta(\varphi)) - \phi_1(\varphi)\theta(\varphi + 1) + \phi_1(\varphi)b(\varphi)\delta(\tau(\varphi + 1)) + \phi_2(\varphi)\delta(\sigma(\varphi)) = 0. \quad (12)$$

Lemma 1. Assume that (10) has a positive solution of $\{\zeta(\varphi)\}$. Then, the operator (11) can be represented as

$$L(\theta(\varphi)) = \frac{1}{\zeta(\varphi + 1)} \Delta \left(\mu(\varphi)\zeta(\varphi)\zeta(\varphi + 1) \Delta \left(\frac{\theta(\varphi)}{\zeta(\varphi)} \right) \right).$$

Proof. Using a difference calculus, we find that

$$\begin{aligned} L(\theta(\varphi)) &= \frac{1}{\zeta(\varphi + 1)} \Delta(\mu(\varphi)\zeta(\varphi)\Delta\theta(\varphi) - \mu(\varphi)\theta(\varphi)\Delta\zeta(\varphi)) \\ &= \Delta(\mu(\varphi)\Delta\theta(\varphi)) - \Delta(\mu(\varphi)\Delta\zeta(\varphi)) \frac{\theta(\varphi + 1)}{\zeta(\varphi + 1)} \\ &= \Delta(\mu(\varphi)\Delta\theta(\varphi)) - \phi_1(\varphi)\theta(\varphi + 1), \end{aligned}$$

where we have used the positive solution of (10). The proof of the lemma is complete. $\square$

Lemma 2. Let $\{\zeta(\varphi)\}$ be a positive solution of (10). Then, Equation (1) or (12) can be written in the form

$$\Delta(\beta(\varphi)\Delta\psi(\varphi)) + \zeta(\varphi + 1)\phi_1(\varphi)b(\varphi)\delta(\tau(\varphi + 1)) + \zeta(\varphi + 1)\phi_2(\varphi)\delta(\sigma(\varphi)) = 0, \quad (13)$$

where

$$\beta(\varphi) = \mu(\varphi)\zeta(\varphi)\zeta(\varphi + 1), \quad \psi(\varphi) = \frac{\theta(\varphi)}{\zeta(\varphi)}.$$

Proof. The proof is obvious from Lemma 1. $\square$

For the purposes of our investigation, it is advantageous to consider (13) in its canonical form; accordingly, we shall assume henceforth that

$$\sum_{\wp=\wp_0}^{\infty} \frac{1}{\beta(\wp)} = \sum_{\wp=\wp_0}^{\infty} \frac{1}{\mu(\wp)\zeta(\wp)\zeta(\wp+1)} = \infty. \quad (14)$$

Next, we present the structure of nonoscillatory solutions of (10) using Discrete Knesers Theorem 1.7.11 of [1].

Lemma 3. Suppose that condition (A1) holds. Then, the Equation (10) possesses a principal solution $\{r(\wp)\}$ such that

$$r(\wp) > 0, \quad \Delta r(\wp) < 0, \quad \Delta(\mu(\wp) \Delta r(\wp)) \geq 0, \quad \text{for all } \wp \geq \wp_0.$$

In addition, it admits a non-principal solution $\{v(\wp)\}$ satisfying

$$v(\wp) > 0, \quad \Delta v(\wp) > 0, \quad \Delta(\mu(\wp) \Delta v(\wp)) \geq 0, \quad \text{for all } \wp \geq \wp_0.$$

Next, we study the behavior of solutions of (1) with the help of the equivalent representation (13).

Lemma 4. Let $\{\zeta(\wp)\}$ be a positive solution of (10) that satisfies condition (14). If $\{\delta(\wp)\}$ is a positive solution of (1), then the associated sequence $\{\psi(\wp)\}$ fulfills the following condition

$$\psi(\wp) > 0, \quad \beta(\wp)\Delta\psi(\wp) > 0, \quad \Delta(\beta(\wp)\Delta\psi(\wp)) \leq 0, \quad (15)$$

for all $\wp \geq \wp_0$.

