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The Vertex-Disjoint and Edge-Disjoint Ramsey Numbers of a Set of Graphs

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Abstract: The Ramsey number $R(F)$ of a graph F without isolated vertices is the smallest positive integer n such that every red–blue coloring of K_n produces a subgraph isomorphic to F all of whose edges are colored the same. Let $\mathcal{F}$ be a set of graphs without isolated vertices. For a positive integer t , the vertex-disjoint Ramsey number $VR_t(\mathcal{F})$ is the smallest positive integer n such that every red–blue coloring of the complete graph K_n of order n results in at least t pairwise vertex-disjoint monochromatic graphs in $\mathcal{F}$; while the edge-disjoint Ramsey number $ER_t(\mathcal{F})$ is the smallest positive integer n such that every red–blue coloring of K_n produces at least t pairwise edge-disjoint monochromatic graphs in $\mathcal{F}$. If $t = 1$ and $\mathcal{F}$ consists of a single graph F , then $VR_1(\mathcal{F}) = ER_1(\mathcal{F}) = R(F)$ is the Ramsey number of the graph F . Thus, the concepts of vertex-disjoint and edge-disjoint Ramsey numbers provide a generalization of the standard Ramsey number. Upper and lower bounds for $VR_t(\mathcal{F})$ and $ER_t(\mathcal{F})$ are established for sets $\mathcal{F}$ of graphs without isolated vertices and the sharpness of these bounds is discussed. The primary goal of this paper is to investigate the values of $VR_t(\mathcal{F})$ and $ER_t(\mathcal{F})$ for sets $\mathcal{F}$ of graphs of size 2 or 3 without isolated vertices. The exact values of $VR_t(\mathcal{F})$ are determined for all such sets $\mathcal{F}$ and all integers $t \geq 2$. The exact values of $ER_t(\mathcal{F})$ of certain such sets $\mathcal{F}$ with prescribed conditions for all integers $t \geq 2$ are determined. For some special sets $\mathcal{F}$ of graphs of size 2 or 3 without isolated vertices, the exact values of $ER_t(\mathcal{F})$ are determined for $2 \leq t \leq 4$. Additional results, problems, and conjectures are also presented dealing with these two Ramsey concepts for graphs in general.



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1. Introduction

Ramsey theory, though a relatively young branch of mathematics, has captivated the attention of graph theorists, combinatorialists, and theoretical computer scientists alike through its beauty, versatility, and applications (see [1–4]). As described in [4], before its emergence as a branch of mathematics, the central idea of Ramsey theory appeared in the form of three lemmas in three separate papers by three different mathematicians working on three distinct areas of research. The first such lemma was published by Hilbert [5] in 1892, followed by the second lemma published by Schur [6] in 1916. However, it was Ramsey’s renowned lemma [7], published in 1930, which compelled mathematicians to establish its study shortly afterwards. In essence, Ramsey’s lemma, now called Ramsey’s theorem, shows that total disorder is impossible in large structures. Given a sufficiently

large structure, some predefined configuration always appears in the structure (see [1]). While the existence of a sufficiently large structure is guaranteed, the question “how large is sufficient to guarantee the existence of a prescribed configuration?” remains. The search for the minimum such number which answers this question is the quest of Ramsey theory and such numbers are aptly dubbed Ramsey numbers. In the past few decades, numerous results have been obtained giving the values of various Ramsey numbers (see [1,8–13]).

We now introduce the formal definition of the Ramsey number of a graph. A *red–blue coloring* of a graph is an assignment of the two colors red and blue to the edges of the graph (one color to each edge). For two graphs F and H , the *Ramsey number* $R(F, H)$ of F and H is the smallest positive integer n such that for every red–blue coloring of the complete graph K_n of order n , there is either a subgraph isomorphic to F each of whose edges is colored red (a *red* F), or a subgraph isomorphic to H each of whose edges is colored blue (a *blue* H). When $F = H$, the Ramsey number $R(F, F)$ of F is commonly denoted by $R(F)$. In this case, $R(F)$ is the smallest positive integer n such that every red–blue coloring of K_n produces a subgraph isomorphic to F all of whose edges are colored the same (a *monochromatic* F). Over the years, many variations and extensions of Ramsey numbers have been introduced and studied (see [14–19], for example).

Here, we are interested in one of the most recent generalized Ramsey concepts, introduced in [20]. For a graph F and a positive integer t , the *vertex-disjoint Ramsey number* $VR_t(F)$ is the minimum positive integer n such that every red–blue coloring of K_n produces at least t pairwise vertex-disjoint monochromatic copies of a subgraph isomorphic to F . As expected, the *edge-disjoint Ramsey number* $ER_t(F)$ is the minimum positive integer n such that every red–blue coloring of the edges of the complete graph K_n of order n results in t pairwise edge-disjoint monochromatic copies of a subgraph isomorphic to F . Consequently, $VR_1(F) = ER_1(F)$ is the Ramsey number $R(F)$ of F . Therefore, these concepts extend the standard concept of Ramsey numbers and have produced some unexpected and interesting results (see [20–22], for example).

We now extend the vertex-disjoint and edge-disjoint Ramsey numbers from a single graph F to a set $\mathcal{F}$ of non-isomorphic graphs. More precisely, let $\mathcal{F}$ be a set of (non-isomorphic) graphs without isolated vertices and let t be a positive integer. The *vertex-disjoint Ramsey number* $VR_t(\mathcal{F})$ of the set $\mathcal{F}$ is the minimum positive integer n such that every red–blue coloring of K_n produces at least t pairwise vertex-disjoint monochromatic graphs in $\mathcal{F}$; while the *edge-disjoint Ramsey number* $ER_t(\mathcal{F})$ is the minimum positive integer n such that every red–blue coloring of K_n produces at least t pairwise edge-disjoint monochromatic graphs in $\mathcal{F}$. If $t = 1$ and $\mathcal{F}$ consists of a single graph F , then $VR_1(\mathcal{F}) = ER_1(\mathcal{F}) = R(F)$ is the well-known Ramsey number of the graph F . Thus, the concepts of vertex-disjoint and edge-disjoint Ramsey numbers of a set of graphs provide a general framework for the standard Ramsey numbers and give rise to a new perspective of the classical Ramsey concepts. There are many applications of Ramsey theory in the sciences and technology, such as number theory, set theory, logic, information theory, and theoretical computer science. It is our hope that this general framework of Ramsey concepts will open up a host of intriguing research problems as well as potential applications in modern society.

First, we make an observation on vertex-disjoint and edge-disjoint Ramsey numbers of a set of graphs.

Observation 1. Let $\mathcal{F}$ and $\mathcal{F}'$ be two sets of graphs without isolated vertices. If $\mathcal{F} \subseteq \mathcal{F}'$, then $VR_t(\mathcal{F}') \leq VR_t(\mathcal{F})$ and $ER_t(\mathcal{F}') \leq ER_t(\mathcal{F})$ for every positive integer t .

As expected, the vertex-disjoint and edge-disjoint Ramsey numbers of a set $\mathcal{F}$ of graphs are related to the vertex-disjoint and edge-disjoint Ramsey numbers of graphs in $\mathcal{F}$.

Proposition 1. For a positive integer t and a set $\mathcal{F}$ of graphs without isolated vertices,

$$t \min\{|V(F)| : F \in \mathcal{F}\} \leq VR_t(\mathcal{F}) \leq \min\{VR_t(F) : F \in \mathcal{F}\} \quad (1)$$

$$t \min\{|E(F)| : F \in \mathcal{F}\} \leq ER_t(\mathcal{F}) \leq \min\{ER_t(F) : F \in \mathcal{F}\}. \quad (2)$$

Proof. Let $VR_t(\mathcal{F}) = n$. To verify the lower bound, let $\min\{|V(F)| : F \in \mathcal{F}\} = n_1$. Since every red–blue coloring of K_n produces at least t pairwise vertex-disjoint monochromatic graphs in $\mathcal{F}$, it follows that $n \geq tn_1$. To verify the upper bound, let $F_1 \in \mathcal{F}$ such that $\min\{VR_t(F) : F \in \mathcal{F}\} = VR_t(F_1) = p$. Let there be given a red–blue coloring of K_p . Since $VR_t(F_1) = p$, there are at least t pairwise vertex-disjoint monochromatic copies of F_1 in K_p . Since $F_1 \in \mathcal{F}$, there are at least t pairwise vertex-disjoint monochromatic graphs in $\mathcal{F}$ and so $VR_t(\mathcal{F}) \leq p$. This verifies (1). The proof of (2) is similar. $\square$

