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Weighted Lorentz Spaces, Variable Exponent Analysis, and Operator Extensions

Saeed Hashemi Sababe [1,*] and Ismail Nikoufar [2,*]

[1] Campus Saint-Jean, University of Alberta, 8406 91 Street, Edmonton, AB T6C 4G9, Canada
[2] Department of Mathematics, Payame Noor University, Tehran 19395-4697, Iran
* Correspondence: sa.hashemi@ualberta.ca or hashemi_1365@yahoo.com (S.H.S.); nikoufar@pnu.ac.ir (I.N.)

Abstract

We develop novel extensions in the theory of weighted Lorentz spaces. In particular, we generalize classical results by introducing variable-exponent Lorentz spaces, establish sharp constants and quantitative bounds for maximal operators, and extend the framework to encompass fractional maximal operators. Moreover, we analyze endpoint cases through the study of oscillation operators and reveal new connections with weighted Hardy spaces. These results provide a unifying approach that not only refines existing inequalities but also opens new avenues in harmonic analysis and partial differential equations.

Keywords: weighted Lorentz spaces; variable-exponent spaces; Hardy–Littlewood maximal operator; fractional maximal operator; endpoint estimates; oscillation operators; weighted Hardy spaces

MSC: 42B25; 42B20; 42B35; 46E30; 26D10

1. Introduction

Weighted function spaces have long played a pivotal role in harmonic analysis and partial differential equations. In particular, the study of weighted Lorentz spaces has provided refined tools to capture the delicate behavior of various operators. The classical work by Muckenhoupt [1] established that the Hardy–Littlewood maximal operator is bounded on weighted Lebesgue spaces $L^p(u)$ if and only if the weight u belongs to the A_p class. Building on these ideas, Lorentz [2] introduced what are now known as Lorentz spaces, which offer a finer scale than the classical Lebesgue spaces.

Subsequent investigations by Arino and Muckenhoupt [3] extended these results to classical Lorentz spaces, and later works such as [4] provided a unified approach that characterizes the strong-type boundedness of the Hardy–Littlewood maximal operator on weighted Lorentz spaces. More recently, the weak-type boundedness for $p > 1$ was resolved in [5], completing the picture initiated in earlier studies.

Despite these advances, several natural extensions remain unexplored. First, the theory of *variable-exponent spaces* has attracted considerable attention in recent years due to its flexibility in modeling media with non-homogeneous properties [6]. Introducing variable exponents into the weighted Lorentz space framework, denoted here as $\Lambda_u^{p(\cdot)}(w)$, poses new challenges and promises to broaden the applicability of these spaces.

A second direction involves the derivation of *sharp constants* and quantitative bounds in maximal inequalities. Although classical estimates are well established, determining

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optimal constants in the context of weighted Lorentz spaces is a problem of both theoretical and practical interest [7].

A third avenue for research is the extension of these results to the *fractional maximal operator* M_α. Such operators, which generalize the Hardy–Littlewood maximal operator, are central to the study of fractional integration and potential theory, and their behavior in weighted Lorentz spaces remains largely unexplored [8].

Endpoint estimates represent another critical aspect of the theory. Standard maximal operators often fail to capture the full range of endpoint phenomena. By introducing and analyzing *oscillation operators*, one can obtain refined endpoint estimates that shed light on these subtle behaviors [9].

Finally, there is a growing interest in exploring the interplay between weighted Lorentz spaces and *weighted Hardy spaces*. The atomic decomposition techniques in Hardy spaces offer powerful methods to address singular integrals, and establishing robust connections between these spaces could lead to significant advancements in harmonic analysis [10].

Variable- exponent Lorentz spaces arise naturally in several concrete settings:

Nonlocal elliptic equations with variable growth. Models of anomalous diffusion in heterogeneous media require fine control of solution tails. The Lorentz-variable norm captures this more precisely than standard Lebesgue spaces.

Image restoration. In denoising algorithms where the fidelity term's exponent varies with local image features, formulating the energy in a Lorentz-variable norm better handles impulsive noise and outliers.

Electrorheological fluids. Energy functionals for fluids whose viscosity depends on the electric field strength naturally lead to Sobolev–Lorentz spaces with a variable exponent.

Background and Novelty.

1. *What is known?* Classical Lorentz spaces $\Lambda^p(w)$ and variable-exponent Lebesgue spaces $L^{p(\cdot)}$ are well studied: the former capture fine tail decay via rearrangements [3], while the latter handle spatially varying growth in applications such as electrorheological fluids.
2. *What needs to be carried out?* A unified framework combining both features tail control and variable growth is missing. In particular, no one has yet developed a full theory of operators (maximal, fractional, singular integrals) on *variable-exponent Lorentz spaces*.
3. *Our contribution.* We introduce $\Lambda_u^{p(\cdot)}(w)$, prove sharp bounds for Hardy–Littlewood and fractional maximal operators, establish endpoint and interpolation estimates, and outline extensions to singular integrals and Sobolev settings.

In this paper, we pursue these research directions by developing a unified framework that extends the classical results in several innovative ways. Our contributions include the introduction and analysis of weighted Lorentz spaces with variable exponents; the derivation of sharp constants and quantitative bounds for the boundedness of the Hardy–Littlewood maximal operator; the study of the fractional maximal operator M_α within the weighted Lorentz space setting; new endpoint estimates via the analysis of oscillation operators; and establishing connections between weighted Lorentz spaces and weighted Hardy spaces.

The remainder of the paper is organized as follows: In Section 2, we review the necessary background of weighted Lorentz spaces, variable-exponent spaces, and maximal operators. Section 3 is devoted to the study of weighted Lorentz spaces with variable exponents, including embedding theorems and illustrative examples. In Section 4, we derive sharp constants and quantitative bounds for the maximal operator. Section 5 examines the fractional maximal operator and its boundedness properties. In Section 6, we present new endpoint estimates through the analysis of oscillation operators. Section 7 explores the

connections with weighted Hardy spaces and discusses applications to partial differential equations. We conclude in Section 8 with remarks and suggestions for future research.

2. Preliminaries

Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function. The *distribution function* of f is defined as

$$d_f(\lambda) = |\{x \in \mathbb{R}^d : |f(x)| > \lambda\}|, \quad \lambda > 0.$$

The *decreasing rearrangement* f^* of f is given by

$$f^*(t) = \inf\{\lambda > 0 : d_f(\lambda) \le t\}, \quad t > 0.$$

Given a weight function $w : [0, \infty) \to [0, \infty)$ and its primitive

$$W(t) = \int_0^t w(s)\, ds,$$

the *classical Lorentz space* $\Lambda^p(w)$ is defined as

$$\Lambda^p(w) = \left\{ f \text{ measurable} : \|f\|_{\Lambda^p(w)} := \left(\int_0^\infty t^{p-1} W\big(d_f(t)\big)\, dt \right)^{1/p} < \infty \right\}.$$

When a weight u on $\mathbb{R}^d$ is incorporated, the *weighted Lorentz space* $\Lambda^p_u(w)$ is defined by

$$\|f\|_{\Lambda^p_u(w)} := \left(\int_0^\infty t^{p-1} W\Big(u\big(\{x \in \mathbb{R}^d : |f(x)| > t\}\big)\Big)\, dt \right)^{1/p}.$$

Similarly, the corresponding weak-type space $\Lambda^{p,\infty}_u(w)$ is defined via the quasi-norm:

$$\|f\|_{\Lambda^{p,\infty}_u(w)} := \sup_{t>0}\, t\, W\Big(u\big(\{x \in \mathbb{R}^d : |f(x)| > t\}\big)\Big)^{1/p}.$$

Let $p : \mathbb{R}^d \to [1, \infty)$ be a measurable function. The *variable Lebesgue space* $L^{p(\cdot)}(\mathbb{R}^d)$ consists of all measurable functions f for which the modular

$$\rho_{p(\cdot)}(f) = \int_{\mathbb{R}^d} |f(x)|^{p(x)}\, dx$$

is finite (up to a normalization). Its (Luxemburg) norm is defined by

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^d)} := \inf\left\{ \lambda > 0 : \rho_{p(\cdot)}\Big(\frac{f}{\lambda}\Big) \le 1 \right\}.$$

Analogously, one may define *variable-exponent Lorentz spaces* $\Lambda^{p(\cdot)}_u(w)$ by adapting the construction of classical Lorentz spaces using the decreasing rearrangement f^* in combination with a variable exponent. For a detailed treatment of such spaces, see, e.g., [6].

For a locally integrable function f on $\mathbb{R}^d$, the *Hardy–Littlewood maximal operator* is defined as

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^d$ containing x. It is well known that M satisfies the weak-type $(1,1)$ inequality and, when w belongs to the Muckenhoupt A_p class, M is bounded on $L^p(w)$ [1].

For $0 < \alpha < d$, the *fractional maximal operator* is defined by

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/d}} \int_Q |f(y)|\, dy.$$

Given a measurable function f and a cube $Q \subset \mathbb{R}^d$, the *oscillation* of f over Q is defined as

$$\mathrm{Osc}(f; Q) = \sup_{x,y \in Q} |f(x) - f(y)|.$$

A related maximal operator, the *maximal oscillation operator*, is defined by

$$\mathcal{O}f(x) = \sup_{Q \ni x} \mathrm{Osc}(f; Q).$$

These operators are useful in obtaining refined endpoint estimates.

For $0 < p \leq 1$ and a weight w, the *weighted Hardy space* $H^p(w)$ is often defined via maximal functions or atomic decompositions. One common definition employs a smooth approximation of the identity ϕ and the associated maximal function

$$M_\phi f(x) = \sup_{t>0} |f * \phi_t(x)|, \quad \phi_t(x) = t^{-d}\phi(x/t).$$

Then,

$$H^p(w) = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|M_\phi f\|_{L^p(w)} < \infty\}.$$

Atomic decomposition results for $H^p(w)$ can be found in [10].

We write χ_Q for the characteristic function of a measurable set $Q \subset \mathbb{R}^n$, i.e.,

$$\chi_Q(x) = \begin{cases} 1, & x \in Q, \\ 0, & x \notin Q. \end{cases}$$

The following classical results will be instrumental in our analysis.

Lemma 1 (Hardy's Inequality [11]). *Let f be a nonnegative measurable function on $(0, \infty)$ and $1 < p < \infty$. Then, there exists a constant $C_p > 0$ such that*

$$\int_0^\infty \left(\frac{1}{t} \int_0^t f(s)\, ds \right)^p dt \leq C_p \int_0^\infty f(t)^p\, dt.$$

Lemma 2 (Rearrangement Inequality [11]). *Let f and g be measurable functions on $\mathbb{R}^d$ with decreasing rearrangements f^* and g^*, respectively. Then,*

$$\int_{\mathbb{R}^d} |f(x)g(x)|\, dx \leq \int_0^\infty f^*(t)g^*(t)\, dt.$$

Theorem 1 (Boundedness of the Hardy–Littlewood Maximal Operator [11]). *Let $1 < p \leq \infty$ and let w be a weight in the Muckenhoupt A_p class. Then, there exists a constant $C > 0$ such that, for all $f \in L^p(w)$,*

$$\|Mf\|_{L^p(w)} \leq C \|f\|_{L^p(w)}.$$

Theorem 2 (Weak-Type $(1,1)$ Estimate [11]). *If $w \in A_1$, then, for every $\lambda > 0$ and all $f \in L^1(w)$,*

$$w(\{x \in \mathbb{R}^d : Mf(x) > \lambda\}) \leq \frac{C}{\lambda} \|f\|_{L^1(w)},$$

where $C > 0$ is a constant independent of f and λ.

These definitions, lemmas, and theorems provide the essential framework for our subsequent analysis. In the next sections, we extend these classical concepts to variable-exponent settings, derive sharp quantitative bounds for maximal operators on weighted Lorentz spaces (Table 1), and study related operators such as the fractional maximal and oscillation operators.

Table 1. Classical vs. variable-exponent Lorentz spaces.

Property	$\Lambda^p(w)$ **(Classical)**	$\Lambda_u^{p(\cdot)}(w)$ **(This Paper)**
Exponent	constant p	function $p(x)$
Norm	$\|f\| = \left(\int_0^\infty [t^{1/p} f^*(t)]^p w(t)\, \mathrm{d}t\right)^{1/p}$	Luxemburg type
Tail behavior	fixed decay rate	spatially varying decay
Key applications	harmonic analysis, interpolation	nonlocal PDEs, image processing

3. Weighted Lorentz Spaces with Variable Exponents

In this section, we introduce a new class of function spaces that extend the classical weighted Lorentz spaces by allowing the exponent to vary. These spaces, which we denote by $\Lambda_u^{p(\cdot)}(w)$, serve as a natural setting in which to study maximal operators and related operators under nonstandard growth conditions.

Definition 1. *Let u be a weight on $\mathbb{R}^d$ and let $w : [0,\infty) \to [0,\infty)$ be a weight function on $(0,\infty)$ with primitive*

$$W(t) = \int_0^t w(s)\, ds.$$

Let $p : (0,\infty) \to (0,\infty)$ be a measurable function. For a measurable function f on $\mathbb{R}^d$, define its distribution function *with respect to u by*

$$d_{f,u}(\lambda) = u\Big(\{x \in \mathbb{R}^d : |f(x)| > \lambda\}\Big), \quad \lambda > 0,$$

and its decreasing rearrangement *with respect to u by*

$$f_u^*(s) = \inf\Big\{\lambda > 0 : d_{f,u}(\lambda) \le s\Big\}, \quad s > 0.$$

We then define the variable-exponent weighted Lorentz space $\Lambda_u^{p(\cdot)}(w)$ by

$$\Lambda_u^{p(\cdot)}(w) = \Big\{f \text{ measurable on } \mathbb{R}^d : \|f\|_{\Lambda_u^{p(\cdot)}(w)} < \infty\Big\},$$

where the (Luxemburg) quasi-norm is given by

$$\|f\|_{\Lambda_u^{p(\cdot)}(w)} := \|f_u^*\|_{L^{p(\cdot)}(0,\infty;dW)} := \inf\Big\{\lambda > 0 : \int_0^\infty \left(\frac{f_u^*(s)}{\lambda}\right)^{p(s)} dW(s) \le 1\Big\}.$$

This definition generalizes the classical setting. In fact, if $p(s) = p$ is constant, then $\Lambda_u^{p(\cdot)}(w)$ coincides with the standard weighted Lorentz space $\Lambda_u^p(w)$.

Proposition 1 (Basic Properties)**.** *Let $\Lambda_u^{p(\cdot)}(w)$ be defined as above. Then, the following apply:*

1. *$\Lambda_u^{p(\cdot)}(w)$ is a quasi-normed space.*
2. *If $p(s) = p$ for almost every $s > 0$, then*

$$\Lambda_u^{p(\cdot)}(w) = \Lambda_u^p(w).$$

3. *If $u \equiv 1$ and $w \equiv 1$ (with the Lebesgue measure in place of W), then $\Lambda_u^{p(\cdot)}(w)$ reduces to the variable-exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^d)$.*

Proof. We prove each item in turn.

(1) By definition, the quasi-norm is

$$\|f\|_{\Lambda_u^{p(\cdot)}(w)} = \inf\left\{\lambda > 0 : \int_0^\infty \left(\frac{f_u^*(s)}{\lambda}\right)^{p(s)} dW(s) \le 1\right\}.$$

Since the integrand is nonnegative, it follows that $\|f\|_{\Lambda_u^{p(\cdot)}(w)} \ge 0$. If $\|f\|_{\Lambda_u^{p(\cdot)}(w)} = 0$. Then, for every $\epsilon > 0$,

$$\int_0^\infty \left(\frac{f_u^*(s)}{\epsilon}\right)^{p(s)} dW(s) \le 1.$$

Letting $\epsilon \to 0$ forces $f_u^*(s) = 0$ for almost every s; hence, $f = 0$ almost everywhere. Conversely, if $f = 0$ almost everywhere, then $f_u^*(s) = 0$ for all $s > 0$, and clearly $\|f\|_{\Lambda_u^{p(\cdot)}(w)} = 0$.

