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Judgment Criteria for Reliability Comparison of Three-Terminal Graphs with High Edge Failure Probability

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Abstract: A three-terminal graph is defined as a simple graph comprising three specified target vertices. The reliability of three-terminal graphs represents the probability that these three target vertices remain connected, given that each edge fails independently with a constant probability q. In this paper, we focus on exploring the characteristics of more reliable three-terminal graphs when the edge failure probability approaches 1. Three reliability comparison criteria are proposed to characterize the locally most reliable three-terminal graph progressively when the number of edges m is in the range of $[5, 4n-10]$ and $[\binom{n}{2}-n+4, \binom{n}{2}]$. At the same time, the locally optimal structures in the range of the edge number m with $(4n-10, \binom{n}{2}-n+4)$ are restricted to six specific classes of graphs. Furthermore, based on these criteria, a method is introduced to search local optimal structures and offer a theoretical foundation for constructing optimal networks and repairing faulty ones.

Keywords: three-terminal graph; locally more reliable; judgment criteria; locally most reliable graph

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1. Introduction

Network reliability is a popular research topic in the fields of network performance analysis, combinatorial mathematics, and graph theory. These studies usually explore the network by transforming it into its topology, which is a graph (where the nodes in the network correspond to the vertices in the graph and the links in the network correspond to the edges between the corresponding vertices in the graph). Therefore, in this paper, we will not make a clear distinction between networks and graphs.

Currently, research on reliability under edge failure but vertex reliability in networks has focused on the following areas. Firstly, from the number of key vertices in the network, studies have been differentiated into two cases: k-terminal networks (where $1 < k < n$) [1,2] and all-terminal networks (i.e., $k = n$) [3–5]. Secondly, in terms of research direction, reliability research is split into two primary areas: reliability analysis [1,5–7] and reliability design [8–12]. Reliability analysis focuses on calculating the probability that a network will be able to work under conditions of edge failure, which is usually quantified by a reliability polynomial, while reliability design aims to construct a network structure that maximizes the value of this reliability polynomial.

Reliability studies have focused mainly on the field of analysis and design of all-terminal networks [1,3,4,6,8], while relatively few investigations on k-terminal reliability

have focused on developing efficient algorithms for computing k-terminal reliability polynomials [1,7,13]. However, in recent years, research on k-terminal reliability design has shown a gradual increase [2,9–12]. In 2018, Bertrand et al. [9] studied the existence of a uniformly optimal structure under edge failure for two-terminal graphs when $20 \leq m \leq 3n-9$ and when $\binom{n}{2} - \lfloor (n-2)/2 \rfloor \leq m \leq \binom{n}{2}$. In other words, they explored whether there exists a structure, regardless of the value of the edge failure probability, which can maximize reliability of two-terminal graphs. In addition, they characterize several locally most reliable two-terminal graphs that are maximized when the probability of edge failure approaches 0 or 1. In 2021, Xie et al. [10] analyzed the properties of two types of locally most reliable two-terminal graphs, and accordingly determined that there exists no uniformly most reliable two-terminal graph when $3n-6 < m \leq \binom{n}{2} - 2$, which further enhances the theoretical framework of the consistent optimality problem for two-terminal graphs. Building on this foundation, in 2024, Gong et al. [2] inscribed the locally most reliable two-terminal graph for satisfying $2n-3 \leq m \leq \binom{n}{2}$ and m not close to $\frac{1}{2}\binom{n-2}{2} + 2n - 3$ under the condition that the probability of edge failure tends to 1.

Based on the research results on the optimality of two-terminal networks, Xie et al. [11,12], in 2021, explored the reliability problem of three-terminal networks by inscribing locally most reliable three-terminal graphs with edge numbers m in $9 < m < 4n-10$ and $\binom{n}{2} - n + 4 < m < \binom{n}{2}$.