In general, it is often challenging to determine the exact value of the Ramsey number $R(F)$ of a graph F even when F has relatively small order or size. In fact, the exact value $R(K_5)$ of the complete graph K_5 of order 5 is still unknown. Consequently, it is not surprising that the exact values of $VR_t(\mathcal{F})$ and $ER_t(\mathcal{F})$ of a set $\mathcal{F}$ of graphs are difficult to determine in general. Hence, we investigate $VR_t(\mathcal{F})$ and $ER_t(\mathcal{F})$ where $\mathcal{F}$ consists of graphs of size 2 or 3. For a positive integer n , let P_n denote the path of order n and size $n - 1$. For a positive integer m , the star of size m is denoted by $K_{1,m}$ and the matching of size m is denoted by mK_2 . For two vertex-disjoint graphs G and H , the union $G + H$ is the graph with vertex set $V(G + H) = V(G) \cup V(H)$ and edge set $E(G + H) = E(G) \cup E(H)$. For an integer $k \geq 2$, the union of k vertex-disjoint copies of a graph G is denoted by kG . For convenience, let X denote the set of all graphs of size 2 or 3 without isolated vertices, namely,

$$X = \{P_3, 2K_2, K_3, P_4, K_{1,3}, P_3 + P_2, 3K_2\}. \quad (3)$$

If $t = 1$ or $\mathcal{F} = \{F\}$, then

- (a) $VR_1(\mathcal{F}) = \min\{VR_1(F) : F \in \mathcal{F}\}$ and $ER_1(\mathcal{F}) = \min\{ER_1(F) : F \in \mathcal{F}\};$
- (b) $VR_t(\{F\}) = VR_t(F)$ and $ER_t(\{F\}) = ER_t(F).$

Hence, we will assume that $t \geq 2$ and $|\mathcal{F}| \geq 2$. In Section 2, we determine the exact values $VR_t(\mathcal{F})$ of all subsets $\mathcal{F}$ of X for all integers $t \geq 2$. In Section 3, we determine the exact values $ER_t(\mathcal{F})$ of certain subsets $\mathcal{F}$ of X with prescribed conditions for all integers $t \geq 2$, and some other sets $\mathcal{F}$ for $2 \leq t \leq 4$. In Section 4, we introduce some related Ramsey concepts for further investigation. We refer to the book [16] for notation and terminology not defined here.

2. On Vertex-Disjoint Ramsey Numbers

In this section, we determine the exact values of $VR_t(\{\mathcal{F}\})$ of all sets $\mathcal{F}$ consisting of two or more graphs of size 2 or 3 in the set X given in (3) for all integers $t \geq 2$. The following information dealing with the vertex-disjoint Ramsey numbers of all graphs in X will be useful.

Theorem 1 ([20–22]). For every integer $t \geq 2$,

$$VR_t(P_3) = 3t, VR_t(2K_2) = 4t + 1, VR_t(K_3) = 3t + 2, VR_t(P_4) = 4t + 1, \\ VR_t(K_{1,3}) = 4t, VR_t(P_3 + P_2) = 5t + 1, VR_t(3K_2) = 6t + 2.$$

For every graph F (without isolated vertices), it follows that $VR_t(F) \geq t|V(F)|$. If $VR_t(F) = t|V(F)|$, then F is referred to as a *tight* graph. For example, since $VR_t(P_3) = 3t$ and $VR_t(K_{1,3}) = 4t$ by Theorem 1, it follows that P_3 and $K_{1,3}$ are tight graphs. The following observation is a consequence of Proposition 1.

Observation 2. Let $t \geq 2$ be an integer and let $\mathcal{F}$ be a set of graphs without isolated vertices. If $\mathcal{F}$ contains a tight graph F^* such that $|V(F^*)| = \min\{|V(F)| : F \in \mathcal{F}\}$, then

$$VR_t(\mathcal{F}) = VR_t(F^*).$$

Consequently, $t \min\{|V(F)| : F \in \mathcal{F}\} = VR_t(\mathcal{F}) = \min\{VR_t(F) : F \in \mathcal{F}\}$.

Since P_3 and $K_{1,3}$ are tight graphs, the following observation is a consequence of Theorem 2 and Observation 2.

Observation 3. Let $\mathcal{F}$ be a set of two or more graphs without isolated vertices.

- (a) If $P_3 \in \mathcal{F}$ and $|V(F)| \geq 3$ for each $F \in \mathcal{F}$, then $VR_t(\mathcal{F}) = 3t$.
- (b) If $K_{1,3} \in \mathcal{F}$ and $|V(F)| \geq 4$ for each $F \in \mathcal{F}$, then $VR_t(\mathcal{F}) = 4t$.

By Observation 3, we may assume that $P_3 \notin \mathcal{F}$. Thus,

$$\mathcal{F} \subseteq \{2K_2, K_3, P_4, K_{1,3}, P_3 + P_2, 3K_2\} \text{ and } |\mathcal{F}| \geq 2. \quad (4)$$

First, we consider those sets $\mathcal{F}$ satisfying (4) such that $K_3 \notin \mathcal{F}$.

Theorem 2. Let $t \geq 2$ be an integer and let $\mathcal{F}$ be a set of two or more graphs.

- (a) If $\mathcal{F} \subseteq \{2K_2, P_4, K_{1,3}, P_3 + P_2, 3K_2\}$ such that $K_{1,3} \in \mathcal{F}$, then $VR_t(\mathcal{F}) = 4t$.
- (b) If $\mathcal{F} \subseteq \{2K_2, P_4, P_3 + P_2, 3K_2\}$ and $\mathcal{F} \neq \{P_3 + P_2, 3K_2\}$, then $VR_t(\mathcal{F}) = 4t + 1$.
- (c) If $\mathcal{F} = \{P_3 + P_2, 3K_2\}$, then $VR_t(\mathcal{F}) = 5t + 1$.

Proof. First, suppose that $\mathcal{F}$ satisfies the conditions in (a). Since $K_{1,3}$ is a tight graph, it follows by Theorem 1 and Observation 3 that $VR_t(\mathcal{F}) = 4t$.

Next, suppose that $\mathcal{F}$ satisfies the conditions in (b). We consider two possibilities, namely, $2K_2 \in \mathcal{F}$ or $2K_2 \notin \mathcal{F}$. Suppose that $2K_2 \in \mathcal{F}$. Since $VR_t(\mathcal{F}) \leq VR_t(2K_2) = 4t + 1$ by Proposition 1 and Theorem 1, it remains to show that $VR_t(\mathcal{F}) \geq 4t + 1$. By Observation 1, we may assume that $\mathcal{F} = \{2K_2, P_4, P_3 + P_2, 3K_2\}$. Consider the red–blue coloring of $G = K_{4t}$ with the red subgraph $G_r = K_{1,4t-1}$ and the blue subgraph $G_b = K_{4t-1} + K_1$. Then, G_r is $(2K_2)$ -free and F -free for each $F \in \{P_4, P_3 + P_2, 3K_2\}$. Since the nontrivial component K_{4t-1} of G_b has order $4t - 1$ and each graph in $\mathcal{F}$ has order 4 or more, it follows that G_b does not have t pairwise vertex-disjoint subgraphs in $\mathcal{F}$. Therefore, $VR_t(\mathcal{F}) \geq 4t + 1$ and so $VR_t(\mathcal{F}) = 4t + 1$. If $2K_2 \notin \mathcal{F}$, then the proof is similar.

Finally, suppose that $\mathcal{F} = \{P_3 + P_2, 3K_2\}$. Since $VR_t(\mathcal{F}) \leq VR_t(P_3 + P_2) = 5t + 1$ by Proposition 1 and Theorem 1, it remains to show that $VR_t(\mathcal{F}) \geq 5t + 1$. Consider the red–blue coloring of $G = K_{5t}$ with the red subgraph $G_r = K_{1,5t-1}$ and the blue subgraph $G_b = K_{5t-1} + K_1$. Then, G_r is F -free for each $F \in \mathcal{F}$. Since the nontrivial component K_{5t-1} of G_b has order $5t - 1$ and each graph in $\mathcal{F}$ has order 5 or more, it follows that G_b does not have t pairwise vertex-disjoint subgraphs in $\mathcal{F}$. Therefore, $VR_t(\mathcal{F}) \geq 5t + 1$ and so $VR_t(\mathcal{F}) = 5t + 1$. $\square$

We now turn our attention to those sets $\mathcal{F}$ satisfying (4) such that $K_3 \in \mathcal{F}$.