For any scalar α, note that

$$(\alpha f)_u^*(s) = |\alpha|\, f_u^*(s).$$

Thus, for any $\lambda > 0$,

$$\int_0^\infty \left(\frac{(\alpha f)_u^*(s)}{\lambda}\right)^{p(s)} dW(s) = \int_0^\infty \left(\frac{|\alpha|\, f_u^*(s)}{\lambda}\right)^{p(s)} dW(s).$$

It follows that

$$\|\alpha f\|_{\Lambda_u^{p(\cdot)}(w)} = |\alpha|\, \|f\|_{\Lambda_u^{p(\cdot)}(w)}.$$

We must show that there exists a constant $K \ge 1$ such that, for all f, g,

$$\|f+g\|_{\Lambda_u^{p(\cdot)}(w)} \le K\left(\|f\|_{\Lambda_u^{p(\cdot)}(w)} + \|g\|_{\Lambda_u^{p(\cdot)}(w)}\right).$$

A key tool is the subadditivity of the decreasing rearrangement:

$$(f+g)_u^*(s) \le f_u^*(s/2) + g_u^*(s/2) \quad \text{for all } s > 0.$$

Using this inequality and the properties of the modular function

$$\rho_{p(\cdot)}(h) = \int_0^\infty (h(s))^{p(s)} dW(s),$$

one can show (by a standard argument in the theory of Luxemburg norms; see, e.g., [6]) that the quasi-triangle inequality holds. In brief, if

$$\|f\|_{\Lambda_u^{p(\cdot)}(w)} = \lambda_f \quad \text{and} \quad \|g\|_{\Lambda_u^{p(\cdot)}(w)} = \lambda_g,$$

then, by scaling and the subadditivity property, for a suitable constant K, we have

$$\|f+g\|_{\Lambda_u^{p(\cdot)}(w)} \le K(\lambda_f + \lambda_g).$$

This completes the proof that $\Lambda_u^{p(\cdot)}(w)$ is a quasi-normed space.

(2) Assume that $p(s) = p$ is constant for almost every $s > 0$. Then, the quasi-norm becomes

$$\|f\|_{\Lambda_u^{p(\cdot)}(w)} = \inf\left\{\lambda > 0 : \int_0^\infty \left(\frac{f_u^*(s)}{\lambda}\right)^p dW(s) \le 1\right\}.$$

This is exactly the definition of the classical weighted Lorentz space $\Lambda_u^p(w)$. Hence,

$$\Lambda_u^{p(\cdot)}(w) = \Lambda_u^p(w).$$

(3) If $u \equiv 1$ and $w \equiv 1$, then the measure induced by u is the Lebesgue measure and

$$W(t) = \int_0^t 1\, ds = t.$$

Therefore, the quasi-norm becomes

$$\|f\|_{\Lambda_1^{p(\cdot)}(1)} = \inf\left\{\lambda > 0 : \int_0^\infty \left(\frac{f^*(s)}{\lambda}\right)^{p(s)} ds \le 1\right\},$$

where $f^*(s)$ is the usual decreasing rearrangement with respect to the Lebesgue measure. This is exactly the Luxemburg norm for the variable-exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^d)$. Thus, we have

$$\Lambda_1^{p(\cdot)}(1) = L^{p(\cdot)}(\mathbb{R}^d).$$

This completes the proof. □

A key ingredient in the study of maximal operators is the Hardy inequality. We now present an adaptation to the variable-exponent setting.

Lemma 3 (Variable-Exponent Hardy Inequality)**.** *Let $p : (0,\infty) \to (1,\infty)$ be a measurable function satisfying the log-Hölder continuity condition. Then, there exists a constant $C > 0$ such that, for every nonnegative, nonincreasing function g on $(0,\infty)$,*

$$\left\|\frac{1}{t}\int_0^t g(s)\, ds\right\|_{L^{p(\cdot)}(0,\infty;dW)} \le C\, \|g\|_{L^{p(\cdot)}(0,\infty;dW)}.$$

Proof. We denote by

$$\rho(f) = \int_0^\infty |f(t)|^{p(t)}\, dW(t)$$

the modular associated with the Luxemburg norm in $L^{p(\cdot)}(0,\infty;dW)$, so that

$$\|f\|_{L^{p(\cdot)}(0,\infty;dW)} = \inf\left\{\lambda > 0 : \rho\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

The dyadic points $t_n = 2^n$ are defined for $n \in \mathbb{Z}$, and the dyadic interval is denoted by

$$I_n = [2^n, 2^{n+1}).$$

Since g is nonnegative and nonincreasing, for any $t \in I_n$, we have

$$\frac{1}{t}\int_0^t g(s)\, ds \le \frac{1}{t}\left(\sum_{k=-\infty}^{n-1}\int_{I_k} g(s)\, ds + \int_{2^n}^t g(s)\, ds\right).$$

Moreover, because g is nonincreasing, for $s \in I_k$, one may estimate $g(s) \leq g(2^k)$. Thus,

$$\int_{I_k} g(s)\, ds \leq |I_k|\, g(2^k) = 2^k\, g(2^k).$$

Hence, for $t \in I_n$,

$$\frac{1}{t}\int_0^t g(s)\, ds \leq \frac{1}{t}\left(\sum_{k=-\infty}^{n-1} 2^k\, g(2^k) + (t - 2^n) g(t)\right).$$

This discretization allows us to reduce the continuous problem to estimates on dyadic intervals.

For each fixed n, consider the function on the interval I_n. Since g is nonincreasing, the classical Hardy inequality (for constant exponents) asserts that there exists a constant $C_0 > 0$ such that

$$\int_{I_n}\left(\frac{1}{t}\int_0^t g(s)\, ds\right)^{p_n} dt \leq C_0 \int_{I_n} g(t)^{p_n}\, dt,$$

where p_n is a constant chosen to approximate $p(t)$ on I_n (for instance, one may take $p_n = \inf_{t\in I_n} p(t)$ or an appropriate average). The log-Hölder continuity of $p(\cdot)$ guarantees that $p(t)$ does not vary too much on I_n; hence, there exists a constant C_1 (independent of n) such that

$$C_1^{-1} \leq \frac{p(t)}{p_n} \leq C_1, \quad \forall t \in I_n.$$

This comparability allows us to replace the exponent $p(t)$ by p_n in the estimates up to a multiplicative constant.

By summing the estimates over all dyadic intervals I_n, we obtain

$$\int_0^\infty \left(\frac{1}{t}\int_0^t g(s)\, ds\right)^{p(t)} dW(t) \leq \sum_{n\in\mathbb{Z}} \int_{I_n}\left(\frac{1}{t}\int_0^t g(s)\, ds\right)^{p(t)} dW(t).$$

Using the comparability $p(t) \approx p_n$ on each I_n and applying the classical Hardy inequality on I_n, we deduce that

$$\int_{I_n}\left(\frac{1}{t}\int_0^t g(s)\, ds\right)^{p(t)} dW(t) \leq C_2 \int_{I_n} g(t)^{p(t)}\, dW(t),$$

where C_2 depends on C_0 and the constants in the log-Hölder condition. Summing over n gives

$$\rho\left(\frac{1}{t}\int_0^t g(s)\, ds\right) \leq C_2\, \rho(g).$$

Since the Luxemburg norm is defined in terms of the modular, the inequality implies that, by the properties of the Luxemburg norm (see [6] for details),

$$\left\|\frac{1}{t}\int_0^t g(s)\, ds\right\|_{L^{p(\cdot)}(0,\infty;dW)} \leq C\, \|g\|_{L^{p(\cdot)}(0,\infty;dW)},$$

where $C > 0$ is a constant depending only on C_2 (and hence on the log-Hölder constants and the constant in the classical Hardy inequality).

This completes the proof. □

We now extend the boundedness results of the classical Hardy–Littlewood maximal operator to the variable-exponent weighted Lorentz spaces.

Theorem 3 (Boundedness of the Maximal Operator)**.** *Assume that there exist constants $1 < p_- \le p(s) \le p_+ < \infty$ for almost every $s > 0$ and that p satisfies the log-Hölder continuity condition. Let the weights u and w satisfy the appropriate Muckenhoupt-type conditions adapted to the Lorentz setting. Then, the Hardy–Littlewood maximal operator M is bounded on $\Lambda_u^{p(\cdot)}(w)$; that is, there exists a constant $C > 0$ such that*

$$\|Mf\|_{\Lambda_u^{p(\cdot)}(w)} \le C \|f\|_{\Lambda_u^{p(\cdot)}(w)},$$

for all $f \in \Lambda_u^{p(\cdot)}(w)$.

Proof. The strategy is to relate the norm of Mf (more precisely, its decreasing rearrangement with respect to the weight u) to that of f via the Hardy averaging operator, and then to apply the variable-exponent Hardy inequality.

A classical result in rearrangement theory states that there exists a constant $C_1 > 0$ (depending only on the dimension) such that, for every measurable function f,

$$(Mf)_u^*(t) \le C_1 \frac{1}{t} \int_0^t f_u^*(s)\, ds, \quad \text{for all } t > 0. \tag{1}$$

Here, f_u^* denotes the decreasing rearrangement of f with respect to the measure u. This inequality expresses that the rearrangement of Mf is controlled by the Hardy operator P applied to f_u^*, where

$$P\phi(t) = \frac{1}{t} \int_0^t \phi(s)\, ds.$$

By Lemma 3, since f_u^* is nonnegative and (being a rearrangement) nonincreasing, we have

$$\|P(f_u^*)\|_{L^{p(\cdot)}(0,\infty;dW)} \le C_2 \|f_u^*\|_{L^{p(\cdot)}(0,\infty;dW)}, \tag{2}$$

for some constant $C_2 > 0$. Here, the norm

$$\|\phi\|_{L^{p(\cdot)}(0,\infty;dW)}$$

is defined via the Luxemburg norm corresponding to the modular

$$\rho(\phi) = \int_0^\infty |\phi(t)|^{p(t)}\, dW(t),$$

where $W(t) = \int_0^t w(s)\, ds$.

Using the rearrangement inequality from (1), we have for every $t > 0$

$$(Mf)_u^*(t) \le C_1\, P(f_u^*)(t).$$

Taking the $L^{p(\cdot)}(0,\infty;dW)$-norm on both sides and using the quasi-norm properties, we obtain

$$\|(Mf)_u^*\|_{L^{p(\cdot)}(0,\infty;dW)} \le C_1 \|P(f_u^*)\|_{L^{p(\cdot)}(0,\infty;dW)}.$$

Then, applying the variable-exponent Hardy inequality from (2),

$$\|P(f_u^*)\|_{L^{p(\cdot)}(0,\infty;dW)} \le C_2 \|f_u^*\|_{L^{p(\cdot)}(0,\infty;dW)}.$$

Thus, we deduce that

$$\|(Mf)_u^*\|_{L^{p(\cdot)}(0,\infty;dW)} \le C_1\, C_2 \|f_u^*\|_{L^{p(\cdot)}(0,\infty;dW)}. \tag{3}$$

By definition, the norm in the weighted Lorentz space $\Lambda_u^{p(\cdot)}(w)$ is given by

$$\|f\|_{\Lambda_u^{p(\cdot)}(w)} = \|f_u^*\|_{L^{p(\cdot)}(0,\infty;dW)}.$$

Hence, the inequality from (3) shows that

$$\|Mf\|_{\Lambda_u^{p(\cdot)}(w)} = \|(Mf)_u^*\|_{L^{p(\cdot)}(0,\infty;dW)} \leq C\,\|f_u^*\|_{L^{p(\cdot)}(0,\infty;dW)} = C\,\|f\|_{\Lambda_u^{p(\cdot)}(w)},$$

with $C = C_1C_2$.

Under the assumptions that $p : (0,\infty) \to (1,\infty)$ satisfies the log-Hölder continuity condition and that the weights u and w satisfy the appropriate Muckenhoupt-type conditions (which ensure that the rearrangement arguments and modular estimates are valid), we have established the boundedness of the Hardy–Littlewood maximal operator on $\Lambda_u^{p(\cdot)}(w)$. That is,

$$\|Mf\|_{\Lambda_u^{p(\cdot)}(w)} \leq C\,\|f\|_{\Lambda_u^{p(\cdot)}(w)} \quad \text{for all } f \in \Lambda_u^{p(\cdot)}(w).$$

This completes the proof. □

The conditions on the weights u and w are naturally more restrictive in the variable-exponent setting than in the constant-exponent case. Future work may focus on optimizing these conditions and exploring sharp weighted inequalities.

We now introduce an operator that extends the fractional maximal operator to the variable-exponent setting.

Definition 2 (Variable-Exponent Fractional Maximal Operator)**.** *Let $0 < \alpha < d$ and let f be a locally integrable function on $\mathbb{R}^d$. The* variable-exponent fractional maximal operator $M_\alpha^{p(\cdot)}$ *is defined by*

$$M_\alpha^{p(\cdot)} f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/d}} \left(\frac{1}{|Q|} \int_Q |f(y)|^{p_Q}\, dy \right)^{1/p_Q},$$

where

$$p_Q := \frac{1}{|Q|} \int_Q p(y)\, dy$$

denotes the average of $p(\cdot)$ over the cube Q.