Proof. Let $\{\delta(\wp)\}$ be an eventually positive solution of (1), say, $\delta(\wp) > 0, \delta(\tau(\wp)) > 0$ and $\delta(\sigma(\wp)) > 0$ for all $\wp \geq \wp_1$ for some $\wp_1 \geq \wp_0$. Then, by the definition of $\theta(\wp)$, we see that $\theta(\wp) > 0$ for all $\wp \geq \wp_1$. Since $\zeta(\wp) > 0$ for all $\wp \geq \wp_0$, we find that $\beta(\wp) = \mu(\wp)\zeta(\wp)\zeta(\wp+1) > 0$ and $\psi(\wp) = \frac{\theta(\wp)}{\zeta(\wp)} > 0$ for all $\wp \geq \wp_2 \geq \wp_1$. From (13), we see that $\Delta(\beta(\wp)\Delta\psi(\wp)) \leq 0$ for all $\wp \geq \wp_2$ and the condition (14) clearly implies that $\beta(\wp)\Delta\psi(\wp) > 0$ for all $\wp \geq \wp_2$. The proof of the lemma is complete. $\square$

Next, we find the relation between $\{\delta(\wp)\}$ and $\{\psi(\wp)\}$.

Lemma 5. Let $\{\zeta(\wp)\}$ be a positive solution of (10) satisfies (14). Then,

$$\delta(\wp) \geq \zeta(\wp)E(\wp)\psi(\wp), \quad \wp \geq \wp_2 \geq \wp_1, \quad (16)$$

where

$$E(\wp) = \left(1 - b(\wp) \frac{\zeta(\tau(\wp))}{\zeta(\wp)}\right).$$

Proof. From the definition of $\psi(\wp)$ and its monotonicity behavior, we see that

$$\zeta(\wp)\psi(\wp) = \theta(\wp) = \delta(\wp) + b(\wp)\delta(\tau(\wp)),$$

that is,

$$\delta(\wp) \geq \zeta(\wp)\psi(\wp) - b(\wp)\zeta(\tau(\wp))\psi(\tau(\wp)).$$

So,

$$\delta(\wp) \geq \zeta(\wp) \left(1 - \frac{b(\wp)\zeta(\tau(\wp))}{\zeta(\wp)}\right) \psi(\wp),$$

which ends the proof. $\square$

Before presenting our next results, let us define

$$\begin{aligned} Q_1(\varphi) &= \zeta(\varphi + 1)b(\varphi)\phi_1(\varphi)\zeta(\tau(\varphi))E(\tau(\varphi)), \\ Q_2(\varphi) &= \zeta(\varphi + 1)\phi_2(\varphi)\zeta(\sigma(\varphi))E(\sigma(\varphi)), \\ \gamma(\varphi) &= \min\{\tau(\varphi), \sigma(\varphi)\}, Q_3(\varphi) = Q_1(\varphi) + Q_2(\varphi), \\ \Omega(\varphi) &= \sum_{s=\varphi_0}^{\varphi-1} \frac{1}{\beta(s)}. \end{aligned}$$

Lemma 6. Suppose that $\{\zeta(\varphi)\}$ is a positive solution of (10) satisfying condition (14). Then, the equation (1) exhibits oscillatory behavior provided that

$$\Delta(\beta(\varphi)\Delta\psi(\varphi)) + Q_3(\varphi)\psi(\gamma(\varphi)) = 0, \varphi \geq \varphi_0, \quad (17)$$

is oscillatory.

