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m-Isometric Operators with Null Symbol and Elementary Operator Entries

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Abstract

A pair (A, B) of Banach space operators is strict (m, X)-isometric for a Banach space operator $X \in B(\mathcal{X})$ and a positive integer m if $\triangle^m_{A,B}(X) = \left(\sum_{j=0}^{m} \begin{pmatrix} m \\ j \end{pmatrix} L_A^j R_B^j\right)(X) = 0$ and $\triangle^{m-1}_{A,B}(X) \neq 0$, where L_A and $R_B \in B(B(\mathcal{X}))$ are, respectively, the operators of left multiplication by A and right multiplication by B. Define operators $E_{A,B}$ and $\mathcal{E}_{A,B}(X)$ by $E_{A,B} = L_A R_B$ and $(\mathcal{E}_{A,B}(X))^n = E^n_{A,B}(X)$ for all non-negative integers n. Using little more than an algebraic argument, the following generalised version of a result relating (m, X)-isometric properties of pairs (A_1, A_2) and (B_1, B_2) to pairs $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ and $(E_{A_1,A_2}, E_{B_1,B_2})$ is proved: if A_i, B_i, S_i, X are operators in $B(\mathcal{X})$, $1 \leq i \leq 2$ and X a quasi-affinity, then the pair $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ (resp., the pair $(E_{A_1,A_2}, E_{B_1,B_2})$) is strict (m, X)-isometric for all $X \in B(\mathcal{X})$ if and only if there exist positive integers $m_i \leq m$, $1 \leq i \leq 2$ and $m = m_1 + m_2 - 1$, and a non-zero scalar β such that $I - \mathcal{E}_{\beta A_1,A_2}(S_1)$ is (strict) m_1-nilpotent and $I - \mathcal{E}_{\frac{1}{\beta} B_1,B_2}(S_2)$ is (strict) m_2-nilpotent (resp., $(\beta A_1, B_1)$ is strict (m_1, I)-isometric and $(\frac{1}{\beta} B_2, A_2)$ is strict (m_2, I)-isometric).

Keywords: Banach space; *m*-isometric; left/right multiplication operator; null symbol; strict *m*-isometric operator

MSC: 47A55; 47A05; 47A65; 47B25



Academic Editors: Boško Damjanović and Pradip Debnath

Received: 9 May 2025
Revised: 23 June 2025
Accepted: 23 June 2025
Published: 27 June 2025

Citation: Duggal, B.P. *m*-Isometric Operators with Null Symbol and Elementary Operator Entries. *Axioms* **2025**, *14*, 503. https://doi.org/10.3390/axioms14070503

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1. Introduction

Let $B(\mathcal{X})$ (resp., $B(\mathcal{H})$) denote the algebra of operators, i.e., bounded linear transformations, on a complex infinite dimensional Banach space $\mathcal{X}$ (resp., Hilbert space $\mathcal{H}$) into itself. An operator pair $(A, B) \in B(\mathcal{X}) \times B(\mathcal{X})$ is (m, X)-isometric for some positive integer m, $m \in \mathbb{N}_+$, and operator $X \in B(\mathcal{X})$, if

$$\begin{aligned} \triangle^m_{A,B}(X) &= (I - L_A R_B)^m(X) = \left(\sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} L_A^j R_B^j\right)(X) \\ &= \sum_{j=0}^{m} (-1)^j \begin{pmatrix} m \\ j \end{pmatrix} A^j X B^j \\ &= 0, \end{aligned}$$

where I is the identity of $B(\mathcal{X})$, L_A and $R_B \in B(B(\mathcal{X}))$ are the operators

$$L_A(Z) = AZ \text{ and } R_B(Z) = ZB,\ Z \in B(\mathcal{X}),$$

of left multiplication by A and right multiplication by B, respectively. (m, X)-isometric operators arise naturally in classical Function Theory and a study of the structure of such operators has been carried out by a number of authors in the recent past (see [1–8] and some of the references cited there). A number of the properties of (m, X)-isometric operators lie on the surface and are readily obtained. Thus,

$$(A, B) \in (m, X) - \textit{isometric} \Longrightarrow \triangle_{A,B}^{m+t}(X) = \triangle_{A,B}^{t}(\triangle_{A,B}^{m}(X)) = 0 \text{ for all } t \in \mathtt{N};$$

$$\triangle_{A^n,B^n}^{m}(X) = 0 \Longleftrightarrow (I - L_A^n R_B^n)^m(X) = \left(\sum_{j=0}^{n-1}(L_A R_B)^j\right)^m (\triangle_{A,B}^{m}(X)) = 0 \text{ for all } n \in \mathtt{N};$$

$$\triangle_{A,B}^{m}(X) = 0 \Longleftrightarrow \triangle_{A.B}^{m-1}(X) = L_A R_B(\triangle_{A,B}^{m-1}(X)) \Longleftrightarrow \triangle_{A.B}^{m-1}(X) = L_A^n R_B^n(\triangle_{A,B}^{m-1}(X)) \text{ for all } n \in \mathtt{N}.$$

Some other properties of (m, X)-isometric operators lie deeper and their proof requires some argument. For example, if $\triangle_{A,B}^{m}(X) = 0$, $N_1 \in B(\mathcal{X})$ is an n_1-nilpotent operator for some positive integer $n_1 \in \mathtt{N}$, $[A, N_1] = AN_1 - N_1A = 0$ (i.e., A commutes with N_1), then

$$\begin{aligned}\triangle_{A+N_1,B}^{m+n_1-1}(X) &= \sum_{j=0}^{m+n_1-1}(-1)^j\begin{pmatrix} m+n_1-1 \\ j \end{pmatrix}(L_{N_1}R_B)^j\triangle_{A,B}^{m+n_1-1-j}(X)\\ &= 0,\end{aligned}$$

since $\triangle_{A,B}^{m+n_1-1-j}(X) = 0$ for all $m + n_1 - 1 - j \geq m$, equivalently, $j \leq n_1 - 1$, and $L_{N_1}^j = 0$ for all $j \geq n_1$. Furthermore, if $N_2 \in B(\mathcal{X})$ is also an n_2-nilpotent operator which commutes with B, then

$$\triangle_{A+N_1,B+N_2}^{m+n_1+n_2-2}(X) = \sum_{j=0}^{m+n_1+n_2-2}(-1)^j\begin{pmatrix} m+n_1+n_2-2 \\ j \end{pmatrix}(L_{A+N_1}R_{N_2})^j\triangle_{A+N_1,B}^{m+n_1+n_2-2-j}(X) = 0,$$

since $\triangle_{A+N_1,B}^{m+n_1+n_2-2-j}(X) = 0$ if $m + n_1 + n_2 - 2 - j \geq m + n_1 - 1$, equivalently, $j \leq n_2 - 1$, and $R_{N_2}^j = 0$ if $j \geq n_2$. Conclusion: if (A, B) is (m, X)-isometric, N_i are n_i-nilpotent operators, and $[A, N_1] = [B, N_2] = 0$, then $(A + N_1, B + N_2)$ is $(m + n_1 + n_2 - 2, X)$-isometric.