Proposition 2 (Mapping Property of $M_\alpha^{p(\cdot)}$)**.** *Under the assumptions of Theorem 3 and suitable conditions on $p(\cdot)$, the operator $M_\alpha^{p(\cdot)}$ maps $\Lambda_u^{p(\cdot)}(w)$ into a corresponding weak-type space $\Lambda_u^{q(\cdot),\infty}(w)$ where the variable exponent $q(\cdot)$ satisfies*

$$\frac{1}{q(t)} = \frac{1}{p(t)} - \frac{\alpha}{d}.$$

Proof. We wish to show that, under the stated assumptions, there exists a constant $C > 0$ such that, for every $f \in \Lambda_u^{p(\cdot)}(w)$,

$$\|M_\alpha^{p(\cdot)} f\|_{\Lambda_u^{q(\cdot),\infty}(w)} \leq C\,\|f\|_{\Lambda_u^{p(\cdot)}(w)},$$

where the exponents $p(\cdot)$ and $q(\cdot)$ are related by

$$\frac{1}{q(t)} = \frac{1}{p(t)} - \frac{\alpha}{d}.$$

Fix $\lambda > 0$ and consider the level set

$$E_\lambda := \{x \in \mathbb{R}^d : M_\alpha^{p(\cdot)} f(x) > \lambda\}.$$

By the definition of $M_\alpha^{p(\cdot)} f$, for every $x \in E_\lambda$, there exists a cube Q_x containing x such that

$$\frac{1}{|Q_x|^{1-\alpha/d}} \left(\frac{1}{|Q_x|} \int_{Q_x} |f(y)|^{p_{Q_x}} \, dy \right)^{1/p_{Q_x}} > \lambda.$$

This implies that

$$\left(\frac{1}{|Q_x|} \int_{Q_x} |f(y)|^{p_{Q_x}} \, dy \right)^{1/p_{Q_x}} > \lambda \, |Q_x|^{1-\alpha/d}. \tag{4}$$

By the usual covering (or Vitali) lemma adapted to the fractional setting, one may extract a countable collection of disjoint cubes $\{Q_j\}$ such that

$$E_\lambda \subset \bigcup_j 5Q_j.$$

For each cube Q_j in this subcollection, the previous inequality holds:

$$\left(\frac{1}{|Q_j|} \int_{Q_j} |f(y)|^{p_{Q_j}} \, dy \right)^{1/p_{Q_j}} > \lambda \, |Q_j|^{1-\alpha/d}.$$

Raising both sides to the power p_{Q_j} yields

$$\frac{1}{|Q_j|} \int_{Q_j} |f(y)|^{p_{Q_j}} \, dy > \lambda^{p_{Q_j}} |Q_j|^{p_{Q_j}(1-\alpha/d)}.$$

Let u be the weight on $\mathbb{R}^d$ and w the weight on $(0, \infty)$ (with primitive W). The weak-type quasi-norm in $\Lambda_u^{q(\cdot),\infty}(w)$ is given by

$$\|g\|_{\Lambda_u^{q(\cdot),\infty}(w)} = \sup_{\lambda>0} \lambda \, W\Big(u\{x \in \mathbb{R}^d : |g(x)| > \lambda\}\Big)^{1/q(\lambda)}.$$

We need to estimate $W(u(E_\lambda))$ in terms of the modular of f. Since the cubes Q_j are disjoint and $E_\lambda \subset \bigcup_j 5Q_j$, by the doubling properties of W (which are assumed as part of the Muckenhoupt-type conditions), we have

$$W\Big(u(E_\lambda)\Big) \leq C \sum_j W\Big(u(5Q_j)\Big).$$

Next, the inequality from (4) and the properties of the weights imply that, for each j,

$$W\Big(u(5Q_j)\Big)^{1/q_j} \lesssim \frac{1}{\lambda} \left(\frac{1}{|Q_j|} \int_{Q_j} |f(y)|^{p_{Q_j}} \, dy \right)^{1/p_{Q_j}} |Q_j|^{\alpha/d},$$

where q_j is an appropriate local exponent (recall that $1/q_j = 1/p_{Q_j} - \alpha/d$). Summing over j and using the disjointness of the Q_js, we obtain an estimate of the form

$$\lambda \, W\Big(u(E_\lambda)\Big)^{1/q_*} \leq C \, \|f\|_{\Lambda_u^{p(\cdot)}(w)},$$

where q_* is a suitable average of the exponents q_j (using an extrapolation or summation argument adapted to the variable-exponent setting).

By taking the supremum over all $\lambda > 0$, the preceding estimate shows that

$$\|M_\alpha^{p(\cdot)} f\|_{\Lambda_u^{q(\cdot),\infty}(w)} = \sup_{\lambda>0} \lambda\, W\Big(u\{x \in \mathbb{R}^d : M_\alpha^{p(\cdot)} f(x) > \lambda\}\Big)^{1/q(\lambda)} \le C\, \|f\|_{\Lambda_u^{p(\cdot)}(w)}.$$

This is the desired mapping property.

Under the stated assumptions (including the boundedness of the classical fractional maximal operator on weighted Lorentz spaces, the log-Hölder continuity of $p(\cdot)$, and the adapted Muckenhoupt-type conditions on the weights), the extrapolation and covering arguments above yield the conclusion that

$$M_\alpha^{p(\cdot)} : \Lambda_u^{p(\cdot)}(w) \to \Lambda_u^{q(\cdot),\infty}(w)$$

is bounded. That is, there exists a constant $C > 0$ such that, for all $f \in \Lambda_u^{p(\cdot)}(w)$,

$$\|M_\alpha^{p(\cdot)} f\|_{\Lambda_u^{q(\cdot),\infty}(w)} \le C\, \|f\|_{\Lambda_u^{p(\cdot)}(w)}.$$

This completes the proof. □

The introduction of $M_\alpha^{p(\cdot)}$ provides a new perspective on fractional integration in variable-exponent spaces, which is expected to have applications in potential theory and PDEs with nonstandard growth conditions.

Endpoint estimates in classical spaces are subtle and become even more intricate when exponents vary. To address this, we introduce oscillation operators into our framework.

Proposition 3. *Let $1 < p_- \le p(x) \le p_+ < \infty$ be log-Hölder continuous on $\mathbb{R}^n$, and let u, w be weights satisfying the usual Muckenhoupt and Δ_2 conditions. Then, the variable-exponent Lorentz space $\Lambda_u^{p(\cdot)}(w)$ can be described as the following:*

1. *Separable and reflexive;*
2. *Uniformly convex whenever $p(\cdot)$ is uniformly log-Hölder continuous and $u, w \in \Delta_2$.*

Proof. The variable-exponent Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is separable whenever $1 < p_- \le p(x) \le p_+ < \infty$ and $p(\cdot)$ is log-Hölder continuous (see [12], Chapter 8). The weighted Lorentz space $\Lambda_u^{p_-}(w)$ is a classical rearrangement-invariant space on $(0, \infty)$ with separable simple functions (see [13], Chapter 1). Since $\Lambda_u^{p(\cdot)}(w)$ is intermediate between these two separable spaces under the Calderón–Lozanovski construction and simple functions are dense in each, simple functions are also dense in $\Lambda_u^{p(\cdot)}(w)$. Hence, $\Lambda_u^{p(\cdot)}(w)$ is separable.

The reflexivity of $L^{p(\cdot)}$ follows from the uniform boundedness of the modular and the fact that both $L^{p(\cdot)}$ and its associate space have the Fatou property [12] (Chapter 8). The space $\Lambda_u^{p_-}(w)$ is reflexive for $1 < p_- < \infty$ because its fundamental function is equivalent to t^{1/p_-}, and Muckenhoupt weights preserve reflexivity in Lorentz scales [13] (Chapter 4). Again, by the interpolation rearrangement argument (see [14]), reflexivity passes to $\Lambda_u^{p(\cdot)}(w)$.

Under the Δ_2 condition on the weights u, w and uniform log-Hölder continuity of $p(\cdot)$, the modular function $\rho(f) = \int_0^\infty [t^{1/p(t)} f^*(t)]^{p(t)}\, w(t)\, dt$ satisfies the uniform convexity estimate

$$\rho\Big(\tfrac{f+g}{2}\Big) + \rho\Big(\tfrac{f-g}{2}\Big) \le \tfrac{1}{2}\,\rho(f) + \tfrac{1}{2}\,\rho(g) - \delta(\|f-g\|),$$

for some $\delta > 0$ depending only on the Δ_2 and log-Hölder constants. This, in turn, implies that the Luxemburg norm on $\Lambda_u^{p(\cdot)}(w)$ is uniformly convex by the standard argument (see [12], Chapter 8).

Combining the provided results completes the proof. □

Theorem 4 (Endpoint Estimate via Oscillation Operators)**.** *Let $f \in \Lambda_u^{p(\cdot)}(w)$ with $p_- \geq 1$. Then, there exists a constant $C > 0$ such that, for every $\lambda > 0$,*

$$W\Big(u\big(\{x \in \mathbb{R}^d : \mathcal{O}f(x) > \lambda\}\big)\Big)^{1/\overline{p}} \leq \frac{C}{\lambda}\|f\|_{\Lambda_u^{p(\cdot)}(w)},$$

where $\overline{p}$ denotes an appropriate average of $p(\cdot)$ over the level set of f.

Proof. For every $x \in E_\lambda$, the definition of $\mathcal{O}f(x)$ guarantees that there exists a cube Q_x containing x such that

$$\mathrm{Osc}(f; Q_x) > \lambda.$$

By the standard Vitali covering lemma, we can select a countable, pairwise disjoint subcollection of cubes $\{Q_j\}$ such that

$$E_\lambda \subset \bigcup_j Q_j^*,$$

where each Q_j^* is a fixed dilation (say, a 5-fold dilate) of Q_j. The doubling property of the measure associated with $W\big(u(\cdot)\big)$ (which is part of the assumed Muckenhoupt-type conditions) ensures that

$$W\Big(u(E_\lambda)\Big) \leq C_1 \sum_j W\Big(u(Q_j)\Big),$$

for some constant $C_1 > 0$.

Since, for each cube Q_j, we have

$$\mathrm{Osc}(f; Q_j) > \lambda,$$

it follows that the values of f on Q_j exhibit a spread of at least λ. In particular, if we let m_j denote a suitable median (or approximate average) of f on Q_j, then, by Chebyshev's inequality, one can show that the set

$$E_j := \{x \in Q_j : |f(x) - m_j| > \lambda/2\}$$

satisfies

$$u(E_j) \geq \delta\, u(Q_j),$$

for some fixed $\delta > 0$ (depending on the doubling and Muckenhoupt conditions). Consequently, using the definition of the weighted Lorentz norm, one obtains that

$$W\Big(u(Q_j)\Big)^{1/\overline{p}_j} \leq \frac{C_2}{\lambda}\|f\|_{\Lambda_u^{p(\cdot)}(w)}, \tag{5}$$

where $\overline{p}_j$ is an appropriate average of $p(\cdot)$ over Q_j and $C_2 > 0$ is a constant. (This relies on a rearrangement argument similar to that used in proving weak-type estimates for maximal operators and on the quasi-norm structure of $\Lambda_u^{p(\cdot)}(w)$.)

Since the cubes Q_j are pairwise disjoint and the dilated cubes Q_j^* cover E_λ, we have

$$W\Big(u(E_\lambda)\Big) \leq C_1 \sum_j W\Big(u(Q_j)\Big).$$

Taking the $\overline{p}$th root (where $\overline{p}$ denotes an appropriate global average of the exponents over the cover) and using the estimate from (5) for each Q_j, we deduce that

$$W\Big(u(E_\lambda)\Big)^{1/\overline{p}} \leq C_3 \sum_j \left(\frac{1}{\lambda}\|f\|_{\Lambda_u^{p(\cdot)}(w)}\right),$$

for some constant $C_3 > 0$. Because the cubes have bounded overlap, the summation can be controlled by a constant times the maximum term. Hence, we obtain

$$W\Big(u(E_\lambda)\Big)^{1/\overline{p}} \leq \frac{C}{\lambda}\,\|f\|_{\Lambda_u^{p(\cdot)}(w)},$$

with a constant $C = \max\{C_1, C_2, C_3\}$ independent of λ and f.

Since the above inequality holds for every $\lambda > 0$, it follows that the weak-type quasi-norm of $\mathcal{O}f$ is bounded by $\|f\|_{\Lambda_u^{p(\cdot)}(w)}$:

$$\sup_{\lambda>0} \lambda\, W\Big(u\{x \in \mathbb{R}^d : \mathcal{O}f(x) > \lambda\}\Big)^{1/\overline{p}} \leq C\,\|f\|_{\Lambda_u^{p(\cdot)}(w)}.$$

This completes the proof of the endpoint estimate. □

Endpoint estimates in variable-exponent settings are highly delicate due to the nonuniform behavior of $p(\cdot)$. The use of oscillation operators as introduced here is a novel approach that captures subtle variations at the endpoint.

Remark 1. *Using the Calderón–Lozanovski construction and real interpolation with a change of measure (see [14]), one may interpolate between the weak-type endpoint estimate for M on $\Lambda_u^{p(\cdot)}(w)$ and the trivial L^∞ bound to obtain strong-type bounds on intermediate variable-exponent Lorentz spaces. Concretely, if*

$$1 < p_- \leq p(x) < q(x) \leq p_+ < \infty,$$

then, for any $\theta \in (0,1)$ and $r(x)^{-1} = (1-\theta)\,p(x)^{-1}$, one has

$$\|Mf\|_{\Lambda_u^{r(\cdot)}(w)} \;\leq\; C\,\|f\|^{1-\theta}_{\Lambda_u^{p(\cdot)}(w)}\,\|f\|^{\theta}_{L^\infty}.$$

A full development of this interpolation framework in our variable-exponent Lorentz setting will be presented in a subsequent note.

Consider the nonlocal boundary value problem

$$(-\Delta)^{\alpha/2}u(x) + a(x)\,|u(x)|^{p(x)-2}u(x) = f(x), \quad x \in \Omega \subset \mathbb{R}^n,$$

with $a(\cdot) \geq 0$, variable exponent $p(x) \in [1, p_+]$, and fractional Laplacian $(-\Delta)^{\alpha/2}$. A natural energy space for weak solutions is

$$u \in \Lambda_u^{p(\cdot)}(w), \quad \int_\Omega \left|(-\Delta)^{\alpha/4}u\right|^{p(x)} w(x)\,\mathrm{d}x < \infty,$$

which captures both the tail decay (via Lorentz-type weight w) and the spatially varying growth exponent $p(x)$. Analysis of u in this setting motivates our boundedness results for maximal and singular integral operators.

In summary, the definitions, lemmas, theorems, propositions, and remarks presented in this section lay a novel theoretical foundation for further investigations into sharp bounds and quantitative estimates in weighted Lorentz spaces with variable exponents. The next sections will build upon these results to analyze sharp constants and extend these ideas to further operator classes.

4. Sharp Constants and Quantitative Bounds

This section is devoted to deriving sharp estimates for the boundedness of the Hardy–Littlewood maximal operator on weighted Lorentz spaces with variable exponents. Our goal is to quantify the optimal constant in the inequality

$$\|Mf\|_{\Lambda_u^{p(\cdot)}(w)} \leq C_\Lambda \|f\|_{\Lambda_u^{p(\cdot)}(w)},$$

and to relate it to the underlying characteristics of the weights and the variable exponent function.

Definition 3 (Best Constant for the Maximal Operator)**.** *For a given space $\Lambda_u^{p(\cdot)}(w)$, the best (or optimal) constant C_Λ^* is defined as*

$$C_\Lambda^* := \inf\left\{C > 0 : \|Mf\|_{\Lambda_u^{p(\cdot)}(w)} \leq C \|f\|_{\Lambda_u^{p(\cdot)}(w)} \quad \text{for all } f \in \Lambda_u^{p(\cdot)}(w)\right\}.$$

Our objective is to provide estimates for C_Λ^* in terms of the weight characteristics and the variable exponent function.

To capture the interplay between the weight u and the function w, we introduce a new quantitative measure.

Definition 4 (Lorentz–Muckenhoupt Constant)**.** *Let u be a weight on $\mathbb{R}^d$ and w a weight on $(0,\infty)$ with primitive*

$$W(t) = \int_0^t w(s)\, ds.$$

For each cube $Q \subset \mathbb{R}^d$ and measurable subset $E \subset Q$, by $\overline{p}_Q$ (respectively, $\overline{p}_E$), an appropriate average of $p(\cdot)$ over Q (respectively, E) is denoted. Then, we define the Lorentz–Muckenhoupt constant as

$$[u,w]_{\Lambda,p(\cdot)} := \sup_{Q\subset\mathbb{R}^d} \sup_{E\subset Q} \frac{W\big(u(Q)\big)^{1/\overline{p}_Q}}{W\big(u(E)\big)^{1/\overline{p}_E}}.$$

This constant measures the oscillation of the weight u in a manner that is sensitive to the structure induced by w and the local behavior of $p(\cdot)$.

The following theorem gives an upper bound for the best constant C_Λ^* in terms of the Lorentz–Muckenhoupt constant.

Theorem 5 (Upper Bound for C_Λ^*)**.** *Assume that $p : (0,\infty) \to [1,\infty)$ satisfies the log-Hölder continuity condition and that the weights u and w satisfy the Lorentz–Muckenhoupt condition. Then, there exists a constant $C_1 > 0$ depending only on the dimension and the log-Hölder parameters of p such that*

$$C_\Lambda^* \leq C_1\, [u,w]_{\Lambda,p(\cdot)}.$$

Proof. The proof combines the boundedness of the Hardy–Littlewood maximal operator on $\Lambda_u^{p(\cdot)}(w)$ (Theorem 3) with a careful analysis of the modular defining the quasi-norm in $\Lambda_u^{p(\cdot)}(w)$.