Theorem 3. Let $t \geq 2$ be an integer and let $\mathcal{F}$ be a set of two or more graphs such that $K_3 \in \mathcal{F}$.

- (a) If $\mathcal{F} \subseteq \{2K_2, K_3, P_4\}$, then $VR_t(\mathcal{F}) = 3t + 1$.
- (b) If $\mathcal{F} \subseteq \{K_3, K_{1,3}, P_3 + P_2, 3K_3\}$, then $VR_t(\mathcal{F}) = 3t + 2$.

Proof. To verify (a), let $\mathcal{F} \subseteq \{2K_2, K_3, P_4\}$ such that $K_3 \in \mathcal{F}$. To show that $VR_t(\mathcal{F}) = 3t + 1$, we proceed by induction on $t \geq 2$. We begin by showing that $VR_2(\mathcal{F}) = 7$. Let $A = \{2K_2, K_3, P_4\}$. First, consider the red–blue coloring of $G = K_6$ with the red

subgraph $G_r = K_{1,5}$ and the blue subgraph $G_b = K_5 + K_1$. Then, G_r is F -free for each $F \in A$. Since the nontrivial component K_5 of G_b has order 5, it follows that G_b does not have two vertex-disjoint subgraphs in $\mathcal{F}$. Thus, $VR_2(\mathcal{F}) \geq 7$. Next, let there be given a red–blue coloring of $H = K_7$. We show that there are two vertex-disjoint monochromatic copies of graphs in $\mathcal{F}$. Since $R(3,3) = 6$, there is a monochromatic triangle $T = K_3$. Then, $H' = H - V(T) \cong K_4$. Let $V(H') = \{v_1, v_2, v_3, v_4\}$. Since $\deg_{H'} v_1 = 3$, there are two edges incident with v_1 in H' that are colored the same; say v_1v_2 and v_1v_3 are red. If v_2v_3 is red, then (v_1, v_2, v_3, v_1) is a red K_3 that is vertex-disjoint from T . Thus, we may assume that v_2v_3 is blue. If one of v_2v_4 and v_3v_4 is red, say v_2v_4 is red, then H' contains a red $2K_2$ with vertex set $\{v_1v_3, v_2v_4\}$ and a red $P_4 = (v_4, v_2, v_1, v_3)$, each of which is vertex-disjoint from T . On the other hand, if both v_2v_4 and v_3v_4 are blue, then (v_2, v_3, v_4, v_2) is a blue K_3 that is vertex-disjoint from T . In any case, there are two vertex-disjoint monochromatic copies of graphs in $\mathcal{F}$. Therefore, $VR_2(\mathcal{F}) \leq 7$ and so $VR_2(\mathcal{F}) = 7$.

Now, suppose that $VR_t(\mathcal{F}) = 3t + 1$ for some integer $t \geq 2$. We show that $VR_{t+1}(\mathcal{F}) = 3t + 4$. First, consider the red–blue coloring of $G = K_{3t+3}$ with red subgraph $G_r = K_{1,3t+2}$ and blue subgraph $G_b = K_{3t+2} + K_1$. Then, G_r is F -free for each $F \in A$. Since the nontrivial component K_{3t+2} of G_b has order $3t + 2$ and each graph in X has order at least 3, it follows that G_b does not have $t + 1$ pairwise vertex-disjoint subgraphs in A . Thus, $VR_{t+1}(\mathcal{F}) \geq 3t + 4$. Next, let there be a red–blue coloring of $H = K_{3t+4}$. We show that there are $t + 1$ pairwise vertex-disjoint monochromatic copies of graphs in $\mathcal{F}$. Since $R(3,3) = 6$, there is a monochromatic triangle $T = K_3$. Since $H' = H - V(T) \cong K_{3t+1}$, it follows by the induction hypothesis that H' contains t pairwise vertex-disjoint monochromatic copies $A_1, A_2, \dots, A_t$ of subgraphs in $\mathcal{F}$ that are vertex-disjoint from T . Hence, $T, A_1, A_2, \dots, A_t$ are $t + 1$ pairwise vertex-disjoint monochromatic copies of graphs in $\mathcal{F}$. Therefore, $VR_{t+1}(\mathcal{F}) \leq 3t + 4$ and so $VR_{t+1}(\mathcal{F}) = 3t + 4$.

To verify (b), let $\mathcal{F} \subseteq \{K_3, K_{1,3}, P_3 + P_2, 3K_3\}$ such that $K_3 \in \mathcal{F}$. We show that $VR_t(\mathcal{F}) = 3t + 2$. Since $VR_t(\mathcal{F}) \leq VR_t(K_3) = 3t + 2$ by Proposition 1, it remains to show that $VR_t(\mathcal{F}) \geq 3t + 2$. By Observation 1, we may assume that $\mathcal{F} = \{K_3, K_{1,3}, P_3 + P_2, 3K_3\}$. Consider the red–blue coloring of $G = K_{3t+1}$ with red subgraph $G_r = K_{2,3t-1}$ and blue subgraph $G_b = K_{3t-1} + K_2$. Consequently, G_r is K_3 -free. Since the edge independence number of $G_r = K_{2,3t-1}$ is 2, it follows that G_r is also $(3K_2)$ -free. We show that G does not have t pairwise vertex-disjoint monochromatic copies of graphs in $\mathcal{F}$. Assume, to the contrary, that G has t pairwise vertex-disjoint monochromatic subgraphs $A_1, A_2, \dots, A_t$ in $\mathcal{F}$. Since the order of G is $3t + 1$, it follows that at least $t - 1$ of the subgraphs A_i ($1 \leq i \leq t$) are K_3 and none of these subgraphs A_i ($1 \leq i \leq t$) can have order 5 or more. We may assume that $A_i = K_3$ for $i = 1, 2, \dots, t - 1$ and A_t has order at most 4. Thus, $A_t \in \{K_3, K_{1,3}\}$. Since $G_r = K_{2,3t-1}$ is K_3 -free and $G_b = K_{3t-1} + K_2$ has only $t - 1$ pairwise vertex-disjoint copies of K_3 , it follows that $A_i = K_3 \subseteq G_b$ for $1 \leq i \leq t - 1$ and $A_t = K_{1,3}$. Let $H = G - [V(A_1) \cup V(A_2) \cup \dots \cup V(A_{t-1})] = K_4$ be the subgraph of order 4 obtained from $G = K_{3t+1}$ by deleting all vertices in $V(A_i)$ for each integer i with $1 \leq i \leq t - 1$. Then, the red subgraph of H is $H_r = C_4$. Since the $t - 1$ pairwise vertex-disjoint blue copies $A_1, A_2, \dots, A_{t-1}$ of K_3 are all subgraphs of the component K_{3t-1} in G_b , there is a copy of K_2 remaining in the component K_{3t-1} of G_b after deleting each vertex of A_i for $1 \leq i \leq t - 1$; that is, $K_{3t-1} - [V(A_1) \cup V(A_2) \cup \dots \cup V(A_{t-1})] = K_2$. Thus, the blue subgraph of H is $H_b = 2K_2$. Hence, there is no monochromatic copy of $K_{1,3}$, which is a contradiction. Therefore, $VR_t(\mathcal{F}) \geq 3t + 2$ and so $VR_t(\mathcal{F}) = 3t + 2$. $\square$

With the aid of Theorems 2 and 3, we can now present a result that gives the exact value of the vertex-disjoint Ramsey numbers of every set of two or more graphs of size 2 or 3.

Corollary 1. Let $t \geq 2$ be an integer and let $\mathcal{F}$ be a set of two or more graphs of size 2 or 3.

