

Article

On Jensen-Related Inequalities for Various Types of Convexity via a Unified Approach

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Abstract

The focus of this paper is on three types of convexity: generalized uniform convexity, Φ -convexity and superquadracity. The similar structures of these types of convexity are such that the same processes can be applied to each one of them to obtain further refinements of known inequalities.

Keywords: convexity; uniform convexity; Φ -convexity; strong convexity; superquadracity; Jensen's differences; Jensen inequality; Jensen–steffensen inequality

MSC: 26D15

1. Introduction

Jensen's inequality is a central inequality in many areas of mathematics. Of the numerous applications of Jensen's inequality, some are in the recent papers [1,2]. The aim of this work is to further develop offshoots of this pivotal inequality.

The focus of this paper is on three types of convexity: generalized uniform convexity, Φ -convexity and superquadracity. The similar structure of these types of convexity is

$$f(x) - f(u) \geq \varphi(u)(x - u) + \Phi(|x - u|). \quad (1)$$

This common structure is such that when we apply the same processes on these types of convexity, we get further refinements of known inequalities.

We start with quoting some known definitions and theorems.

The first type of convexity we deal with is **generalized uniform convexity**:

Definition 1 ([3]). Let $[a, b] \subset \mathbb{R}$ be an interval. A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be generalized Φ -uniformly convex if there exists a function $\Phi : [0, b - a] \rightarrow \mathbb{R}$ such that

$$\begin{aligned} tf(x) + (1 - t)f(y) &\geq f(tx + (1 - t)y) + t(1 - t)\Phi(|x - y|) \\ \text{for } x, y &\in [a, b] \text{ and } t \in [0, 1]. \end{aligned} \quad (2)$$

holds. If, in addition, $\Phi \geq 0$, then f is said to be Φ -uniformly convex or uniformly convex with modulus Φ . In the special case where $\Phi(|x - y|) = c(x - y)^2$, $c > 0$, f is called **strongly convex function**.

Remark 1. In this paper we apply a more restricted definition of uniform convexity (used in [Theorem 2.1] in [4]). A function $f : [a, b] \rightarrow \mathbb{R}$ is said to be Φ -uniformly convex if there exists a function $\Phi : [0, b - a] \rightarrow \mathbb{R}_+$ where Φ is increasing and vanishes only at $\Phi(0) = 0$, such that (2) holds.



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Also, it is shown in [3,5] that the inequality

$$f\left(\sum_{r=1}^n \lambda_r x_r\right) \leq \sum_{r=1}^n \lambda_r \left(f(x_r) - \Phi\left(\left|x_r - \sum_{j=1}^n \lambda_j x_j\right|\right)\right) \quad (3)$$

holds when $x_r \in [a, b]$, $0 \leq \lambda_r \leq 1$, $r = 1, \dots, n$, $\sum_{r=1}^n \lambda_r = 1$. It is easy to verify that (3) holds for generalized uniformly convex functions and that in these cases $\Phi(0) \leq 0$.

Another inequality satisfied by uniformly convex functions is proved in [Theorem 2.3] in [4] as follows:

Let $f : [a, b] \rightarrow \mathbb{R}$ be a uniformly convex function with modulus $\Phi : [0, b - a] \rightarrow \mathbb{R}_+$, $\{x_k\}_{k=1}^n \subseteq [a, b]$ be a sequence and π be a permutation on $\{1, \dots, n\}$ such that $x_{\pi(1)} \leq x_{\pi(2)} \leq \dots \leq x_{\pi(n)}$. Then, the inequality

$$\sum_{k=1}^n p_k f(x_k) - f\left(\sum_{k=1}^n p_k x_k\right) \geq \sum_{k=1}^{n-1} p_{\pi(k)} p_{\pi(k+1)} \Phi(x_{\pi(k+1)} - x_{\pi(k)}),$$

holds for every convex combination $\sum_{k=0}^n p_k x_k$ of points $x_k \in [a, b]$.

Remark 2. The functions $f(x) = x^n$, $x \geq 0$, $n = 2, 3, \dots$ are uniformly convex on $x \geq 0$ with a modulus $\Phi(x) = x^n$, $x \geq 0$ (see [6]), (these functions are also superquadratic for any real $p \geq 2$).

Remark 3. In [Theorem 1, Inequality (23)] in [3], it is proved that the set of generalized uniformly convex functions f defined on $[a, b] \in \mathbb{R}$ and Φ defined on $[0, b - a]$ which are both continuously differentiable, satisfy the inequality

$$f(y) - f(x) \geq f'(x)(y - x) + \Phi(|y - x|), \quad \Phi : [0, (b - a)] \rightarrow \mathbb{R}, \quad (4)$$

for $x, y \in [a, b]$. In [Theorem 2.1] in [5], (4) is proved for uniformly convex functions.

Theorem 1 deals with strongly convex functions, that is, functions f for which $\Phi = cx^2$, $c > 0$:

Theorem 1 ([7]). Let $f : (a, b) \rightarrow \mathbb{R}$ be a differentiable and strongly convex function. Suppose $\mathbf{x} = (x_1, \dots, x_n) \in (a, b)^n$ and $\mathbf{a} = (a_1, \dots, a_n)$ is a nonnegative n -tuple with $A_n = \sum_{i=1}^n a_i > 0$. Let $\bar{x} = \frac{1}{A_n} \sum_{i=1}^n a_i x_i$ and $\hat{x}_i = (1 - \lambda_i)\bar{x} + \lambda_i x_i$, $\lambda_i \in [0, 1]$, $i \in \{1, \dots, n\}$. Then

$$\begin{aligned} 0 &\leq \left| \frac{1}{A_n} \sum_{i=1}^n a_i \left| f(x_i) - f(\hat{x}_i) - c(1 - \lambda_i)^2 (\bar{x} - x_i)^2 \right| \right. \\ &\quad \left. - \frac{1}{A_n} \sum_{i=1}^n a_i (1 - \lambda_i) \left| f'(\hat{x}_i) \right| |\bar{x} - x_i| \right| \\ &\leq \frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) - \frac{1}{A_n} \sum_{i=1}^n a_i f(\hat{x}_i) \\ &\quad - \frac{1}{A_n} \sum_{i=1}^n a_i (1 - \lambda_i) f'(\hat{x}_i) (\bar{x} - x_i) - c \sum_{i=1}^n a_i (1 - \lambda_i)^2 (\bar{x} - x_i)^2. \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) - \frac{1}{A_n} \sum_{i=1}^n a_i f((1-\lambda_i)\bar{x} + \lambda_i x_i) \\ & \leq \frac{1}{A_n} \sum_{i=1}^n a_i (1-\lambda_i) f'(x_i) (x_i - \bar{x}) - \frac{c}{A_n} \sum_{i=1}^n a_i (1-\lambda_i)^2 (\bar{x} - x_i)^2 \end{aligned}$$

hold.

