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Stability and Hyperstability of Ternary Hom-Multiplier on Ternary Banach Algebra

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Abstract

In this article, we investigate the 3D additive-type functional equation. Next, we introduce the ternary hom-multiplier in ternary Banach algebras using the concepts of ternary homomorphisms and ternary multipliers. We first establish proof that solutions to the 3D additive-type functional equation are additive mappings. We further demonstrate that these solutions are $\mathbb{C}$ -linear mappings. The final portion of our work examines both the stability and hyperstability properties of the 3D additive-type functional equation, ternary hom-multiplier, and ternary Jordan hom-multiplier on ternary Banach algebras. Our analysis employs the fixed-point theorem using control functions developed by Găvruta and Rassias.

Keywords: Hyers–Ulam stability; hyperstability; fixed point method; 3D additive-type functional equation; ternary hom-multiplier

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1. Introduction

The stability of functional equations is a fundamental concept in mathematics with far-reaching applications. It was first proposed by Ulam in 1940 [1] who posed the essential question: under what conditions does an approximate solution to a functional equation imply the existence of a nearby exact solution? This idea was further developed by Hyers [2] in 1941 within the framework of Banach spaces, marking a major milestone in the study of functional equation stability. In 1978, Rassias [3] extended this theory by introducing a generalized version of Hyers–Ulam stability, incorporating a control function of the form $\varepsilon(\|a\|^r + \|b\|^r)$, where $\varepsilon > 0$ and $r < 1$, particularly for additive mappings. Later, in 1994, Găvruta [4] refined this approach by replacing Rassias’ control function with a more general function $\varphi(a, b)$, further broadening the scope of stability results. More researchers have been studying Hyers–Ulam stability in various structures, for example, non-Abelian semigroups [5,6], Banach algebras [7], commutative normed algebras [8], and C^* -algebras [9–11]. In addition, many researchers have investigated Ulam–Hyers–Rassias stability via fixed point theory (see [12–15]).

Hyers–Ulam stability refers to a property of equations, where small changes in the equation result in only minor changes in its solutions. In 1993, Obloza [16] extended Hyers–Ulam stability to linear differential equations. In 2003, Miura et al. [17] studied this stability for first-order and higher-order linear constant coefficient differential equations. Following that, researchers examined Hyers–Ulam stability for linear operators on Hilbert spaces [18] and differential operators on weighted Hardy spaces [19].

The concept of hyperstability was first introduced in [20], with an initial focus on specific ring homomorphisms. A functional equation is said to be hyperstable if every approximate solution is, in fact, an exact solution. This notion marks a notable development in the theory of functional equation stability, highlighting cases where stability is so strong that any deviation from the equation necessarily implies the function already satisfies it exactly. Extensive research has since been conducted in this area; for further details, see [21–23].

In the 19th century, Cayley made pioneering contributions by introducing 3-ary operations through cubic matrices. This innovation paved the way for the development of n -ary algebraic structures across various fields. Notably, the exploration of n -ary algebras, with a specific focus on ternary algebraic structures, gained significant attention. In 2008, Amyari and Moslehian [24] introduced the concept of ternary algebras, defining a ternary algebra $\mathfrak{B}$ as a complex space equipped with a ternary product $(u, v, w) \rightarrow [u, v, w]$ from $\mathfrak{B}^3$ into $\mathfrak{B}$. This product is required to be $\mathbb{C}$ -linear in the outer variables, conjugate $\mathbb{C}$ -linear in the middle variable, and associative. Additionally, it must satisfy certain norm properties, such as $\|[u, v, w]\| \leq \|u\| \cdot \|v\| \cdot \|w\|$ and $\|[u, u, u]\| = \|u\|^3$. When a ternary algebra $\mathfrak{B}$ is also a Banach space, it is referred to as a ternary Banach algebra. These extended operations satisfy key properties like associativity and linearity, enabling more complex algebraic structures. Both ternary and n -ary algebras have applications in coding theory, computer science, physics, quantum mechanics, and Nambu mechanics. For more details, see [25–27].

In the following, we solve an example of the ternary structure property.

Example 1 (Matrix Algebra with Ternary Product). *Let $\mathbb{M}_n(\mathbb{C})$ be the algebra of all $n \times n$ complex matrices, equipped with the norm $\|\cdot\|$ (e.g., the Frobenius norm). Define a ternary product $[A, B, C]$ on $\mathbb{M}_n(\mathbb{C})$ by*

$$[A, B, C] := ABC + BCA + CAB,$$

for all $A, B, C \in \mathbb{M}_n(\mathbb{C})$. This defines a ternary Banach algebra, since the product is linear in each variable, and $\mathbb{M}_n(\mathbb{C})$ is complete.

Now, define a linear mapping $T : \mathbb{M}_n(\mathbb{C}) \rightarrow \mathbb{M}_n(\mathbb{C})$ by

$$T(X) = \lambda X,$$

for some fixed scalar $\lambda \in \mathbb{C}$. Then, for all $A, B, C \in \mathbb{M}_n(\mathbb{C})$, we compute

$$\begin{aligned} T([A, B, C]) &= \lambda(ABC + BCA + CAB), \\ [T(A), B, C] + [A, T(B), C] + [A, B, T(C)] &= \lambda(ABC + BCA + CAB). \end{aligned}$$

Consequently,

$$T([A, B, C]) = [T(A), B, C] + [A, T(B), C] + [A, B, T(C)],$$

showing that T is a ternary derivation.