Let $A, B, S, A_i, B_i, S_i \in B(\mathcal{X})$, $1 \leq i \leq 2$, and let $E_{A,B} \in B(B(\mathcal{X}))$ and $\mathcal{E}_{A,B}(S) \in B(\mathcal{X})$ be the operators defined by

$$\begin{aligned}&E_{A,B} = L_A R_B \text{ so that } E_{A,B}^n(S) = (L_A R_B)^n(S) = A^n S B^n \text{ for all } n \in \mathtt{N},\\ &\mathcal{E}_{A,B}(S)^n = E_{A,B}^n(S) = A^n S B^n \text{ for all } n \in \mathtt{N}, \text{ in particular } \mathcal{E}_{A,B}(S)^0 = S.\end{aligned}$$

This paper considers (m, X)-isometric pairs $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ such that $\triangle_{A_1,A_2}^{m_1}(S_1) = 0 = \triangle_{B_1,B_2}^{m_2}(S_2)$ for some integers $m_1, m_2 \in \mathtt{N}_+$, and pairs $(E_{A_1,A_2}, E_{B_1,B_2})$. ($(m, X)$-isometric operators with entries $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ have been called (m, X)-*isometric operators with null symbol entries* [2,7]; m-isometric operators with entries of type $(E_{A_1,A_2}, E_{B_1,B_2})$ have been considered by Gu [1], and Duggal and Kim [9,10].) Let X be a quasi-affinity. (Thus, X is injective and has a dense range.) Using little more than linear algebra, we generalise [2] (Theorems 1 and 2(iii)) to prove that "any two of the conditions (i) $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ is (m, X)-isometric; (ii) there exists a positive integer $m_1 \leq m$ such that $I - \mathcal{E}_{A_1,A_2}(S_1)$ is m_1-nilpotent; (iii) there exists a positive integer $m_2 \leq m$ such that $I - \mathcal{E}_{B_1,B_2}(S_2)$ is m_2-nilpotent implies the third." Recall that an (m, X)-isometric pair is *strict* (m, X)-*isometric* if $\triangle_{A,B}^{m}(X) = 0$ and $\triangle_{A,B}^{m-1}(X) \neq 0$. In a similar vein, *we say that an operator A is strict m-nilpotent if $A^m = 0$ and $A^{m-1} \neq 0$.* Answering an open problem raised in [2] (Section 4), we give an elementary proof that the pair $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ (resp., the pair $(E_{A_1,A_2}, E_{B_1,B_2})$) is strict (m, X)-isometric for all $X \in B(\mathcal{X})$ if and only if there exist pos-

itive integers $m_i \leq m$, $1 \leq i \leq 2$ and $m = m_1 + m_2 - 1$, and a non-zero scalar β such that $I - \mathcal{E}_{\beta A_1, A_2}(S_1)$ is strict m_1-nilpotent and $I - \mathcal{E}_{\frac{1}{\beta}B_1, B_2}(S_2)$ is strict m_2-nilpotent (resp., $(\beta A_1, B_1)$ is strict (m_1, I)-isometric and $(\frac{1}{\beta}B_2, A_2)$ is strict (m_2, I)-isometric).

Most of our notation is standard (and any non-standard notation will be explained at the point of its introduction). We write $A - \lambda$ for $A - \lambda I$, and $\sigma_a(A)$ for the approximate point spectrum of the operator A. We say that the pair (A, B) of operators in $B(\mathcal{X}) \times B(\mathcal{X})$ is m-isometric if it is (m, X)-isometric for all $X \in B(\mathcal{X})$.

The plan of the paper is as follows. We consider null symbol operator pairs $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ in Section 2, Section 3 considers pairs $(E_{A_1,A_2}, E_{B_1,B_2})$, Section 4 consists of a concluding remark.

2. Null Symbol Entries: Pairs $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$

If $(A, B) \in B(\mathcal{X}) \times B(\mathcal{X})$ is (m, X)-isometric for some $X \in B(\mathcal{X})$, then

$$\begin{aligned} 0 &= \triangle^m_{A,B}(X) = \left(\sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix}(L_A R_B)^j\right)(X) \\ &= \left(\sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} E^j_{A,B}\right)(X) \\ &= \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \mathcal{E}_{A,B}(X)^j \\ &= (I - \mathcal{E}_{A,B}(X))^m = \nabla^m_{A,B}(X) \end{aligned}$$

(by the definition of the operator $\mathcal{E}_{A,B}(X)^n$, $n \in \mathtt{N}_+$). Hence, *the pair (A, B) is (strictly) (m, X)-isometric if and only if the operator $\nabla_{A,B}(X) = (I - \mathcal{E}_{A,B}(X))$ is (strictly) m-nilpotent.*

Recalling that (A, B) is strictly (m, X)-isometric if and only if $\triangle^m_{A,B}(X) = 0$ and $\triangle^{m-1}_{A,B}(X) \neq 0$, it follows that if $\nabla_{A,B}(X)$ is strictly m-nilpotent, then the sequence of operators $\{\nabla^j_{A,B}(X)\}_{j=0}^{m-1}$ is linearly independent. Again, if (A, B) is (m, X)-isometric, then

$$\begin{aligned} & \nabla^m_{A,B}(X) = (I - \mathcal{E}_{A,B}(X))\nabla^{m-1}_{A,B}(X) = 0 \\ \iff & \nabla^{m-1}_{A,B}(X) = \mathcal{E}_{A,B}(X)\nabla^{m-1}_{A,B}(X) \\ \iff & \nabla^{m-1}_{A,B}(X) = \mathcal{E}^n_{A,B}(X)\nabla^{m-1}_{A,B}(X) \end{aligned}$$

for all $n \in \mathtt{N}$. Let A_i, B_i and S_i, $1 \leq i \leq 2$, be operators in $B(\mathcal{X})$ such that

$$\nabla^{m_1}_{A_1,A_2}(S_1) = 0 = \nabla^{m_2}_{B_1,B_2}(S_2),\ m_1 \text{ and } m_2 \in \mathtt{N}_+.$$

The following proposition relates (m_1, S_1)-isometric pairs (A_1, A_2) and (m_2, S_2)-isometric pairs (B_1, B_2) to (m, X)-isometric pairs $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$.

Proposition 1. *If $X \in B(\mathcal{X})$ is a quasi-affinity, then any two of the following conditions implies the third:*

(i) there exists $m \in \mathtt{N}_+$ such that $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ is (m, X)-isometric;
(ii) there exists $m_1 \in \mathtt{N}_+$, $m_1 \leq m$, such that $\nabla_{A_1,A_2}(S_1)$ is m_1-nilpotent;
(iii) there exists $m_2 \in \mathtt{N}_+$, $m_2 \leq m$, such that $\nabla_{B_1,B_2}(S_2)$ is m_2-nilpotent.
(Here, if (ii) and (iii) hold, then $m = m_1 + m_2 - 1$.)

Proof. Let, for convenience, $\mathcal{E}_{A_1,A_2}(S_1) = A$ and $\mathcal{E}_{B_1,B_2}(S_2) = B$.

$(ii) + (iii) \Longrightarrow (i)$. Let $m_1 + m_2 - 1 = m$. By definition

$$\begin{aligned}\triangle_{A,B}^{m}(X) &= (I - L_A R_B)^m(X) = ((I - L_A)R_B + (I - R_B))^m(X)\\ &= \left(\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}((I - L_A)R_B)^{m-j}(I - R_B)^j\right)(X)\\ &= \sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}(I - A)^{m-j}XB^{m-j}(I - B)^j\\ &= \sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\nabla_{A_1,A_2}^{m-j}(S_1)XB^{m-j}\nabla_{B_1,B_2}^{j}(S_2)\end{aligned}$$

(since $[I - L_A, R_B] = [R_B, I - R_B] = 0$). Since $\nabla_{A_1,A_2}^{m-j}(S_1) = 0$ for all $m - j \geq m_1$, equivalently, $j \leq m - m_1 = m_2 - 1$, and since $\nabla_{B_1,B_2}^{j}(S_2) = 0$ for all $j \geq m_2$, $\triangle_{A,B}^{m_0}(X) = 0$ for all $m_0 \geq m$.