The second type of convexity dealt with in this paper is Φ -convexity.

Definition 2 ([8]). A real value function f defined on a real interval $[a, b]$ is called Φ -convex if for all $x, y \in [a, b]$, $t \in [0, 1]$ it satisfies

$$\begin{aligned} & tf(x) + (1-t)f(y) + t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|) \\ & \geq f(tx + (1-t)y), \end{aligned}$$

where $\Phi : [0, b-a] \rightarrow \mathbb{R}_+$, is called the **error function**.

In [Theorem 3.1] in [8] it is proved that

Corollary 1. Let f be a Φ -convex function on $[a, b]$, then there exists a function $\varphi : [a, b] \rightarrow \mathbb{R}$ such that for all $x, u \in [a, b]$,

$$f(u) + (x-u)\varphi(u) \leq f(x) + \Phi(|u-x|).$$

Also, when $x_r \in [a, b]$, $r = 1, \dots, n$, $0 \leq \lambda_r \leq 1$, $r = 1, \dots, n$, $\sum_{j=1}^n \lambda_j = 1$,

$$f\left(\sum_{r=1}^n \lambda_r x_r\right) \leq \sum_{r=1}^n \lambda_r \left(f(x_r) + \Phi\left(\left|x_r - \sum_{j=1}^n \lambda_j x_j\right|\right)\right),$$

holds.

The third type of convexity is superquadracity. It should be mentioned that there are two different concepts of superquadracity, see for instance [9,10] and the references cited there. The basic properties of superquadracity as used here were proved in [11] in 2004. Since then, hundreds of works have been published on this subject.

Definition 3 ([11]). A function $\varphi : I = [0, b] \rightarrow \mathbb{R}$, $b \leq \infty$ is **superquadratic** provided that for all $x \in I$ there exists a constant $C_x \in \mathbb{R}$ such that

$$\varphi(y) \geq \varphi(x) + C_x(y-x) + \varphi(|y-x|) \quad (5)$$

for all $y \in I$. If the reverse of (5) holds, then φ is called **subquadratic**.

Remark 4 ([11]). If φ is superquadratic, then the inequality on $I = [0, b] \rightarrow \mathbb{R}$,

$$\varphi\left(\sum_{r=1}^n \lambda_r x_r\right) \leq \sum_{r=1}^n \lambda_r \left(\varphi(x_r) - \varphi\left(\left|x_r - \sum_{j=1}^n \lambda_j x_j\right|\right)\right)$$

holds for $x_r \in I$, $\lambda_r \geq 0$, $r = 1, \dots, n$ and $\sum_{r=1}^n \lambda_r = 1$.

Lemma 1 and Theorem 2 are about superquadracity and its relation to superadditivity. In [12], Beckenbach deals with superadditivity inequalities, and in [13] Bruckner and Ostrow deal with some function classes related to the class of convex functions. Following these two important works, the authors of [11] deal with conditions that constitute the scale of superquadracity. In this scale, the following is proved in [Lemma 3.1] in [11].

Suppose $\varphi : [0, \infty) \rightarrow \mathbb{R}$ is continuously differentiable and $\varphi(0) \leq 0$. If φ' is superadditive or $\varphi'(x)/x$ is non-decreasing, then φ is superquadratic.

Also in [Lemma 4.1] in [11], it is proved that a non-positive, non-increasing, superadditive function is superquadratic.

In Section 2, by adding the definitions of H -superadditivity, we get theorems for generalized uniformly convex functions and for Φ -convex functions.

Lemma 1 ([Lemma 1] in [14]). *Let f be continuously differentiable on $[0, b)$ and f' be superadditive on $[0, b)$. Then, the function $D : [0, b) \rightarrow \mathbb{R}$ is defined by*

$$D(y) = f(y) - f(z) - f'(z)(y - z) - f(|y - z|) + f(0)$$

is nonnegative on $[0, b)$, nonincreasing on $[0, z)$ and nondecreasing on $[z, b)$ for $0 \leq z < b$.

If also $f(0) \leq 0$, we have

$$f(y) - f(z) - f(|y - z|) \geq f'(z)(y - z) - f(0) \geq f'(z)(y - z).$$

Taking $C(x) = f'(z)$, we see that f is superquadratic.

Theorem 2 ([Theorem 1] in [14]). *Let f be continuously differentiable on $[0, b)$ and f' be superadditive on $[0, b)$. Let $\mathbf{a}$ be a real n -tuple satisfying $0 \leq \sum_{i=1}^j a_i \leq \sum_{i=1}^n a_i$, $1 \leq j \leq n$, $\sum_{i=1}^n a_i > 0$ and $x_i \in [0, b)$, $i = 1, \dots, n$ be such that $x_1 \leq x_2 \leq \dots \leq x_n$. Then,*

a.

$$f(c) - f(0) + f'(c)(\bar{x} - c) + \frac{1}{A_n} \sum_{i=1}^n a_i f(|x_i - c|) \leq \frac{1}{A_n} \sum_{i=1}^n a_i f(x_i)$$

holds for all $c \in [0, b)$, where $\bar{x} = \frac{1}{A_n} \sum_{i=1}^n a_i x_i$.

b. If, in addition, $f(0) \leq 0$, then f is superquadratic and

$$\frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) \geq f(c) + f'(c)(\bar{x} - c) + \frac{1}{A_n} \sum_{i=1}^n a_i f(|x_i - c|).$$

In particular

$$\frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) \geq f(\bar{x}) + \frac{1}{A_n} \sum_{i=1}^n a_i f(|x_i - \bar{x}|). \quad (6)$$

c. If, in addition, $f \geq 0$ and $f(0) = f'(0) = 0$, then f is convex increasing and superquadratic and

$$\frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) - f(c) - f'(c)(\bar{x} - c) \geq \frac{1}{A_n} \sum_{i=1}^n a_i f(|x_i - c|) \geq 0.$$

We get the results in Sections 2 and 3 by using the following definition:

Definition 4. *Let the functions $g : [a, b] \rightarrow \mathbb{R}$ and $H : [0, b - a] \rightarrow \mathbb{R}$ be continuously differentiable. The function g is named H -superadditive if for all $x, y \in [a, b]$ with $x \leq y$, when the inequality*

$$g(y) - g(x) \geq H(y - x)$$

holds.