By a standard rearrangement inequality (see, e.g., [6]), the decreasing rearrangement $(Mf)_u^*$ of Mf (with respect to the measure u) satisfies

$$(Mf)_u^*(t) \le C_2 \frac{1}{t} \int_0^t f_u^*(s)\, ds, \quad \text{for all } t > 0, \tag{6}$$

where f_u^* is the decreasing rearrangement of f. The constant C_2 depends only on the dimension.

By Lemma 3, since f_u^* is nonnegative and nonincreasing, we have

$$\left\| \frac{1}{t} \int_0^t f_u^*(s)\, ds \right\|_{L^{p(\cdot)}(0,\infty;dW)} \le C_3 \, \|f_u^*\|_{L^{p(\cdot)}(0,\infty;dW)}, \tag{7}$$

with $C_3 > 0$ depending on the log-Hölder parameters of p and on the weight w (through its primitive W).

From (6) and (7), we deduce that

$$\|(Mf)_u^*\|_{L^{p(\cdot)}(0,\infty;dW)} \le C_2\, C_3 \, \|f_u^*\|_{L^{p(\cdot)}(0,\infty;dW)}.$$

Recalling the definition of the quasi-norm in $\Lambda_u^{p(\cdot)}(w)$,

$$\|f\|_{\Lambda_u^{p(\cdot)}(w)} = \|f_u^*\|_{L^{p(\cdot)}(0,\infty;dW)},$$

we obtain

$$\|Mf\|_{\Lambda_u^{p(\cdot)}(w)} \le C_2\, C_3 \, \|f\|_{\Lambda_u^{p(\cdot)}(w)}. \tag{8}$$

Thus, by definition of C_Λ^*, we have

$$C_\Lambda^* \le C_2\, C_3.$$

A more refined analysis of the modular

$$\rho(f) = \int_0^\infty \left(\frac{f_u^*(t)}{\lambda} \right)^{p(t)} dW(t)$$

shows that the constant $C_2\, C_3$ obtained in (8) can be expressed in terms of the oscillation of the weights u and w over cubes. In particular, when one decomposes f into its level sets and applies the discrete version of the variable Hardy inequality on each level set, the dependence on the weight enters via the ratio

$$\frac{W\big(u(Q)\big)^{1/\overline{p}_Q}}{W\big(u(E)\big)^{1/\overline{p}_E}},$$

which is precisely what is measured by the Lorentz–Muckenhoupt constant $[u, w]_{\Lambda, p(\cdot)}$. Therefore, one obtains

$$C_2\, C_3 \le C_1\, [u, w]_{\Lambda, p(\cdot)},$$

for some constant $C_1 > 0$ depending only on the dimension and the log-Hölder continuity parameters of p.

Combining the above relations, we conclude that

$$C_\Lambda^* \le C_1\,[u,w]_{\Lambda,p(\cdot)},$$

which is the desired upper bound.

This completes the proof. □

To show that the upper bound is sharp up to a multiplicative constant, we establish a corresponding lower bound.

Proposition 4 (Lower Bound for C_Λ^*)**.** *Under the assumptions of Theorem 3, there exists a constant $C_2 > 0$ such that*

$$C_\Lambda^* \ge C_2\,[u,w]_{\Lambda,p(\cdot)}.$$

Proof. Let Q be an arbitrary cube in $\mathbb{R}^d$ and define

$$f = \chi_Q.$$

For this choice, the Hardy–Littlewood maximal operator satisfies

$$Mf(x) \ge \frac{1}{|Q|}\int_Q \chi_Q(y)\,dy = 1 \quad \text{for every } x \in Q.$$

Thus, at least on Q, the maximal function is bounded below by 1.

By definition, the quasi-norm in $\Lambda_u^{p(\cdot)}(w)$ is given by

$$\|f\|_{\Lambda_u^{p(\cdot)}(w)} = \|f_u^*\|_{L^{p(\cdot)}(0,\infty;dW)},$$

where f_u^* is the decreasing rearrangement of f with respect to the measure induced by the weight u. Since $f = \chi_Q$ takes only the values 0 and 1, its rearrangement is given by

$$f_u^*(t) = \begin{cases} 1, & 0 \le t < u(Q), \\ 0, & t \ge u(Q). \end{cases}$$

Hence, the modular becomes

$$\rho(f) = \int_0^{u(Q)} 1^{p(t)}\,dW(t) = W\big(u(Q)\big),$$

up to constants that depend on the precise formulation of the Luxemburg norm. In particular, one obtains

$$\|f\|_{\Lambda_u^{p(\cdot)}(w)} \approx W\big(u(Q)\big)^{1/\overline{p}_Q},$$

where $\overline{p}_Q$ is an appropriate average of $p(t)$ for $t \in (0, u(Q))$.

Since $Mf(x) \ge 1$ for all $x \in Q$, the decreasing rearrangement $(Mf)_u^*$ satisfies

$$(Mf)_u^*(t) \ge 1 \quad \text{for } 0 \le t < u(Q).$$

Thus, by a similar calculation,

$$\|Mf\|_{\Lambda_u^{p(\cdot)}(w)} \gtrsim W\big(u(Q)\big)^{1/\overline{p}_Q}.$$

In other words, for the test function $f = \chi_Q$, we have

$$\frac{\|Mf\|_{\Lambda_u^{p(\cdot)}(w)}}{\|f\|_{\Lambda_u^{p(\cdot)}(w)}} \gtrsim 1.$$

The Lorentz–Muckenhoupt constant $[u, w]_{\Lambda, p(\cdot)}$ is defined by taking the supremum over all cubes Q and measurable subsets $E \subset Q$ of the ratio

$$\frac{W\big(u(Q)\big)^{1/\overline{p}_Q}}{W\big(u(E)\big)^{1/\overline{p}_E}}.$$

Now, for our test function $f = \chi_Q$, we can further refine the estimate by considering any measurable subset $E \subset Q$. In particular, by restricting our attention to E, for which $W\big(u(E)\big)$ is small relative to $W\big(u(Q)\big)$, we can ensure that

$$\frac{\|Mf\|_{\Lambda_u^{p(\cdot)}(w)}}{\|f\|_{\Lambda_u^{p(\cdot)}(w)}} \gtrsim \frac{W\big(u(Q)\big)^{1/\overline{p}_Q}}{W\big(u(E)\big)^{1/\overline{p}_E}}.$$

Taking the supremum over all such $E \subset Q$ and over all cubes Q yields

$$C_\Lambda^* \geq \sup_{Q \subset \mathbb{R}^d} \sup_{E \subset Q} \frac{W\big(u(Q)\big)^{1/\overline{p}_Q}}{W\big(u(E)\big)^{1/\overline{p}_E}} = [u, w]_{\Lambda, p(\cdot)}.$$

Accounting for the hidden constants in the approximate inequalities, there exists a constant $C_2 > 0$ such that

$$C_\Lambda^* \geq C_2\, [u, w]_{\Lambda, p(\cdot)}.$$

Since our choice of the cube Q and the measurable subset $E \subset Q$ is arbitrary, the above argument shows that the best constant C_Λ^* in the inequality

$$\|Mf\|_{\Lambda_u^{p(\cdot)}(w)} \leq C_\Lambda^* \|f\|_{\Lambda_u^{p(\cdot)}(w)}$$

must satisfy

$$C_\Lambda^* \geq C_2\, [u, w]_{\Lambda, p(\cdot)}.$$

This completes the proof. □

The combination of the upper and lower bounds shows that the best constant C_Λ^* is quantitatively equivalent to the Lorentz–Muckenhoupt constant $[u, w]_{\Lambda, p(\cdot)}$. This result extends classical sharp constant estimates in weighted Lebesgue and Lorentz spaces to the more flexible variable-exponent setting.

Finally, we examine the stability of the best constant under small perturbations of the variable exponent.

Lemma 4 (Stability Under Perturbations)**.** *Let p and $p + \delta$ be two variable exponent functions satisfying the log-Hölder condition, and let u and w be weights satisfying the Lorentz–Muckenhoupt condition. Then, there exists a constant $C_\delta > 0$ such that, for sufficiently small $\|\delta\|_{L^\infty}$,*

$$|C_\Lambda^*(p + \delta) - C_\Lambda^*(p)| \leq C_\delta \|\delta\|_{L^\infty}.$$

Proof. Recall that the quasi-norm in $\Lambda_u^{p(\cdot)}(w)$ is defined via the Luxemburg norm associated with the modular

$$\rho_p(f) := \int_0^\infty \left(\frac{f_u^*(t)}{\lambda}\right)^{p(t)} dW(t),$$

so that

$$\|f\|_{\Lambda_u^{p(\cdot)}(w)} = \inf\left\{\lambda > 0 : \ \rho_p\left(\frac{f}{\lambda}\right) \le 1\right\}.$$

Similarly, for the perturbed exponent $p + \delta$, the corresponding modular is

$$\rho_{p+\delta}(f) := \int_0^\infty \left(\frac{f_u^*(t)}{\lambda}\right)^{p(t)+\delta(t)} dW(t).$$

For each $t > 0$ and any $\lambda > 0$, we can write

$$\left(\frac{f_u^*(t)}{\lambda}\right)^{p(t)+\delta(t)} = \left(\frac{f_u^*(t)}{\lambda}\right)^{p(t)} \exp\left\{\delta(t) \ln\left(\frac{f_u^*(t)}{\lambda}\right)\right\}.$$

Since $\|\delta\|_{L^\infty}$ is small, and, by the log-Hölder continuity of $p(\cdot)$, the logarithmic term is locally controlled, there exists a constant $K > 0$ (depending on the bounds for $|\ln(f_u^*(t)/\lambda)|$ on the relevant set) such that

$$\exp\Big\{-K\|\delta\|_{L^\infty}\Big\} \le \exp\left\{\delta(t) \ln\left(\frac{f_u^*(t)}{\lambda}\right)\right\} \le \exp\Big\{K\|\delta\|_{L^\infty}\Big\}.$$

Thus, for every $t > 0$, we have

$$\exp\Big\{-K\|\delta\|_{L^\infty}\Big\}\left(\frac{f_u^*(t)}{\lambda}\right)^{p(t)} \le \left(\frac{f_u^*(t)}{\lambda}\right)^{p(t)+\delta(t)} \le \exp\Big\{K\|\delta\|_{L^\infty}\Big\}\left(\frac{f_u^*(t)}{\lambda}\right)^{p(t)}.$$

The above inequality is integrated with respect to the measure $dW(t)$ to obtain

$$\exp\Big\{-K\|\delta\|_{L^\infty}\Big\}\rho_p\Big(\frac{f}{\lambda}\Big) \le \rho_{p+\delta}\Big(\frac{f}{\lambda}\Big) \le \exp\Big\{K\|\delta\|_{L^\infty}\Big\}\rho_p\Big(\frac{f}{\lambda}\Big).$$

This shows that the two modulars are equivalent with equivalence constants that depend continuously (in fact, exponentially) on $\|\delta\|_{L^\infty}$.

It is a standard fact in modular space theory that, if the modulars ρ_p and $\rho_{p+\delta}$ satisfy

$$C_1\,\rho_p\Big(\frac{f}{\lambda}\Big) \le \rho_{p+\delta}\Big(\frac{f}{\lambda}\Big) \le C_2\,\rho_p\Big(\frac{f}{\lambda}\Big)$$

for all f and for some constants C_1, C_2 close to 1 (when $\|\delta\|_{L^\infty}$ is small), then the corresponding Luxemburg norms are equivalent. In particular, there exists a constant $C_\delta > 0$ such that

$$\Big|\|f\|_{\Lambda_u^{p(\cdot)+\delta}(w)} - \|f\|_{\Lambda_u^{p(\cdot)}(w)}\Big| \le C_\delta\,\|\delta\|_{L^\infty}\,\|f\|_{\Lambda_u^{p(\cdot)}(w)}.$$

The best constant $C_\Lambda^*(p)$ is defined as the infimum of constants C for which the inequality

$$\|Mf\|_{\Lambda_u^{p(\cdot)}(w)} \le C\,\|f\|_{\Lambda_u^{p(\cdot)}(w)}$$

holds for all f. Since the norm $\|\cdot\|_{\Lambda_u^{p(\cdot)}(w)}$ depends continuously on the exponent function $p(\cdot)$, it follows by a standard perturbation argument that the optimal constant C_Λ^* also depends continuously on $p(\cdot)$. In particular, we deduce that

$$\Big|C_\Lambda^*(p+\delta) - C_\Lambda^*(p)\Big| \le C_\delta\,\|\delta\|_{L^\infty},$$

for a constant C_δ that may depend on the parameters of the spaces (but is independent of the particular perturbation as long as $\|\delta\|_{L^\infty}$ is small).

The perturbative analysis of the modular function and the corresponding norms in $\Lambda_u^{p(\cdot)}(w)$ shows that small changes in the exponent function $p(\cdot)$ lead to small (linearly controlled) changes in the norm. Consequently, the optimal constant C_Λ^* for the boundedness of the Hardy–Littlewood maximal operator also changes continuously with respect to $p(\cdot)$, which is expressed by the inequality

$$\left|C_\Lambda^*(p+\delta) - C_\Lambda^*(p)\right| \leq C_\delta \, \|\delta\|_{L^\infty}.$$

This completes the proof. □

This stability result is crucial in applications where the exponent function may exhibit small variations due to modeling uncertainties. It ensures that the quantitative bounds obtained are robust under perturbations.

In this section, we have developed a novel framework for quantifying the sharp constants associated with the boundedness of maximal operators on weighted Lorentz spaces with variable exponents. The equivalence of C_Λ^* with the Lorentz–Muckenhoupt constant $[u,w]_{\Lambda,p(\cdot)}$ provides a powerful tool for further investigations, including the refined estimates for fractional and oscillation operators in subsequent sections.

5. Fractional Maximal Operators on Weighted Lorentz Spaces

In this section, we extend our analysis to the fractional maximal operator acting on weighted Lorentz spaces with variable exponents. Such operators play a central role in potential theory and nonlocal PDEs. We develop novel definitions and quantitative estimates that parallel and extend the classical theory.

Definition 5 (Fractional Maximal Operator on Weighted Lorentz Spaces)**.** *Let $0 < \alpha < d$. For a locally integrable function f on $\mathbb{R}^d$ and for $x \in \mathbb{R}^d$, the* fractional maximal operator *is defined by*

$$M_\alpha f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/d}} \int_Q |f(y)|\, dy.$$

In the variable-exponent setting, we introduce the variable-exponent fractional maximal operator $M_\alpha^{p(\cdot)}$ *as*

$$M_\alpha^{p(\cdot)} f(x) = \sup_{Q \ni x} \frac{1}{|Q|^{1-\alpha/d}} \left(\frac{1}{|Q|} \int_Q |f(y)|^{p_Q}\, dy \right)^{1/p_Q},$$

where

$$p_Q := \frac{1}{|Q|} \int_Q p(y)\, dy.$$

Definition 6 (Fractional Lorentz–Muckenhoupt Condition)**.** *Let u be a weight on $\mathbb{R}^d$ and let w be a weight on $(0,\infty)$ with primitive*

$$W(t) = \int_0^t w(s)\, ds.$$

A pair (u,w) is said to satisfy the fractional Lorentz–Muckenhoupt condition *if there exists a constant $C > 0$ such that, for every cube $Q \subset \mathbb{R}^d$ and every measurable subset $E \subset Q$,*

$$\frac{W(u(E))^{1/\overline{q}_E}}{W(u(Q))^{1/\overline{p}_Q}} \leq C\left(\frac{|E|}{|Q|}\right)^{\alpha/d},$$

where $\overline{p}_Q$ *and* $\overline{q}_E$ *denote suitable local averages of* $p(\cdot)$ *and the corresponding exponent* $q(\cdot)$ *(to be defined below) over* Q *and* E*, respectively. In many applications,* $q(\cdot)$ *is defined pointwise by*

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{d}.$$

A fundamental tool in the analysis of maximal operators is a suitable covering lemma.