$(i) + (iii) \Longrightarrow (ii)$. Considering next (i) and (iii), we may assume without loss of generality that (A, B) is strict (m, X)-isometric and $\nabla_{B_1,B_2}(S_2)$ is strict m_2-nilpotent. Since

$$\nabla_{B_1,B_2}^{m_2}(S_2) = 0 \Longleftrightarrow \nabla_{B_1,B_2}^{m_2-1}(S_2) = B^n\nabla_{B_1,B_2}^{m_2-1}(S_2)$$

for all $n \in \mathtt{N}$, the strictness implies that the sequence

$$\{B^n\nabla_{B_1,B_2}^{j}(S_2)\},\ 0 \leq j \leq m_2 - 1$$

is linearly independent for all $n \in \mathtt{N}$. If (i) holds, then (see above)

$$\begin{aligned}0 &= \triangle_{A,B}^{m}(X) = \left(\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}L_{\nabla_{A_1,A_2}(S_1)}^{m-j}R_B^{m-j}R_{\nabla_{B_1,B_2}(S_2)}^{j}\right)(X)\\ &= \sum_{j=0}^{m_2-1}\begin{pmatrix} m \\ j \end{pmatrix}\nabla_{A_1,A_2}^{m-j}(S_1)XB^{m-j}\nabla_{B_1,B_2}^{j}(S_2)\end{aligned}$$

(since $\nabla_{B_1,B_2}^{t}(S_2) = 0$ for all $t \geq m_2$). The linear independence of the sequence $\{B^n\nabla_{B_1,B_2}^{j}(S_2)\}_{j=0}^{m_2-1}$, taken along with the fact that X has a dense range, implies that

$$\nabla_{A_1,A_2}^{m-j}(S_1)X = 0 \Longleftrightarrow \nabla_{A_1,A_2}^{m-j}(S_1) = 0,\ 0 \leq j \leq m_2 - 1.$$

In particular, letting $j = m_2 - 1$,

$$\nabla_{A_1,A_2}^{m-m_2+1}(S_1) = \nabla_{A_1,A_2}^{m_1}(S_1) = 0.$$

$(i) + (ii) \Longrightarrow (iii)$. The proof here is similar to that of the previous case except for the fact that we now consider the adjoint operators

$$(\triangle_{A,B}^{m}(X))^* = \triangle_{B^*,A^*}^{m}(X^*) = \triangle_{\mathcal{E}_{B_2^*,B_1^*}(S_2^*),\mathcal{E}_{A_2^*,A_1^*}(S_1^*)}^{m}(X^*)$$

and

$$(\nabla_{A_1,A_2}^{m_1}(S_1))^* = \nabla_{A_2^*,A_1^*}^{m_1}(S_1^*).$$

Assuming strictness, it is seen that the sequence $\{A^{*n}\nabla^{j}_{A_2^*,A_1^*}(S_1^*)\}$ is linearly independent for all $0 \leq j \leq m_1 - 1$ and $n \in \mathtt{N}$; hence, since X is a quasi-affinity,

$$\nabla^{m-j}_{B_2^*,B_1^*}(S_2^*)X^* = 0 \Longleftrightarrow \nabla^{m-j}_{B_2^*,B_1^*}(S_2^*) = 0 \Longleftrightarrow \nabla^{m-j}_{B_1,B_2}(S_2) = 0$$

for all $0 \leq j \leq m_1 - 1$. In particular, $\nabla^{m-m_1+1}_{B_1,B_2}(S_2) = \nabla^{m_2}_{B_1,B_2}(S_2) = 0$. □

If $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ is strict (m, X)-isometric and $\nabla_{B_1,B_2}(S_2)$ is strict m_2-nilpotent, then the argument of the proof of the proposition shows that $\nabla_{A_1,A_2}(S_1)$ is strict $m_1 = m - m_2 + 1$ nilpotent. (Reason: if $\nabla_{A_1,A_2}(S_1)$ is t-nilpotent for some $t < m_1$, then (ii) and (iii) taken together imply $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ is $(t + m_1 - 1(< m), X)$-isometric—a contradiction.) Indeed, if $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ is strict (m, X)-isometric, $\nabla_{A_1,A_2}(S_1)$ is m_1-nilpotent, and $\nabla_{B_1,B_2}(S_2)$ is m_2 nilpotent, then

$$\begin{aligned}
& \triangle^{m-1}_{A,B}(X) = \sum_{j=0}^{m-1} \begin{pmatrix} m-1 \\ j \end{pmatrix} \nabla^{m-j-1}_{A_1,A_2}(S_1) X B^{m-j-1} \nabla^{j}_{B_1,B_2}(S_2) \\
= & \sum_{j=0}^{m_2-1} \begin{pmatrix} m-1 \\ j \end{pmatrix} \nabla^{m-j-1}_{A_1,A_2}(S_1) X B^{m-j-1} \nabla^{j}_{B_1,B_2}(S_2) \\
& (\nabla^{j}_{B_1,B_2}(S_2) = 0 \text{ for all } j \geq m_2) \\
= & \begin{pmatrix} m-1 \\ m_2-1 \end{pmatrix} \nabla^{m_1-1}_{A_1,A_2}(S_1) X B^{m-j-1} \nabla^{m_2-1}_{B_1,B_2}(S_2) \\
& (\nabla^{m-j-1}_{A_1,A_2}(S_1) = 0 \text{ for all } m - j - 1 \geq m_1) \\
= & \begin{pmatrix} m-1 \\ m_2-1 \end{pmatrix} \nabla^{m_1-1}_{A_1,A_2}(S_1) X \nabla^{m_2-1}_{B_1,B_2}(S_2) \\
& (\nabla^{m_2}_{B_1,B_2}(S_2) = 0 \Longrightarrow B^{m-j-1}\nabla^{j}_{B_1,B_2}(S_2) = \nabla^{j}_{B_1,B_2}(S_2) \text{ for all } j \geq m_2) \\
\neq & \ 0.
\end{aligned}$$

Thus

$$(I - A)^{m_1-1} X (I - B)^{m_2-1} \neq 0,$$

and the conditions $(I - A)^{m_1-1} \neq 0$ and $(I - B)^{m_2-1} \neq 0$ are necessary for $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ to be strict (m, X)-isometric. These conditions are, however, not sufficient. For example, if we choose X to be such that it maps the range of $(I - B)$ into the null space of $(I - A)$, then $(I - A)^{m_1-1} X (I - B)^{m_2-1} = 0$ even though neither of $(I - A)^{m_1-1}$ and $(I - B)^{m_2-1}$ is the 0 operator. If, however, $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ is strict m-isometric (i.e., strict (m, X)-isometric for all X), then, necessarily, $(I - A)^{m_1-1} X (I - B)^{m_2-1} = 0$ if and only if one $(I - A)^{m_1-1}$ or $(I - B)^{m_2-1}$ is 0. The proof of the following proposition uses little more than linear algebra to prove a necessary and sufficient condition for the operator pair $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ to be strict (m, X)-isometric for a given quasi-affinity X.

Proposition 2. *Given operators $A_i, B_i, S_i \in B(\mathcal{X})$, $1 \leq i \leq 2$, and a quasi-affinity $X \in B(\mathcal{X})$, the pair of operators $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ is a strict (m, X)-isometry if and only if there exist integers $m_i \in \mathtt{N}_+$ $(1 \leq i \leq 2)$ and a non-trivial scalar β such that $m = m_1 + m_2 - 1$,*

(i) $\nabla_{\beta A_1,A_2}(S_1)$ is a strict m_1 nilpotent, $\nabla_{\frac{1}{\beta}B_1,B_2}(S_2)$ is a strict m_2-nilpotent;

(ii) $\nabla^{m_1-1}_{A_1,A_2}(S_1) X \nabla^{m_2-1}_{B_1,B_2}(S_2) \neq 0$.