Next, consider the three-dimensional additive-type functional equation

$$f(x + y + z) = f(x) + f(y) + f(z).$$

Suppose a mapping $f : \mathbb{M}_n(\mathbb{C}) \rightarrow \mathbb{M}_n(\mathbb{C})$ satisfies the inequality

$$\|f(x + y + z) - f(x) - f(y) - f(z)\| \leq \varepsilon(\|x\| + \|y\| + \|z\|),$$

for some small $\varepsilon > 0$ and all $x, y, z \in \mathbb{M}_n(\mathbb{C})$. Define

$$f(X) = \lambda X + E(X),$$

where $E : \mathbb{M}_n(\mathbb{C}) \rightarrow \mathbb{M}_n(\mathbb{C})$ is a perturbation satisfying $\|E(X)\| \leq \varepsilon \|X\|$. Then,

$$\begin{aligned} f(x + y + z) &= \lambda(x + y + z) + E(x + y + z), \\ f(x) + f(y) + f(z) &= \lambda(x + y + z) + E(x) + E(y) + E(z), \end{aligned}$$

and thus,

$$\begin{aligned} &\|f(x + y + z) - f(x) - f(y) - f(z)\| \\ &= \|E(x + y + z) - E(x) - E(y) - E(z)\| \leq 3\varepsilon(\|x\| + \|y\| + \|z\|). \end{aligned}$$

Therefore, f is an approximate solution of the 3D additive-type functional equation.

In 2015, Park [28] introduced additive ρ -functional inequalities and provided a comprehensive analysis of the Hyers–Ulam stability of these inequalities in both Banach spaces and non-Archimedean Banach spaces. Subsequently, leveraging the concept of additive-type functional equations, we establish a 3D additive-type functional equation between normed spaces. In this context, let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ denote normed spaces. The mapping ζ from $(X, \|\cdot\|_X)$ to $(Y, \|\cdot\|_Y)$ is called a 3D additive-type functional equation, if ζ satisfies

$$\zeta(a + b + c) + \zeta(a) - \zeta(a + b) - \zeta(a + c) = \zeta(a - b) + \zeta(b - c) - \zeta(a - c). \quad (1)$$

In 2012, Eshaghi Gordji et al. [29], defined a 3-Lie multiplier on a 3-Lie Banach algebra. In this sequel, by paying attention to the concept of ternary homomorphisms and ternary multipliers, we investigate a ternary hom-multiplier between ternary Banach algebras. For this work, let $\mathfrak{B}$ be a ternary Banach algebra, and let the mappings γ and θ from $\mathfrak{B}$ to $\mathfrak{B}$ be called a ternary homomorphism and a ternary multiplier, respectively. Assume γ and θ satisfy $\gamma([a, b, c]) = [\gamma(a), \gamma(b), \gamma(c)]$, and $[\theta(a), b, c] = [a, \theta(b), c] = [a, b, \theta(c)]$ for all $a, b, c \in \mathfrak{B}$.

Let mappings γ and θ from $\mathfrak{B}$ to $\mathfrak{B}$ be a ternary homomorphism and a ternary multiplier, respectively. Then,

$$[\gamma\theta(a), \gamma(b), \gamma(c)] = [\gamma(a), \gamma\theta(b), \gamma(c)] = [\gamma(a), \gamma(b), \gamma\theta(c)]$$

$$[\theta\gamma(a), \gamma(b), \gamma(c)] = [\gamma(a), \theta\gamma(b), \gamma(c)] = [\gamma(a), \gamma(b), \theta\gamma(c)]$$

for all $a, b, c \in \mathfrak{B}$.

As noted above, the study of higher-order algebraic structures such as ternary (or more generally, n -ary) algebras has gained attention due to their natural appearance in various branches of physics and abstract algebra, particularly in modeling systems with non-binary interactions. While binary (or double) hom-multipliers have been widely studied and serve as foundational tools in functional equations and homomorphism theory, they are often insufficient for capturing the full complexity of interactions inherent in ternary systems.

This motivates the aim to extend classical notions like the hom-multiplier and Jordan hom-multiplier into the ternary setting. A ternary hom-multiplier is designed to capture the behavior of mappings that preserve ternary operations under specific structural constraints. Unlike their binary counterparts, these mappings interact with a three-variable product, which is characteristic of ternary Banach algebras.

To illustrate the distinction, a double hom-multiplier deals with structures where associativity or compatibility is considered between two elements, whereas a ternary hom-multiplier extends this compatibility to ternaries of elements, making them more suitable for analyzing systems governed by ternary operations. Similarly, the ternary Jordan hom-multiplier extends the symmetric structure of the Jordan product to a ternary context, allowing for broader applications in areas where symmetry in ternary operations is central.

These extensions are not merely formal generalizations—they provide new insights and tools for studying stability phenomena in ternary functional equations, as well as modeling physical systems where ternary interactions are fundamental (e.g., in quark models in quantum chromodynamics or in ternary logic systems).

In the following section, we define a ternary hom-ternary multiplier (briefly, ternary hom-multiplier) and a ternary Jordan hom-Jordan multiplier (briefly, (ternary Jordan hom-multiplier) between ternary Banach algebras.

Definition 1. Let a mapping γ from $\mathcal{B}$ to $\mathcal{B}$ be a ternary homomorphism. A $\mathbb{C}$ -linear mapping θ from $\mathcal{B}$ to $\mathcal{B}$ is called a ternary hom-multiplier if θ satisfies

$$[\theta(a), \gamma(b), \gamma(c)] = [\gamma(a), \theta(b), \gamma(c)] = [\gamma(a), \gamma(b), \theta(c)]$$

for all $a, b, c \in \mathcal{B}$.

Definition 2. Let a mapping γ from $\mathcal{B}$ to $\mathcal{B}$ be a ternary homomorphism. A $\mathbb{C}$ -linear mapping θ from $\mathcal{B}$ to $\mathcal{B}$ is called a ternary Jordan hom-multiplier if θ satisfies

$$[\theta(a), \gamma(a), \gamma(a)] = [\gamma(a), \theta(a), \gamma(a)] = [\gamma(a), \gamma(a), \theta(a)]$$

for all $a \in \mathcal{B}$.