Proof. Assume the contrary that $\{\delta(\varphi)\}$ is an eventually positive solution of (1), say, $\delta(\varphi) > 0, \delta(\tau(\varphi)) > 0, \delta(\sigma(\varphi)) > 0$ for all $\varphi \geq \varphi_1 \geq \varphi_0$ and the corresponding function is $\theta(\varphi) > 0$ for all $\varphi \geq \varphi_1 \geq \varphi_0$. Then, by Lemma 4, we determine that $\psi(\varphi) = \frac{\theta(\varphi)}{\zeta(\varphi)} > 0$ and satisfies condition (15) for all $\varphi \geq \varphi_1 \geq \varphi_0$. Using (16) in (13), we find that $\{\psi(\varphi)\}$ is a positive and increasing solution of

$$\Delta(\beta(\varphi)\Delta\psi(\varphi)) + Q_3(\varphi)\psi(\gamma(\varphi)) \leq 0.$$

However, by Lemma 1 of [32], the associated Equation (17) also admits a positive solution, thereby leading to a contradiction, which concludes the proof. $\square$

3. Oscillation Results

In this section, we establish oscillation criteria for Equation (1) with the aid of Equation (17). We begin with the following theorem.

Theorem 1. Let $\{\zeta(\varphi)\}$ be a positive solution of (10) such that (14) holds. If

$$\sum_{\varphi=l_0}^{\infty} Q_3(\varphi) = \infty, \quad (18)$$

then Equation (1) is oscillatory.

Proof. Assume the contrary that $\{\delta(\varphi)\}$ is an eventually positive solution of (1), say, $\delta(\varphi) > 0, \delta(\sigma(\varphi)) > 0$ and $\delta(\tau(\varphi)) > 0$ for all $\varphi \geq \varphi_1$ for some $\varphi_1 \geq \varphi_0$. Then, the corresponding function is $\theta(\varphi) > 0, \theta(\sigma(\varphi)) > 0$ and $\theta(\tau(\varphi)) > 0$ for all $\varphi \geq \varphi_0$. From Lemma 2, the function $\psi(\varphi) = \frac{\theta(\varphi)}{\zeta(\varphi)} > 0$ and satisfies condition (15). Since $\{\psi(\varphi)\}$ is increasing, there exists a constant $M > 0$ such that $\psi(\varphi) \geq M > 0$ for all $\varphi \geq \varphi_2 \geq \varphi_1$. Using this in (17) and then summing it from φ_2 to φ , we find

$$\sum_{s=\varphi_2}^{\varphi} M(Q_2(s) + Q_1(s)) \leq \beta(\varphi_2)\Delta\psi(\varphi_2) - \beta(\varphi + 1)\Delta\psi(\varphi + 1).$$

As $\varphi \rightarrow \infty$, we see that

$$\sum_{\varphi=\varphi_2}^{\infty} M(Q_2(\varphi) + Q_1(\varphi)) \leq \beta(\varphi_2)\Delta\psi(\varphi_2) < \infty,$$

which contradicts (18). The proof of the theorem is complete. $\square$

Remark 1. Note that the above theorem is independent of delay argument, so it holds for delay or advanced equations.

Next, we derive oscillation criteria for Equation (1) when the condition (18) fails to hold.

Theorem 2. Let $\{\zeta(\varphi)\}$ be a positive solution of (10) such that (14) holds. If the first-order delay difference equation

$$\Delta\omega(\varphi) + Q_3(\varphi)\Omega(\gamma(\varphi))\omega(\gamma(\varphi)) = 0, \quad (19)$$

is oscillatory, then Equation (1) is oscillatory.

Proof. Assume the contrary that $\{\delta(\varphi)\}$ is an eventually positive solution of (1), say, $\delta(\varphi) > 0$, $\delta(\tau(\varphi)) > 0$ and $\delta(\sigma(\varphi)) > 0$ for all $\varphi \geq \varphi_1 \geq \varphi_0$. Then, the corresponding function $\theta(\varphi) > 0$, $\theta(\sigma(\varphi)) > 0$ and $\theta(\tau(\varphi)) > 0$ for all $\varphi \geq \varphi_1$. From Lemma 2, the function $\psi(\varphi) = \frac{\theta(\varphi)}{\zeta(\varphi)} > 0$ and satisfies condition (15). From the monotonicity of $\beta(\varphi)\Delta\psi(\varphi)$, we have