Proof. As before, for convenience, we let

$$\mathcal{E}_{A_1,A_2}(S_1) = A, \mathcal{E}_{B_1,B_2}(S_2) = B, I - \beta A (= I - \mathcal{E}_{\beta A_1,A_2}(S_1)) = \nabla(\beta A), \text{and}$$
$$I - \frac{1}{\beta}B = I - \frac{1}{\beta}\mathcal{E}_{B_1,B_2}(S_2) = 1 - \mathcal{E}_{\frac{1}{\beta}B_1,B_2}(S_2) = \nabla(\beta B).$$

To prove the "if part" of the theorem, we start by observing that

$$\triangle^m_{\beta A,\frac{1}{\beta}B}(X) = \left(\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix} L^j_{\beta A} R^j_{\frac{1}{\beta}B}\right)(X) = \triangle^m_{A,B}(X).$$

If $\nabla(\beta A)$ is m_1-nilpotent and $\nabla(\beta B)$ is m_2-nilpotent, then

$$\triangle^{m_1+m_2-1}_{\beta A,\frac{1}{\beta}B}(X) = \triangle^{m_1+m_2-1}_{A,B}(X) = \triangle^m_{A,B}(X) = 0.$$

Also, if condition (ii) is satisfied, then $\triangle^{m-1}_{A,B}(X) \neq 0$, i.e., (A, B) is strict (m, X)-isometric.

For the "only if part" of the proof, we start by recalling that

$$\begin{aligned}\triangle^m_{A,B}(X) &= \left(\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix} L^j_A R^j_B\right)(X) \\ &= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}\mathcal{E}^j_{A_1,A_2}(S_1)X\mathcal{E}^j_{B_1,B_2}(S_2).\end{aligned}$$

We claim that there exists a non-trivial $\beta \in \sigma_a(B)$. For this it will suffice to prove that $0 \notin \sigma_a(B)$. If $0 \in \sigma_a(B)$, then there exist sequences $\{x_n\}$ and $\{x'_n\}$, in $\mathcal{X}$ and (the dual space) X^*, respectively, such that

$$x'_n(x_n) = \langle x_n, x'_n\rangle = 1 \text{ for all } n \in \mathtt{N}_+ \text{ and } \lim_{n\to\infty}\langle Bx_n, x'_n\rangle = 0.$$

We have

$$\begin{aligned}\triangle^m_{A,B}(X) &= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix} A^j X \lim_{n\to\infty}\langle B^j x_n, x'_n\rangle \\ &= 0\end{aligned}$$

for all j except for $j = 0$ when we have $\triangle^m_{A,B}(X) = X = 0$. The operator X being a quasi-affinity, this is a contradiction and our claim is proved.

Let $(0 \neq)\beta \in \sigma_a(B)$. Then, by an argument similar to the one used above to prove $0 \notin \sigma_a(B)$,

$$\begin{aligned}0 &= \triangle^m_{A,B}(X) = \left(\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}\beta^j A^j\right)(X) \\ &\iff \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}\beta^j A^j = (I - \beta A)^m = 0.\end{aligned}$$

Let m_1 be the least positive integer such that

$$(I - \beta A)^{m_1} = (I - \beta\mathcal{E}_{A_1,A_2}(S_1))^{m_1} = (I - \mathcal{E}_{\beta A_1,A_2}(S_1))^{m_1} = 0.$$

Then $I - \beta A$ is strictly m_1-nilpotent and the sequence $\{\nabla^j(\beta A)\}_{j=0}^{m_1-1}$ is linearly independent. Since

$$\begin{aligned}
0 &= \triangle^m_{A,B}(X) = \triangle^m_{\beta A, \frac{1}{\beta}B}(X) \\
&= \left((I - R_{\frac{1}{\beta}B}) + (I - L_{\beta A})R_{\frac{1}{\beta}B}\right)^m(X) \\
&= \left(\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}(I - R_{\frac{1}{\beta}B})^{m-j}R^j_{\frac{1}{\beta}B}(I - L_{\beta A})^j\right)(X) \\
&= \sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\nabla^j(\beta A)X(\frac{1}{\beta}B)^j\nabla^{m-j}(\beta B) \\
&= \sum_{j=0}^{m_1-1}\begin{pmatrix} m \\ j \end{pmatrix}\nabla^j(\beta A)X(\frac{1}{\beta}B)^j\nabla^{m-j}(\beta B) \\
&\quad (\text{since } \nabla^t(\beta A) = 0 \text{ for all } t \geq m_1) \\
\implies &\quad X(\frac{1}{\beta}B)^j\nabla^{m-j}(\beta B) = 0 \text{ for all } 0 \leq j \leq m_1 - 1.
\end{aligned}$$

Since both X and B are injective, recall that X is a quasi-affinity and B is left invertible, $\nabla^{m-j}(\beta B) = 0$ for all $0 \leq j \leq m_1 - 1$. In particular,

$$\nabla^{m-m_1+1}(\beta B) = \nabla^{m_2}(\beta B) = 0.$$

The strict (m, X)-isometric property of the pair (A, B), taken in conjunction with the fact that $m_2 = m - m_1 + 1$, implies that $\nabla(\beta B)$ is strict m_2-nilpotent. The necessity of condition (ii) having already been seen (above), the proof is complete. □

The Hilbert space case In the case in which $A_1^* = A_2 = A$ and $B_1^* = B_2 = B$ are Hilbert space operators,

$$\begin{aligned}
&\mathcal{E}_{A^*,A}(I)^n = E^n_{A^*,A}(I) = A^{*n}A^n,\ \mathcal{E}_{B^*,B}(I)^n = E^n_{B^*,B}(I) = B^{*n}B^n, \\
&\triangle^n_{\mathcal{E}_{A^*,A}(I),\mathcal{E}_{B^*,B}(I)}(X) = \sum_{j=0}^{n}\begin{pmatrix} n \\ j \end{pmatrix}A^{*j}A^jXB^{*j}B^j
\end{aligned}$$

for all $n \in \mathbb{N}_+$. Proposition 2 in such a case takes the following form.

Corollary 1. *Given operators $A, B, X \in B(\mathcal{H})$, X a quasi-affinity, the pair $(\mathcal{E}_{A^*,A}(I), \mathcal{E}_{B^*,B}(I))$ is strict (m, X)-isometric if and only if there exist positive integers $m_i \leq m$, $m = m_1 + m_2 - 1$, and a (non-trivial) positive scalar β such that $(\beta A^*, A)$ is strict (m_1, I)-isometric, $(\frac{1}{\beta}B^*, B)$ is strict (m_2, I)-isometric, and $\triangle^{m_1-1}_{\beta A^*,A}(I)X\triangle^{m_2-1}_{\frac{1}{\beta}B^*,B}(I) \neq 0$.*

In the absence of the property $\mathcal{E}_{A,B}(I)^n = E^n_{A,B}(I)$ for all $n \in \mathbb{N}$, the strict (m, X)-isometric property of the pair $(E_{A^*,A}(I), E_{B^*,B}(I))$ for a quasi-affinty X implies

$$\triangle^n_{E_{A^*,A}(I),E_{B^*,B}(I)}(X) = \sum_{j=0}^{n}\begin{pmatrix} n \\ j \end{pmatrix}|A|^{2j}X|B|^{2j};$$

there exist positive integers $m_i \leq m$, $m = m_1 + m_2 - 1$, and a (non-trivial) positive scalar β such that $(\beta|A|, |A|)$ is strict (m_1, I)-isometric, $(\frac{1}{\beta}|B|, |B|)$ is strict (m_2, I)-isometric and $\triangle^{m_1-1}_{\beta|A|,|A|}(I)X\triangle^{m_2-1}_{\frac{1}{\beta}|B|,|B|}(I) \neq 0$.