Theorem 1 ([30]). Consider a complete generalized metric space (A, d) equipped with a strictly contractive mapping $F : A \rightarrow A$ characterized by a Lipschitz constant $\beta < 1$. For any element $u \in A$, we examine the behavior of the sequence $\{d(F^i u, F^{i+1} u)\}$, where i ranges over non-negative integers. We observe that this sequence either diverges to infinity for all non-negative integers i , or there exists a positive integer i_0 such that

- (1) $d(F^i u, F^{i+1} u) < \infty$ for all $i \geq i_0$;
- (2) The sequence $\{F^i u\}$ converges to a fixed point v^* of F ;
- (3) v^* is the unique fixed point of F within the set $B = \{v \in A \mid d(F^{i_0} u, v) < \infty\}$;
- (4) $d(v, v^*) \leq \frac{1}{1-\beta} d(v, Fv)$ for all $v \in B$.

In Section 2, we delve into the intricate details of the 3D additive-type functional equation and establish its properties as an additive mapping. Subsequently, we embark on an investigation into its $\mathbb{C}$ -linearity for each $\mu \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$. Finally, employing a fixed-point method, we rigorously demonstrate the Hyers–Ulam stability and hyperstability of the 3D additive-type functional equation within the realm of ternary Banach algebras.

In Section 3, by using Theorem 1, we prove that both the ternary hom-multiplier and ternary Jordan hom-multiplier can be stable and hyperstable associated with the 3D additive-type functional equation under the control functions of Găvruta and Rassias. For this reason, we will mix the idea of control functions with the bounded/unbounded Cauchy difference.

2. Results of 3D Additive-Type Functional Equation

In this section, let μ be a member of the set T^1 , let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces, and let $\mathfrak{B}$ be a ternary Banach algebra. In the following lemma, we solve the 3D additive-type functional equation and investigate when it is an additive mapping. Please note that X , Y , and $\mathfrak{B}$ are Banach spaces endowed with norms (denoted by $\|\cdot\|$); the context will clarify which space is meant.

Lemma 1. *If a mapping ζ from $(X, \|\cdot\|_X)$ to $(Y, \|\cdot\|_Y)$ satisfies (1), then ζ is an additive mapping.*

Proof. Suppose the mapping ζ satisfies Equation (1). If $a = b = c = 0$ in (1), then $\zeta(0) = 0$. Letting $a = c = 0$ in (1), we get

$$\zeta(-b) = -\zeta(b). \quad (2)$$

In the subsequent step, by using (2) and letting $a = 0$ and $b = c$ in (1), we have

$$\zeta(2b) = 2\zeta(b). \quad (3)$$

Finally, letting $a = 0$ once again and using Equations (2) and (3) in Equation (1), we get

$$\zeta(a + b) = \zeta(a) + \zeta(b).$$

□

In the subsequent lemma, we introduce the element $\mu \in T^1$ into the 3D additive-type functional equation. By doing so, we proceed to confirm that the mapping ζ , is a $\mathbb{C}$ -linear mapping. This implies that ζ satisfies the lemma above and is homogeneous over the complex numbers (i.e., for any complex number μ and any element $a \in X$,

$$\zeta(\mu a) = \mu \zeta(a).$$

Lemma 2. *If a mapping ζ from $(X, \|\cdot\|_X)$ to $(Y, \|\cdot\|_Y)$ satisfies the equation*

$$\mu \zeta(a + b + c) + \mu \zeta(a) - \zeta(\mu a + \mu b) - \zeta(\mu a + \mu c) = \zeta(\mu a - \mu b) + \mu \zeta(b - c) - \zeta(\mu a - \mu c) \quad (4)$$

for all $a, b, c \in X$ and $\mu \in T^1$, then ζ is a $\mathbb{C}$ -linear mapping.

Proof. By utilizing Lemma 1, it follows that ζ is an additive mapping. Setting $a = c = 0$ in (4), we obtain the following

$$\zeta(\mu b) = \mu \zeta(b). \quad (5)$$

By applying ([31], Lemma 1), or ([32], Corollary 2.2), we obtain that the mapping J is linear with respect to complex numbers. □

Remark 1. *Actually, it follows from ([31], Lemma 1), or ([32], Corollary 2.2) that it is enough to assume in Lemma 2 that α belongs to some "big" subsets of T^1 .*

In the upcoming theorem, we establish that the 3D additive-type functional equation can achieve stability by employing Găvruta's control function.

Theorem 2. *Let a function $\eta : \mathfrak{B}^3 \rightarrow [0, \infty)$ satisfy the inequality*

$$\eta\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right) \leq \frac{\delta}{2} \eta(a, b, c) \quad (6)$$

for some $0 < \delta < 1$. If $\zeta : \mathfrak{B} \rightarrow \mathfrak{B}$ satisfies

$$\left\| \mu\zeta(a+b+c) + \mu\zeta(a) - \zeta(\mu a + \mu b) - \zeta(\mu a + \mu c) - \left(\zeta(\mu a - \mu b) + \mu\zeta(b-c) - \zeta(\mu a - \mu c) \right) \right\| \leq \eta(a, b, c), \quad (7)$$

there exists a unique additive mapping $F : \mathfrak{B} \rightarrow \mathfrak{B}$ such that

$$\|\zeta(a) - F(a)\| \leq \frac{\delta}{2(1-\delta)} \eta(0, a, a)$$

for all $a \in \mathfrak{B}$.

Proof. Using (6) and taking $a = b = c = 0$, we have $\eta(0, 0, 0) = 0$. Additionally from (6) we get

$$\lim_{n \rightarrow \infty} 2^n \eta\left(\frac{a}{2^n}, \frac{b}{2^n}, \frac{c}{2^n}\right) = 0. \quad (8)$$

Setting $\mu = 1$, $a = 0$ and $b = c$ in (7), we obtain the relation

$$\|\zeta(2a) - 2\zeta(a)\| \leq \eta(0, a, a)$$

for all $a \in \mathfrak{B}$.