In Section 4, we prove the results on the difference between two normalized Jensen functionals for superquadratic and uniformly convex functions.

The Jensen functional is

$$J_n(f, \mathbf{x}, \mathbf{p}) = \sum_{i=1}^n p_i f(x_i) - f\left(\sum_{i=1}^n p_i x_i\right),$$

about which S. S. Dragomir proved the following.

Theorem 3 ([15]). Consider the normalized Jensen functional where $f : C \rightarrow \mathbb{R}$ is a convex function on the convex set C in a real linear space, $\mathbf{x} = (x_1, \dots, x_n) \in C^n$, and $\mathbf{p} = (p_1, \dots, p_n)$, $\mathbf{q} = (q_1, \dots, q_n)$ are non-negative n -tuples satisfying $\sum_{i=1}^n p_i = 1$, $\sum_{i=1}^n q_i = 1$, $p_i, q_i > 0$, $i = 1, \dots, n$. Then

$$MJ_n(f, \mathbf{x}, \mathbf{q}) \geq J_n(f, \mathbf{x}, \mathbf{p}) \geq mJ_n(f, \mathbf{x}, \mathbf{q}), \quad (7)$$

provided that

$$m = \min_{1 \leq i \leq n} \left(\frac{p_i}{q_i} \right), \quad M = \max_{1 \leq i \leq n} \left(\frac{p_i}{q_i} \right). \quad (8)$$

In [16], the following two theorems on normalized Jensen functionals are proven.

Theorem 4. Under the same conditions and definitions on $\mathbf{p}, \mathbf{q}, \mathbf{x}, m$ and M as in Theorem 3, if I is $[0, a)$ or $[0, \infty)$ and f is a superquadratic function on I , then

$$\begin{aligned} & J_n(f, \mathbf{x}, \mathbf{p}) - mJ_n(f, \mathbf{x}, \mathbf{q}) \\ & \geq mf \left(\left| \sum_{i=1}^n (q_i - p_i)x_i \right| \right) + \sum_{i=1}^n (p_i - mq_i) f \left(\left| x_i - \sum_{j=1}^n p_j x_j \right| \right) \end{aligned} \quad (9)$$

and

$$\begin{aligned} & J_n(f, \mathbf{x}, \mathbf{p}) - MJ_n(f, \mathbf{x}, \mathbf{q}) \\ & \leq - \sum_{i=1}^n (Mq_i - p_i) f \left(\left| x_i - \sum_{j=1}^n q_j x_j \right| \right) - f \left(\left| \sum_{i=1}^n (p_i - q_i)x_i \right| \right). \end{aligned} \quad (10)$$

Theorem 5. Let $\mathbf{p} = (p_1, \dots, p_n)$, where $0 \leq \sum_{j=1}^i p_j \leq 1$, $i = 1, \dots, n$, $\sum_{i=1}^n p_i = 1$, $\mathbf{q} = (q_1, \dots, q_n)$, $0 < \sum_{j=1}^i q_j < 1$, $i = 1, \dots, n-1$, $\sum_{i=1}^n q_i = 1$ and $\mathbf{p} \neq \mathbf{q}$. Denote

$$m_i = \frac{\sum_{j=1}^i p_j}{\sum_{j=1}^i q_j}, \quad \bar{m}_i = \frac{\sum_{j=i}^n p_j}{\sum_{j=i}^n q_j}, \quad i = 1, \dots, n \quad (11)$$

and

$$m^* = \min_{1 \leq i \leq n} \{m_i, \bar{m}_i\}, \quad M^* = \max_{1 \leq i \leq n} \{m_i, \bar{m}_i\}. \quad (12)$$

If $\mathbf{x} = (x_1, \dots, x_n)$ is an increasing n -tuple in I^n , where I is an interval in $\mathbb{R}$, then

$$M^* J_n(f, \mathbf{x}, \mathbf{q}) \geq J_n(f, \mathbf{x}, \mathbf{p}) \geq m^* J_n(f, \mathbf{x}, \mathbf{q}), \quad (13)$$

where $f : I \rightarrow \mathbb{R}$ is a convex function on the interval I .

In Section 4, we replace the coefficients $p_i \geq 0$, $q_i > 0$, with $0 \leq \sum_{j=1}^i p_j \leq 1$, $0 < \sum_{j=1}^i q_j < 1$, $i = 1, \dots, n$ and m and M with m^* and M^* . In this way, we get results resembling (9) and (10) in Theorem 4. There, we also extend Theorem 5 for functions that are offshoots of convexity.

The next sections are arranged as follows:

In Section 2, we extend the results in Theorem 2 to include the three sets of functions that satisfy (1).

In Section 3, we extend the results in Theorem 1, there only on strongly convex functions, to include also generalized uniform convexity, Φ -convexity and superquadratic functions.

In Section 4, we improve the result of Theorem 4 regarding the difference between two Jensen's functionals.

Section 5 shows some examples of the relations between superquadracity and some other types of convex functions.

2. Improvement of Jensen–Steffensen Inequality for Generalized Uniformly Convex Functions and for Φ -Convex Functions

Lemma 2 and Theorem 6 are related to generalized uniform convexity, where in Theorem 6 f' is Φ' -superadditive. We prove this lemma and this theorem similarly to the proofs of Lemma 1 and Theorem 2.

In Corollary 2, we deal with Φ -convexity.

We prove, in detail, the results related to generalized uniform convexity.