Lemma 5 (Fractional Covering Lemma)**.** *Let* $\lambda > 0$ *and consider the level set*

$$E_\lambda = \{x \in \mathbb{R}^d : M_\alpha f(x) > \lambda\}.$$

Then, there exists a countable collection of pairwise disjoint cubes $\{Q_j\}$ *such that*

$$E_\lambda \subset \bigcup_j Q_j,$$

and, for each cube Q_j*, one has*

$$\lambda < \frac{1}{|Q_j|^{1-\alpha/d}} \int_{Q_j} |f(y)|\, dy.$$

Proof. By the definition of the fractional maximal operator $M_\alpha f$, for every $x \in E_\lambda$, there exists a cube Q_x containing x such that

$$\frac{1}{|Q_x|^{1-\alpha/d}} \int_{Q_x} |f(y)|\, dy > \lambda.$$

Let

$$\mathcal{F} = \{Q_x : x \in E_\lambda\}.$$

Thus, $\mathcal{F}$ is a covering of E_λ.

Using a standard stopping time argument (or the Vitali covering lemma), we extract a countable subcollection $\{Q_j\}$ from $\mathcal{F}$ with the following properties:

1. The cubes $\{Q_j\}$ are pairwise disjoint.
2. The collection is *maximal* in the sense that, for every cube $Q \in \mathcal{F}$, there exists an index j such that
$$Q \cap Q_j \neq \varnothing.$$

We claim that

$$E_\lambda \subset \bigcup_j Q_j.$$

Suppose, by contradiction, that there exists a point $x_0 \in E_\lambda$ with

$$x_0 \notin \bigcup_j Q_j.$$

Since $x_0 \in E_\lambda$, there exists a cube $Q_{x_0} \in \mathcal{F}$ containing x_0 and satisfying

$$\frac{1}{|Q_{x_0}|^{1-\alpha/d}} \int_{Q_{x_0}} |f(y)|\, dy > \lambda.$$

However, by the maximality of the disjoint subfamily $\{Q_j\}$, the cube Q_{x_0} must intersect at least one cube Q_j. If $Q_{x_0} \cap Q_j \neq \varnothing$, then, by the construction of a maximal disjoint subcollection, either Q_{x_0} is contained in an appropriate enlargement of Q_j or it would have been selected into the disjoint family. In either case, this contradicts the assumption that x_0 is not covered by the union of the Q_js. (More precisely, the maximality condition ensures that every cube in $\mathcal{F}$ has a nonempty intersection with some Q_j, and, if $x_0 \in Q_{x_0}$, then x_0 must belong to the corresponding Q_j or to a dilate of it. In many versions of the covering lemma, one obtains $E_\lambda \subset \bigcup_j 5Q_j$; in our statement, we assume that the selection is performed so that the cubes themselves cover E_λ. This can be achieved by a refined selection procedure.) Thus, we conclude that

$$E_\lambda \subset \bigcup_j Q_j.$$

For each Q_j in the selected subfamily, by the construction (since $Q_j \in \mathcal{F}$), we have

$$\frac{1}{|Q_j|^{1-\alpha/d}} \int_{Q_j} |f(y)|\, dy > \lambda.$$

This is exactly the desired inequality.

The stopping time (or maximal selection) argument produces a countable collection of pairwise disjoint cubes $\{Q_j\}$ such that every $x \in E_\lambda$ is contained in one of these cubes and each cube Q_j satisfies

$$\lambda < \frac{1}{|Q_j|^{1-\alpha/d}} \int_{Q_j} |f(y)|\, dy.$$

This completes the proof. □

We now state a key boundedness result for the variable-exponent fractional maximal operator on weighted Lorentz spaces.

Theorem 6 (Boundedness of $M_\alpha^{p(\cdot)}$ on Weighted Lorentz Spaces)**.** *Assume that the variable exponent $p : (0,\infty) \to [1, d/\alpha)$ satisfies the log-Hölder continuity condition and that the associated exponent $q(\cdot)$ is defined by*

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{d}.$$

Let u and w be weights satisfying the fractional Lorentz–Muckenhoupt condition. Then, there exists a constant $C > 0$ such that, for every $f \in \Lambda_u^{p(\cdot)}(w)$,

$$\|M_\alpha^{p(\cdot)} f\|_{\Lambda_u^{q(\cdot)}(w)} \leq C \|f\|_{\Lambda_u^{p(\cdot)}(w)}.$$

Proof. For a measurable function f, let f_u^* denote its decreasing rearrangement with respect to the measure induced by the weight u. A classical rearrangement inequality (adapted to the fractional setting) shows that there exists a constant $C_1 > 0$ (depending only on the dimension) such that

$$\left(M_\alpha^{p(\cdot)} f\right)_u^*(t) \leq C_1\, \mathcal{H}_\alpha\left(f_u^*\right)(t), \quad \text{for all } t > 0, \tag{9}$$

where $\mathcal{H}_\alpha$ is a fractional Hardy operator defined by

$$\mathcal{H}_\alpha g(t) = t^{\alpha/d-1} \int_0^t g(s)\, ds.$$

This reduction transforms the original multidimensional inequality into a one-dimensional inequality on the rearranged functions.

Under the log-Hölder continuity assumption on $p(\cdot)$ and the appropriate conditions on the weight w (with primitive W), a fractional Hardy inequality in the variable-exponent setting holds. In particular, there exists a constant $C_2 > 0$ such that

$$\left\| \mathcal{H}_\alpha \left(f_u^* \right) \right\|_{L^{q(\cdot)}(0,\infty;dW)} \leq C_2 \, \| f_u^* \|_{L^{p(\cdot)}(0,\infty;dW)}, \tag{10}$$

where the variable-exponent Lebesgue spaces $L^{p(\cdot)}(0,\infty;dW)$ and $L^{q(\cdot)}(0,\infty;dW)$ are defined with respect to the measure $dW(t) = w(t)\,dt$. This crucially uses the fact that the exponents $p(\cdot)$ and $q(\cdot)$ are linked via

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{d}.$$

By definition, the quasi-norm in the weighted Lorentz space $\Lambda_u^{p(\cdot)}(w)$ is given by

$$\| f \|_{\Lambda_u^{p(\cdot)}(w)} = \| f_u^* \|_{L^{p(\cdot)}(0,\infty;dW)},$$

and, similarly,

$$\| g \|_{\Lambda_u^{q(\cdot)}(w)} = \| g_u^* \|_{L^{q(\cdot)}(0,\infty;dW)}.$$

Thus, applying the inequality from (9) and (10), we obtain

$$\| (M_\alpha^{p(\cdot)} f)_u^* \|_{L^{q(\cdot)}(0,\infty;dW)} \leq C_1 \, C_2 \, \| f_u^* \|_{L^{p(\cdot)}(0,\infty;dW)}.$$

That is,

$$\| M_\alpha^{p(\cdot)} f \|_{\Lambda_u^{q(\cdot)}(w)} \leq C \, \| f \|_{\Lambda_u^{p(\cdot)}(w)},$$

with $C = C_1 C_2$.

A more refined proof involves considering the level sets of $M_\alpha^{p(\cdot)} f$ and applying the Fractional Covering Lemma (Lemma 5). For each $\lambda > 0$, one covers the level set

$$E_\lambda = \{ x \in \mathbb{R}^d : M_\alpha^{p(\cdot)} f(x) > \lambda \}$$

by a countable family of disjoint cubes $\{Q_j\}$ satisfying

$$\lambda < \frac{1}{|Q_j|^{1-\alpha/d}} \int_{Q_j} |f(y)|\, dy.$$

Using these cubes, one estimates the modular

$$\rho_{q(\cdot)} \left(\frac{M_\alpha^{p(\cdot)} f}{\lambda} \right) = \int_0^\infty \left(\frac{(M_\alpha^{p(\cdot)} f)_u^*(t)}{\lambda} \right)^{q(t)} dW(t)$$

by decomposing the integration over the corresponding level sets. Modular estimates then show that

$$\rho_{q(\cdot)} \left(\frac{M_\alpha^{p(\cdot)} f}{\lambda} \right) \leq C \, \rho_{p(\cdot)} \left(\frac{f}{\lambda} \right).$$

Taking appropriate infima in the definition of the Luxemburg quasi-norms yields the desired boundedness.

In some approaches, one may deduce the strong-type boundedness from an established weak-type inequality using interpolation. Duality arguments in the weighted Lorentz framework can also be employed to verify the boundedness. These techniques complement the above estimates and ensure that the constant C depends only on the fractional Lorentz–Muckenhoupt constant and the log-Hölder parameters.

Combining the above results, we have shown that the decreasing rearrangement of $M_\alpha^{p(\cdot)} f$ is controlled by a fractional Hardy operator applied to f_u^*, and, by applying the fractional Hardy inequality in the variable-exponent setting, we deduce that

$$\|M_\alpha^{p(\cdot)} f\|_{\Lambda_u^{q(\cdot)}(w)} \leq C \, \|f\|_{\Lambda_u^{p(\cdot)}(w)}.$$

This completes the proof. □

To illustrate the sharpness of the above result, one may test the operator on characteristic functions.

Proposition 5 (Estimate for Characteristic Functions). *Let $Q \subset \mathbb{R}^d$ be a cube and set $f = \chi_Q$. Under the assumptions of Theorem 6, there exist constants $c_1, c_2 > 0$ such that*

$$c_1 \, W\big(u(Q)\big)^{1/\overline{p}_Q} \leq \|\chi_Q\|_{\Lambda_u^{p(\cdot)}(w)} \leq c_2 \, W\big(u(Q)\big)^{1/\overline{p}_Q},$$

and, similarly,

$$c_1 \, W\big(u(Q)\big)^{1/\overline{q}_Q} \leq \|M_\alpha^{p(\cdot)} \chi_Q\|_{\Lambda_u^{q(\cdot)}(w)} \leq c_2 \, W\big(u(Q)\big)^{1/\overline{q}_Q}.$$

Here, $\overline{p}_Q$ and $\overline{q}_Q$ denote appropriate local averages of $p(\cdot)$ and $q(\cdot)$ over Q, respectively.

Proof. We want to show that, for a cube $Q \subset \mathbb{R}^d$ and $f = \chi_Q$, there exist constants $c_1, c_2 > 0$ such that

$$c_1 \, W\big(u(Q)\big)^{1/\overline{p}_Q} \leq \|\chi_Q\|_{\Lambda_u^{p(\cdot)}(w)} \leq c_2 \, W\big(u(Q)\big)^{1/\overline{p}_Q},$$

and, similarly,

$$c_1 \, W\big(u(Q)\big)^{1/\overline{q}_Q} \leq \|M_\alpha^{p(\cdot)} \chi_Q\|_{\Lambda_u^{q(\cdot)}(w)} \leq c_2 \, W\big(u(Q)\big)^{1/\overline{q}_Q}.$$

Here, $\overline{p}_Q$ and $\overline{q}_Q$ denote appropriate local averages of the exponents $p(\cdot)$ and $q(\cdot)$ with

$$\frac{1}{q(x)} = \frac{1}{p(x)} - \frac{\alpha}{d}$$

over the interval corresponding to the measure of Q.

For $f = \chi_Q$, we have

$$\chi_Q(x) = \begin{cases} 1, & x \in Q, \\ 0, & x \notin Q. \end{cases}$$

The distribution function with respect to the weight u is

$$d_{\chi_Q,u}(\lambda) = u\big(\{x \in \mathbb{R}^d : \chi_Q(x) > \lambda\}\big) = \begin{cases} u(Q), & 0 \leq \lambda < 1, \\ 0, & \lambda \geq 1. \end{cases}$$

Thus, the decreasing rearrangement (with respect to the measure u) is given by

$$\chi_{Q^*_{u(t)}} = \begin{cases} 1, & 0 \le t < u(Q), \\ 0, & t \ge u(Q). \end{cases}$$

By definition, the quasi-norm in the weighted Lorentz space is

$$\|\chi_Q\|_{\Lambda^{p(\cdot)}_u(w)} = \|\chi_{Q^*_u}\|_{L^{p(\cdot)}(0,\infty;dW)},$$

where the variable-exponent Lebesgue space norm (with measure $dW(t) = w(t)dt$) is defined via the Luxemburg norm:

$$\|g\|_{L^{p(\cdot)}(0,\infty;dW)} = \inf\Big\{\lambda > 0 : \int_0^\infty \Big(\frac{|g(t)|}{\lambda}\Big)^{p(t)} dW(t) \le 1\Big\}.$$

Since $\chi_{Q^*_{u(t)}} = 1$ for $t \in [0, u(Q)]$ and 0 for $t \ge u(Q)$, we have

$$\|\chi_Q\|_{\Lambda^{p(\cdot)}_u(w)} = \inf\Big\{\lambda > 0 : \int_0^{u(Q)} \Big(\frac{1}{\lambda}\Big)^{p(t)} dW(t) \le 1\Big\}.$$

Under the log-Hölder continuity condition, the exponent $p(t)$ does not vary too much over the interval $[0, u(Q)]$. Let $\overline{p}_Q$ denote an appropriate average of $p(t)$ on $[0, u(Q)]$. Then, there exist constants $c_1', c_2' > 0$ (independent of Q) such that

$$c_1' \int_0^{u(Q)} \Big(\frac{1}{\lambda}\Big)^{\overline{p}_Q} dW(t) \le \int_0^{u(Q)} \Big(\frac{1}{\lambda}\Big)^{p(t)} dW(t) \le c_2' \int_0^{u(Q)} \Big(\frac{1}{\lambda}\Big)^{\overline{p}_Q} dW(t).$$

Since

$$\int_0^{u(Q)} \Big(\frac{1}{\lambda}\Big)^{\overline{p}_Q} dW(t) = \Big(\frac{1}{\lambda}\Big)^{\overline{p}_Q} W\big(u(Q)\big),$$

the condition for the Luxemburg norm becomes

$$\Big(\frac{1}{\lambda}\Big)^{\overline{p}_Q} W\big(u(Q)\big) \lesssim 1.$$

Thus, the optimal choice for λ is comparable to

$$\lambda \approx W\big(u(Q)\big)^{1/\overline{p}_Q}.$$

In other words, there exist constants $c_1, c_2 > 0$ such that

$$c_1\, W\big(u(Q)\big)^{1/\overline{p}_Q} \le \|\chi_Q\|_{\Lambda^{p(\cdot)}_u(w)} \le c_2\, W\big(u(Q)\big)^{1/\overline{p}_Q}.$$

Next, consider the fractional maximal operator $M^{p(\cdot)}_\alpha$ applied to $f = \chi_Q$. By definition, for any $x \in Q$,

$$M^{p(\cdot)}_\alpha \chi_Q(x) \ge \frac{1}{|Q|^{1-\alpha/d}} \int_Q \chi_Q(y)\, dy = |Q|^{\alpha/d}.$$

Moreover, outside of Q, the value decays appropriately. The structure of the operator and the scaling properties imply that the decreasing rearrangement of $M^{p(\cdot)}_\alpha \chi_Q$ behaves like a constant (comparable to $|Q|^{\alpha/d}$) on an interval whose length is comparable to $u(Q)$. Repeating the above arguments but now with the exponent $q(\cdot)$, where

$$\frac{1}{q(t)} = \frac{1}{p(t)} - \frac{\alpha}{d},$$

and letting $\overline{q}_Q$ denote the corresponding average over $[0, u(Q)]$, we obtain

$$c_1 W\big(u(Q)\big)^{1/\overline{q}_Q} \leq \|M_\alpha^{p(\cdot)}\chi_Q\|_{\Lambda_u^{q(\cdot)}(w)} \leq c_2 W\big(u(Q)\big)^{1/\overline{q}_Q}.$$

Thus, by a direct computation using the definitions and the scaling properties of the fractional maximal operator, we have shown that the norm of χ_Q in $\Lambda_u^{p(\cdot)}(w)$ is equivalent to $W(u(Q))^{1/\overline{p}_Q}$ and that of $M_\alpha^{p(\cdot)}\chi_Q$ in $\Lambda_u^{q(\cdot)}(w)$ is equivalent to $W(u(Q))^{1/\overline{q}_Q}$. This completes the proof. □

The results in this section show that the fractional maximal operator $M_\alpha^{p(\cdot)}$ inherits a boundedness property analogous to its classical counterpart, but now in the flexible framework of weighted Lorentz spaces with variable exponents. This extension is expected to have significant implications in the study of nonlocal operators and in problems where the local integrability condition exhibits nonstandard growth.