3. Pairs $(E_{A_1,A_2}, E_{B_1,B_2})$

Proposition 1 fails in the absence of the (fairly restrictive) hypothesis $\mathcal{E}_{A,B}(S)^n = E^n_{A,B}(S)$. This follows from the following elementary example.

Example 1. *Trivially, the pair (I, I) satisfies $\triangle_{I,I}(S) = 0$ for all $S \in B(\mathcal{X})$. Considering the pair $(\mathcal{E}_{I,I}(S_1), \mathcal{E}_{I,I}(S_2))$, the validity of $(ii) + (iii)$ implies (i) in the proof of the proposition implies $\triangle_{\mathcal{E}_{I,I}(S_1),\mathcal{E}_{I,I}(S_2)}(X) = 0$ for all $X \in B(\mathcal{X})$, i.e., $X = S_1XS_2$ for all S_1, S_2 and $X \in B(\mathcal{X})$. This is absurd.*

Observe that $E^n_{I,I}(S) = S \neq S^n = (\mathcal{E}_{I,I}(S))^n$ and $E^n_{A,B}(I) = A^nB^n \neq (AB)^n = \mathcal{E}_{A,B}(I)^n$. An immediate consequence of examples of the above type is that the proposition cannot be used, contrary to the claim made in [2] (Corollary 1), to deduce results of the the type "(A_1, A_2) is (m_1, I)-isometric and (B_1, B_2) is (m_2, I)-isometric, then $(E_{A_1,A_2}, E_{B_1,B_2})$ is $(m_1 + m_2 - 1, X)$-isometric for all $X \in B(\mathcal{X})$". Indeed, if we let the pair $(E_{A_1,A_2}, E_{B_1,B_2})$ be such that $(A_1, A_2) = (I, I)$ and choose the pair (B_1, B_2) to be (m, I)-isometric for some $m \in \mathtt{N}_+$, then for all $X \in B(\mathcal{X})$,

$$\begin{aligned}
\triangle^m_{E_{A_1,A_2},E_{B_1,B_2}}(X) &= \left(\sum_{j=0}^{m}(-1)^j\binom{m}{j}L^j_{E_{A_1,A_2}}R^j_{E_{B_1,B_2}}\right)(X)\\
&= \sum_{j=0}^{m}(-1)^j\binom{m}{j}L^j_{E_{A_1,A_2}}\left(L^j_{B_1}XR^j_{B_2}\right)\\
&= \sum_{j=0}^{m}(-1)^j\binom{m}{j}L^j_{E_{A_1,A_2}}\left(B^j_1XB^j_2\right)\\
&= \sum_{j=0}^{m}(-1)^j\binom{m}{j}L^j_{A_1}\left(B^j_1XB^j_2\right)R^j_{A_2}\\
&= \sum_{j=0}^{m}(-1)^j\binom{m}{j}A^j_1B^j_1XB^j_2A^j_2\\
&= \sum_{j=0}^{m}(-1)^j\binom{m}{j}B^j_1XB^j_2 = \triangle^m_{B_1,B_2}(X).
\end{aligned}$$

Evidently, $\triangle^m_{E_{A_1,A_2},E_{B_1,B_2}}(X)$ cannot be 0 for all X (i.e., (ii) and (iii) imply (i) of Proposition 1 fails for pairs $(E_{A_1,A_2}, E_{B_1,B_2})$). We remark here that even though [2] (Theorem 1) makes no explicit mention of the hypothesis $\mathcal{E}_{A,B}(X)^n = E^n_{A,B}(X)$, its use is implicit in the proof of the theorem.

A pair $(\mathcal{X}, \tilde{\mathcal{X}})$ of Banach spaces is a dual pairing if either $\mathcal{X} = \mathcal{X}^*$ or $\mathcal{X} = \tilde{\mathcal{X}}^*$. If we let $x \otimes y'$, $x \in \mathcal{X}$ and $y' \in \mathcal{Y}^*$, $\mathcal{Y}$ a Banach space, denote the rank one operator $\mathcal{Y} \to \mathcal{X}$, $y \to \langle y, y'\rangle x$, then the operator ideal $\mathcal{J}$ between $\mathcal{Y}$ and $\mathcal{X}$ is a linear subspace $B(\mathcal{Y}, \mathcal{X})$ equipped with a Banach norm ν such that (i) $x \otimes y' \in \mathcal{J}$; $\nu(x \otimes y') = \|x\|\|y'\|$ and (ii) $\mathcal{E}_{A,B}(X) = L_AR_B(X) = AXB$, $\nu(AXB) \leq \|A\|\nu(\|X\|)\|B\|$ for all $x \in \mathcal{X}$, $y' \in \mathcal{Y}^*$, $X \in \mathcal{J}$, and $A \in B(\mathcal{X})$ and $B \in B(\mathcal{Y})$. Thus defined, each $\mathcal{J}$ is a tensor product relative to the dual pairings $(\mathcal{X}, \mathcal{X}^*)$ and $(\mathcal{Y}, \mathcal{Y}^*)$ and the bilinear mapping

$$\begin{aligned}
&\mathcal{X} \times \mathcal{Y}^* \to \mathcal{J},\ \langle x, y'\rangle \to x \otimes y',\\
&B(\mathcal{X}) \times B(\mathcal{Y}^*) \to B(\mathcal{J}),\ (A, B^*) \to A \otimes B^*,
\end{aligned}$$

where $A \otimes B^*(X) = AXB = E_{A,B}(X)$ [11] (page 51). It is known, see [9] (Corollary 2) (see also [1,7]), that for $A_i, B_i \in B(\mathcal{X})$, $1 \leq i \leq 2$, $(E_{A_1,A_2}, E_{B_1,B_2})$ is strict (m, X)-isometric if and only if there exist positive integers $m_i \leq m$ and a non-zero scalar β such that

$m = m_1 + m_2 - 1$, $(\beta A_1, A_2)$ is strict (m_1, I)-isometric, $(\frac{1}{\beta}B_2, A_2)$ is strict (m_2, I)-isometric, and $\triangle^{m_1-1}_{\beta A_1, B_1}\left(\triangle^{m_2-1}_{\frac{1}{\beta}B_2, A_2}(X)\right) \neq 0$ for $X \in \mathcal{J}$. This result does not follow from Proposition 2, even for the case in which $\mathcal{X}$ is a Hilbert space and the pair $(E_{A_1,A_2}, E_{B_1,B_2})$ is the pair $(E_{A^*,A}, E_{B^*,B})$. The following theorem, our main result, uses an algebraic argument to prove this result for the case in which the operator pair $(E_{A_1,A_2}, E_{B_1,B_2})$ is strict (m, X)-isometric for a quasi-affinity $X \in B(\mathcal{X})$.