Lemma 2. *Let f be continuously differentiable on $[a, b]$ and f' be Φ' -superadditive. Let $\Phi : [0, b - a] \rightarrow \mathbb{R}$ be continuously differentiable and $\Phi(0) \leq 0$. Then, the function $D : [a, b] \rightarrow \mathbb{R}$ defined by*

$$D(y) = f(y) - f(z) - f'(z)(y - z) - \Phi(|y - z|) \quad (14)$$

is nonnegative on $[a, b]$, nonincreasing on $[a, z]$ and nondecreasing on $[z, b]$ for $a \leq z < b$, and f satisfies (4) (which includes the set of generalized uniformly convex functions).

Proof. It is given that f' is Φ' -superadditive, and $\Phi(0) \leq 0$, therefore, if $a \leq z \leq y < b$, we get that

$$0 \leq \int_z^y (f'(t) - f'(z) - \Phi'(t - z)) dt = f(y) - f(z) - f'(z)(y - z) - \Phi(y - z)$$

and if $0 \leq a \leq y \leq z < b$, then we get that

$$0 \leq \int_y^z (f'(z) - f'(t) - \Phi'(z - t)) dt = f'(z)(z - y) - f(z) + f(y) - \Phi(z - y).$$

Together, these show that for any $y, z \in [a, b]$,

$$f(y) - f(z) - f'(z)(y - z) - \Phi(|y - z|) \geq 0,$$

so, we conclude that D is nonnegative on $[a, b]$ and, according to Remark 3 f satisfies (4), which includes generalized uniformly convex functions with $\Phi : [0, b - a] \rightarrow \mathbb{R}$.

From

$$D'(y) = f'(y) - f'(z) - \Phi'(|y - z|) \operatorname{sgn}(y - z),$$

as f' is Φ' -superadditive for $a \leq y \leq z$ we have

$$D'(y) = f'(y) - f'(z) + \Phi'(z - y) \leq 0$$

and similarly for $z \leq y < b$, we have

$$D'(y) = f'(y) - f'(z) - \Phi'(y - z) \geq 0.$$

This completes the proof. $\square$

Now, we present the main results of this section, where we show that inequality (3) is satisfied, not only for nonnegative coefficients but also when

$$0 \leq \sum_{i=1}^j a_i \leq \sum_{i=1}^n a_i, \quad j = 1, \dots, n, \quad \sum_{i=1}^n a_i > 0 \quad (15)$$

is satisfied (called Jensen–Steffensen coefficients) when $x_1 \leq x_2 \leq \dots \leq x_n$.

Theorem 6. Let f be continuously differentiable on $[a, b]$ and f' be Φ' -superadditive. Let also $\Phi : [0, b - a] \rightarrow \mathbb{R}$ be continuously differentiable and $\Phi(0) \leq 0$. Let $\mathbf{a}$ be a real n -tuple satisfying (15) and $x_i \in [a, b]$, $i = 1, \dots, n$ be such that $x_1 \leq x_2 \leq \dots \leq x_n$. Then, f satisfies (4) and

$$f(c) + f'(c)(\bar{x} - c) + \frac{1}{A_n} \sum_{i=1}^n a_i \Phi(|x_i - c|) \leq \frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) \quad (16)$$

holds for all $c \in [a, b]$, where $A_i = \sum_{j=1}^i a_j$, $\bar{x} = \frac{1}{A_n} \sum_{i=1}^n a_i x_i$.

In particular, when $c = \frac{1}{A_n} \sum_{i=1}^n a_i x_i$, we get the inequality

$$\frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) \geq f(\bar{x}) + \frac{1}{A_n} \sum_{i=1}^n a_i \Phi(|x_i - \bar{x}|). \quad (17)$$

If in addition f is uniformly convex and Φ is convex, then

$$\begin{aligned} & \frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) - f(c) - f'(c)(\bar{x} - c) \\ & \geq \frac{1}{A_n} \sum_{i=1}^n a_i \Phi(|x_i - c|) \geq 0. \end{aligned} \quad (18)$$

Proof. It was proved in [14] that $x_1 \leq \bar{x} \leq x_n$. From Lemma 2, we get that f satisfies (4).

Let $D(x_i) = f(x_i) - f(c) - f'(c)(x_i - c) - \Phi(|x_i - c|)$, $i = 1, \dots, n$.

From Lemma 2, we know that $D(x_i) \geq 0$ for all $i = 1, \dots, n$. Comparing c with $x_1, \dots, x_n$, we must consider three cases.

Case 1. $x_n < c < b$:

In this case, $x_i \in [0, c]$ for all $i = 1, \dots, n$, hence, according to Lemma 2, $D(x_1) \geq D(x_2) \geq \dots \geq D(x_n) \geq 0$.

Denoting $A_0 = 0$, we get that

$$a_i = A_i - A_{i-1}, \quad i = 1, \dots, n$$

and therefore

$$\begin{aligned} \sum_{i=1}^n a_i D(x_i) &= \sum_{i=1}^n (A_i - A_{i-1}) D(x_i) \\ &= A_1 D(x_1) + (A_2 - A_1) D(x_2) + \dots + (A_n - A_{n-1}) D(x_n) \\ &= \sum_{i=1}^{n-1} A_i (D(x_i) - D(x_{i+1})) + A_n D(x_n) \geq 0. \end{aligned}$$

Case 2. $a \leq c \leq x_1$

In this case, $x_i \in (c, b)$ for all $i = 1, \dots, n$, hence

$$0 \leq D(x_1) \leq D(x_2) \leq \dots \leq D(x_n).$$

Denoting $\bar{A}_{n+1} = 0$, we get that

$$\bar{A}_k = \sum_{i=k}^n a_i = A_n - A_{k-1}, \quad k = 1, \dots, n,$$

and

$$a_i = \bar{A}_i - \bar{A}_{i+1}, \quad i = 1, \dots, n.$$

Therefore

$$\begin{aligned} \sum_{i=1}^n a_i D(x_i) &= \sum_{i=1}^n (\bar{A}_i - \bar{A}_{i+1}) D(x_i) \\ &= \bar{A}_1 D(x_1) + \sum_{i=2}^n \bar{A}_i (D(x_i) - D(x_{i-1})) \geq 0. \end{aligned}$$

Case 3. $x_1 \leq c \leq x_n$

In this case, there exists $k \in \{1, \dots, n-1\}$ such that $x_k \leq c \leq x_{k+1}$.