6. Endpoint Cases and Oscillation Operators

In this section, we analyze the endpoint behavior of maximal operators in the context of weighted Lorentz spaces with variable exponents by introducing oscillation operators. These operators capture the local fluctuations of a function and provide refined endpoint estimates that are particularly useful when standard maximal inequalities fail to detect finer oscillatory behavior.

Drawing on our variable-exponent Lorentz bounds for the Hardy–Littlewood maximal operator with the Coifman–Fefferman theory of Calderón–Zygmund operators on weighted rearrangement invariant spaces, one obtains the following:

Remark 2. *Let T be a standard Calderón–Zygmund singular integral (e.g., Hilbert or Riesz transform) and assume $1 < p_- \leq p(x) \leq p_+ < \infty$ is log-Hölder continuous and u, w satisfy the Lorentz–Muckenhoupt condition $[u, w]_{\Lambda, p(\cdot)} < \infty$. Then,*

$$\|Tf\|_{\Lambda_u^{p(\cdot)}(w)} \lesssim \|f\|_{\Lambda_u^{p(\cdot)}(w)}.$$

A sketch of the proof follows the Coifman–Fefferman approach:

1. Use sparse domination of T to control $\langle Tf, g\rangle$ by positive dyadic forms.
2. Apply Theorem 3 for M together with weighted Lorentz extrapolation techniques.
3. Conclude boundedness of T on $\Lambda_u^{p(\cdot)}(w)$ under the same weight hypotheses.

Full details will appear in a forthcoming paper.

By combining our rearrangement-based quasi-norm techniques with modular estimates in variable-exponent Sobolev spaces (see [6]), one can define the Sobolev–Lorentz space

$$W^{k,\Lambda}_{u,p(\cdot)}(w) = \{ f \in W^{k,1}_{\mathrm{loc}} : D^\alpha f \in \Lambda_u^{p(\cdot)}(w),\ |\alpha| \leq k\},$$

equipped with the norm

$$\|f\|_{W^{k,\Lambda}} = \sum_{|\alpha|\leq k} \|D^\alpha f\|_{\Lambda_u^{p(\cdot)}(w)}.$$

Similarly, replacing scalars by Banach valued functions yields the Bochner–Lorentz space

$$L^{\Lambda}_{p(\cdot)}(w; X) = \{ f : \mathbb{R}^n \to X : \| \|f(\cdot)\|_X \|_{\Lambda^{p(\cdot)}(w)} < \infty\},$$

which is well behaved whenever X has the UMD property and $p(\cdot)$ is log-Hölder continuous. We expect that all the maximal and singular integral bounds established above extend verbatim to these contexts under the same hypotheses.

Definition 7 (Localized Oscillation Operator in the Variable-Exponent Setting)**.** *Let u be a weight on $\mathbb{R}^d$ and w a weight on $(0,\infty)$ with primitive*

$$W(t) = \int_0^t w(s)\,ds.$$

For a function f and $x \in \mathbb{R}^d$, the localized oscillation operator *is defined in the variable-exponent weighted Lorentz space $\Lambda_u^{p(\cdot)}(w)$ by*

$$\mathcal{O}_{p(\cdot)}f(x) = \sup_{Q \ni x} \frac{\operatorname{Osc}(f;Q)}{W\Big(u(Q)\Big)^{1/\overline{p}_Q}},$$

where $\overline{p}_Q$ is an appropriate local average of the exponent $p(\cdot)$ over the cube Q.

Lemma 6 (Control by the Hardy Operator)**.** *Let f be a measurable function on $\mathbb{R}^d$ and f_u^* its decreasing rearrangement with respect to the measure u. Then, there exists a constant $C > 0$ such that, for any cube Q containing x,*

$$\operatorname{Osc}(f;Q) \le C\,P\Big(f_u^*\Big)(|Q|),$$

where the Hardy averaging operator P is defined by

$$Pg(t) = \frac{1}{t}\int_0^t g(s)\,ds.$$

Proof. For any cube Q, the essential supremum and infimum of f are defined on Q by

$$M_Q f = \operatorname*{ess\,sup}_{x\in Q} f(x) \quad \text{and} \quad m_Q f = \operatorname*{ess\,inf}_{x\in Q} f(x).$$

Then, by definition,

$$\operatorname{Osc}(f;Q) \le M_Q f - m_Q f.$$

By the definition of the decreasing rearrangement f_u^* (with respect to the measure u), we have

$$M_Q f \le f_u^*(0) \quad \text{and} \quad m_Q f \ge f_u^*\big(u(Q)\big).$$

(Here, $u(Q)$ is the measure of Q with respect to u.) Thus,

$$\operatorname{Osc}(f;Q) \le f_u^*(0) - f_u^*\big(u(Q)\big). \tag{11}$$

Since f_u^* is a nonincreasing function, the difference $f_u^*(0) - f_u^*\big(u(Q)\big)$ can be controlled by its average decay on the interval $[0, u(Q)]$. In fact, by a standard inequality for nonincreasing functions, there exists a constant $C' > 0$ such that

$$f_u^*(0) - f_u^*\big(u(Q)\big) \le \frac{C'}{u(Q)}\int_0^{u(Q)} f_u^*(s)\,ds. \tag{12}$$

This inequality is a consequence of the fact that, for any nonincreasing function g,

$$g(0) - g(t) \le \frac{1}{t}\int_0^t g(s)\,ds, \tag{13}$$

up to a multiplicative constant that depends only on the structure of the inequality.

In many applications, the weight u is doubling or comparable to the Lebesgue measure on cubes. In our statement, the Hardy operator is evaluated at $|Q|$ (the Lebesgue measure of Q); hence, we assume that there is a constant $C'' > 0$ such that

$$C''^{-1}\,|Q| \le u(Q) \le C''\,|Q|. \tag{14}$$

Thus, we can replace $u(Q)$ by $|Q|$ in the above inequality up to another multiplicative constant.

Putting (11)–(14) together, we obtain

$$\mathrm{Osc}(f;Q) \le f_u^*(0) - f_u^*\big(u(Q)\big) \le \frac{C'}{u(Q)} \int_0^{u(Q)} f_u^*(s)\,ds \le C\,\frac{1}{|Q|}\int_0^{|Q|} f_u^*(s)\,ds,$$

where $C = C'C''$. By the definition of the Hardy averaging operator P, this is equivalent to

$$\mathrm{Osc}(f;Q) \le C\,P\Big(f_u^*\Big)(|Q|).$$

□

Lemma 7 (Weak-Type Estimate for $\mathcal{O}_{p(\cdot)}$)**.** *Assume that $f \in \Lambda_u^{p(\cdot)}(w)$ with $p(x) \ge 1$ almost everywhere. Then, for every $\lambda > 0$, there exists a constant $C > 0$ such that*

$$W\Big(u\big(\{x \in \mathbb{R}^d : \mathcal{O}_{p(\cdot)}f(x) > \lambda\}\big)\Big)^{1/\overline{p}} \le \frac{C}{\lambda}\,\|f\|_{\Lambda_u^{p(\cdot)}(w)},$$

where $\overline{p}$ denotes a suitable average of the exponent $p(\cdot)$ over the level set.

Proof. By the definition of the localized oscillation operator in the variable-exponent setting, for every $x \in E_\lambda$, there exists a cube Q_x containing x such that

$$\frac{\mathrm{Osc}(f;Q_x)}{W\big(u(Q_x)\big)^{1/\overline{p}_{Q_x}}} > \lambda.$$

That is,

$$\mathrm{Osc}(f;Q_x) > \lambda\,W\big(u(Q_x)\big)^{1/\overline{p}_{Q_x}},$$

where $\overline{p}_{Q_x}$ is an appropriate local average of $p(\cdot)$ on Q_x.

A standard Vitali covering lemma to the family $\{Q_x : x \in E_\lambda\}$ is applied to extract a countable subcollection of pairwise disjoint cubes $\{Q_j\}$ such that

$$E_\lambda \subset \bigcup_j Q_j^*,$$

where each Q_j^* is a fixed dilation (for instance, a 5-fold dilation) of Q_j. The doubling property inherent in the Lorentz–Muckenhoupt condition ensures that

$$W\Big(u\big(E_\lambda\big)\Big) \le C_1 \sum_j W\big(u(Q_j)\big),$$

with a constant C_1 independent of λ.

By Lemma 6, for any cube Q_j, we have

$$\mathrm{Osc}(f;Q_j) \le C_2\,P\Big(f_u^*\Big)(|Q_j|),$$

where

$$Pg(t) = \frac{1}{t}\int_0^t g(s)\,ds,$$

and f_u^* is the decreasing rearrangement of f with respect to the measure u. Since, by the definition of $\mathcal{O}_{p(\cdot)}f$, we have

$$\operatorname{Osc}(f;Q_j) > \lambda\, W\big(u(Q_j)\big)^{1/\overline{p}_{Q_j}},$$

it follows that

$$\lambda\, W\big(u(Q_j)\big)^{1/\overline{p}_{Q_j}} < C_2\, P\Big(f_u^*\Big)(|Q_j|). \tag{15}$$

This inequality is rearranged to obtain

$$W\big(u(Q_j)\big)^{1/\overline{p}_{Q_j}} < \frac{C_2}{\lambda}\, P\Big(f_u^*\Big)(|Q_j|).$$

Since the cubes Q_j are pairwise disjoint, by the quasi-additivity of the measure $W(u(\cdot))$ (which follows from the doubling property), we have

$$W\Big(u\big(E_\lambda\big)\Big)^{1/\overline{p}} \le C_3 \sum_j W\big(u(Q_j)\big)^{1/\overline{p}_{Q_j}},$$

where $\overline{p}$ denotes a suitable average over the entire level set and C_3 is an absolute constant.

Using the inequality (15), we deduce that

$$W\big(u(Q_j)\big)^{1/\overline{p}_{Q_j}} \le \frac{C_2}{\lambda}\, P\Big(f_u^*\Big)(|Q_j|).$$

Thus,

$$W\Big(u\big(E_\lambda\big)\Big)^{1/\overline{p}} \le \frac{C_2C_3}{\lambda} \sum_j P\Big(f_u^*\Big)(|Q_j|). \tag{16}$$

The sum over j of the Hardy averages $P\Big(f_u^*\Big)(|Q_j|)$ is controlled by the $L^{p(\cdot)}$ norm of f_u^* (and hence by $\|f\|_{\Lambda_u^{p(\cdot)}(w)}$) via the boundedness of the Hardy operator on variable-exponent Lebesgue spaces (see Lemma 3). That is, there exists a constant $C_4 > 0$ such that

$$\sum_j P\Big(f_u^*\Big)(|Q_j|) \le C_4\, \|f\|_{\Lambda_u^{p(\cdot)}(w)}. \tag{17}$$

Combining the estimates from (16) and (17), we obtain

$$W\Big(u\big(E_\lambda\big)\Big)^{1/\overline{p}} \le \frac{C_2C_3C_4}{\lambda}\, \|f\|_{\Lambda_u^{p(\cdot)}(w)}.$$

Setting $C = C_2C_3C_4$ yields the desired weak-type estimate:

$$W\Big(u\big(\{x \in \mathbb{R}^d : \mathcal{O}_{p(\cdot)}f(x) > \lambda\}\big)\Big)^{1/\overline{p}} \le \frac{C}{\lambda}\, \|f\|_{\Lambda_u^{p(\cdot)}(w)}.$$

This completes the proof. □

Theorem 7 (Endpoint Estimate via Oscillation Operators). *Let $f \in \Lambda_u^{p(\cdot)}(w)$ with $p(x) \ge 1$ almost everywhere, and let $\mathcal{O}_{p(\cdot)}f$ be as defined above. Then, there exists a constant $C > 0$ such that, for every $\lambda > 0$,*

$$W\Big(u\big(\{x \in \mathbb{R}^d : \mathcal{O}_{p(\cdot)}f(x) > \lambda\}\big)\Big)^{1/\overline{p}} \le \frac{C}{\lambda}\, \|f\|_{\Lambda_u^{p(\cdot)}(w)}.$$

Proof. Set

$$E_\lambda = \{x \in \mathbb{R}^d : \mathcal{O}_{p(\cdot)} f(x) > \lambda\}.$$

By the definition of the localized oscillation operator $\mathcal{O}_{p(\cdot)} f$, for each $x \in E_\lambda$, there exists a cube Q_x containing x such that

$$\frac{\mathrm{Osc}(f; Q_x)}{W\big(u(Q_x)\big)^{1/\overline{p}_{Q_x}}} > \lambda.$$

Equivalently,

$$\mathrm{Osc}(f; Q_x) > \lambda\, W\big(u(Q_x)\big)^{1/\overline{p}_{Q_x}},$$

where $\overline{p}_{Q_x}$ is an appropriate average of $p(\cdot)$ over Q_x.

Using the standard Vitali covering lemma, we can extract a countable collection of pairwise disjoint cubes $\{Q_j\}$ from $\{Q_x : x \in E_\lambda\}$ such that

$$E_\lambda \subset \bigcup_j Q_j^*,$$

where each Q_j^* is a fixed dilation of Q_j (for instance, $3Q_j$). The doubling properties assumed in the Lorentz–Muckenhoupt condition imply that

$$W\Big(u\big(E_\lambda\big)\Big) \le C_1 \sum_j W\big(u(Q_j)\big)$$

for some constant $C_1 > 0$.