- ★ If $P_3 \in \mathcal{F}$, then $VR_t(\mathcal{F}) = VR_t(P_3) = 3t$.
- ★ If $P_3 \notin \mathcal{F}$ and $K_3 \notin \mathcal{F}$, then

$$VR_t(\mathcal{F}) = \begin{cases} 4t & \text{if } K_{1,3} \in \mathcal{F} \\ 4t + 1 & \text{if } K_{1,3} \notin \mathcal{F} \text{ and } \mathcal{F} \neq \{P_3 + P_2, 3K_2\} \\ 5t + 1 & \text{if } \mathcal{F} = \{P_3 + P_2, 3K_2\}. \end{cases}$$

- ★ If $P_3 \notin \mathcal{F}$ and $K_3 \in \mathcal{F}$, then

$$VR_t(\mathcal{F}) = \begin{cases} 3t + 1 & \text{if } \mathcal{F} \subseteq \{K_3, 2K_2, P_4\} \\ 3t + 2 & \text{if } \mathcal{F} \subseteq \{K_3, K_{1,3}, P_3 + P_2, 3K_3\}. \end{cases}$$

By Corollary 1, if $\mathcal{F} \subseteq \{K_3, 2K_2, P_4\}$ such that $K_3 \in \mathcal{F}$ and $|\mathcal{F}| \geq 2$, then $VR_t(\mathcal{F}) = 3t + 1$. Since $VR_t(K_3) = 3t + 2$, $VR_t(2K_2) = 4t + 1$, and $VR_t(P_4) = 4t + 1$, it follows that

$$t \min\{|V(F)| : F \in \mathcal{F}\} < VR_t(\mathcal{F}) < \min\{VR_t(F) : F \in \mathcal{F}\}.$$

Hence, both the upper and lower bounds in (1) of Proposition 1 can be strict. We saw in Observation 2 that if $\mathcal{F}$ is a set of two or more graphs without isolated vertices such that $\mathcal{F}$ contains a tight graph F of minimum order in $\mathcal{F}$, then

$$t \min\{|V(F)| : F \in \mathcal{F}\} = VR_t(\mathcal{F}) = \min\{VR_t(F) : F \in \mathcal{F}\}.$$

Hence, both the upper and lower bounds in (1) of Proposition 1 are sharp. These observations and the results obtained thus far lead us to the following question.

Question 1. Let $\mathcal{F}$ be a set of two or more graphs without isolated vertices and let $t \geq 2$ be an integer. Under what conditions is either of the following true:

- (a) $t \min\{|V(F)| : F \in \mathcal{F}\} < VR_t(\mathcal{F}) = \min\{VR_t(F) : F \in \mathcal{F}\}$;
- (b) $t \min\{|V(F)| : F \in \mathcal{F}\} = VR_t(\mathcal{F}) < \min\{VR_t(F) : F \in \mathcal{F}\}$.

3. On Edge-Disjoint Ramsey Numbers

We now turn our attention to the edge-disjoint Ramsey number $ER_t(\mathcal{F})$ of a set $\mathcal{F}$ for which $|\mathcal{F}| \geq 2$ and $\mathcal{F} \subseteq X$, where X is the set of graphs of size 2 or 3 without isolated vertices described in (3). It is useful to present some information on $ER_t(F)$ for each $F \in X$.

Theorem 4 ([20–22]). For each integer $t \geq 2$, $ER_t(P_3) = \lceil 2\sqrt{t} + 1 \rceil$ and

$$ER_t(P_3) = \min\{ER_t(F) : F \in X\}, \quad (5)$$

where X is the set of graphs of size 2 or 3 without isolated vertices.

With the aid of the exact value of $ER_t(P_3)$ of P_3 given Equation (5) in Theorem 4, we are able to determine the exact values of $ER_t(\mathcal{F})$ for several sets $\mathcal{F} \subseteq X$, where $P_3 \in \mathcal{F}$. We begin with the situation when $t \geq 2$ is a perfect square.

Theorem 5. Let $t \geq 2$ be an integer. If t is a perfect square and $\mathcal{F}$ is a set of two or more graphs of size 2 or 3 such that $P_3 \in \mathcal{F}$, then $ER_t(\mathcal{F}) = \lceil 2\sqrt{t} + 1 \rceil$.

Proof. Let $\mathcal{F}$ be a set of two or more graphs of size 2 or 3 such that $P_3 \in \mathcal{F}$. By Proposition 1 and Theorem 4, it follows that $ER_t(\mathcal{F}) \leq ER_t(P_3) = \lceil 2\sqrt{t} + 1 \rceil$. Thus, it remains to show that $ER_t(\mathcal{F}) \geq \lceil 2\sqrt{t} + 1 \rceil$. Since t is a perfect square, $t = k^2$ for some positive integer k and

so $\lceil 2\sqrt{t} + 1 \rceil = 2k + 1$. To show that $ER_{k^2}(\mathcal{F}) \geq 2k + 1$, let there be a red–blue coloring of $G = K_{2k}$. Since $|E(K_{2k})| = 2k^2 - k$ and each set of t pairwise vertex-disjoint copies of graphs in $\mathcal{F}$ requires at least $2t = 2k^2$ edges, there are not t pairwise vertex-disjoint monochromatic copies of graphs in $\mathcal{F}$ in G . Therefore, $ER_t(\mathcal{F}) \geq 2k + 1 = \lceil 2\sqrt{t} + 1 \rceil$ and so $ER_t(\mathcal{F}) = \lceil 2\sqrt{t} + 1 \rceil$. $\square$

We now turn to the situation when $t \geq 2$ is not a perfect square. First, we present a lemma.

Lemma 1. *Let $t \geq 2$ be an integer. If t is not a perfect square, then*

$$2t > \frac{(\lceil 2\sqrt{t} + 1 \rceil - 1)(\lceil 2\sqrt{t} + 1 \rceil - 2)}{2} - \left\lfloor \frac{\lceil 2\sqrt{t} + 1 \rceil - 1}{2} \right\rfloor. \quad (6)$$

Proof. Since $t \geq 2$ is an integer that is not a perfect square, $2\sqrt{t}$ is not an integer and so $2\sqrt{t} + 1$ is not an integer. Thus, $\lceil 2\sqrt{t} + 1 \rceil < 2\sqrt{t} + 1$ and so $\lceil 2\sqrt{t} + 1 \rceil - 1 = \lfloor 2\sqrt{t} + 1 \rfloor$. Furthermore, $\left\lfloor \frac{\lceil 2\sqrt{t} + 1 \rceil - 1}{2} \right\rfloor = \left\lfloor \sqrt{t} + \frac{1}{2} \right\rfloor$. Thus, the inequality (6) becomes

$$2t > \frac{\lfloor 2\sqrt{t} + 1 \rfloor (\lfloor 2\sqrt{t} + 1 \rfloor - 1)}{2} - \left\lfloor \sqrt{t} + \frac{1}{2} \right\rfloor. \quad (7)$$

We now verify (7). Since $\lfloor 2\sqrt{t} + 1 \rfloor - 1 < 2\sqrt{t}$, it follows that

$$\frac{\lfloor 2\sqrt{t} + 1 \rfloor (\lfloor 2\sqrt{t} + 1 \rfloor - 1)}{2} < \frac{(2\sqrt{t} + 1)(2\sqrt{t})}{2} = \frac{4t + 2\sqrt{t}}{2} = 2t + \sqrt{t}. \quad (8)$$

Because $\lfloor \sqrt{t} + \frac{1}{2} \rfloor \geq \lfloor \sqrt{t} \rfloor$, either $\lfloor \sqrt{t} + \frac{1}{2} \rfloor \geq \sqrt{t}$ or $\lfloor \sqrt{t} + \frac{1}{2} \rfloor < \sqrt{t}$. We now consider these two cases.

Case 1. $\lfloor \sqrt{t} + \frac{1}{2} \rfloor \geq \sqrt{t}$. It follows by (8) that

$$\begin{aligned} \frac{\lfloor 2\sqrt{t} + 1 \rfloor (\lfloor 2\sqrt{t} + 1 \rfloor - 1)}{2} - \left\lfloor \sqrt{t} + \frac{1}{2} \right\rfloor &< 2t + \sqrt{t} - \left\lfloor \sqrt{t} + \frac{1}{2} \right\rfloor \\ &< 2t + \sqrt{t} - \sqrt{t} = 2t. \end{aligned}$$

Thus, the inequality (7) holds in Case 1.