By Lemma 2, we get that

$$D(x_1) \geq D(x_2) \geq \dots \geq D(x_k) \geq 0$$

and

$$0 \leq D(x_{k+1}) \leq D(x_{k+2}) \leq \dots \leq D(x_n),$$

and

$$\begin{aligned} \sum_{i=1}^n a_i D(x_i) &= \sum_{i=1}^k a_i D(x_i) + \sum_{i=k+1}^n a_i D(x_i) \\ &= \sum_{i=1}^{k-1} A_i (D(x_i) - D(x_{i+1})) + A_k D(x_k) \\ &\quad + \bar{A}_{k+1} D(x_{k+1}) + \sum_{i=k+2}^n \bar{A}_i (D(x_i) - D(x_{i-1})) \\ &\geq 0. \end{aligned}$$

From these three cases, we get that

$$\begin{aligned} \sum_{i=1}^n a_i D(x_i) &= \sum_{i=1}^n a_i [f(x_i) - f(c) - f'(c)(x_i - c) - \Phi(|x_i - c|)] \\ &\geq 0 \end{aligned}$$

and, therefore, (16) holds. When $c = \bar{x}$, then (17) holds.

It is given in this paper that if f is uniformly convex, then, by Remark 1, Φ is increasing and $\Phi(0) = 0$. If Φ is also convex, we only need to show that under our conditions

$$\frac{1}{A_n} \sum_{i=1}^n a_i \Phi(|x_i - c|) \geq 0 \quad (19)$$

holds.

We show that (19) holds in the case that $x_k \leq c \leq x_{k+1}$, $k = 1, \dots, n-1$.

We use the identity

$$\begin{aligned} & \sum_{i=1}^n a_i \Phi(|x_i - c|) \\ &= \sum_{i=1}^{k-1} A_i (\Phi(c - x_i) - \Phi(c - x_{i+1})) + A_k \Phi(c - x_k) + \bar{A}_{k+1} \Phi(x_{k+1} - c) \\ &+ \sum_{i=k+2}^n \bar{A}_i (\Phi(x_i - c) - \Phi(x_{i-1} - c)). \end{aligned} \quad (20)$$

As Φ is nonnegative, convex and $\Phi(0) = 0$ it follows that for $a \leq x_i \leq x_{i+1} \leq c, i=1, \dots, k-1$,

$$\Phi(c - x_i) - \Phi(c - x_{i+1}) \geq \Phi(x_{i+1} - x_i) - \Phi(0) = \Phi(x_{i+1} - x_i) \geq 0$$

and for $c \leq x_{i-1} \leq x_i < b, i = k+2, \dots, n$,

$$\Phi(x_i - c) - \Phi(x_{i-1} - c) \geq \Phi(x_i - x_{i-1}) - \Phi(0) = \Phi(x_i - x_{i-1}) \geq 0.$$

Therefore, (19) is satisfied and, together with (16), we get that (18) holds.

Thus, the proof of Theorem 6 is complete. $\square$

In order to show when Lemma 2 and Theorem 6 hold for Φ -convex functions, we prove the following theorem:

Theorem 7. Let f be a Φ -convex function on $I = [a, b]$ with an error function Φ , f and Φ both be continuously differentiable and $\Phi(0) = \Phi'(0) = 0$. Then, f satisfies the inequality

$$f(x) - f(u) \geq \varphi(u)(x - u) + \Phi(|x - u|), \quad (21)$$

where $\varphi = f'$.

Proof. It is easy to verify from (21) that, because of the differentiability of f and Φ , the inequality

$$\lim_{y \rightarrow x^-} \left(\frac{\varphi(x) - \varphi(y)}{x - y} + \frac{\varphi(x - y)}{x - y} \right) \leq \varphi(x) \leq \lim_{y \rightarrow x^+} \left(\frac{\varphi(y) - \varphi(x)}{y - x} - \frac{\varphi(y - x)}{y - x} \right) \quad (22)$$

holds. As $\Phi(0) = 0$, then $\lim_{y \rightarrow x} \frac{\varphi(x - y)}{x - y} = 0$ and, therefore, we get from (22) that

$$f'(x) = \lim_{y \rightarrow x^-} \left(\frac{\varphi(x) - \varphi(y)}{x - y} \right) \leq \varphi(x) \leq \lim_{y \rightarrow x^+} \left(\frac{\varphi(y) - \varphi(x)}{y - x} \right) = f'(x). \quad (23)$$

Hence, $f' = \varphi$. The proof of the theorem is complete. $\square$

Corollary 2. The results of Lemma 2 and (16) and (17) in Theorem 6 hold also when the function f is Φ -convex in case that Theorem 7 holds and, therefore, Corollary 1 holds for $\varphi = f'$ and $\Phi(x) \leq 0$.

3. Jensen-Type Results on Generalized Uniformly Convex, Φ -Convex and Superquadratic Functions

In this section, we deal with the same functions $f : [a, b] \rightarrow \mathbb{R}$ as in Section 2, which satisfy the following inequality:

$$f(x) - f(u) \geq \varphi(u)(x - u) + \Phi(|x - u|), \quad \Phi : [0, b - a] \rightarrow \mathbb{R} \quad (24)$$

We use an analogous technique to the proof of Theorem 1 and apply it on (24) to obtain new results. Theorem 1 becomes a special case of the theorems proved in this section.

The cases included in (24) are

- Superquadratic functions f where in (24) we replace Φ with f and $[a, b]$ with $[0, b]$ (see Definition 3);
- Generalized uniformly convex functions f where in (24) we replace φ with f' (see Remark 3);
- Φ -convex functions f where in (24) $\Phi \leq 0$ (see Corollary 1).