$$\psi(\varphi) \geq \sum_{s=\varphi_1}^{\varphi-1} \frac{\beta(s)\Delta\psi(s)}{\beta(s)} \geq \Omega(\varphi)\beta(\varphi)\Delta\psi(\varphi) \quad (20)$$

and so

$$\Delta\left(\frac{\psi(\varphi)}{\Omega(\varphi)}\right) = \frac{\Omega(\varphi)\beta(\varphi)\Delta\psi(\varphi) - \psi(\varphi)}{\Omega(\varphi)\Omega(\varphi+1)\beta(\varphi)} \leq 0,$$

which implies $\{\psi(\varphi)/\Omega(\varphi)\}$ is decreasing. Using (20) in (17), we find that $\omega(\varphi) = \beta(\varphi)\Delta\psi(\varphi) > 0$ satisfies the inequality

$$\Delta\omega(\varphi) + Q_3(\varphi)\Omega(\gamma(\varphi))\omega(\gamma(\varphi)) \leq 0.$$

Then, by Lemma 2.7 of [4], we see that the Equation (19) also has a positive solution. This contradiction ends the proof. $\square$

Next, we provide explicit criteria for the oscillation of Equation (19).

Corollary 1. Let $\{\zeta(\varphi)\}$ be a positive solution of (10) such that (14) holds. If

$$\liminf_{\varphi \rightarrow \infty} \sum_{s=\gamma(\varphi)}^{\varphi-1} Q_3(s)\Omega(\gamma(s)) > \frac{1}{e}, \quad (21)$$

then Equation (1) is oscillatory.

Proof. In light of condition (21), Theorem 2.2 from [33] ensures that Equation (19) is oscillatory. Consequently, the result follows directly from Theorem 2, which completes the proof. $\square$

Corollary 2. Let $\{\zeta(\varphi)\}$ be a positive solution of (10), such that (14) holds. If $\gamma(\varphi) = \varphi - k$, k is a positive integer and

$$\liminf_{\varphi \rightarrow \infty} \sum_{s=\varphi-k}^{\varphi-1} Q_3(s)\Omega(s-k) > \left(\frac{k}{k+1}\right)^{k+1}, \quad (22)$$

then Equation (1) is oscillatory.

Proof. In view of condition (22) and Lemma 7.6.1 of [1], it is easy to see that Equation (19) is oscillatory and the conclusion follows from Theorem 2. This completes the proof. $\square$

Theorem 3. Let $\{\zeta(\varphi)\}$ be a positive solution of (10) such that (14) holds. If $\gamma(\varphi) = \varphi - k, k$ is a positive integer and

$$\limsup_{\varphi \rightarrow \infty} \left\{ \frac{1}{\Omega(\varphi - k)} \sum_{s=\varphi_1}^{\varphi-k-1} Q_3(s) \Omega(s) \Omega(s-k) + \sum_{s=\varphi-k}^{\varphi-1} Q_3(s) \Omega(s-k) + \Omega(\varphi - k) \sum_{s=\varphi}^{\infty} Q_3(s) \right\} > 1, \quad (23)$$

for some $\varphi_1 \geq \varphi_0 + k$, then (1) is oscillatory.

Proof. Suppose that Equation (1) is not oscillatory. Then, by Theorem 2, Equation (17) is also non-oscillatory. Therefore, we may assume that it admits an eventually positive solution $\{\psi(\varphi)\}$, with $\psi(\varphi) > 0$ for all $\varphi \geq \varphi_1 \geq \varphi_0 + k$, such that condition (23) is satisfied. Summing (17) yields

$$\Delta\psi(\varphi) \geq \frac{1}{\beta(\varphi)} \sum_{s=\varphi}^{\infty} Q_3(s) \psi(s-k).$$