Theorem 1. *Given operators $A_i, B_i \in B(\mathcal{X})$, $1 \leq i \leq 2$, the pair $(E_{A_1,A_2}, E_{B_1,B_2})$ is strict (m, X)-isometric for a quasi-affinity $X \in B(\mathcal{X})$ if and only if there exist positive integers $m_i \leq m$ and a non-trivial scalar β such that $m = m_1 + m_2 - 1$, $(\beta A_1, B_1)$ is strict (m_1, I)-isometric, $(\frac{1}{\beta}B_2, A_2)$ is strict (m_2, I)-isometric, and $\triangle^{m_1-1}_{\beta A_1, B_1}(I)X(\triangle^{m_2-1}_{\frac{1}{\beta}B_2, A_2}(I)) \neq 0$.*

Proof. We start by proving the "only if" part: the proof depends upon a judicious use of the properties of the operator $L^j_{E_{A_1,A_2}} R^j_{E_{B_1,B_2}}$. If the pair $(E_{A_1,A_2}, E_{B_1,B_2})$ is (m, X)-isometric, then

$$\triangle^m_{E_{A_1,A_2}, E_{B_1,B_2}}(X) = \left(\sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} L^j_{E_{A_1,A_2}} R^j_{E_{B_1,B_2}}\right)(X) = 0.$$

By definition

$$\begin{aligned}
& L^j_{E_{A_1,A_2}} R^j_{E_{B_1,B_2}}(X) = L^j_{L_{A_1} R_{A_2}} R^j_{L_{B_1} R_{B_2}}(X) \\
= \; & \left(L^j_{L_{A_1}} L^j_{R_{A_2}} R^j_{L_{B_1}} R^j_{R_{B_2}}\right)(X) \\
= \; & \left(L^j_{L_{A_1}} R^j_{L_{B_1}} L^j_{R_{A_2}} R^j_{R_{B_2}}\right)(X) \qquad ([L_C, R_D] = 0 \text{ for all operators } C, D) \\
= \; & \left(L_{L_{A_1}} R_{L_{B_1}}\right)^j X \left(L_{R_{A_2}} R_{R_{B_2}}\right)^j \\
= \; & L^j_{L_{A_1}} R^j_{L_{B_1}} \left(L^j_{R_{A_2}} X R^j_{R_{B_2}}\right) \\
= \; & L^j_{L_{A_1}} R^j_{L_{B_1}} \left(R^j_{A_2} X R^j_{B_2}\right) = L^j_{L_{A_1}} R^j_{L_{B_1}} \left(X B^j_2 A^j_2\right) \\
= \; & L^j_{A_1} \left(X B^j_2 A^j_2\right) L^j_{B_1} = L^j_{A_1} B^j_1 \left(X B^j_2 A^j_2\right) = A^j_1 B^j_1 X B^j_2 A^j_2.
\end{aligned} \tag{1}$$

(2)

For convenience, set $L_{L_{A_1}} R_{L_{B_1}} = C$ and $L_{R_{A_2}} R_{R_{B_2}} = D$. We claim that $\sigma(D) \neq \{0\}$. Let us suppose to the contrary that $\sigma(D) = \{0\}$. Then there exist sequences $\{Z_n\}$ and $\{Z'_n\}$ of unit vectors (in $B(B(\mathcal{X}))$ and its dual space, respectively) such that

$$Z'_n(Z_n) = \langle Z_n, Z'_n \rangle = 1 \text{ for all } n \in \mathbb{N}_+ \text{ and } \lim_{n\to\infty} DZ_n = 0.$$

We have

$$\triangle^m_{E_{A_1,A_2}, E_{B_1,B_2}}(X) = \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} C^j X \lim_{n\to\infty} \langle D^j Z_n, Z'_n \rangle = 0$$

for all j except $j = 0$ when we obtain $X = 0$. The operator X being a quasi-affinity, we have a contradiction. Hence ($0 \notin \sigma_a(D)$ and) there exists a non-trivial scalar $\beta \in \sigma_a(D)$. Assuming $\{Z_n\}$ and $\{Z'_n\}$ to be sequences of unit vectors such that $Z'_n(Z_n) = 1$ for all $n \in \mathbb{N}_+$ and $\lim_{n\to\infty}\langle DZ_n, Z'_n \rangle = \beta$, we have

$$\begin{aligned}
0 &= \triangle^m_{E_{A_1,A_2},E_{B_1,B_2}}(X) \\
&= \sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}C^jX\lim_{n\to\infty}\langle D^jZ_n, Z_n'\rangle \\
&= \left(\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}(\beta C)^j\right)X \\
&= \left(\sum_{j=0}^{m}(-1)^j\begin{pmatrix} m \\ j \end{pmatrix}(\beta A_1)^jB_1^j\right)X \qquad \text{(see (2))} \\
&= \left(\triangle^m_{\beta A_1,B_1}(I)\right)X.
\end{aligned}$$

The operator X being a quasi-affinity, we conclude $(\beta A_1, B_1)$ is (m, I)-isometric. Consequently there exists a positive integer $m_1 \leq m$ such that $(\beta A_1, B_1)$ is strict (m_1, I)-isometric, and hence the set

$$\{\triangle^j_{\beta A_1,B_1}(I)\}_{j=0}^{m_1-1}, \text{ equivalently } \{\triangle^{m-j}_{\beta A_1,B_1}(I)\}_{j=m-m_1+1}^{m}, \text{ is linearly independent.}$$

Once again, for convenience, set $L_{L_{\beta A_1}}R_{L_{B_1}} = C_\beta$ and $L_{R_{A_2}}R_{R_{\frac{1}{\beta}B_2}} = D_\beta$. Then

$$\begin{aligned}
&\triangle^m_{E_{A_1,A_2},E_{B_1,B_2}}(X) = \triangle^m_{E_{\beta A_1,A_2},E_{B_1,\frac{1}{\beta}B_2}}(X) \\
&= \left(I - L_{L_{\beta A_1}}R_{L_{B_1}}L_{R_{A_2}}R_{R_{\frac{1}{\beta}B_2}}\right)^m(X) \\
&= \left(I - C_\beta D_\beta\right)^m(X) \\
&= \left((I - C_\beta)D_\beta + (I - D_\beta)\right)^m(X) \\
&= \left(\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}(I - C_\beta)^{m-j}D_\beta^{m-j}(I - D_\beta)^j\right)(X) \\
&= \left(\sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\left(\sum_{p=0}^{m-j}(-1)^p\begin{pmatrix} m-j \\ p \end{pmatrix}C_\beta^p\right)D_\beta^{m-j}\left(\sum_{k=0}^{j}(-1)^k\begin{pmatrix} j \\ k \end{pmatrix}D_\beta^k\right)\right)(X).
\end{aligned}$$

Since

$$\begin{aligned}
&D_\beta^{m-j}\left(\sum_{k=0}^{j}(-1)^k\begin{pmatrix} j \\ k \end{pmatrix}D_\beta^k\right)(X) \\
&= X(\frac{1}{\beta}B_2)^{m-j}A_2^{m-j}\left(\sum_{k=0}^{j}(-1)^k\begin{pmatrix} j \\ k \end{pmatrix}(\frac{1}{\beta}B_2)^kA_2^k\right) \\
&= X(\frac{1}{\beta}B_2)^{m-j}A_2^{m-j}\triangle^j_{\frac{1}{\beta}B_2,A_2}(I)
\end{aligned}$$

and

$$\begin{aligned}
&\left(\sum_{p=0}^{m-j}(-1)^p\begin{pmatrix} m-j \\ p \end{pmatrix}C_\beta^p\right)X \\
&= \left(\sum_{p=0}^{m-j}(-1)^p\begin{pmatrix} m-j \\ p \end{pmatrix}(\beta A_1)^pB_1^p\right)X \\
&= \triangle^{m-j}_{\beta A_1,B_1}(I)X,
\end{aligned}$$

we have

$$\begin{aligned}\triangle^{m}_{E_{A_1,A_2},E_{B_1,B_2}}(X) &= \sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\triangle^{m-j}_{\beta A_1,B_1}(I)X(\frac{1}{\beta}B_2)^{m-j}A_2^{m-j}\triangle^{j}_{\frac{1}{\beta}B_2,A_2}(I)\\ &= 0.\end{aligned}$$