Theorem 8. Let $f : (a, b) \rightarrow \mathbb{R}$ satisfy (24) with $\Phi : [0, b - a] \rightarrow \mathbb{R}$. Suppose $\mathbf{x} = (x_1, \dots, x_n) \in (a, b)^n$ and $\mathbf{a} = (a_1, \dots, a_n)$ is a nonnegative n -tuple with $A_n = \sum_{i=1}^n a_i > 0$. Let $\bar{x} = \frac{1}{A_n} \sum_{i=1}^n a_i x_i$, $i \in \{1, \dots, n\}$. Then

$$\begin{aligned} 0 &\leq \left| \frac{1}{A_n} \sum_{i=1}^n a_i |f(x_i) - f(\bar{x}) - \Phi(|\bar{x} - x_i|)| - \frac{1}{A_n} \sum_{i=1}^n a_i |\varphi(\bar{x})| |\bar{x} - x_i| \right| \\ &\leq \frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) - f(\bar{x}) - \frac{1}{A_n} \sum_{i=1}^n a_i \Phi(|\bar{x} - x_i|). \end{aligned} \quad (25)$$

Proof. Applying the triangle inequality $||u| - |v|| \leq |u - v|$ to

$$f(x) - f(y) - \Phi(|x - y|) - \varphi(y)(x - y) \geq 0$$

satisfied by cases a., b. and c. above, we get

$$\begin{aligned} &||f(x) - f(y) - \Phi(|x - y|)| - |\varphi(y)| |x - y|| \\ &\leq |f(x) - f(y) - \Phi(|x - y|) - \varphi(y)(x - y)| \\ &= f(x) - f(y) - \Phi(|x - y|) - \varphi(y)(x - y). \end{aligned} \quad (26)$$

Setting $y = \bar{x}$ and $x = x_i$, $i \in \{1, \dots, n\}$, we have

$$\begin{aligned} &||f(x_i) - f(\bar{x}) - \Phi(|\bar{x} - x_i|)| - |\varphi(\bar{x})| |\bar{x} - x_i|| \\ &\leq |f(x_i) - f(\bar{x}) - \varphi(\bar{x})(\bar{x} - x_i) - \Phi(|\bar{x} - x_i|)| \\ &= f(x_i) - f(\bar{x}) - \varphi(\bar{x})(\bar{x} - x_i) - \Phi(|\bar{x} - x_i|) \end{aligned} \quad (27)$$

Now, multiplying (27) by a_i , summing over i , $i = 1, \dots, n$ and dividing by $A_n = \sum_{i=1}^n a_i > 0$, we get

$$\begin{aligned} &\frac{1}{A_n} \sum_{i=1}^n a_i ||f(x_i) - f(\bar{x}) - \Phi(|\bar{x} - x_i|)| - |\varphi(\bar{x})| |\bar{x} - x_i|| \\ &\leq \frac{1}{A_n} \sum_{i=1}^n a_i |f(x_i) - f(\bar{x}) - \varphi(\bar{x})(\bar{x} - x_i) - \Phi(|\bar{x} - x_i|)| \\ &= \frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) - \frac{1}{A_n} \sum_{i=1}^n a_i f(\bar{x}) \\ &\quad - \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(\bar{x})(\bar{x} - x_i) - \frac{1}{A_n} \sum_{i=1}^n a_i \Phi(|\bar{x} - x_i|). \end{aligned} \quad (28)$$

Because $\sum_{i=1}^n (\bar{x} - x_i) = 0$, the inequalities (28) and (25) are the same and the proof of the theorem is complete. $\square$

With the same technique as in Theorem 8 we get the other three theorems about functions that satisfy (24), that is, superquadratic, generalized uniformly convex and Φ -convex functions.

Theorem 9. Let $f : (a, b) \rightarrow \mathbb{R}$ satisfy (24) with $\Phi : [0, b - a] \rightarrow \mathbb{R}$. Suppose $\mathbf{x} = (x_1, \dots, x_n) \in (a, b)^n$ and $\mathbf{a} = (a_1, \dots, a_n)$ is a nonnegative n -tuple with $A_n = \sum_{i=1}^n a_i > 0$. Let $\bar{x} = \frac{1}{A_n} \sum_{i=1}^n a_i x_i$ and $\hat{x}_i = (1 - \lambda_i)\bar{x} + \lambda_i x_i$, $\lambda_i \in [0, 1]$, $i \in \{1, \dots, n\}$. Then

$$\begin{aligned} 0 &\leq \left| \frac{1}{A_n} \sum_{i=1}^n a_i [f(x_i) - f(\hat{x}_i) - \Phi((1 - \lambda_i)(|\bar{x} - x_i|))] \right. \\ &\quad \left. - \frac{1}{A_n} \sum_{i=1}^n a_i (1 - \lambda_i) \varphi(\hat{x}_i) |\bar{x} - x_i| \right| \\ &\leq \frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) - \frac{1}{A_n} \sum_{i=1}^n a_i f(\hat{x}_i) \\ &\quad - \frac{1}{A_n} \sum_{i=1}^n a_i (1 - \lambda_i) \varphi(\hat{x}_i) (\bar{x} - x_i) - \sum_{i=1}^n a_i \Phi((1 - \lambda_i)(|\bar{x} - x_i|)). \end{aligned}$$

hold.

Similarly, we get an inequality that counterparts the Jensen inequality.

Theorem 10. Let $f : (a, b) \rightarrow \mathbb{R}$ satisfy (24) with $\Phi : [0, b - a] \rightarrow \mathbb{R}$. Suppose $\mathbf{x} = (x_1, \dots, x_n) \in (a, b)^n$ and $\mathbf{a} = (a_1, \dots, a_n)$ is a nonnegative n -tuple with $A_n = \sum_{i=1}^n a_i > 0$ and $\bar{x} = \frac{1}{A_n} \sum_{i=1}^n a_i x_i$. Then

$$\begin{aligned} &\frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) - f(\bar{x}) \\ &\leq \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(x_i) (x_i - \bar{x}) - \frac{1}{A_n} \sum_{i=1}^n a_i \Phi(|\bar{x} - x_i|). \end{aligned} \quad (29)$$

Proof. Inequality (24) for $x = \bar{x}$ and $y = x_i$, $i \in \{1, \dots, n\}$, gives

$$f(\bar{x}) - f(x_i) \geq \varphi(x_i)(\bar{x} - x_i) + \Phi(|\bar{x} - x_i|).$$