Let Y be the set of all mappings $g : \mathfrak{B} \rightarrow \mathfrak{B}$ with $g(0) = 0$. Define d on Y as

$$d(\zeta, g) = \inf\{\beta \in (0, \infty) : \|\zeta(a) - g(a)\| \leq \beta \eta(0, a, a) \quad \forall a \in \mathfrak{B}\},$$

where, as usual, $\inf \emptyset = \infty$. It is clear that (Y, d) constitutes a generalized metric space (see [33]). Next, we investigate the linear mappings $\Gamma : Y \rightarrow Y$ satisfying

$$\Gamma\zeta(a) = 2\zeta\left(\frac{a}{2}\right)$$

for all $a \in \mathfrak{B}$.

If ζ and g are elements of Y such that $d(\zeta, g) = \varepsilon$, then $\|\zeta(a) - g(a)\| \leq \varepsilon \eta(0, a, a)$. Since

$$\|\Gamma\zeta(a) - \Gamma g(a)\| \leq 2\varepsilon \frac{\delta}{2} \eta(0, a, a) = \delta\varepsilon \eta(0, a, a)$$

for each $a \in \mathfrak{B}$ and $d(\Gamma\zeta, \Gamma g) \leq \delta\varepsilon$, then

$$d(\Gamma\zeta, \Gamma g) \leq \delta d(\zeta, g), \quad \text{with } \delta \in (0, 1)$$

for all $\zeta, g \in Y$. It follows from (6) that

$$\left\| \zeta(a) - 2\zeta\left(\frac{a}{2}\right) \right\| \leq \eta\left(0, \frac{a}{2}, \frac{a}{2}\right) \leq \frac{\delta}{2} \eta(0, a, a)$$

for all $a \in \mathfrak{B}$. So, $d(\zeta, \Gamma\zeta) \leq \frac{\delta}{2}$. According to Theorem 1, there is a mapping $F : \mathfrak{B} \rightarrow \mathfrak{B}$ that satisfies the following.

(1) F is a fixed point of Γ , i.e.,

$$F(a) = 2F\left(\frac{a}{2}\right) \quad \forall a \in \mathfrak{B}. \quad (9)$$

The mapping F serves as the unique fixed point of Γ in the set

$$\Phi = \left\{ \zeta, g \in Y : d(\zeta, g) \leq \infty \right\}.$$

This implies that F is the unique mapping that satisfies (9) such that there exists a $\beta \in (0, \infty)$ satisfying

$$\|\zeta(a) - F(a)\| \leq \beta \eta(0, a, a) \quad \forall a \in \mathfrak{B}.$$

(2) $d(\Gamma^n \zeta, F) \rightarrow 0$ as $n \rightarrow \infty$. This implies the equality

$$\lim_{n \rightarrow \infty} 2^n \zeta\left(\frac{a}{2^n}\right) = F(a) \quad \forall a \in \mathfrak{B}.$$

(3) $d(\zeta, F) \leq \frac{\delta}{2(1-\delta)} d(\zeta, \Gamma \zeta)$, which implies

$$\|\zeta(a) - F(a)\| \leq \frac{\delta}{2(1-\delta)} \eta(0, a, a).$$

Finally, it follows from (7) that

$$\begin{aligned} & \left\| F(a+b+c) + F(a) - F(a+b) - F(a+c) - (F(a-b) + F(b-c) - F(a-c)) \right\| \\ &= \lim_{n \rightarrow \infty} 2^n \left(\left\| \zeta\left(\frac{a+b+c}{2^n}\right) + \zeta\left(\frac{a}{2^n}\right) - \zeta\left(\frac{a+b}{2^n}\right) - \zeta\left(\frac{a+c}{2^n}\right) \right. \right. \\ & \quad \left. \left. - \left(\zeta\left(\frac{a-b}{2^n}\right) + \zeta\left(\frac{b-c}{2^n}\right) - \zeta\left(\frac{a-c}{2^n}\right) \right) \right\| \right) + \lim_{n \rightarrow \infty} 2^n \eta\left(\frac{a}{2^n}, \frac{b}{2^n}, \frac{b}{2^n}\right) = 0 \end{aligned}$$

for all $a, b, c \in \mathfrak{B}$. Thus,

$$F(a+b+c) + F(a) - F(a+b) - F(a+c) = (F(a-b) + F(b-c) - F(a-c))$$

for all $a, b, c \in \mathfrak{B}$. According to Lemma 2, the mapping F is an additive mapping. $\square$

Considering Theorem 2, we delve into the investigation of the stability of the 3D additive-type functional equation on a ternary Banach algebra in the subsequent corollary, exploring stability using the control function proposed by Rassias.

Corollary 1. Let $r > 1$ and κ be non-negative real numbers. If any mapping $\zeta : \mathfrak{B} \rightarrow \mathfrak{B}$ satisfies

$$\begin{aligned} & \left\| \mu \zeta(a+b+c) + \mu \zeta(a) - \zeta(\mu a + \mu b) - \zeta(\mu a + \mu c) \right. \\ & \quad \left. - (\zeta(\mu a - \mu b) + \mu \zeta(b-c) - \zeta(\mu a - \mu c)) \right\| \leq \kappa (\|a\|^r + \|b\|^r + \|c\|^r), \end{aligned} \quad (10)$$

then there is a unique $\mathbb{C}$ -linear mapping $F : \mathfrak{B} \rightarrow \mathfrak{B}$ such that

$$\|\zeta(a) - F(a)\| \leq \frac{2\kappa}{2^r - 2} \|a\|^r.$$

Proof. Let $r > 1$ and κ be non-negative real numbers. Consider Theorem 2 with $\delta = 2^{1-r} \in (0, 1)$ and $\eta(a, b, c) = \kappa (\|a\|^r + \|b\|^r + \|c\|^r)$. Checking (6), we have

$$\eta\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right) \leq 2^{-r} \eta(a, b, c) = \frac{\delta}{2} \eta(a, b, c).$$

By the assumption in (10), we have that (7) holds; so, the conditions of Theorem 2 are met. Therefore, the conclusion of Theorem 2 holds, which is the stated conclusion of this corollary. $\square$

In the forthcoming theorem, we explore the hyperstability of the 3D additive-type functional equation on a ternary Banach algebra, focusing on Găvruta's control function for a comprehensive analysis.