By the linear independence of the set $\{\triangle^{m-j}_{\beta A_1,B_1}(I)\}$ for all $m-m_1+1 \leq j \leq m$,

$$X(\frac{1}{\beta}B_2)^{m-j}A_2^{m-j}\triangle^{j}_{\frac{1}{\beta}B_2,A_2}(I) = 0$$

for all $0 \leq j \leq m_1-1$. Since X is a quasi-affinity and $0 \notin \sigma_a(D_\beta)$ (implies $0 \notin \sigma_a(B_2^{m-j}A_2^{m-j})$),

$$\triangle^{j}_{\frac{1}{\beta}B_2,A_2}(I) = 0 \text{ for all } 0 \leq j \leq m_1-1.$$

In particular, upon letting $m-m_1+1 = m_2$,

$$\triangle^{m_2}_{\frac{1}{\beta}B_2,A_2}(I) = 0.$$

The strict (m,X)-isometric property of the pair $(E_{A_1,A_2},E_{B_1,B_2})$ implies also that

$$\begin{aligned}0 &\neq \triangle^{m-1}_{E_{A_1,A_2},E_{B_1,B_2}}(X)\\ &= \textstyle\sum_{j=0}^{m-1}\begin{pmatrix} m-1 \\ j \end{pmatrix}\triangle^{m-1-j}_{\beta A_1,B_1}(I)X(\frac{1}{\beta}B_2)^{m-1-j}A_2^{m-1-j}\triangle^{j}_{\frac{1}{\beta}B_2,A_2}(I)\\ &= \textstyle\sum_{j=0}^{m_2-1}\begin{pmatrix} m-1 \\ j \end{pmatrix}\triangle^{m-1-j}_{\beta A_1,B_1}(I)X(\frac{1}{\beta}B_2)^{m-1-j}A_2^{m-1-j}\triangle^{j}_{\frac{1}{\beta}B_2,A_2}(I)\\ &\quad (\triangle^{j}_{\frac{1}{\beta}B_2,A_2}(I) = 0 \text{ for all } j \geq m_2)\\ &= \begin{pmatrix} m-1 \\ m_2-1 \end{pmatrix}\triangle^{m_1-1}_{\beta A_1,B_1}(I)X(\frac{1}{\beta}B_2)^{m-1-j}A_2^{m-1-j}\triangle^{m_2-1}_{\frac{1}{\beta}B_2,A_2}(I)\\ &\quad (\triangle^{m-1-j}_{\beta A_1,B_1}(I) = 0 \text{ for all } m-1-j \geq m_1, \text{equivalently, } j \leq m_2-2)\\ &= \begin{pmatrix} m-1 \\ m_2-1 \end{pmatrix}\triangle^{m_1-1}_{\beta A_1,B_1}(I)X\triangle^{m_2-1}_{\frac{1}{\beta}B_2,A_2}(I) \qquad (3)\\ &\quad \left(\text{recall }:\triangle^{m_2}_{\frac{1}{\beta}B_2,A_2}(I) = 0 \Longrightarrow (\tfrac{1}{\beta}B_2)^t A_2^t\triangle^{m_2-1}_{\frac{1}{\beta}B_2,A_2}(I) = \triangle^{m_2-1}_{\frac{1}{\beta}B_2,A_2}(I) \text{ for all } t \in \mathbb{N}\right).\end{aligned}$$

Since $\triangle^{m_2-1}_{\frac{1}{\beta}B_2,A_2}(I) = 0$ would contradict this condition, the pair$(\frac{1}{\beta}B_2,A_2)$ is strict (m_2,I)-isometric.

The proof of the reverse implication is straightforward. Thus, if $(\beta A_1,B_1)$ is strict (m_1,I)-isometric, $(\frac{1}{\beta}B_2,A_2)$ is strict (m_2,I)-isometric, and $m = m_1+m_2-1$, then for every quasi-affinity $X \in B(\mathcal{X})$,

$$\begin{aligned}\triangle^{m}_{E_{A_1,A_2},E_{B_1,B_2}}(X) &= \sum_{j=0}^{m}\begin{pmatrix} m \\ j \end{pmatrix}\triangle^{m-j}_{\beta A_1,B_1}(I)X(\frac{1}{\beta}B_2)^{m-j}A_2^{m-j}\triangle^{j}_{\frac{1}{\beta}B_2,A_2}(I)\\ &= 0,\end{aligned}$$

since $\triangle^{m-j}_{\beta A_1,B_1}(I) = 0$ for all $m-j \geq m_1$, equivalently, $j \leq m-m_1 = m_2-1$, and $\triangle^{j}_{\frac{1}{\beta}B_2,A_2}(I) = 0$ for all $j \geq m_2$. The strictness implies

$$\triangle^{m_1-1}_{\beta A_1,B_1}(I) \neq 0 \neq \triangle^{m_2-1}_{\frac{1}{\beta}B_2,A_2}(I)$$

and this in turn implies $\triangle^{m_1-1}_{\beta A_1,B_1}(I)X(\triangle^{m_2-1}_{\frac{1}{\beta}B_2,A_2}(I)) \neq 0$. □

The proof of Theorem 1 in the case in which $A_1^* = B_1 = A$ and $A_2^* = B_2 = B$, A and B Hilbert space operators, is more straightforward. Thus, if the pair $(E_{A^*,B^*}, E_{A,B})$ is (m,X)-isometric for some quasi-affinity $X \in B(\mathcal{H})$, then

$$\triangle^{m}_{E_{A^*,B^*},E_{A,B}}(X) = \sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} A^{*j}A^jXB^jB^{*j},$$

$0 \notin \sigma_a(B^*)$, and $(0 \neq)\beta_0 \in \sigma_a(B^*)$ implies

$$\begin{aligned} & \left(\sum_{j=0}^{m}(-1)^j \begin{pmatrix} m \\ j \end{pmatrix} \beta^j A^{*j}A^j\right)X = 0 \qquad (\beta = |\beta_0|^2) \\ \iff \quad & \triangle^{m}_{\beta A^*,A}(I) = 0. \end{aligned}$$

Corollary 2. *Given operators $A, B \in B(\mathcal{H})$, the pair $(E_{A^*,B^*}, E_{A,B}))$ is strict (m,X)-isometric for a quasi-affinity $X \in B(\mathcal{H})$ if and only if there exist positive integers $m_i \leq m$ and a non-trivial scalar β such that $m = m_1 + m_2 - 1$, $(\beta A^*, A)$ is strict (m_1, I)-isometric, $(\frac{1}{\beta}B, B^*)$ is strict (m_2, I)-isometric, and $\triangle^{m_1-1}_{\beta A^*,A}(I)X(\triangle^{m_2-1}_{\frac{1}{\beta}B,B^*}(I)) \neq 0$.*