Case 2. $\lfloor \sqrt{t} + \frac{1}{2} \rfloor < \sqrt{t}$. Then, there is a positive integer n such that $n < \sqrt{t} < n + \frac{1}{2}$. Thus, $t > n^2$ and $2n + 1 < 2\sqrt{t} + 1 < 2n + 2$. Hence, $\lfloor 2\sqrt{t} + 1 \rfloor = 2n + 1$. Then,

$$\lfloor 2\sqrt{t} + 1 \rfloor (\lfloor 2\sqrt{t} + 1 \rfloor - 1) + 2 = 4n^2 + 2n + 2. \quad (9)$$

Let $g(t) = 4t + 2\sqrt{t} - (4n^2 + 2n + 2)$ be a function of t where $t > n^2$. Since (a) $g(t)$ is an increasing function of t , (b) $t \geq n^2 + 1$ is an integer, and (c) n is a positive integer, it follows that

$$g(t) \geq g(n^2 + 1) = 2n + 2\sqrt{n^2 + 1} - 2 > 0$$

and so $4t + 2\sqrt{t} > 4n^2 + 2n + 2$. It then follows by (9) that

$$\lfloor 2\sqrt{t} + 1 \rfloor (\lfloor 2\sqrt{t} + 1 \rfloor - 1) + 2 < 4t + 2\sqrt{t}.$$

Therefore,

$$\lfloor 2\sqrt{t} + 1 \rfloor (\lfloor 2\sqrt{t} + 1 \rfloor - 1) < 4t + 2\sqrt{t} - 2 \quad (10)$$

for each integer $t > n^2$. Since $\lfloor \sqrt{t} + \frac{1}{2} \rfloor \geq \lfloor \sqrt{t} \rfloor \geq \sqrt{t} - 1$, it follows by (10) that

$$\frac{\lfloor 2\sqrt{t} + 1 \rfloor (\lfloor 2\sqrt{t} + 1 \rfloor - 1)}{2} - \left\lfloor \sqrt{t} + \frac{1}{2} \right\rfloor < \frac{4t + 2\sqrt{t} - 2}{2} - (\sqrt{t} - 1) = 2t.$$

Thus, the inequality (7) holds in Case 2. $\square$

We are now prepared to determine the exact values $ER_t(\mathcal{F})$ for all integers $t \geq 2$ when $\mathcal{F} \subseteq X$ that contains P_3 but neither $2K_2$ nor $3K_2$.

Theorem 6. *For every integer $t \geq 2$, if $\mathcal{F}$ is a set of two or more graphs of size 2 or 3 such that $P_3 \in \mathcal{F}$ and $\{2K_2, 3K_3\} \cap \mathcal{F} = \emptyset$, then $ER_t(\mathcal{F}) = \lfloor 2\sqrt{t} + 1 \rfloor$.*

Proof. By Theorem 5, we may assume that t is not a perfect square. Let $\mathcal{F}$ be a set of two or more graphs of size 2 or 3 such that $\mathcal{F}$ contains P_3 but neither $2K_2$ nor $3K_2$. Since $ER_t(\mathcal{F}) \leq ER_t(P_3) = \lfloor 2\sqrt{t} + 1 \rfloor$ by Proposition 1 and Theorem 4, it remains to show that $ER_t(\mathcal{F}) \geq \lfloor 2\sqrt{t} + 1 \rfloor$. Let $\ell = \lfloor 2\sqrt{t} + 1 \rfloor - 1$ and consider the red–blue coloring of $G = K_\ell$, where the red subgraph is the matching $G_r = \lfloor \frac{\ell}{2} \rfloor K_2$ of size $\lfloor \frac{\ell}{2} \rfloor$. Since G_r is F -free for each graph $F \in \{P_3, K_3, P_4, K_{1,3}, P_3 + P_2\}$ and $\{2K_2, 3K_3\} \cap \mathcal{F} = \emptyset$, it follows that G_r is F -free for each $F \in \mathcal{F}$. We now consider the blue subgraph G_b . Since $G_b \cong K_\ell - E(\lfloor \frac{\ell}{2} \rfloor K_2)$, the size of G_b is

$$m_b = \frac{\ell(\ell - 1)}{2} - \left\lfloor \frac{\ell}{2} \right\rfloor = \frac{(\lfloor 2\sqrt{t} + 1 \rfloor - 1)(\lfloor 2\sqrt{t} + 1 \rfloor - 2)}{2} - \left\lfloor \frac{\lfloor 2\sqrt{t} + 1 \rfloor - 1}{2} \right\rfloor.$$

Since t is not a perfect square, $m_b < 2t$ by Lemma 1. Any t pairwise vertex-disjoint copies of graphs in $\mathcal{F}$ require at least $2t$ edges. Hence, G_b does not contain t pairwise vertex-disjoint copies of graphs in $\mathcal{F}$. Therefore, $ER_t(\mathcal{F}) \geq \ell + 1 = \lfloor 2\sqrt{t} + 1 \rfloor$ and so $ER_t(\mathcal{F}) = \lfloor 2\sqrt{t} + 1 \rfloor$. $\square$

By Theorem 5, if $t \geq 2$ is a perfect square, then $ER_t(\{P_3, 2K_2\}) = \lfloor 2\sqrt{t} + 1 \rfloor$. While it can be shown that $ER_t(\{P_3, 2K_2\}) = \lfloor 2\sqrt{t} + 1 \rfloor$ for infinitely many positive integers t that are not perfect squares, there are positive integers t for which $ER_t(\{P_3, 2K_2\}) = \lfloor 2\sqrt{t} + 1 \rfloor - 1$. For example, it can be shown that $ER_t(\{P_3, 2K_2\}) = \lfloor 2\sqrt{t} + 1 \rfloor - 1$ if $t = 7, 10, 13$. Similarly, it can be shown that $ER_t(\{P_3, 3K_2\}) = \lfloor 2\sqrt{t} + 1 \rfloor$ for infinitely many positive integers t that are not perfect squares and there are positive integers t for which $ER_t(\{P_3, 3K_2\}) = \lfloor 2\sqrt{t} + 1 \rfloor - 1$. Furthermore, for every set $\mathcal{F} \subseteq X$ that contains P_3 for which $ER_t(\mathcal{F})$ has been determined, the value of $ER_t(\mathcal{F})$ is one of the two numbers $\lfloor 2\sqrt{t} + 1 \rfloor$ and $\lfloor 2\sqrt{t} + 1 \rfloor - 1$, that is,

$$ER_t(\mathcal{F}) = \min\{ER_t(F) : F \in \mathcal{F}\} \text{ or } ER_t(\mathcal{F}) = \min\{ER_t(F) : F \in \mathcal{F}\} - 1. \quad (11)$$

In fact, (11) is also true for every set $\mathcal{F} \subseteq X$ that does not contain P_3 for which $ER_t(\mathcal{F})$ can be determined. In order to verify this fact, it is useful to present some information on the edge-disjoint numbers of graphs in $X - \{P_3\}$. For each $F \in X - \{P_3\}$, the exact value of $ER_t(F)$ is only known for small values of t (see [21,22]). For example, these values for $1 \leq t \leq 4$ are listed below.

t	$ER_t(2K_2)$	$ER_t(K_3)$	$ER_t(P_4)$	$ER_t(K_{1,3})$	$ER_t(P_3 + P_2)$	$ER_t(3K_2)$
1	5	6	5	6	6	8
2	5	7	5	7	6	8
3	5	9	6	9	7	8
4	6	10	7	10	7	8

For all integers $t \geq 8$, the exact values of $ER_t(F)$ are not known if $F \in X - \{P_3\}$. With the aid of the known values of $ER_t(F)$ where $F \in X - \{P_3\}$ for $2 \leq t \leq 4$, we now determine the exact values of $ER_t(\mathcal{F})$ for several sets $\mathcal{F} \subseteq X$ when $2 \leq t \leq 4$.

Theorem 7. *If $\mathcal{F} = \{K_3, P_3 + P_2\}$, then*

- (a) $ER_t(\mathcal{F}) = \min\{ER_t(F) : F \in \mathcal{F}\}$ if $t = 2, 4$;
- (b) $ER_3(\mathcal{F}) = \min\{ER_3(F) : F \in \mathcal{F}\} - 1$.

Proof. Let $\mathcal{F} = \{K_3, P_3 + P_2\}$. First, we consider $ER_2(\mathcal{F})$. Since $ER_2(K_3) = 7$ and $ER_2(P_3 + P_2) = 6$, it follows that $\min\{ER_2(F) : F \in \mathcal{F}\} = ER_2(P_3 + P_2) = 6$. Thus, we show that $ER_2(\mathcal{F}) = 6$. By Proposition 1, $ER_2(\mathcal{F}) \leq 6$. On the other hand, the red–blue coloring of K_5 in which both the red subgraph and blue subgraph are the 5-cycle C_5 does not produce a copy of a monochromatic graph in $\mathcal{F}$. Therefore, $ER_2(\mathcal{F}) \geq 6$ and so $ER_2(\mathcal{F}) = 6$.