Theorem 3. Let a function $\eta : \mathfrak{B}^3 \rightarrow [0, \infty)$ satisfy

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} \eta(2^n a, 0, 0) = 0. \quad (11)$$

Additionally, if the mapping $\zeta : \mathfrak{B} \rightarrow \mathfrak{B}$ satisfies

$$\left\| \begin{aligned} &\mu\zeta(a+b+c) + \mu\zeta(a) - \zeta(\mu a + \mu b) - \zeta(\mu a + \mu c) \\ &- \left(\zeta(\mu a - \mu b) + \mu\zeta(b-c) - \zeta(\mu a - \mu c) \right) \end{aligned} \right\| \leq \eta(a, 0, 0) \quad (12)$$

for all $a, b, c \in \mathfrak{B}$, then ζ is an additive mapping.

Proof. If $\mu = 1$, $a = 0$, and $b = c$ in (12), then $\zeta(2a) = 2\zeta(a)$, and by using induction on $n \in \mathbb{N}$, we get

$$\zeta(a) = 2^n \zeta\left(\frac{a}{2^n}\right).$$

Hence,

$$2^n \left\| \begin{aligned} &\mu\zeta(a+b+c) + \mu\zeta(a) - \zeta(\mu a + \mu b) - \zeta(\mu a + \mu c) \\ &- \left(\zeta(\mu a - \mu b) + \mu\zeta(b-c) - \zeta(\mu a - \mu c) \right) \end{aligned} \right\| \leq 2^n \eta\left(\frac{a}{2^n}, 0, 0\right), \quad (13)$$

for all $a, b, c \in \mathfrak{B}$ and $n \in \mathbb{N}$. Therefore, as $n \rightarrow \infty$ in (13) and using (11), we get

$$\mu\zeta(a+b+c) + \mu\zeta(a) - \zeta(\mu a + \mu b) - \zeta(\mu a + \mu c) = \zeta(\mu a - \mu b) + \mu\zeta(b-c) - \zeta(\mu a - \mu c)$$

for all $a, b, c \in \mathfrak{B}$. Therefore, ζ is an additive mapping. $\square$

Referring to Theorem 3, in the subsequent corollary we investigate hyperstability concerning the 3D additive-type functional equation, focusing on the utilization of Rassias' control function for a comprehensive analysis.

Corollary 2. Let $r > 1$ and κ be non-negative real numbers. If a mapping $\zeta : \mathfrak{B} \rightarrow \mathfrak{B}$ satisfies

$$\left\| \begin{aligned} &\mu\zeta(a+b+c) + \mu\zeta(a) - \zeta(\mu a + \mu b) - \zeta(\mu a + \mu c) \\ &- \left(\zeta(\mu a - \mu b) + \mu\zeta(b-c) - \zeta(\mu a - \mu c) \right) \end{aligned} \right\| \leq \kappa(\|a\|^r) \quad (14)$$

for all $a, b, c \in \mathfrak{B}$, then ζ is a $\mathbb{C}$ -linear mapping.

3. Results of Ternary Hom-Multipliers

In the forthcoming theorem, we rigorously establish the stability of a ternary hom-multiplier on a ternary Banach algebra. Our proof methodology involves leveraging the fixed point method while carefully considering the insights provided by Theorem 2.

Theorem 4. Let a function η satisfy

$$\eta\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right) \leq \frac{\delta}{2^3} \eta(a, b, c) \quad (15)$$

for all $a, b, c \in \mathfrak{B}$. Assume that mappings ζ and h satisfy (7) and the inequalities

$$\|h([a, b, c]) - [h(a), h(b), h(c)]\| \leq \eta(a, b, c), \quad (16)$$

$$\|[\zeta(a), h(b), h(c)] - [h(a), \zeta(b), h(c)]\| \leq \eta(a, b, c), \quad (17)$$

and

$$\|[\zeta(a), h(b), h(c)] - [h(a), h(b), \zeta(c)]\| \leq \eta(a, b, c). \quad (18)$$

Then, there exist a unique ternary homomorphism $\gamma : \mathfrak{B} \rightarrow \mathfrak{B}$ and a unique ternary hom-multiplier $\theta : \mathfrak{B} \rightarrow \mathfrak{B}$ such that

$$\|h(a) - \gamma(a)\| \leq \frac{\delta}{1-\delta} \eta(0, a, a)$$

and

$$\|\zeta(a) - \theta(a)\| \leq \frac{\delta}{1-\delta} \eta(0, a, a).$$

Proof. Utilizing a methodology akin to the one elucidated in the proof of Theorem 2, we can define mappings $\gamma : \mathfrak{B} \rightarrow \mathfrak{B}$ and $\theta : \mathfrak{B} \rightarrow \mathfrak{B}$ in the manner of

$$\gamma(a) = \lim_{n \rightarrow \infty} 2^n h\left(\frac{a}{2^n}\right) \quad \text{and} \quad \theta(a) = \lim_{n \rightarrow \infty} 2^n \zeta\left(\frac{a}{2^n}\right) \quad \forall a \in \mathfrak{B}. \quad (19)$$

It follows from (15), (16), and (19) that

$$\begin{aligned} \|\gamma([a, b, c]) - [\gamma(a), \gamma(b), \gamma(c)]\| &= \lim_{n \rightarrow \infty} 2^{3n} \left\| h\left(\frac{[a, b, c]}{2^{3n}}\right) - \left[h\left(\frac{a}{2^n}\right), h\left(\frac{b}{2^n}\right), h\left(\frac{c}{2^n}\right) \right] \right\| \\ &\leq \lim_{n \rightarrow \infty} 2^{3n} \eta\left(\frac{a}{2^n}, \frac{b}{2^n}, \frac{c}{2^n}\right) \\ &= 0. \end{aligned}$$

So, the mapping γ is a ternary homomorphism.