Now, multiplying by a_i , summing over i , $i = 1, \dots, n$ and then dividing by $A_n > 0$, we get

$$f(\bar{x}) - \frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) \geq \frac{1}{A_n} \sum_{i=1}^n a_i \varphi(x_i) (\bar{x} - x_i) + \frac{1}{A_n} \sum_{i=1}^n a_i \Phi(|\bar{x} - x_i|),$$

which is equivalent to (29). $\square$

Also, we get

Theorem 11. Let $f : (a, b) \rightarrow \mathbb{R}$ satisfy (24) with $\Phi : [0, b - a] \rightarrow \mathbb{R}$. Suppose $\mathbf{x} = (x_1, \dots, x_n) \in (a, b)^n$ and $\mathbf{a} = (a_1, \dots, a_n)$ is a nonnegative n -tuple with $A_n = \sum_{i=1}^n a_i > 0$ and $\bar{x} = \frac{1}{A_n} \sum_{i=1}^n a_i x_i$. Let $\lambda_i \in [0, 1]$, $i \in \{1, \dots, n\}$. Then the inequality

$$\begin{aligned} &\frac{1}{A_n} \sum_{i=1}^n a_i f(x_i) - \frac{1}{A_n} \sum_{i=1}^n a_i f((1 - \lambda_i)\bar{x} + \lambda_i x_i) \\ &\leq \frac{1}{A_n} \sum_{i=1}^n a_i (1 - \lambda_i) \varphi(x_i) (x_i - \bar{x}) - \frac{1}{A_n} \sum_{i=1}^n a_i \Phi((1 - \lambda_i)(|\bar{x} - x_i|)), \end{aligned}$$

holds.

Comment. Theorems 8, 9, 10 and 11 hold in the special case when f is a strongly convex function where $\Phi(|x - y|) = c(x - y)^2$, $c > 0$, as proved in ([Theorem 4] in [7]) and quoted in Theorem 1.

4. Improvement of Jensen's Functional for Superquadratic and Generalized Uniformly Convex Functions

Theorem 12. Let f be continuously differentiable on $[a, b]$, f' be Φ' -superadditive and $\Phi(0) \leq 0$. Let also $\Phi : [0, b - a] \rightarrow \mathbb{R}$, be continuously differentiable. Then, under the same conditions and definitions on $\mathbf{p}, \mathbf{q}, \mathbf{x}, m^*$ and M^* as in Theorem 5, and assuming that $x_i \leq x_{i+1}$, $i = 1, \dots, n - 1$, we get, for generalized uniformly convex functions, the inequalities

$$\begin{aligned} & J_n(f, \mathbf{x}, \mathbf{p}) - m^* J_n(f, \mathbf{x}, \mathbf{q}) \\ & \geq m^* \Phi \left(\left| \sum_{i=1}^n (q_i - p_i) x_i \right| \right) + \sum_{i=1}^n (p_i - m^* q_i) \Phi \left(\left| x_i - \sum_{j=1}^n p_j x_j \right| \right) \end{aligned} \quad (30)$$

and

$$\begin{aligned} & J_n(f, \mathbf{x}, \mathbf{p}) - M^* J_n(f, \mathbf{x}, \mathbf{q}) \\ & \leq - \sum_{i=1}^n (M^* q_i - p_i) \Phi \left(\left| x_i - \sum_{j=1}^n q_j x_j \right| \right) - \Phi \left(\left| \sum_{i=1}^n (p_i - q_i) x_i \right| \right). \end{aligned} \quad (31)$$

Proof. As $\sum_{i=1}^n q_i = 1$ and $1 \geq \sum_{i=1}^j q_i > 0$ and $x_i \leq x_{i+1}$, $i = 1, \dots, n - 1$ then, according to Jensen–Steffensen theorem (see [14]), there is an integer k , $2 \leq k \leq n$ such that $x_{k-1} \leq \sum_{i=1}^n q_i x_i \leq x_k$.

The increasing $(n + 1)$ -tuple $\mathbf{y} = (y_1, \dots, y_{n+1})$ is defined by

$$y_i = \begin{cases} x_i, & i = 1, \dots, k - 1 \\ \sum_{j=1}^n q_j x_j, & i = k \\ x_{i-1}, & i = k + 1, \dots, n + 1 \end{cases} \quad (32)$$

and

$$a_i = \begin{cases} p_i - m^* q_i, & i = 1, \dots, k - 1 \\ m^*, & i = k \\ p_{i-1} - m^* q_{i-1}, & i = k + 1, \dots, n + 1 \end{cases} \quad (33)$$

where m^* is defined in (11) and (12).

It is clear that $(n + 1)$ tuple $\mathbf{a}$ satisfies the inequalities

$$0 \leq \sum_{i=1}^j a_i \leq \sum_{i=1}^{n+1} a_i = 1, \quad j = 1, \dots, n + 1. \quad (34)$$

Therefore, (17) in Theorem 6 holds for the increasing $(n + 1)$ -tuple $\mathbf{y}$ and a generalized uniformly convex function f that its derivative is Φ' -superadditive.

Hence

$$\begin{aligned} & \sum_{i=1}^n (p_i - m^* q_i) f(x_i) + m^* f \left(\sum_{i=1}^n q_i x_i \right) - f \left(\sum_{i=1}^n p_i x_i \right) \\ & = \sum_{i=1}^{n+1} d_i f(y_i) - f \left(\sum_{i=1}^{n+1} d_i y_i \right) \end{aligned}$$

$$\begin{aligned} &\geq \sum_{i=1}^{n+1} d_i \Phi \left(\left| y_i - \sum_{j=1}^{n+1} d_j y_j \right| \right) \\ &= \sum_{i=1}^n (p_i - m q_i) \Phi \left(\left| x_i - \sum_{j=1}^n p_j x_j \right| \right) + m \Phi \left(\left| \sum_{i=1}^n (p_i - q_i) x_i \right| \right) \end{aligned}$$

This completes the proof of (30).