Next, we consider $ER_4(\mathcal{F})$. Since $ER_4(K_3) = 10$ and $ER_4(P_3 + P_2) = 7$, it follows that $\min\{ER_4(F) : F \in \mathcal{F}\} = ER_4(P_3 + P_2) = 7$. Thus, we show that $ER_4(\mathcal{F}) = 7$. By Proposition 1, $ER_4(\mathcal{F}) \leq 7$. Next, consider a red–blue coloring of $G = K_6$ such that the red subgraph $G_r = C_4$ and the blue subgraph $G_b = K_6 - E(C_4)$. Then, G_r is F -free for each $F \in \mathcal{F}$. Furthermore, since the size of G_b is $\binom{6}{2} - 4 = 11$ and four pairwise edge-disjoint monochromatic copies of graphs in $\mathcal{F}$ require 12 edges, it follows that G_b does not contain four pairwise edge-disjoint monochromatic copies of graphs in $\mathcal{F}$. Therefore, this red–blue coloring does not produce four pairwise edge-disjoint monochromatic copies of graphs in $\mathcal{F}$. Therefore, $ER_4(\mathcal{F}) \geq 7$ and so $ER_4(\mathcal{F}) = 7$.

Finally, we consider $ER_3(\mathcal{F})$. Since $ER_3(K_3) = 9$ and $ER_3(P_3 + P_2) = 7$, it follows that $\min\{ER_3(F) : F \in \mathcal{F}\} = ER_3(P_3 + P_2) = 7$. Thus, we show that $ER_3(\mathcal{F}) = 6$. To show $ER_3(\mathcal{F}) \geq 6$, consider the red–blue coloring of K_5 such that both the red subgraph and the blue subgraph are 5-cycles C_5 . Since C_5 is K_3 -free and contains at most one copy of $P_3 + P_2$, this coloring of K_5 does not produce three pairwise edge-disjoint monochromatic copies of graphs in $\mathcal{F}$. Hence, $ER_3(\mathcal{F}) \geq 6$.

To show that $ER_3(\mathcal{F}) \leq 6$, let there be a red–blue coloring of $G = K_6$. We show that there are three pairwise edge-disjoint monochromatic copies of graphs in $\mathcal{F}$. Since $ER_2(P_3 + P_2) = 6$, there are two edge-disjoint monochromatic copies F_1 and F_2 of $P_3 + P_2$ in G . Let $H = G - (E(F_1) \cup E(F_2))$. We show that H contains a monochromatic copy of $F_3 \in \mathcal{F}$. Let H_r and H_b be the red and blue subgraphs of H , respectively. We may assume that $\Delta(H_r) \geq \Delta(H_b)$. Since H has order 6 and size 9, it follows that $\Delta(H) \geq 3$ and so $\Delta(H_r) \geq 2$. We consider two cases, according to whether $\Delta(H_r) \geq 3$ or $\Delta(H_r) = 2$.

Case 1. $\Delta(H_r) \geq 3$. Let $v \in V(H)$ with $\deg_{H_r} v = \Delta(H_r)$ and let $V(G) - \{v\} = U$. Then, $G[U] \cong K_5$. If H contains a red edge that belongs to $G[U]$, then there is a red $F_3 \in \mathcal{F}$ in H . Thus, we may assume that every edge of H that belongs to $G[U]$ is blue. Let B be the set of all blue edges of H that belong to $G[U]$. Since $|E(F_1 \cup F_2)| = 6$ and $G[U] \cong K_5$, it follows that $|B| \geq 4$. Thus, the subgraph $H[B]$ of H induced by B has order 5 and size at least 4. We claim that $H[B]$ contains $F_3 \in \mathcal{F}$. Assume, to the contrary, that $H[B]$ does not contain $F_3 \in \mathcal{F}$. Then, $|B| = 4$ and $H[B] \in \{K_{1,4}, C_4 + K_1\}$. Thus, $E(G[U] - B) = E(F_1) \cup E(F_2)$. If $H[B] = K_{1,4}$, then $G[U] - E(B) \cong K_4 + K_1$; while if

$H[B] = C_4 + K_1$, then $G[U] - E(B) \cong 2K_2 \vee K_1$ (the join of $2K_2$ and K_1). In either case, $G[U] - E(B)$ cannot be decomposed into two copies of $P_3 + P_2$, which is a contradiction. Hence, $H[B]$ contains $F_3 \in \mathcal{F}$, as claimed. Therefore, G contains three pairwise edge-disjoint monochromatic copies of graphs in $\mathcal{F}$ if $\Delta(H_r) \geq 3$.

Case 2. $\Delta(H_r) = 2$. Thus, $\Delta(H) = 4$ or $\Delta(H) = 3$. We consider these two subcases.

Subcase 2.1. $\Delta(H) = 4$. Let $v \in V(H)$ with $\deg_H v = 4$. Thus, $\deg_{H_r} v = \deg_{H_b} v = 2$. Let $V(G) - \{v\} = \{v_1, v_2, v_3, v_4, v_5\}$. We may assume that vv_1, vv_2 are red edges in H and vv_3, vv_4 are blue edges in H . Thus, $vv_5 \notin E(H)$. First, suppose that one of v_1v_2 and v_3v_4 belongs to H , say $v_1v_2 \in E(H)$. If v_1v_2 is red, then $F_3 = K_3 = (v_1, v_2, v_3, v_1)$; while if v_1v_2 is blue, then $F_3 = P_3 + P_2$, where $P_3 = (v_3, v, v_4)$ and $P_2 = (v_1, v_2)$. Therefore, we may assume that $v_1v_2, v_3v_4 \notin E(H)$ and so

$$\{vv_5, v_1v_2, v_3v_4\} \cap E(H) = \emptyset. \quad (12)$$

We now consider the edge v_2v_3 . Here, there are two subcases, according to whether $v_2v_3 \in E(H)$ or $v_2v_3 \notin E(H)$.

Subcase 2.1.1. $v_2v_3 \in E(H)$. By symmetry, we may assume that v_2v_3 is red. If one of v_1v_5 and v_2v_5 belongs to H , then regardless of its color, there is a monochromatic copy F_3 of $P_3 + P_2$ in H . Thus, we may assume that $v_1v_5, v_2v_5 \notin E(H)$. By (12), $\{vv_5, v_1v_2, v_3v_4, v_1v_5, v_2v_5\} \cap E(H) = \emptyset$. Since $|E(F_1 \cup F_2)| = 6$, at most one edge in $\{v_1v_4, v_2v_4, v_5v_4\}$ belongs to $E(F_1 \cup F_2)$ and so at least two edges e and f in $\{v_1v_4, v_2v_4, v_5v_4\}$ belong to H . Since vv_4 is blue and $\Delta(H_b) \leq 2$, at most one of e and f is blue and so at least one of e and f is red; say e is red and $e \in \{v_1v_4, v_2v_4, v_5v_4\}$. Regardless of the choice of e , there is a red copy F_3 of $P_3 + P_2$ in H .

Subcase 2.1.2. $v_2v_3 \notin E(H)$. By (12),

$$\{vv_5, v_1v_2, v_3v_4, v_2v_3\} \cap E(H) = \emptyset. \quad (13)$$

Thus, at most two edges in $\{v_1v_5, v_2v_5, v_3v_5, v_4v_5\}$ are not in H and so at least two edges in $\{v_1v_5, v_2v_5, v_3v_5, v_4v_5\}$ belong to H .