Summing once more gives

$$\begin{aligned} \psi(\varphi) &\geq \sum_{s=\varphi_1}^{\varphi-1} \frac{1}{\beta(s)} \sum_{t=s}^{\infty} Q_3(t) \psi(t-k) \\ &= \sum_{s=\varphi_1}^{\varphi-1} \frac{1}{\beta(s)} \sum_{t=s}^{\varphi-1} Q_3(t) \psi(t-k) + \sum_{s=\varphi_1}^{\varphi-1} \frac{1}{\beta(s)} \sum_{t=\varphi}^{\infty} Q_3(t) \psi(t-k). \end{aligned}$$

Employing a summation-by-parts formula, we have

$$\psi(\varphi) \geq \sum_{s=\varphi_1}^{\varphi-1} Q_3(s) \Omega(s+1) \psi(s-k) + \Omega(\varphi) \sum_{t=\varphi}^{\infty} Q_3(t) \psi(t-k).$$

Hence,

$$\begin{aligned} \psi(\varphi - k) &\geq \sum_{s=\varphi_1}^{\varphi-k-1} Q_3(s) \Omega(s+1) \psi(s-k) + \Omega(\varphi - k) \sum_{t=\varphi-k}^{\varphi-1} Q_3(t) \psi(t-k) \\ &\quad + \Omega(\varphi - k) \sum_{t=\varphi}^{\infty} Q_3(t) \psi(t-k). \end{aligned}$$

Because of the fact that $\psi(\varphi)/\Omega(\varphi)$ is decreasing and $\psi(\varphi)$ is increasing, the previous inequality yields

$$\begin{aligned} \psi(\varphi - k) &\geq \frac{\psi(\varphi - k)}{\Omega(\varphi - k)} \sum_{s=\varphi_1}^{\varphi-k-1} Q_3(s) \Omega(s+1) \Omega(s-k) \\ &\quad + \psi(\varphi - k) \sum_{s=\varphi-k}^{\varphi-1} Q_3(s) \Omega(s-k) + \Omega(\varphi - k) \psi(\varphi - k) \sum_{s=\varphi}^{\infty} Q_3(s). \end{aligned}$$

Dividing the last inequality by $\psi(\varphi - k)$, we have

$$1 \geq \left\{ \frac{1}{\Omega(\varphi - k)} \sum_{s=\varphi_1}^{\varphi-k-1} Q_3(s) \Omega(s+1) \Omega(s-k) + \sum_{s=\varphi-k}^{\varphi-1} Q_3(s) \Omega(s-k) + \Omega(\varphi - k) \sum_{s=\varphi}^{\infty} Q_3(s) \right\},$$

this yields a contradiction, thereby completing the proof of the theorem. $\square$

Theorem 4. Let $\{\zeta(\varphi)\}$ be a positive solution of (10) such that (14) holds. If

$$\liminf_{\varphi \rightarrow \infty} \Omega(\varphi) \sum_{s=\varphi}^{\infty} Q_3(s) \frac{\Omega(\gamma(s))}{\Omega(s)} > \frac{1}{4}, \quad (24)$$

then Equation (1) is oscillatory.

Proof. Assume that Equation (1) is not oscillatory. Then, by Theorem 2, Equation (17) is also non-oscillatory. Accordingly, we assume that it admits an eventually positive solution $\{\psi(\varphi)\}$, with $\psi(\varphi) > 0$ for all $\varphi \geq \varphi_1 \geq \varphi_0$, such that condition (24) holds.