The proof of (31) is similar:

We define an increasing $(n+1)$ -tuple $\mathbf{z}$

$$z_i = \begin{cases} x_i, & i = 1, \dots, s-1 \\ \sum_{j=1}^n p_j x_j, & i = s \\ x_{i-1}, & i = s+1, \dots, n+1 \end{cases} \quad (35)$$

and

$$b_i = \begin{cases} q_i - \frac{p_i}{M^*}, & i = 1, \dots, s-1 \\ \frac{1}{M^*}, & i = s \\ q_{i-1} - \frac{p_{i-1}}{M^*}, & i = s+1, \dots, n+1, \end{cases} \quad (36)$$

where s satisfies $x_{s-1} \leq \sum_{j=1}^n p_j x_j \leq x_s$. As b_i satisfies

$$0 \leq \sum_{i=1}^j b_i \leq \sum_{i=1}^{n+1} b_i = 1, \quad j = 1, \dots, n+1,$$

and $\sum_{i=1}^{n+1} b_i = 1$. By using inequality (17), we get from (35) and (36) inequality (31). $\square$

Corollary 3. Theorem 12 holds also for Φ -convex functions that satisfy Theorem 7, that is, f and Φ are continuously differentiable and $\Phi(0) = \Phi'(0) = 0$.

Similarly we get

Theorem 13. Under the same conditions and definitions on $\mathbf{p}, \mathbf{q}, \mathbf{x}, m^*$ and M^* as in Theorem 12, if I is $[0, a)$ or $[0, \infty)$ and f is a superquadratic function on I , such that f' is superadditive, $f(0) \leq 0$, and $x_1 \leq x_2 \leq \dots \leq x_n$, then

$$\begin{aligned} &J_n(f, \mathbf{x}, \mathbf{p}) - m^* J_n(f, \mathbf{x}, \mathbf{q}) \\ &\geq m^* f \left(\left| \sum_{i=1}^n (q_i - p_i) x_i \right| \right) + \sum_{i=1}^n (p_i - m^* q_i) f \left(\left| x_i - \sum_{j=1}^n p_j x_j \right| \right) \end{aligned} \quad (37)$$

and

$$\begin{aligned} &J_n(f, \mathbf{x}, \mathbf{p}) - M^* J_n(f, \mathbf{x}, \mathbf{q}) \\ &\leq - \sum_{i=1}^n (M^* q_i - p_i) f \left(\left| x_i - \sum_{j=1}^n q_j x_j \right| \right) - f \left(\left| \sum_{i=1}^n (p_i - q_i) x_i \right| \right). \end{aligned} \quad (38)$$

Remark 5. It is clear that $m^* < 1$, and $M^* > 1$ when $\mathbf{p} \neq \mathbf{q}$ and $p_i \geq 0, q_i > 0, i = 1, \dots, n$. Therefore, Theorem 13 refines Theorem 4.

5. Examples: Relations Between Superquadracity and Some Other Types of Convex Functions

The following examples, in addition to Remark 2, emphasize the relations between superquadracity and other extensions of convexity. As shown in the examples, none of

the three sets of convexity is completely included in another set. However, part of the uniformly convex functions are superquadratic (Remark 2) and some of the Φ -convex functions are also superquadratic.

Example 1. Let $f(x) = x^2 \ln x$, $x \geq 0$, $\lim_{x \rightarrow 0^+} x^2 \ln x = 0$. This function is superquadratic (see [11]) and negative on $[0, 1]$. Hence, f on $I = [a, a + 1]$, $a \geq 0$ is ϕ -convex where $\phi = -f$ is defined on $[0, 1]$.

Example 2. From Definitions 1 and 3 it is obvious that when a superquadratic function f is negative, the function f is ϕ -convex where $\phi = -f$.

Therefore, according to [Example 4.2] in [11], the functions

$$f_p(x) = -(1 + x^p)^{\frac{1}{p}}, \quad p > 0,$$

are superquadratic and negative. Hence, these functions are also ϕ -convex where $\phi = -f$.

Example 3. As in Example 2, the same holds for the functions $f(x) = -x^p$, $0 \leq p \leq 2$, $x \geq 0$, which are ϕ -convex where $\phi = -f$.

Example 4. Let $f(x) = -x^p$, $x \geq 1$, $1 < p \leq 2$, $0 \leq a \leq x \leq a + 1$. Then, f is ϕ -convex where $\phi(x) = x^q$, $x \in [0, 1]$ and $1 < q \leq p \leq 2$.

Example 5. The function

$$f(x) = \frac{1}{2} \ln(1 + x^2) - x \arctan(x), \quad f(0) = 0$$

satisfies

$$f'(x) = -\arctan(x) \quad f'(0) = 0$$

and is superquadratic (see [11]).

Therefore, this function is superquadratic and because it is negative on $[0, \infty)$ it is also ϕ -convex for $\phi = -f$.

Example 6. The function

$$\begin{aligned} f(x) &= \int_0^x \frac{t(t-2)}{\sqrt{t^2+1}} dt \\ &= \frac{1}{2} x \sqrt{x^2+1} - 2 \sqrt{x^2+1} - \frac{1}{2} \ln(x + \sqrt{x^2+1}) + 2, \quad x \geq 0 \end{aligned}$$

is superquadratic and $f(x) \leq 0$, $x \in [0, T]$, where T satisfies $2 < T < 3$ and $f(x) \geq 0$, $x \geq T$. Therefore, on $[a, a + 1]$, $a \geq 0$ is ϕ -convex where $\phi = -f$, $0 \leq x \leq T$.

Another set of functions f that are either ϕ -convex or strongly convex are

Example 7. Let $f(x) = x\varphi(x)$, $x \geq 0$, where φ is convex. It is easy to verify that

$$tf(x) + (1-t)f(y) \geq f(tx + (1-t)y) + t(1-t)\varphi'(0)(x-y)^2$$

holds. Therefore, if $\varphi'(0) < 0$ it means that $f : \mathbb{R}_+ \rightarrow \mathbb{R}$ is ϕ -convex, $\phi = -\varphi'(0)x^2$ and $x \geq 0$. If $\varphi'(0) > 0$, it means that f is strongly convex and uniformly convex where $\Phi = \varphi'(0)x^2$.

In particular, the function $f(x) = x(x-1)^{2n} = x\varphi(x)$, $x \geq 0$ where the function $\varphi(x) = (x-1)^{2n}$, $n = 1, 2, \dots$ is convex and satisfies $\varphi'(0) = -2n$. Therefore, f is ϕ -convex where $\phi = 2nx^2$.

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