- ★ First, suppose that $\{v_1v_5, v_2v_5\} \subseteq E(H)$ or $\{v_3v_5, v_4v_5\} \subseteq E(H)$. By symmetry, we may assume that $\{v_1v_5, v_2v_5\} \subseteq E(H)$. Then, v_1v_5, v_2v_5 are red, for otherwise there is a blue copy F_3 of $P_3 + P_2$ in H with $P_3 = (v_3, v, v_4)$. Let $Y = \{v_1v_3, v_3v_5, v_1v_4, v_2v_4, v_4v_5\}$. If there is a red edge of H belonging to Y , then there is a red $F_3 = P_3 + P_2$ in H . Thus, each edge in Y is either blue or is not in H . Let $Z = \{v_1v_4, v_2v_4, v_4v_5\}$ be the set of the three edges in Y that are incident with v_4 . Since at most two edges in Y do not belong to H by (13), at least one edge in Z belongs to H . Furthermore, if an edge of Z belongs to H , then this edge is blue. First, suppose that at least two edges in Z belong to H . If $v_1v_4, v_2v_4 \in E(H)$, then there is a blue copy $F_3 = P_3 + P_2$ with $P_3 = (v_1, v_4, v_2)$ and $P_2 = (v, v_3)$. If $v_1v_4, v_4v_5 \in E(H)$ or $v_2v_4, v_4v_5 \in E(H)$, say the former, then there is a blue copy F_3 of $P_3 + P_2$ with $P_3 = (v_1, v_4, v_5)$ and $P_2 = (v, v_3)$. Next, suppose that exactly one edge in Z belongs to H . Then, two edges in Z do not belong to H . Since $|E(F_1) \cup E(F_2)| = 6$, it follows by (13) that the remaining two edges v_1v_3, v_3v_5 in $Y - Z$ belong to H and so v_1v_3, v_3v_5 are blue. Then, there is a blue copy F_3 of $P_3 + P_2$ with $P_3 = (v_1, v_3, v_5)$ and $P_2 = (v, v_4)$.
- ★ Next, suppose that at least one edge in $\{v_1v_5, v_2v_5\}$ does not belong to H and at least one edge in $\{v_3v_5, v_4v_5\}$ does not belong to H . Again, since $|E(F_1) \cup E(F_2)| = 6$, it follows by (13) that exactly one edge in $\{v_1v_5, v_2v_5\}$ does not belong to H and exactly one edge in $\{v_3v_5, v_4v_5\}$ does not belong to H . Hence, exactly one edge in $\{v_1v_5, v_2v_5\}$ belongs to H and exactly one edge in $\{v_3v_5, v_4v_5\}$ belongs to H . We may assume that $v_1v_5, v_4v_5 \notin E(H)$ and $v_2v_5, v_3v_5 \in E(H)$. By (13), then $E(F_1) \cup E(F_2) =$

$\{vv_5, v_1v_2, v_3v_4, v_2v_3, v_1v_5, v_4v_5\}$. We may further assume that v_2v_5 is a red edge in H and v_3v_5 is a blue edge in H , for otherwise there is a monochromatic copy F_3 of $P_3 + P_2$ in H . We now consider the edge v_1v_3 . If v_1v_3 is red, then there is a red copy F_3 of $P_3 + P_2$ in H with $P_3 = (v, v_2, v_5)$ and $P_2 = (v_1, v_3)$. If v_1v_3 is blue, then there is a blue copy F_3 of $P_3 + P_2$ in H with $P_3 = (v_1, v_3, v_5)$ and $P_2 = (v, v_4)$.

Subcase 2.2. $\Delta(H) = 3$. Then, $G[E(F_1) \cup E(F_2)] \cong C_6$ or $G[E(F_1) \cup E(F_2)] \cong 2K_3$. We consider these two possibilities.

Subcase 2.2.1. $G[E(F_1) \cup E(F_2)] \cong C_6$. Let $v \in V(H)$ with $\deg_H v = 3$. Thus, $\deg_{H_r} v = 2$ and $\deg_{H_b} v = 1$. We may assume that $C_6 = (v, v_1, v_2, v_3, v_4, v_5, v)$. Thus, exactly two of the three edges vv_2, vv_3, vv_4 in H are red.

- ★ First, suppose that vv_2, vv_4 are red and vv_3 is blue. Then, v_2v_4, v_1v_3, v_3v_5 are blue; for otherwise, there is a red $F_3 \in \mathcal{F}$ in H . Then, there is a blue copy F_3 of $P_3 + P_2$, where $P_3 = (v_1, v_3, v_5)$ and $P_2 = (v_2, v_4)$.
- ★ Next, suppose that vv_2 and vv_3 are red or vv_3 and vv_4 are red. By symmetry, we may assume that vv_2 and vv_3 are red and so vv_4 is blue. Thus, v_1v_4 and v_1v_5 are blue, for otherwise, there is a red copy F_3 of $P_3 + P_2$ in H , where $P_3 = (v_2, v, v_3)$. Then, v_2v_5 and v_3v_5 are red, for otherwise, there is a blue copy F_3 of $P_3 + P_2$ in H , where $P_3 = (v, v_4, v_1)$. We now consider v_2v_4 . If v_2v_4 is blue, then $\deg_{H_b} v_4 = 3$. Since $\Delta(H_b) \leq \Delta(H_r) = 2$, this is impossible. Hence, v_2v_4 is red and so there is a red copy F_3 of $P_3 + P_2$ in H , where $P_3 = (v, v_3, v_5)$.

Subcase 2.2.2. $G[E(F_1) \cup E(F_2)] \cong 2K_3$. We may assume that $G[E(F_1) \cup E(F_2)]$ consists of two triangles (v_1, v_2, v_3, v_1) and (v_4, v_5, v_6, v_4) . Since $\Delta(H_r) = 2$, it follows that two of v_1v_4, v_1v_5, v_1v_6 are red and one is blue; say v_1v_4 and v_1v_5 are red and v_1v_6 is blue. Then, v_2v_6 and v_3v_6 are blue, for otherwise, there is a red copy F_3 of $P_3 + P_2$ in H , where $P_3 = (v_5, v_1, v_6)$. However then $\deg_{H_b} v_6 = 3$, which is impossible.

Hence, H contains a monochromatic $F_3 \in \mathcal{F}$ and so G contains three pairwise edge-disjoint monochromatic copies of graphs in $\mathcal{F}$ if $\Delta(H_r) = 2$. Thus, $ER_3(\mathcal{F}) \leq 6$ and so $ER_3(\mathcal{F}) = 6$. $\square$

By Theorem 7, it is possible that if t_1 and t_2 are two distinct positive integers, then $ER_{t_1}(\mathcal{F}) = \min\{ER_{t_1}(F) : F \in \mathcal{F}\}$, while $ER_{t_2}(\mathcal{F}) = \min\{ER_{t_2}(F) : F \in \mathcal{F}\} - 1$. While this is also the case if $\mathcal{F} = \{K_{1,3}, P_3 + P_2\}$, the situation is a bit different.

Theorem 8. *If $\mathcal{F} = \{K_{1,3}, P_3 + P_2\}$, then*

- (a) $ER_t(\mathcal{F}) = \min\{ER_t(F) : F \in \mathcal{F}\} - 1$ for $t = 2, 3$;
- (b) $ER_4(\mathcal{F}) = \min\{ER_4(F) : F \in \mathcal{F}\}$.

Proof. Let $\mathcal{F} = \{K_{1,3}, P_3 + P_2\}$. First, we consider $ER_2(\mathcal{F})$. Since $ER_2(K_{1,3}) = ER_2(P_3 + P_2) = 6$, it follows that $\min\{ER_2(F) : F \in \mathcal{F}\} = 6$. Thus, we show that $ER_2(\mathcal{F}) = 5$. The red–blue coloring of K_4 in which both the red subgraph and blue subgraph are the 4-cycle C_4 does not produce a copy of a monochromatic graph in $\mathcal{F}$. Therefore, $ER_2(\mathcal{F}) \geq 5$. It remains to show that $ER_2(\mathcal{F}) \leq 5$. Let there be a red–blue coloring of $G = K_5$ with red subgraph G_r and blue subgraph G_b . First, suppose that $G_r \cong G_b \cong C_5$. Since C_5 is $K_{1,3}$ -free and contains exactly one copy of $P_3 + P_2$, it follows that G contains two edge-disjoint monochromatic copies of $P_3 + P_2 \in \mathcal{F}$. Thus, we may assume that some vertex v in G is incident with three edges of the same color. Then, G_r contains a monochromatic copy of $F_1 = K_{1,3}$ centered at v . Let $V(G) - \{v\} = \{v_1, v_2, v_3, v_4\}$. We may assume that $E(F_1) = \{vv_1, vv_2, vv_3\}$. Then, $H = G - v \cong K_4$. Since $\deg_H v_4 = 3$, there are at least two edges incident with v_4 that are colored the same; say v_2v_4, v_3v_4 are colored red. If either v_4v or v_4v_1 is colored red, then there is a red copy F_2 of $K_{1,3}$ that is edge-disjoint from F_1 .