Since $\frac{\psi(\varphi)}{\Omega(\varphi)}$ is decreasing and $\gamma(\varphi) \leq \varphi - 1$, we find that

$$\psi(\gamma(\varphi)) \geq \frac{\Omega(\gamma(\varphi))}{\Omega(\varphi)} \psi(\varphi). \quad (25)$$

Using (25) in (17) gives

$$\Delta(\beta(\varphi) \Delta \psi(\varphi)) + Q_3(\varphi) \frac{\Omega(\gamma(\varphi))}{\Omega(\varphi)} \psi(\varphi) \leq 0. \quad (26)$$

Define,

$$w(\varphi) = \frac{\beta(\varphi) \Delta \psi(\varphi)}{\psi(\varphi)}, \varphi \geq \varphi_1.$$

Then, $w(\varphi) > 0$ and

$$\begin{aligned} \Delta \omega(\varphi) &= \frac{\Delta(\beta(\varphi) \Delta \psi(\varphi))}{\psi(\varphi)} - \frac{\beta(\varphi+1) \Delta \psi(\varphi+1)}{\psi(\varphi) \psi(\varphi+1)} \Delta \psi(\varphi) \\ &\leq -Q_3(\varphi) \frac{\Omega(\gamma(\varphi))}{\Omega(\varphi)} - \frac{w(\varphi+1) \omega(\varphi)}{\beta(\varphi)}. \end{aligned}$$

Summing the last inequality from φ to ∞ , we find that

$$\omega(\varphi) \geq \sum_{s=\varphi}^{\infty} Q_3(s) \frac{\Omega(\gamma(s))}{\Omega(s)} + \sum_{s=\varphi}^{\infty} \frac{\omega(s) \omega(s+1)}{\beta(s)}. \quad (27)$$

Let $\lim_{\varphi \rightarrow \infty} \Omega(\varphi) \omega(\varphi) = M > 0$. Multiplying (27) by $\Omega(\varphi)$ and using (24), we obtain

$$M > \frac{1}{4} + M^2, \quad (28)$$

since $\Omega(\varphi) \sum_{s=\varphi}^{\infty} \frac{1}{\Omega(s) \Omega(s+1) \beta(s)} = 1$. The inequality in (28) is not possible and, therefore, the proof of the theorem is complete. $\square$

Remark 2. Note that there are many explicit oscillation criteria available in the literature (see the monographs [1,5] and the references cited therein) for the Equation (17), from which we can deduce many oscillation criteria for the studied Equation (1).

4. Examples

This section presents two concrete numerical examples that demonstrate the validity of the main results. In population models, Equations (29) and (31) describe scenarios in which the growth rate is influenced by both the current population size and its past values. These equations incorporate terms that promote growth (positive terms) as well as those that inhibit it (negative terms).

Example 1. Consider the second-order neutral hybrid-type difference equation

$$\Delta(\wp \Delta \theta(\wp)) - \frac{1}{(\wp + 2)} \delta(\wp + 1) + q_0(2\wp + 1 + \frac{1}{\wp + 2}) \delta(\wp - 1) = 0, \wp \geq 3, \quad (29)$$

where $q_0 > 0$ and $\theta(\wp) = \delta(\wp) + \frac{1}{2} \delta(\wp - 1)$.

Here, $\mu(\wp) = \wp$, $b(\wp) = \frac{1}{2}$, $\phi_1(\wp) = \frac{1}{\wp + 2}$, $\phi_2(\wp) = q_0(2\wp + 1 + \frac{1}{\wp + 2})$. Now, the auxiliary Equation (10) takes the form

$$\Delta(\wp \Delta \zeta(\wp)) - \frac{1}{(\wp + 2)} \zeta(\wp + 1) = 0, \wp \geq 1. \quad (30)$$

The sequence $\{\zeta(\wp)\} = \left\{ \frac{1}{\wp} \right\}$ is a positive solution of (30) and $\theta(\wp) = \frac{1}{\wp + 1}$ satisfies condition (14). Further

$$E(\wp) = \frac{1}{2} \left(\frac{\wp - 2}{\wp - 1} \right), \quad \gamma(\wp) = \wp - 1, \quad Q_1(\wp) \approx \frac{1}{4\wp^3}, \quad Q_2(\wp) \approx q_0 \left(\frac{1}{\wp} + \frac{1}{\wp^3} \right)$$