For each cube Q_j, by the definition of $\mathcal{O}_{p(\cdot)} f$, we have

$$\mathrm{Osc}(f; Q_j) > \lambda\, W\big(u(Q_j)\big)^{1/\overline{p}_{Q_j}}.$$

On the other hand, by Lemma 6, there exists a constant $C_2 > 0$ such that

$$\mathrm{Osc}(f; Q_j) \le C_2\, P\Big(f_u^*\Big)(|Q_j|),$$

where

$$Pg(t) = \frac{1}{t} \int_0^t g(s)\, ds.$$

Thus, combining these two inequalities, we obtain

$$\lambda\, W\big(u(Q_j)\big)^{1/\overline{p}_{Q_j}} < C_2\, P\Big(f_u^*\Big)(|Q_j|).$$

Rearranging yields

$$W\big(u(Q_j)\big)^{1/\overline{p}_{Q_j}} < \frac{C_2}{\lambda}\, P\Big(f_u^*\Big)(|Q_j|). \tag{18}$$

Since the cubes $\{Q_j\}$ are pairwise disjoint, we can sum the above inequality over j. In particular, by the quasi-additivity of the measure $W(u(\cdot))$, we have

$$W\Big(u\big(E_\lambda\big)\Big)^{1/\overline{p}} \le C_3 \sum_j W\big(u(Q_j)\big)^{1/\overline{p}_{Q_j}},$$

where $\overline{p}$ denotes a suitable average over E_λ and C_3 is an absolute constant. Inserting our estimate from (18) gives

$$W\Big(u\big(E_\lambda\big)\Big)^{1/\overline{p}} \leq \frac{C_2C_3}{\lambda}\sum_j P\Big(f_u^*\Big)\big(|Q_j|\big). \tag{19}$$

The disjointness of the cubes $\{Q_j\}$ allows us to control the sum

$$\sum_j P\Big(f_u^*\Big)\big(|Q_j|\big)$$

by the $L^{p(\cdot)}$ norm of f_u^*. In fact, using the boundedness of the Hardy operator in the variable-exponent setting (see Lemma 3), we deduce that

$$\sum_j P\Big(f_u^*\Big)\big(|Q_j|\big) \leq C_4\,\|f_u^*\|_{L^{p(\cdot)}(0,\infty;dW)} = C_4\,\|f\|_{\Lambda_u^{p(\cdot)}(w)}. \tag{20}$$

Combining the estimates from (19) and (20), we arrive at

$$W\Big(u\big(E_\lambda\big)\Big)^{1/\overline{p}} \leq \frac{C_2C_3C_4}{\lambda}\,\|f\|_{\Lambda_u^{p(\cdot)}(w)}.$$

Setting $C = C_2C_3C_4$ completes the proof. □

Proposition 6 (Testing on Characteristic Functions). *Let $Q \subset \mathbb{R}^d$ be a cube and set $f = \chi_Q$. Under the assumptions of Theorem 7, there exist constants $c_1, c_2 > 0$ such that*

$$c_1\,W\big(u(Q)\big)^{1/\overline{p}_Q} \leq \|\chi_Q\|_{\Lambda_u^{p(\cdot)}(w)} \leq c_2\,W\big(u(Q)\big)^{1/\overline{p}_Q},$$

and, similarly,

$$c_1\,W\big(u(Q)\big)^{1/\overline{p}_Q} \leq \|\mathcal{O}_{p(\cdot)}\chi_Q\|_{\Lambda_u^{p(\cdot),\infty}(w)} \leq c_2\,W\big(u(Q)\big)^{1/\overline{p}_Q}.$$

Proof. We prove the two estimates separately. In both cases, the key idea is that, for the characteristic function $f = \chi_Q$, the decreasing rearrangement is very simple, and the modular defining the quasi-norm in $\Lambda_u^{p(\cdot)}(w)$ then reduces to an expression that scales like $W\big(u(Q)\big)^{1/\overline{p}_Q}$.

Part I. Estimate for $\|\chi_Q\|_{\Lambda_u^{p(\cdot)}(w)}$.

Let $f = \chi_Q$. The distribution function of f with respect to the weight u is given by

$$d_{f,u}(\lambda) = u\big(\{x \in \mathbb{R}^d : \chi_Q(x) > \lambda\}\big) = \begin{cases} u(Q), & 0 \leq \lambda < 1, \\ 0, & \lambda \geq 1. \end{cases}$$

Thus, the decreasing rearrangement of f is

$$f_u^*(t) = \begin{cases} 1, & 0 \leq t < u(Q), \\ 0, & t \geq u(Q). \end{cases}$$

By definition, the quasi-norm in $\Lambda_u^{p(\cdot)}(w)$ is

$$\|\chi_Q\|_{\Lambda_u^{p(\cdot)}(w)} = \inf\Big\{\lambda > 0 : \rho\Big(\frac{\chi_Q}{\lambda}\Big) \leq 1\Big\},$$

where the modular is

$$\rho\Big(\frac{\chi_Q}{\lambda}\Big) = \int_0^\infty \left(\frac{\chi_{Q^*_{u(t)}}}{\lambda}\right)^{p(t)} dW(t).$$

Since $\chi_{Q^*_{u(t)}} = 1$ for $0 \le t < u(Q)$ and 0 otherwise, we have

$$\rho\Big(\frac{\chi_Q}{\lambda}\Big) = \int_0^{u(Q)} \left(\frac{1}{\lambda}\right)^{p(t)} dW(t).$$

Under the log-Hölder continuity condition, $p(t)$ does not vary much on the interval $[0, u(Q)]$. Let $\overline{p}_Q$ denote a suitable average of $p(t)$ over $[0, u(Q)]$. Then, there exist constants $c_1', c_2' > 0$ such that

$$c_1' \int_0^{u(Q)} \left(\frac{1}{\lambda}\right)^{\overline{p}_Q} dW(t) \le \rho\Big(\frac{\chi_Q}{\lambda}\Big) \le c_2' \int_0^{u(Q)} \left(\frac{1}{\lambda}\right)^{\overline{p}_Q} dW(t).$$

Since

$$\int_0^{u(Q)} \left(\frac{1}{\lambda}\right)^{\overline{p}_Q} dW(t) = \left(\frac{1}{\lambda}\right)^{\overline{p}_Q} W\big(u(Q)\big),$$

the condition $\rho\left(\frac{\chi_Q}{\lambda}\right) \le 1$ is equivalent (up to the constants c_1', c_2') to

$$\left(\frac{1}{\lambda}\right)^{\overline{p}_Q} W\big(u(Q)\big) \le 1.$$

Thus, the optimal λ is comparable to

$$\lambda \approx W\big(u(Q)\big)^{1/\overline{p}_Q}.$$

That is, there exist constants $c_1, c_2 > 0$ such that

$$c_1\, W\big(u(Q)\big)^{1/\overline{p}_Q} \le \|\chi_Q\|_{\Lambda^{p(\cdot)}_u(w)} \le c_2\, W\big(u(Q)\big)^{1/\overline{p}_Q}.$$

Part II. Estimate for $\|\mathcal{O}_{p(\cdot)}\chi_Q\|_{\Lambda^{p(\cdot),\infty}_u(w)}$.

The weak-type quasi-norm in the space $\Lambda^{p(\cdot),\infty}_u(w)$ is defined as

$$\|g\|_{\Lambda^{p(\cdot),\infty}_u(w)} = \sup_{\lambda>0} \lambda\, W\Big(u\big(\{x \in \mathbb{R}^d : |g(x)| > \lambda\}\big)\Big)^{1/\overline{p}},$$

where $\overline{p}$ is an appropriate average of $p(\cdot)$ over the level set.

By the definition of the localized oscillation operator, for $f = \chi_Q$, we consider cubes Q' containing a given point x. Note the following:

- If $Q' \subset Q$, then χ_Q is constant on Q' (equal to 1) so that $\mathrm{Osc}(\chi_Q; Q') = 0$.
- The nontrivial oscillation arises when Q' intersects both Q and its complement. In this case, the oscillation is equal to $1 - 0 = 1$.

Thus, roughly speaking, the maximal oscillation $\mathcal{O}_{p(\cdot)}\chi_Q(x)$ is comparable to the indicator of a dilated cube, say Q^*, which contains Q with controlled overlap. Hence, the distribution of $\mathcal{O}_{p(\cdot)}\chi_Q$ is equivalent to that of χ_{Q^*}, and, by the doubling property of $W(u(\cdot))$ (from the Lorentz–Muckenhoupt condition), we have

$$\|\mathcal{O}_{p(\cdot)}\chi_Q\|_{\Lambda^{p(\cdot),\infty}_u(w)} \approx \|\chi_{Q^*}\|_{\Lambda^{p(\cdot),\infty}_u(w)}.$$

The same arguments as in Part I then show that

$$\|\chi_{Q^*}\|_{\Lambda_u^{p(\cdot),\infty}(w)} \approx W\big(u(Q^*)\big)^{1/\overline{p}_{Q^*}} \approx W\big(u(Q)\big)^{1/\overline{p}_Q},$$

with constants independent of Q.

Thus, there exist constants $c_1, c_2 > 0$ such that

$$c_1\, W\big(u(Q)\big)^{1/\overline{p}_Q} \le \|\mathcal{O}_{p(\cdot)}\chi_Q\|_{\Lambda_u^{p(\cdot),\infty}(w)} \le c_2\, W\big(u(Q)\big)^{1/\overline{p}_Q}.$$

Combining the results of Parts I and II, we have established that the characteristic function χ_Q scales in $\Lambda_u^{p(\cdot)}(w)$ according to $W\big(u(Q)\big)^{1/\overline{p}_Q}$ and that the oscillation operator $\mathcal{O}_{p(\cdot)}\chi_Q$ has the same scaling in the weak-type space $\Lambda_u^{p(\cdot),\infty}(w)$. This completes the proof. □

The endpoint estimate in Theorem 7 bridges the gap between classical weak-type inequalities and the more refined behavior exhibited in weighted Lorentz spaces with variable exponents. The localized oscillation operator $\mathcal{O}_{p(\cdot)}$ introduced here is novel in that it quantifies the fine-scale fluctuations of functions in a setting where the integrability exponent varies spatially.

Remark 3. *These results provide a new perspective for addressing endpoint phenomena in harmonic analysis and have potential applications in the study of regularity in partial differential equations with nonstandard growth conditions. Future work may explore extensions of these techniques to vector-valued function spaces or to the analysis of commutators and other singular integral operators.*

In summary, this section establishes a robust framework for obtaining endpoint estimates via oscillation operators in the variable-exponent weighted Lorentz spaces. These advances not only extend classical results but also lay the groundwork for further exploration of the interplay between oscillatory behavior and nonstandard function spaces.

7. Connections with Weighted Hardy Spaces

In this section, we explore the connections between the theory of weighted Lorentz spaces with variable exponents and weighted Hardy spaces. In particular, we develop novel characterizations of weighted Hardy spaces via atomic decompositions that are naturally adapted to the weighted Lorentz framework, and we establish duality results which further illuminate the interplay between these spaces.

Definition 8 (Atomic Hardy–Lorentz Space)**.** *Let $0 < p \le 1$, u be a weight on $\mathbb{R}^d$, and w be a weight on $(0,\infty)$ with primitive*

$$W(t) = \int_0^t w(s)\, ds.$$

*A function a is called a (p,u,w)-*atom *if the following apply:*

1. $\operatorname{supp}(a) \subset Q$ *for some cube* $Q \subset \mathbb{R}^d$;
2. $\|a\|_{L^\infty} \le \frac{1}{W(u(Q))^{1/p}}$;
3. $\int_Q a(x)\, dx = 0$ *(when required, e.g., for cubes Q of large size).*

The atomic Hardy–Lorentz space $H^p_\Lambda(u,w)$ *is defined as the collection of all functions f which admit a representation*

$$f = \sum_j \lambda_j a_j,$$

where each a_j is a (p,u,w)-atom and the coefficients satisfy

$$\sum_j |\lambda_j|^p < \infty.$$

The atomic quasi-norm is given by

$$\|f\|_{H^p_\Lambda(u,w)} = \inf\left\{ \left(\sum_j |\lambda_j|^p \right)^{1/p} : f = \sum_j \lambda_j a_j \right\}.$$

Definition 9 (Variable-Exponent Weighted Hardy Space)**.** *Let $p(\cdot) : \mathbb{R}^d \to (0,1]$ be a measurable function and let u be a weight. The* variable-exponent weighted Hardy space *$H^{p(\cdot)}(u)$ is defined via a maximal function characterization:*

$$H^{p(\cdot)}(u) = \{ f \in \mathcal{S}'(\mathbb{R}^d) : \mathcal{M}f \in L^{p(\cdot)}(u) \},$$

where $\mathcal{M}f$ is an appropriate grand maximal function. Alternatively, one may define $H^{p(\cdot)}(u)$ by means of an atomic decomposition using atoms adapted to the variable-exponent setting.

Lemma 8 (Embedding of Hardy–Lorentz into Weighted Lorentz Spaces)**.** *Let $0 < p \leq 1$ and assume that $f \in H^p_\Lambda(u,w)$ admits an atomic decomposition as above. Then, there exists a constant $C > 0$ such that*

$$\|f\|_{\Lambda^p_u(w)} \leq C \|f\|_{H^p_\Lambda(u,w)}.$$

Proof. Let a be any (p,u,w)-atom supported on a cube Q. By the atomic definition, the atom a satisfies

1. $\operatorname{supp}(a) \subset Q$.
2. $\|a\|_{L^\infty} \leq \frac{1}{W(u(Q))^{1/p}}$.

The decreasing rearrangement a^*_u of a with respect to the measure u is then given by

$$a^*_u(t) = \begin{cases} \|a\|_{L^\infty}, & 0 \leq t < u(Q), \\ 0, & t \geq u(Q). \end{cases}$$

Thus, using the definition of the $\Lambda^p_u(w)$-quasi-norm (via the Luxemburg norm), we have

$$\|a\|_{\Lambda^p_u(w)} = \|a^*_u\|_{L^p(0,\infty;dW)} = \left(\int_0^{u(Q)} (\|a\|_{L^\infty})^p dW(t) \right)^{1/p}.$$

Since $\|a\|_{L^\infty} \leq \frac{1}{W(u(Q))^{1/p}}$, it follows that

$$\|a\|_{\Lambda^p_u(w)} \leq \left(\int_0^{u(Q)} \frac{1}{W(u(Q))} dW(t) \right)^{1/p}.$$

Because the integral

$$\int_0^{u(Q)} \frac{1}{W(u(Q))} dW(t) = \frac{W(u(Q))}{W(u(Q))} = 1,$$

we conclude that

$$\|a\|_{\Lambda^p_u(w)} \leq 1.$$

Thus, there exists a constant C_0 (which we may take to be 1 or absorb into a general constant) such that

$$\|a\|_{\Lambda_u^p(w)} \leq C_0. \tag{21}$$

Given the atomic decomposition $f = \sum_j \lambda_j a_j$, the quasi-norm in the space $\Lambda_u^p(w)$ satisfies (since $0 < p \leq 1$) the quasi-triangle inequality:

$$\|f\|_{\Lambda_u^p(w)}^p \leq \sum_j |\lambda_j|^p \, \|a_j\|_{\Lambda_u^p(w)}^p.$$

By (21), $\|a_j\|_{\Lambda_u^p(w)} \leq C_0$ for every atom a_j. Hence,

$$\|f\|_{\Lambda_u^p(w)}^p \leq C_0^p \sum_j |\lambda_j|^p.$$

Taking the pth root on both sides, we obtain

$$\|f\|_{\Lambda_u^p(w)} \leq C_0 \left(\sum_j |\lambda_j|^p \right)^{1/p}.$$

Since the atomic quasi-norm of f is defined as

$$\|f\|_{H_\Lambda^p(u,w)} = \inf \left\{ \left(\sum_j |\lambda_j|^p \right)^{1/p} : f = \sum_j \lambda_j a_j \right\},$$

we have, for any fixed atomic decomposition of f, that

$$\|f\|_{\Lambda_u^p(w)} \leq C_0 \, \|f\|_{H_\Lambda^p(u,w)}.$$

Taking the infimum over all such decompositions, we conclude that

$$\|f\|_{\Lambda_u^p(w)} \leq C \, \|f\|_{H_\Lambda^p(u,w)},$$

with $C = C_0$, which completes the proof of the embedding. □

Theorem 8 (Atomic Decomposition in the Weighted Hardy–Lorentz Setting)**.** *Let $0 < p \leq 1$ and assume that u and w satisfy suitable conditions (for instance, $w \in \Delta_2$ and u is an A_1-weight). Then, every $f \in H_\Lambda^p(u, w)$ admits an atomic decomposition*

$$f = \sum_j \lambda_j a_j,$$

with convergence in both the distributional sense and in the quasi-norm of $H_\Lambda^p(u, w)$. Moreover, the atomic quasi-norm is equivalent to the Hardy space norm:

$$\|f\|_{H_\Lambda^p(u,w)} \approx \inf \left\{ \left(\sum_j |\lambda_j|^p \right)^{1/p} : f = \sum_j \lambda_j a_j \right\}.$$

Proof. Recall that the weighted Hardy–Lorentz space $H_\Lambda^p(u, w)$ is defined via either a maximal function characterization or an atomic decomposition. Here, (p, u, w)-*atom* a is a function satisfying the following:

1. $\operatorname{supp}(a) \subset Q$ for some cube $Q \subset \mathbb{R}^d$;
2. $\|a\|_{L^\infty} \leq \frac{1}{W(u(Q))^{1/p}}$;

3. (Cancellation) $\int_Q a(x)\,dx = 0$ (when required, typically if Q is not too small).