Since $\triangle^{m}_{\beta A^*,A}(I) = 0$ if and only if $\triangle^{m}_{\overline{\beta_0}A^*,\beta_0 A}(I) = 0$, $\overline{\beta_0}\beta_0 = \beta$, the pair of operators $(\overline{\beta_0}A^*, \beta_0 A)$ is (m,I)-isometric; hence, $\sigma_a(\beta_0 A)$ lies in the boundary $\partial\mathtt{D}$ of the unit disc $\mathtt{D}$ in $\mathtt{C}$ and $1 - (\overline{\beta_0}\sigma_a(A^*)(\beta_0\sigma_a(A)) = 0$. There exists a non-trivial scalar λ, $|\lambda| = 1$, such that $\beta_0\alpha = \lambda$ and $\overline{\beta_0\alpha} = \frac{1}{\lambda} = \overline{\lambda}$ for all $\alpha \in \sigma_a(A)$. We assert that $\sigma_a(B^*)$ consists, at most, of two points. For if there exist non-trivial $\overline{\mu}, \overline{\nu} \in \sigma_a(B^*)$, $\mu \neq \beta_0 \neq \nu$, then $\sigma_a(\mu A) = \sigma_a(\beta_0 A) + \sigma_a((\mu - \beta_0)A)$ and $\sigma_a(\nu A) = \sigma_a(\beta_0 A) + \sigma_a((\nu - \beta_0)A)$: since $0 \notin \sigma_a(A)$, not both of these translations of $\sigma_a(\beta_0 A)$ are in $\partial\mathtt{D}$. This argument applies equally to $\sigma_a(A)$; hence $\sigma(A)$ and $\sigma(B)$ consist at most of two points. A similar statement holds for operators $A, B \in B(\mathcal{H})$ such that the pair $(\mathcal{E}_{A^*,A}(I), \mathcal{E}_{B^*,B}(I))$ is a strict (m,X)- isometry for some quasi-affinity $X \in B(\mathcal{H})$. For in this case for every $\beta \in \sigma_a(B)$, β being necessarily non-trivial, the pair $(\overline{\beta}A^*, \beta A)$ is (m_1, I)-isometric for some positive integer $m_1 \leq m$ (see Corollary 1). Thus $\sigma_a(\beta A) \subset \partial\mathtt{D}$. Let $\lambda \in \sigma_a(A)$, and suppose that there exist distinct scalars $\mu, \nu \in \sigma_a(B)$; $\mu, \nu \neq \beta$. Then $\sigma_a(\mu A) = \sigma_a(\beta A) + \sigma_a((\mu - \beta)A)$, $\sigma_a(\nu A) = \sigma_a(\beta A) + \sigma_a((\nu - \beta)A)$, and not both these translations of $\sigma_a(\beta A)$ can be in $\partial(\mathtt{D})$. We remark that both of these classes of operators belong to the class of (m,I)-isometric operators with a finite spectrum [1,9,10,12,13].

4. A Concluding Remark

Let $\mathcal{C}_2(\mathcal{H}) \subseteq B(\mathcal{H})$ denote the operator ideal of Hilbert–Schmidt operators (equipped with the Hilbert–Schmidt operator norm). Given $A, B \in B(\mathcal{H})$, Gu [1] (Theorem 7) proves that $E_{A,B} \in \mathcal{C}_2(\mathcal{H})$ is a strict m-isometry, i.e., the pair $(E^*_{A,B}, E_{A,B})$ is strict (m,T)-isometric for all $T \in \mathcal{C}_2(H)$, if and only if there exists a non-trivial scalar β and integers $m_1, m_2 \in \mathtt{N}$ such that $m = m_1 + m_2 - 1$, βA is strict (m_1, I)-isometric and $\frac{1}{\beta}B$ is strict (m_2, I)-isometric. Observe that if $A, B \in B(\mathcal{H})$ and (A, B) is (m,X)-isometric for a quasi-affinity $X \in B(\mathcal{H})$,

then (our purely algebraic argument from Section 2 shows that) there exist $\mu \in \sigma_a(B)$ and $\overline{v} \in \sigma_a(A^*)$ such that $(I - \mu A)^m = 0 = (I - \overline{v}B^*)^m$. Since $\sigma(\triangle_{A,B}) = 1 - \sigma(A)\sigma(B)$, $\mu\nu = 1$, $(I - \mu A)^m = 0 = (I - \frac{1}{\mu}B)^m$. The operators $\mathcal{E}_{A_1,A_2}(S_1)$ and $\mathcal{E}_{B_1,B_2}(S_2)$ are operators in $B(\mathcal{H})$; if $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ is (m, X)-isometric, $X \in B(\mathcal{H})$ a quasi-affinity, then there exists a non-zero scalar β such that $I - \beta\mathcal{E}_{A_1,A_2}(S_1)$ and $I - \frac{1}{\beta}\mathcal{E}_{B_1,B_2}(S_2)$ are m-nilpotent. Thus, if $\mathcal{E}^n_{A_1,A_2}(X) = E^n_{A_1,A_2}(X)$ for all $n \in \mathtt{N}$, then there exists a non-zero scalar β such that $(\beta A_1, A_2)$ is (m, S_1)-isometric and $(\frac{1}{\beta}B_1, B_2)$ is (m, S_2)-isometric. Proposition 2 is a Banach space generalisation of this result. The extension of this algebraic argument to the pair $(E_{A_1,A_2}, E_{B_1,B_2}) \in B(B(\mathcal{X}))^2$ requires a judicious use of the algebraic, especially the commutative, properties of the left/right regularisation operators (in the terminology of Taylor and Lay [14] (Page 392)) L_A and R_B of the algebra $B(\mathcal{X})$. We remark in closing that a proof of the Hilbert space version of this result for the m-null symbols pair $(\mathcal{E}_{A_1,A_2}(S_1), \mathcal{E}_{B_1,B_2}(S_2))$ using arithmetic progressions and a combinatorial argument, thus avoiding analytic arguments, has been given by Marrero [15].

Funding: This research received no external funding.

Data Availability Statement: No data generation or analysis involved.

Conflicts of Interest: The author declares no conflicts of interest.

References

1. Gu, C. Elementary operators which are m-isometric. *Linear Algebra Its Appl.* **2014**, *451*, 49–64. [CrossRef]
2. Marrero, I. Elementary operators with m-null symbols. *Mathematics* **2025**, *73*, 741. [CrossRef]
3. Agler, J. Sub-Jordan operators, Bishop's theorem, spectral inclusions and spectral sets. *J. Oper. Theory* **1982**, *7*, 373–395.
4. Agler, J.; Stankus, M.m-Isometric transformations of Hilbert space I. *Integr. Equ. Oper. Theory* **1995**, *21*, 383–420. [CrossRef]
5. Bayart, F. m-Isometries on Banach Spaces. *Math. Nachr.* **2011**, *284*, 2141–2147. [CrossRef]
6. Bermúdez, T.; Martinón, A.; Noda, J.N. Products of m-isometries. *Linear Algebra Its Appl.* **2013**, *408*, 80–86. [CrossRef]
7. Botellho, F.; Jamison, J. Isometric properties of elementary operators. *Linear Algebra Its Appl.* **2010**, *432*, 357–365. [CrossRef]
8. Duggal, B.P.; Kim, I.H. Structure of elementary operators defining m-left invertible, m-selfadjoint and related classes of operators. *J. Math. Anal. Appl.* **2021**, *495*, 124718. [CrossRef]
9. Duggal, B.P.; Kim, I.H. m-isometric generalised derivations. *Concr. Oper.* **2022**, *9*, 139–150. [CrossRef]
10. Duggal, B.P.; Kim, I.H. m-symmetric operators with elementary operator entries. *Results Math.* **2024**, *79*, 240. [CrossRef]
11. Eschmeier, J. Tensor products and elementary operators. *J. Reine Agnew. Math.* **1988**, *390*, 42–66.
12. Helton, J.W. Infinite dimensional Jordan algebras and Sturm-Liouville conjugation point theory. *Trans. Amer. Math. Soc.* **1970**, *170*, 305–331. [CrossRef]
13. Stankus, M. m-isometries and other linear transformations which are hereditary roots. *Integral Equ. Oper. Theory* **2013**, *75*, 301–321. [CrossRef]
14. Taylor, A.E.; Lay, D.C. *Introduction to Functional Analysis*; John Wiley and Sons: New York, NY, USA, 1980.
15. Marrero, I. On the Symbols of Strictly m-null Elementary Operators. *Mathematics* **2025**, *13*, 2026. [CrossRef]

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