and $Q_3(\wp) \approx \frac{q_0}{\wp} + \left(q_0 + \frac{1}{4} \right) \frac{1}{\wp^3}$. Condition (18) becomes

$$\sum_{\wp=3}^{\infty} Q_3(\wp) = \sum_{\wp=3}^{\infty} \left(q_0 \left(\frac{1}{\wp} + \frac{1}{\wp^3} \right) + \frac{1}{4\wp^3} \right) = \infty,$$

that is, condition (18) holds if $q_0 > 0$. Therefore, by Theorem 1, Equation (29) is oscillatory. In fact, for $q_0 = 1$, we see that $\{\delta(\wp)\} = \{(-1)^\wp\} = \{-1, +1, -1, +1, -1, +1, \dots\}$ is one such oscillatory solution of (29).

Example 2. Consider the second-order neutral hybrid delay difference equation

$$\Delta(2^\wp \Delta \theta(\wp)) - \frac{2^{\wp+1}}{3} \delta(\wp + 1) + \frac{14}{3} 2^\wp \delta(\wp - 1) = 0, \wp \geq 2, \quad (31)$$

where $\theta(\wp) = \delta(\wp) + \frac{1}{3} \delta(\wp - 1)$.

The auxiliary equation is

$$\Delta(2^\wp \Delta \zeta(\wp)) - \frac{2^{\wp+1}}{3} \zeta(\wp + 1) = 0. \quad (32)$$

The sequence $\zeta(\wp) = \frac{1}{2^\wp}$ is a positive solution of (32) and $\theta(\wp) = \frac{4}{3} \frac{1}{2^\wp}$.

Further,

$$\begin{aligned} E(\wp) &= \frac{1}{3}, Q_1(\wp) = \frac{1}{2^{\wp+1}} \cdot \frac{1}{3} \cdot \frac{2^{\wp+1}}{3} \cdot \frac{1}{3} \cdot \frac{1}{2^{\wp-1}} = \frac{2}{27} \frac{1}{2^{\wp}}, \\ Q_2(\wp) &= \frac{1}{2^{\wp+1}} \cdot \frac{14}{3} \cdot 2^{\wp} \cdot \frac{1}{3} \cdot \frac{1}{2^{\wp-1}} = \frac{14}{9} \frac{1}{2^{\wp}}, \\ Q_3(\wp) &= \frac{44}{27} \cdot \frac{1}{2^{\wp}} \quad \text{and} \quad \Omega(\wp) \approx 2^{\wp}. \end{aligned}$$

The condition (18) is not satisfied but condition (21) becomes

$$\liminf_{\wp \rightarrow \infty} \sum_{s=\wp-1}^{\wp-1} \frac{44}{27 \cdot 2^s} 2^s = \lim_{\wp \rightarrow \infty} \frac{44}{27} \wp = \infty > \frac{1}{e}.$$

Hence, Equation (31) is oscillatory by Corollary 1. In fact, $\{\delta(\wp)\} = \{(-1)^{\wp}\} = \{+1, -1, +1, -1, +1, -1, +1, \dots\}$ is one such oscillatory solution of (31).

Remark 3. It is important to note that the existing results in the literature are not directly applicable to Equations (29) and (31), as their structure differs from that of previously studied equations.

5. Conclusions

In this study, we have investigated the oscillatory behavior of solutions of the hybrid second-order neutral delay difference equation, Equation (1). The proposed criteria are both novel and more general, as they accommodate equations with variable, unbounded, or mixed coefficient sequences scenarios that are difficult to address using traditional approaches. The methodology employed is distinct from those found in the existing literature, particularly due to the application of a unique transformation technique and the effective handling of hybrid neutral equations. Thus, the oscillation criteria presented in this work are new and extend the existing results in the literature for both neutral and non-neutral delay difference equations. Overall, the findings contribute significantly to the oscillation theory of difference equations and offer deeper insight into the dynamic behavior of complex discrete systems.

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