In the following proof, by using relations (15) and (17)–(19), we get

$$\begin{aligned} &\|[\theta(a), \gamma(b), \gamma(c)] - [\gamma(a), \theta(b), \gamma(c)]\| \\ &= \lim_{n \rightarrow \infty} 2^{3n} \left\| \left[\zeta\left(\frac{a}{2^n}\right), h\left(\frac{b}{2^n}\right), h\left(\frac{c}{2^n}\right) \right] - \left[h\left(\frac{a}{2^n}\right), \zeta\left(\frac{b}{2^n}\right), h\left(\frac{c}{2^n}\right) \right] \right\| \\ &\leq \lim_{n \rightarrow \infty} 2^{3n} \eta\left(\frac{a}{2^n}, \frac{b}{2^n}, \frac{c}{2^n}\right) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} & \|[\theta(a), \gamma(b), \gamma(c)] - [\gamma(a), \gamma(b), \theta(c)]\| \\ &= \lim_{n \rightarrow \infty} 2^{3n} \left\| \left[\zeta\left(\frac{a}{2^n}\right), h\left(\frac{b}{2^n}\right), h\left(\frac{c}{2^n}\right) \right] - \left[h\left(\frac{a}{2^n}\right), h\left(\frac{b}{2^n}\right), \gamma\left(\frac{c}{2^n}\right) \right] \right\| \\ &\leq \lim_{n \rightarrow \infty} 2^{3n} \eta\left(\frac{a}{2^n}, \frac{b}{2^n}, \frac{c}{2^n}\right) \\ &= 0. \end{aligned}$$

Consequently, the mapping θ is a ternary hom-multiplier. $\square$

In the next corollary, we investigate whether the ternary hom-multiplier can be stable with the control function of Rassias.

Corollary 3. Consider non-negative real numbers κ and r such that $r > 1$. If mappings $h, \zeta : \mathfrak{B} \rightarrow \mathfrak{B}$ satisfy in (10) and

$$\|h([a, b, c]) - [h(a), h(b), h(c)]\| \leq \kappa(\|a\|^r + \|b\|^r + \|c\|^r),$$

$$\|[\zeta(a), h(b), h(c)] - [h(a), \zeta(b), h(c)]\| \leq \kappa(\|a\|^r + \|b\|^r + \|c\|^r),$$

and

$$\|[\zeta(a), h(b), h(c)] - [h(a), h(b), \zeta(c)]\| \leq \kappa(\|a\|^r + \|b\|^r + \|c\|^r),$$

then there exist a unique ternary homomorphism $\gamma : \mathfrak{B} \rightarrow \mathfrak{B}$ and unique ternary hom-multiplier $\theta : \mathfrak{B} \rightarrow \mathfrak{B}$ such that

$$\|h(a) - \gamma(a)\| \leq \frac{2\kappa}{2^r - 2} \|a\|^r$$

and

$$\|\zeta(a) - \theta(a)\| \leq \frac{2\kappa}{2^r - 2} \|a\|^r.$$

Proof. For this, it is enough to put $\eta(a, b, c)$ and δ instead of $\kappa(\|a\|^r + \|b\|^r + \|c\|^r)$ and 2^{1-r} into the proof of Theorem 4, respectively. $\square$

The subsequent theorem and corollary delve into an in-depth exploration of the hyperstability exhibited by the ternary hom-multiplier on a ternary Banach algebra. By using two control functions Găvruta and Rassias, we investigate the hyperstability properties of the ternary hom-multiplier using the fixed point method.

Theorem 5. Assume that function η satisfies (11). If mappings ζ and h from $\mathfrak{B}$ to $\mathfrak{B}$ satisfy (12) and the inequalities

$$\|h([a, b, c]) - [h(a), h(b), h(c)]\| \leq \eta(a, b, c),$$

$$\|[\zeta(a), h(b), h(c)] - [h(a), \zeta(b), h(c)]\| \leq \eta(a, b, c),$$

and

$$\|[\zeta(a), h(b), h(c)] - [h(a), h(b), \zeta(c)]\| \leq \eta(a, b, c),$$

then the mappings γ and θ from $\mathfrak{B}$ to $\mathfrak{B}$ are a ternary homomorphism and a ternary hom-multiplier, respectively.

Proof. Similar to the proof of Theorem 3 and employing the method of mathematical induction on $n \in \mathbb{N}$, we obtain

$$h(a) = 2^n h\left(\frac{a}{2^n}\right), \quad \zeta(a) = 2^n \zeta\left(\frac{a}{2^n}\right)$$

for all $a \in \mathfrak{B}$. The proof continues as in the proof of Theorem 4. $\square$

Corollary 4. Consider non-negative real numbers κ and r such that $r > 1$. If mappings $h, \zeta : \mathfrak{B} \rightarrow \mathfrak{B}$ satisfy (14) and the inequalities

$$\|h([a, b, c]) - [h(a), h(b), h(c)]\| \leq \kappa(\|a\|^r + \|b\|^r + \|c\|^r),$$

$$\|[\zeta(a), h(b), h(c)] - [h(a), \zeta(b), h(c)]\| \leq \kappa(\|a\|^r + \|b\|^r + \|c\|^r),$$

and

$$\|[\zeta(a), h(b), h(c)] - [h(a), h(b), \zeta(c)]\| \leq \kappa(\|a\|^r + \|b\|^r + \|c\|^r),$$

then the mappings h and ζ are a ternary homomorphism and a ternary hom-multiplier, respectively.

In the concluding phase of our study, through the fixed point method, we delve into the exploration of the Hyers–Ulam stability and hyperstability of the ternary Jordan hom-multiplier with two control functions introduced by Găvruta and Rassias.