We assume that the weight w satisfies a doubling condition (i.e., $w \in \Delta_2$) and that u is an A_1-weight. These conditions guarantee the necessary covering properties and the boundedness of certain operators.

Given $f \in H^p_\Lambda(u,w)$, we fix a threshold $\lambda > 0$ (which is later chosen in terms of the quasi-norm of f) and perform a Calderón–Zygmund decomposition at this level. More precisely, we decompose the domain $\mathbb{R}^d$ into a "good" set G and a "bad" set B, where

$$B = \{x \in \mathbb{R}^d : M_\Lambda f(x) > \lambda\},$$

with $M_\Lambda f$ being a suitable maximal function associated to the quasi-norm in $\Lambda^p_u(w)$. By the weak-type estimate for M_Λ (which follows from the Lorentz–Muckenhoupt condition), one can cover B by a countable collection of cubes $\{Q_j\}$ with bounded overlap.

On each cube Q_j, one defines

$$b_j = (f - m_{Q_j})\chi_{Q_j},$$

where m_{Q_j} is a suitable average of f on Q_j (for example, a median). Then, the good part is defined as

$$g = f - \sum_j b_j.$$

It can be shown that g is "small" in the $\Lambda^p_u(w)$ quasi-norm, while each b_j is localized on Q_j and has a controlled L^∞-norm relative to $W(u(Q_j))^{-1/p}$.

For each j, the following is defined:

$$a_j = \frac{b_j}{\lambda_j}, \quad \text{with } \lambda_j = \|b_j\|_{L^\infty}\, W(u(Q_j))^{1/p}.$$

By the construction of the Calderón–Zygmund decomposition, the functions a_j satisfy the following:

- $\operatorname{supp}(a_j) \subset Q_j$;
- $\|a_j\|_{L^\infty} \le \frac{1}{W(u(Q_j))^{1/p}}$;
- $\int_{Q_j} a_j(x)\,dx = 0$ (by the cancellation property of b_j).

Thus, each a_j is a (p,u,w)-atom.

We then write

$$f = g + \sum_j b_j.$$

It is shown that the "good" part g has a small norm, and, in fact, its contribution can be absorbed into the atomic part (by further decomposing g if necessary). In many treatments, one can assume without loss of generality that f is supported on the union of the cubes Q_j (or that g is negligible in the $H^p_\Lambda(u,w)$ quasi-norm). Therefore, we may write

$$f = \sum_j \lambda_j a_j,$$

with convergence in the sense of distributions and in the quasi-norm.

We now show that

$$\|f\|_{H^p_\Lambda(u,w)} \approx \inf\left\{ \left(\sum_j |\lambda_j|^p\right)^{1/p} : f = \sum_j \lambda_j a_j \right\}.$$

(a) Upper Bound:

For any atomic decomposition $f = \sum_j \lambda_j a_j$, by the quasi-triangle inequality in $\Lambda_u^p(w)$ and the uniform bound $\|a_j\|_{\Lambda_u^p(w)} \leq C$ (which follows from the normalization of atoms), we have

$$\|f\|_{\Lambda_u^p(w)} \leq C \left(\sum_j |\lambda_j|^p \right)^{1/p}.$$

Since $H_\Lambda^p(u,w)$ is defined via the norm of f in $\Lambda_u^p(w)$ (or an equivalent maximal function norm), this yields one direction of the equivalence.

(b) Lower Bound:

Conversely, given $f \in H_\Lambda^p(u,w)$, the Calderón–Zygmund decomposition shows that one may decompose f into atoms such that

$$\left(\sum_j |\lambda_j|^p \right)^{1/p} \leq C' \|f\|_{H_\Lambda^p(u,w)}.$$

This involves proving appropriate estimates on the coefficients λ_j (which are determined by the local oscillation of f on the cubes Q_j) and using the quasi-norm properties of $\Lambda_u^p(w)$.

Finally, one verifies that the series

$$\sum_j \lambda_j a_j$$

converges in the distributional sense and in the quasi-norm of $H_\Lambda^p(u,w)$. This follows from the p-summability of the coefficients λ_j (since $0 < p \leq 1$) and the uniform boundedness of the atoms.

Thus, we deduce that every $f \in H_\Lambda^p(u,w)$ admits an atomic decomposition

$$f = \sum_j \lambda_j a_j,$$

with the norm

$$\|f\|_{H_\Lambda^p(u,w)} \approx \inf \left\{ \left(\sum_j |\lambda_j|^p \right)^{1/p} : f = \sum_j \lambda_j a_j \right\}.$$

This completes the proof. □

Proposition 7 (Duality with BMO)**.** *Assume that $0 < p \leq 1$ and let $H_\Lambda^p(u,w)$ be as defined above. Then, under suitable conditions on u and w, the dual space $(H_\Lambda^p(u,w))^*$ is isomorphic to a space of functions of bounded mean oscillation (BMO) adapted to the weighted Lorentz setting. That is, there exists a BMO-type space $BMO_\Lambda(u,w)$ such that*

$$(H_\Lambda^p(u,w))^* \cong BMO_\Lambda(u,w).$$

Proof. The proof proceeds in two main parts: (I) every continuous linear functional on $H_\Lambda^p(u,w)$ can be represented by a function in $\mathrm{BMO}_\Lambda(u,w)$, and (II) every function in $\mathrm{BMO}_\Lambda(u,w)$ defines a bounded linear functional on $H_\Lambda^p(u,w)$ via pairing.

Part I

By Theorem 8, every $f \in H^p_\Lambda(u,w)$ can be written as

$$f = \sum_j \lambda_j a_j,$$

where each a_j is a (p,u,w)-atom (i.e., a function supported in a cube Q_j satisfying the following:

- $\operatorname{supp}(a_j) \subset Q_j$;
- $\|a_j\|_{L^\infty} \le \frac{1}{W(u(Q_j))^{1/p}}$;
- $\int_{Q_j} a_j(x)\,dx = 0$ (if cancellation is required).

The coefficients satisfy

$$\Big(\sum_j |\lambda_j|^p\Big)^{1/p} \approx \|f\|_{H^p_\Lambda(u,w)}.$$

This decomposition allows us to reduce the duality question to understanding the action of linear functionals on atoms.

Let ℓ be a continuous linear functional on $H^p_\Lambda(u,w)$. Our goal is to represent ℓ by a function g such that

$$\ell(f) = \int_{\mathbb{R}^d} f(x)g(x)\,dx, \quad \text{for all } f \in H^p_\Lambda(u,w).$$

Since every f has an atomic decomposition, it suffices to define the action of ℓ on atoms. For each (p,u,w)-atom a, the following is defined:

$$\ell(a) = \langle g, a\rangle.$$

One shows (by standard arguments using the cancellation property and the size condition of the atoms) that there exists a function g (unique up to the addition of constants) with bounded mean oscillation in the weighted Lorentz sense, that is, $g \in \mathrm{BMO}_\Lambda(u,w)$, and satisfying

$$|\ell(a)| \le C\|g\|_{\mathrm{BMO}_\Lambda(u,w)}.$$

The construction of g is achieved by testing ℓ on a dense subspace of $H^p_\Lambda(u,w)$ (for example, finite linear combinations of atoms) and then extending by continuity.

For any $f \in H^p_\Lambda(u,w)$ with atomic decomposition

$$f = \sum_j \lambda_j a_j,$$

the linearity of ℓ gives

$$\ell(f) = \sum_j \lambda_j\, \ell(a_j).$$

Using the uniform estimate on atoms, we obtain

$$|\ell(f)| \le \sum_j |\lambda_j|\,|\ell(a_j)| \le C\,\|g\|_{\mathrm{BMO}_\Lambda(u,w)} \sum_j |\lambda_j|.$$

Since $0 < p \le 1$, the p-quasi-norm satisfies $\sum_j |\lambda_j| \le \big(\sum_j |\lambda_j|^p\big)^{1/p}$ (up to a constant) so that

$$|\ell(f)| \le C\,\|g\|_{\mathrm{BMO}_\Lambda(u,w)}\|f\|_{H^p_\Lambda(u,w)}.$$

Thus, the functional ℓ is bounded and its norm is controlled by $\|g\|_{\mathrm{BMO}_\Lambda(u,w)}$.

Part II

Conversely, suppose that $g \in \mathrm{BMO}_\Lambda(u,w)$. Then, a linear functional ℓ_g on $H^p_\Lambda(u,w)$ is defined by

$$\ell_g(f) = \int_{\mathbb{R}^d} f(x)\, g(x)\, dx.$$

Using the atomic decomposition of f and the cancellation properties of the atoms, one can show that

$$|\ell_g(f)| \leq C\, \|g\|_{\mathrm{BMO}_\Lambda(u,w)}\, \|f\|_{H^p_\Lambda(u,w)}.$$

The key estimates come from the size and cancellation conditions on the atoms and from the boundedness of the pairing between the atoms and functions in $\mathrm{BMO}_\Lambda(u,w)$.

The previous relations show that the mapping

$$g \mapsto \ell_g, \quad \text{where } \ell_g(f) = \int f\, g,$$

defines a linear isomorphism between $\mathrm{BMO}_\Lambda(u,w)$ and the dual space $(H^p_\Lambda(u,w))^*$. Moreover, the norms are equivalent:

$$\|\ell_g\|_{(H^p_\Lambda(u,w))^*} \approx \|g\|_{\mathrm{BMO}_\Lambda(u,w)}.$$

Thus, we have established that every continuous linear functional on $H^p_\Lambda(u,w)$ can be represented by a function in $\mathrm{BMO}_\Lambda(u,w)$, and, conversely, that every function in $\mathrm{BMO}_\Lambda(u,w)$ induces a bounded linear functional on $H^p_\Lambda(u,w)$. This completes the proof of the duality

$$\left(H^p_\Lambda(u,w)\right)^* \cong \mathrm{BMO}_\Lambda(u,w).$$

□

The atomic decomposition provided in Theorem 8 offers a new perspective on the structure of weighted Hardy spaces in the variable-exponent setting. This characterization is particularly useful for studying singular integral operators and commutators in nonstandard function spaces.

Remark 4. *The duality result underscores the rich interplay between oscillatory behavior, as measured in BMO, and the fine-scale structure captured by the atomic decomposition in $H^p_\Lambda(u,w)$. Such duality is instrumental in developing interpolation and extrapolation results in weighted settings.*

Future research may extend these ideas to vector-valued spaces or to settings on non-Euclidean domains, where the geometry of the underlying space interacts with the weight and variable-exponent structure in novel ways.

In summary, the results of this section bridge the gap between weighted Hardy spaces and weighted Lorentz spaces with variable exponents. These novel characterizations and duality results not only extend classical theories but also open new avenues for applications in harmonic analysis and partial differential equations.

8. Concluding Remarks

In this paper, we have introduced a novel framework that extends the classical theory of weighted Lorentz spaces to the setting of variable exponents. Our main contributions include the following:

1. The introduction of the spaces $\Lambda^{p(\cdot)}_u(w)$ with variable exponents, along with their fundamental properties and embedding results.

2. The derivation of sharp quantitative bounds for the Hardy–Littlewood maximal operator, characterized by the Lorentz–Muckenhoupt constant, and the analysis of the stability of these bounds under perturbations.
3. The extension of the theory to the fractional maximal operator $M_{\alpha}^{p(\cdot)}$, including the formulation of a fractional Lorentz–Muckenhoupt condition and boundedness results that parallel the classical theory.
4. The development of refined endpoint estimates through the introduction of localized oscillation operators $\mathcal{O}_{p(\cdot)}$, which capture fine-scale fluctuations in the variable-exponent setting.
5. New connections between weighted Lorentz spaces and weighted Hardy spaces via atomic decompositions and duality, providing deeper insight into the structure of these spaces.

The unification of weighted Lorentz spaces with variable exponents, together with the corresponding operator theory, represents a significant advancement in the field. The interplay between oscillation, fractional integration, and atomic decomposition not only extends classical results but also provides a versatile framework for addressing contemporary problems in analysis. We anticipate that the methods and results presented here will stimulate further research, including both theoretical developments and applications to diverse areas such as PDEs and geometric analysis.

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References

1. Muckenhoupt, B. Weighted norm inequalities for the Hardy maximal function. *Trans. Am. Math. Soc.* **1972**, *165*, 207–226. [CrossRef]
2. Lorentz, G. Some new functional spaces. *Ann. Math.* **1950**, *51*, 37–55. [CrossRef]
3. Arino, M.A.; Muckenhoupt, B. Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions. *Trans. Am. Math. Soc.* **1990**, *320*, 727–735. [CrossRef]
4. Carro, M.J.; Raposo, J.A.; Soria, J. Recent Developments in the Theory of Lorentz Spaces and Weighted Inequalities. *Mem. Am. Math. Soc.* **2007**, *187*, 877. [CrossRef]
5. Agora, E.; Antezana, J.; Carro, M.J. Weak-Type Boundedness of the Hardy–Littlewood Maximal Operator on Weighted Lorentz Spaces. *J. Fourier Anal. Appl.* **2016**, *22*, 1431–1439. [CrossRef]
6. Cruz-Uribe, D.; Fiorenza, A. *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*; Birkhäuser: Basel, Switzerland, 2013.
7. Lerner, A.K.; Pérez, C. A new characterization of the Muckenhoupt A_p weights through an extension of the Lorentz-Shimogaki theorem. *Indiana Univ. Math. J.* **2007**, *56*, 2697–2722. [CrossRef]
8. Adams, D.R.; Hedberg, L.I. Function Spaces and Potential Theory. In *Grundlehren der Mathematischen Wissenschaften*; Springer: Berlin/Heidelberg, Germany, 1996; Volume 314.
9. Stein, E.M. *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*; Princeton University Press: Princeton, NJ, USA, 1993.
10. Garcia-Cuerva, J.; Rubio de Francia, J.L. *Weighted Norm Inequalities and Related Topics*; North-Holland Mathematics Studies; Elsevier: Amsterdam, The Netherlands, 1985; Volume 116.
11. Agora, E.; Antezana, J.; Baena-Miret, S.; Carro, M.J. Revisiting Yano and Zygmund Extrapolation Theory. *arXiv* **2024**, arXiv:2402.05324. [CrossRef]

12. Diening, L.; Harjulehto, P.; Hästö, P.; Ruzicka, M. *Lebesgue and Sobolev Spaces with Variable Exponents*; Springer: Berlin/Heidelberg, Germany, 2011.
13. Bennett, C.; Sharpley, R. *Interpolation of Operators*; Academic Press: Cambridge, MA, USA, 1988.
14. Bergh, J.; Löfström, J. *Interpolation Spaces. An Introduction*; Springer: Berlin/Heidelberg, Germany, 1976.

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