Thus, we may assume that v_4v and v_4v_1 are both blue. We now consider the edge v_2v_3 . If v_2v_3 is blue, then there is a blue copy F_2 of $P_3 + P_2$, where $P_3 = (v, v_4, v_1)$ and $P_2 = (v_2, v_3)$, that is edge-disjoint from F_1 . If v_2v_3 is red, then there is a red copy F_2 of $K_3 = (v_2, v_3, v_4, v_2)$ that is edge-disjoint from F_1 . In either case, G contains two edge-disjoint monochromatic copies of $P_3 + P_2 \in \mathcal{F}$. Therefore, $ER_2(\mathcal{F}) \leq 5$ and so $ER_2(\mathcal{F}) = 5$.

Next, we consider $ER_3(\mathcal{F})$. Since $ER_3(K_{1,3}) = ER_3(P_3 + P_2) = 7$, it follows that $\min\{ER_3(F) : F \in \mathcal{F}\} = 7$. Thus, we show that $ER_3(\mathcal{F}) = 6$. We saw that a red–blue coloring of K_5 (in which the red subgraph and the blue subgraph are both C_5) produces only two edge-disjoint monochromatic copies of $P_3 + P_2 \in \mathcal{F}$, but no monochromatic $K_{1,3}$. Hence, $ER_3(\mathcal{F}) \geq 6$. Let there be a red–blue coloring of $G = K_6$ and let $v \in V(G)$. Since $\deg_G v = 5$, there are at least three edges incident with v that are colored the same, producing a monochromatic copy F_1 of $K_{1,3}$ centered at v . Then, $H = G - v \cong K_5$, each of whose edges are colored red or blue. Since $ER_2(\mathcal{F}) = 5$, it follows that H contains two edge-disjoint monochromatic copies F_2, F_3 of graphs in $\mathcal{F}$. Hence, F_1, F_2, F_3 are three pairwise edge-disjoint monochromatic copies of graphs in $\mathcal{F}$. Therefore, $ER_3(\mathcal{F}) \leq 6$ and so $ER_3(\mathcal{F}) = 6$.

Finally, we consider $ER_4(\mathcal{F})$. Since $ER_4(K_{1,3}) = ER_4(P_3 + P_2) = 7$, it follows that $\min\{ER_4(F) : F \in \mathcal{F}\} = 7$. Thus, we show that $ER_4(\mathcal{F}) = 7$. First, $ER_4(\mathcal{F}) \leq 7$ by Proposition 1. Next, consider the red–blue coloring of $G = K_6$ described in the proof of Theorem 7, in which the red subgraph is $G_r = C_4$ and the blue subgraph is $G_b = K_6 - E(C_4)$. Then, G_r is F -free for each $F \in \mathcal{F}$. Furthermore, since the size of G_b is $\binom{6}{2} - 4 = 11$ and four pairwise edge-disjoint monochromatic copies of graphs in $\mathcal{F}$ require 12 edges, it follows that G_b does not contain four pairwise edge-disjoint monochromatic copies of graphs in $\mathcal{F}$. Therefore, this red–blue coloring does not produce four pairwise edge-disjoint monochromatic copies of graphs in $\mathcal{F}$. Therefore, $ER_4(\mathcal{F}) \geq 7$ and so $ER_4(\mathcal{F}) = 7$. $\square$

With the aid of known values of $ER_t(F)$, where $F \in X - \{P_3\}$ for $2 \leq t \leq 4$, the following result has been obtained, which we state without proof.

Theorem 9. For $t = 2, 3, 4$, if $\mathcal{F}$ is one of the following sets,

- (a) $\{2K_2, F\}$, where $F \in \{P_4, K_{1,3}, P_3 + P_2\}$;
- (b) $\{K_3, F\}$, where $F \in \{2K_2, P_4, K_{1,3}, 3K_2\}$;
- (c) $\{P_4, F\}$, where $F \in \{K_{1,3}, P_3 + P_2, 3K_2\}$;

then $ER_t(\mathcal{F}) = \min\{ER_t(F) : F \in \mathcal{F}\}$.

For every set $\mathcal{F}$ of graphs of size 2 or 3 for which $ER_t(\mathcal{F})$ has been determined, either $ER_t(\mathcal{F}) = \min\{ER_t(F) : F \in \mathcal{F}\}$ or $ER_t(\mathcal{F}) = \min\{ER_t(F) : F \in \mathcal{F}\} - 1$. Therefore, we conclude this section with the following problem.

Problem 1. Let $t \geq 2$ be an integer and let $\mathcal{F}$ be a set of graphs of size 2 or 3 without isolated vertices. Is it true that

$$\min\{ER_t(F) : F \in \mathcal{F}\} - 1 \leq ER_t(\mathcal{F}) \leq \min\{ER_t(F) : F \in \mathcal{F}\}?$$

4. Ramsey Concepts for Further Study

We have introduced two new Ramsey concepts in this paper, namely, the vertex-disjoint and edge-disjoint Ramsey numbers of sets of graphs without isolated vertices. These two Ramsey concepts provide a generalization and a new perspective of the classical Ramsey concepts. We summarize what has been presented here. Let $\mathcal{F}$ be a set of graphs without isolated vertices.

- Upper and lower bounds are established for the vertex-disjoint Ramsey number $VR_t(\mathcal{F})$ and the edge-disjoint Ramsey number $ER_t(\mathcal{F})$ of $\mathcal{F}$. The sharpness of these bounds is discussed.
- Exact values of $VR_t(\mathcal{F})$ are determined for all sets $\mathcal{F}$ of graphs of size 2 or 3 and for all integers $t \geq 2$.
- Exact values of $ER_t(\mathcal{F})$ are determined for certain sets $\mathcal{F}$ of graphs of size 2 or 3 with prescribed conditions and for all integers $t \geq 2$. Furthermore, exact values of $ER_t(\mathcal{F})$ are determined for several other special sets $\mathcal{F}$ of graphs of size 2 or 3 where $2 \leq t \leq 4$.

While many problems on this topic remain unsolved, there are some related and potentially intriguing Ramsey concepts that are worthy of further investigation. Recall that for two graphs F and H , the Ramsey number $R(F, H)$ of F and H is the minimum positive integer n such that for every red–blue coloring of the complete graph K_n of order n , there is either a red F or a blue H . A natural extension of the Ramsey number $R(F, H)$ of two graphs F and H is to require the existence of multiple vertex-disjoint copies of a red F or a blue H . More precisely, let F and H be two non-isomorphic graphs without isolated vertices. For a positive integer t , the *vertex-disjoint Ramsey number* $VR_t(F, H)$ of F and H is the minimum positive integer n such that for every red–blue coloring of K_n , there are at least t pairwise vertex-disjoint subgraphs of K_n , each of which is either a red F or a blue H . Thus, $VR_1(F, H) = R(F, H)$. If it is not required that there exists at least one red F and at least one blue H in a red–blue coloring of a complete graph, then $VR_t(F, H) \leq \min\{VR_t(F), VR_t(H)\}$. On the other hand, if it is required that there must be at least one red F and at least one blue H in a red–blue coloring of a complete graph, then the vertex-disjoint Ramsey number of F and H gives rise to another Ramsey concept. For positive integers t and s , the *vertex-disjoint Ramsey number* $VR_{t,s}(F, H)$ of F and H is the minimum positive integer n such that every red–blue coloring of K_n produces at least $t + s$ pairwise vertex-disjoint subgraphs of K_n , where t of these are a red F and s of these are a blue H .

Similarly, the Ramsey number $R(F, H)$ can be extended to multiple edge-disjoint copies of a red F or a blue H by defining the *edge-disjoint Ramsey number* $ER_t(F, H)$ of F and H and the *edge-disjoint Ramsey number* $ER_{t,s}(F, H)$ of F and H as expected. Furthermore, the concepts of the vertex-disjoint and edge-disjoint Ramsey numbers of two graphs F and H can be extended even further by replacing a single graph F by a set $\mathcal{F}$ of non-isomorphic graphs and replacing a single graph H by a set $\mathcal{H}$ of non-isomorphic graphs. This observation gives rise to the concepts of vertex-disjoint Ramsey numbers $VR_t(\mathcal{F}, \mathcal{H})$ and $VR_{t,s}(\mathcal{F}, \mathcal{H})$ as well as the edge-disjoint Ramsey numbers $ER_t(\mathcal{F}, \mathcal{H})$ and $ER_{t,s}(\mathcal{F}, \mathcal{H})$ of two sets $\mathcal{F}$ and $\mathcal{H}$. It would be of interest to have information about these Ramsey numbers for graphs of small size.

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