Theorem 6. Let a function η satisfy (15). If mappings ζ and h satisfy (7) and the inequalities

$$\|h([a, b, c]) - [h(a), h(b), h(c)]\| \leq \eta(a, b, c),$$

$$\|[\zeta(a), h(a), h(a)] - [h(a), \zeta(a), h(a)]\| \leq \eta(a, a, a),$$

and

$$\|[\zeta(a), h(a), h(a)] - [h(a), h(a), \zeta(a)]\| \leq \eta(a, a, a),$$

then there exist a unique ternary homomorphism $\gamma : \mathfrak{B} \rightarrow \mathfrak{B}$ and unique ternary Jordan hom-multiplier $\theta : \mathfrak{B} \rightarrow \mathfrak{B}$ such that

$$\|h(a) - \gamma(a)\| \leq \frac{\delta}{1 - \delta} \eta(0, a, a)$$

and

$$\|\zeta(a) - \theta(a)\| \leq \frac{\delta}{1 - \delta} \eta(0, a, a).$$

Proof. This is similar to the proof that is employed for Theorem 4. $\square$

Corollary 5. Consider non-negative real numbers κ and r such that $r > 1$. If mappings $h, \zeta : \mathfrak{B} \rightarrow \mathfrak{B}$ satisfy in (10) and

$$\|h([a, b, c]) - [h(a), h(b), h(c)]\| \leq \kappa(\|a\|^r + \|b\|^r + \|c\|^r),$$

$$\|[\zeta(a), h(a), h(a)] - [h(a), \zeta(a), h(a)]\| \leq \kappa(\|a\|^r + \|a\|^r + \|a\|^r),$$

and

$$\|[\zeta(a), h(a), h(a)] - [h(a), h(a), \zeta(a)]\| \leq \kappa(\|a\|^r + \|a\|^r + \|a\|^r),$$

then there exist a unique ternary homomorphism $\gamma : \mathfrak{B} \rightarrow \mathfrak{B}$ and unique ternary Jordan hom-multiplier $\theta : \mathfrak{B} \rightarrow \mathfrak{B}$ such that

$$\|h(a) - \gamma(a)\| \leq \frac{2\kappa}{2^r - 2} \|a\|^r$$

and

$$\|\zeta(a) - \theta(a)\| \leq \frac{2\kappa}{2^r - 2} \|a\|^r.$$

Remark 2. Let a function η satisfy (11). If the mappings ζ and h satisfy (12) and the inequalities

$$\|h([a, b, c]) - [h(a), h(b), h(c)]\| \leq \eta(a, b, c),$$

$$\|[\zeta(a), h(a), h(a)] - [h(a), \zeta(a), h(a)]\| \leq \eta(a, a, a),$$

and

$$\|[\zeta(a), h(a), h(a)] - [h(a), h(a), \zeta(a)]\| \leq \eta(a, a, a),$$

then the mappings γ and θ are a ternary homomorphism and ternary Jordan hom-multiplier, respectively.

Remark 3. Let $r > 1$ and κ be non-negative real numbers. If mappings $h, \zeta : \mathfrak{B} \rightarrow \mathfrak{B}$ satisfy (14) and

$$\|h([a, b, c]) - [h(a), h(b), h(c)]\| \leq \kappa(\|a\|^r + \|b\|^r + \|c\|^r),$$

$$\|[\zeta(a), h(a), h(a)] - [h(a), \zeta(a), h(a)]\| \leq \kappa(\|a\|^r + \|a\|^r + \|a\|^r),$$

and

$$\|[\zeta(a), h(a), h(a)] - [h(a), h(a), \zeta(a)]\| \leq \kappa(\|a\|^r + \|a\|^r + \|a\|^r)$$

for all $a, b, c \in \mathfrak{B}$, then the mappings h and ζ are a ternary homomorphism and ternary Jordan hom-multiplier, respectively.

4. Conclusions

Due to the significance and wide-ranging applications of n -ary structures—particularly ternary algebras and ternary homomorphisms—in quantum mechanics, mathematical physics, and other scientific fields, we focus on the study of such systems. In this work, using the concepts of additive mappings, we define a three-dimensional additive-type functional equation and introduce the notion of a ternary hom-multiplier on ternary Banach algebras. We demonstrate that the solution to this functional equation is both an additive mapping and a $\mathbb{C}$ -linear mapping. Furthermore, employing a fixed point approach, we investigate the stability and hyperstability of the 3D additive-type functional equation, the ternary hom-multiplier, and the ternary Jordan hom-multiplier within the framework of ternary Banach algebras.

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References

1. Ulam, S.M. *Problems in Modern Mathematics (Science Editions)*; Wiley: New York, NY, USA, 1964; Chapter IV.
2. Hyers, D.H. On the stability of the linear functional equation. *Proc. Natl. Acad. Sci. USA* **1941**, *27*, 222–224. [[CrossRef](#)]
3. Rassias, T.M. On the stability of the linear mapping in Banach spaces. *Proc. Am. Math. Soc.* **1978**, *72*, 297–300. [[CrossRef](#)]
4. Găvruta, P. A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings. *J. Math. Anal. Appl.* **1994**, *184*, 431–436.
5. Abdollahpour, M.R.; Rassias, M.T. Hyers–Ulam stability of hypergeometric differential equations. *Aequat. Math.* **2019**, *93*, 691–698. [[CrossRef](#)]

6. Elqorachi, E.; Rassias, M.T. Generalized Hyers–Ulam stability of trigonometric functional equations. *Mathematics* **2018**, *6*, 83. [\[CrossRef\]](#)
7. Jahedi, S.; Keshavarz, V. Approximate generalized additive-quadratic functional equations on ternary Banach algebras. *Math. Ext.* **2022**, *16*, 1–11. [\[CrossRef\]](#)
8. Jung, S.-M.; Lee, K.-S.; Rassias, M.T.; Yang, S.-M. Approximation properties of solutions of a mean value-type functional inequality, II. *Mathematics* **2020**, *8*, 1299. [\[CrossRef\]](#)
9. Gordji, M.E.; Fazeli, A. Stability and superstability of homomorphisms on C^* -ternary algebras. *An. St. Univ. Ovidius Constanta* **2012**, *20*, 173–188. [\[CrossRef\]](#)
10. Keshavarz, V.; Jahedi, S. Orthogonally C^* -ternary Jordan homomorphisms and Jordan derivations: Solution and stability. *J. Math.* **2022**, *2022*, 3482254. [\[CrossRef\]](#)
11. Moghadam, M.R.; Rassias, T.M.; Keshavarz, V.; Park, C.; Park, Y.S. Jordan homomorphisms in C^* -ternary algebras and JB^* -triples. *J. Comput. Anal. Appl.* **2018**, *24*, 416–424.
12. Keshavarz, V.; Lee, J.R.; Park, C. Orthogonal Pexider hom-derivations in Banach algebras. *Nonlinear Funct. Anal. Appl.* **2023**, *28*, 95–105. [\[CrossRef\]](#)
13. Kim, H.-M.; Shin, W.-Y. Approximate Lie $*$ -derivations on ρ -complete convex modular algebras. *J. Appl. Anal. Comput.* **2019**, *9*, 765–776. [\[CrossRef\]](#)
14. Lu, Z.; Lu, G.; Jin, Y.; Park, C. The stability of additive (α, β) -functional equations. *J. Appl. Anal. Comput.* **2019**, *9*, 2295–2307. [\[CrossRef\]](#)
15. Park, C.; Rassias, M.T. Hom-derivations in C^* -ternary algebras. *Acta Math. Sin. Engl. Ser.* **2020**, *36*, 1025–1038.
16. Obloza, M. Hyers stability of the linear differential equation. *Rocznik Nauk.-Dydakt. Prace Mat.* **1993**, *13*, 259–270.
17. Miura, T.; Miyajima, S.; Takahasi, S.E. A characterization of Hyers–Ulam stability of first order linear differential operators. *J. Math. Anal. Appl.* **2003**, *286*, 136–146. [\[CrossRef\]](#)
18. Anderson, D.R.; Onitsuka, M. Best constant for Hyers–Ulam stability of two step sizes linear difference equations. *J. Math. Anal. Appl.* **2021**, *496*, 124807. [\[CrossRef\]](#)
19. Keshavarz, V.; Heydari, M.T.; Anderson, D.R. Hyers–Ulam stabilities for m^{th} differential operators on H^2_β . *Chaos Solitons & Fractals* **2024**, *179*, 114443. [\[CrossRef\]](#)
20. Bourgin, D.G. Approximately isometric and multiplicative transformations on continuous function rings. *Duke Math. J.* **1949**, *16*, 385–397. [\[CrossRef\]](#)
21. Gordji, M.E.; Keshavarz, V. Hyperstability of ternary Jordan homomorphisms on unital ternary C^* -algebras. In *Analysis, Geometry, Nonlinear Optimization and Applications*; Pardalos, P.M., Rassias, T.M., Eds.; World Scientific Publishing Company: Singapore, 2023. [\[CrossRef\]](#)
22. Keshavarz, V.; Jahedi, S. Hyperstability of orthogonally 3-Lie homomorphism: An orthogonally fixed point approach. In *Approximation and Computation in Science and Engineering*; Springer Optimization and Its Applications; Daras, N.J., Rassias, T.M., Eds.; Springer: Cham, Switzerland, 2022; Volume 180. [\[CrossRef\]](#)
23. Kim, S.O.; Bodaghi, A. Hyperstability of Jordan triple derivations on Banach algebras. *Acta Math. Vietnam.* **2016**, *41*, 583–594. [\[CrossRef\]](#)
24. Amyari, M.; Moslehian, M.S. Approximately ternary semigroup homomorphisms. *Lett. Math. Phys.* **2006**, *77*, 1–9. [\[CrossRef\]](#)
25. Abramov, V.; Zappala, E. 3-Lie algebras, ternary Nambu-Lie algebras and the Yang-Baxter equation. *J. Geom. Phys.* **2023**, *183*, 104687. [\[CrossRef\]](#)
26. Kumduang, T.; Leeratanavalee, S. Semigroups of terms, tree languages, menger algebra of n -ary functions and their embedding theorems. *Symmetry* **2021**, *13*, 558. [\[CrossRef\]](#)
27. Nambu, J. Generalized Hamiltonian dynamics. *Phys. Rev.* **1973**, *D7*, 2405–2412. [\[CrossRef\]](#)
28. Park, C. Additive ρ -functional inequalities in non-Archimedean normed spaces. *J. Math. Inequal.* **2015**, *9*, 397–407. [\[CrossRef\]](#)
29. Gordji, M.E.; Fazeli, A.; Park, C. 3-Lie multipliers on Banach 3-Lie algebras. *Int. J. Geom. Methods Mod. Phys.* **2012**, *9*, 1250052. [\[CrossRef\]](#)
30. Margolis, B.; Diaz, J.B. A fixed point theorem of the alternative for contractions on the generalized complete metric space. *Bull. Am. Math. Soc.* **1968**, *126*, 305–309.
31. Brzdek, J.; Fošner, A. Remarks on the stability of Lie homomorphisms. *J. Math. Anal. Appl.* **2013**, *400*, 585–596. [\[CrossRef\]](#)
32. Brzdek, J.; Fošner, A. On approximate generalized Lie derivation. *Glasnik Matematički* **2015**, *50*, 77–99. [\[CrossRef\]](#)
33. Miheţ, D.; Radu, V. On the stability of the additive Cauchy functional equation in random normed spaces. *J. Math. Anal. Appl.* **2008**, *343*, 567–572. [\[CrossRef\]](#)

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