However, most of the research focuses on constructing locally most reliable graphs. When many edges in the graph are failed, it is not clear which edge is repaired priorly to reduce the loss for network failure at the fastest speed. In real life, we need to study the structure of the locally most reliable graph, while also exploring a way of designing a structure with higher reliability. According to actual demand, we build a relatively reliable network under the condition of resource constraints and realize the balance between resource saving and demand satisfaction. Additionally, we provide a theoretical basis for quickly repairing critical edges if the network experiences failure.

In this paper, by comparing the reliability polynomial coefficients of three-terminal graphs step by step, we obtain three comparative judgment criteria for reliability under the condition of high edge failure probability. These criteria provide a scheme to prioritize edge connections and construct a more reliable structure. Meanwhile, according to the judgment criteria, we characterize locally most reliable three-terminal graphs in the range $5 \leq m \leq 4n-10$ and $\binom{n}{2} - n + 4 \leq m \leq \binom{n}{2}$ with edge failure probabilities close to 1. The locally most reliable three-terminal graphs obtained by this method are consistent with the results in [11,12], thereby verifying the validity and accuracy of our method. Furthermore, the study of locally optimal structures for other ranges have been limited to six special classes of graphs, which significantly reduces the complexity. Finally, we propose a search method that identifies the locally most reliable three-terminal graphs or locally more reliable three-terminal graphs based on the judgment criteria.

This paper is organized as follows. In Section 2, some basic definitions and related notations are introduced. In Section 3, three local reliability comparison criteria are stated and several locally most reliable three-terminal graphs by these criteria are characterized. In Section 4, a flowchart is given to find the locally optimal structures by the criteria. Section 5 summarizes the paper and puts forward some interesting problems to be solved.

2. Basic Concepts and Notations

This section introduces some symbolic terms and basic concepts. For those not mentioned, see [14]. The graph without loops or parallel edges is the simple graph. In this paper, we study only simple graphs. For graphs G and H, if there are bijections ϕ: $V(G) \to V(H)$ and φ: $E(G) \to E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\varphi(e)) = \phi(u)\phi(v)$, then G

and H are isomorphic, written as $G \cong H$. Use $G \vee H$ to denote the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{uv | u \in V(G), v \in V(H)\}$. Let $N_G[H]$ denote the number of the induced subgraphs of G isomorphic to H. Let $N_G(H)$ denote the number of subgraphs of G that are isomorphic to H. Using P_i denotes the path in the graph with i vertices and $i-1$ edges.

A simple graph $G = (V(G), E(G))$ is a three-terminal graph if the vertex set $V(G)$ has three specified key vertices (target vertices) r, s and t, and an all-terminal graph if all vertices are key vertices. Denote by $\mathcal{G}_{n,m}$ the set of all three-terminal graphs on n vertices and m edges. Denote by $\mathcal{A}_{n,m}$ the set of all all-terminal graphs on n vertices and m edges. The reliability (or the reliability polynomial) of three-terminal graph G, denoted by $R_3(G; p)$, is the probability that the three specified target vertices r, s, t in graph $G \in \mathcal{G}_{n,m}$ remains connected when its edges fail independently with probability q.

An *rst*-subgraph is a subgraph of G in which r, s, t are connected. This *rst*-subgraph is minimal if it is not an *rst*-subgraph after deleting any of its edges, otherwise it is a non-minimal *rst*-subgraph. If an *rst*-subgraph has i edges, it is denoted as S_i. Since S_0 and S_1 do not exist, the reliability polynomial of three-terminal graph $G \in \mathcal{G}_{n,m}$ can be written as

$$R_3(G; q) = \sum_{i=2}^{m} n_i^{rst}(G)(1-q)^i q^{m-i} = \sum_{i=2}^{m} n_i^{rst}(1-q)^i q^{m-i},$$

where $n_i^{rst}(G)$ or simply n_i^{rst} is the number of *rst*-subgraph S_i in graph G.

Definition 1 ([11]). *A graph G is the locally most reliable graph in $\mathcal{G}_{n,m}$ if $R_3(G; q) \geq R_3(H; q)$ for $\forall H \in \mathcal{G}_{n,m}$ and $q \to 1$.*

Example 1. *The following four graphs given in Figure 1 all show the topology of a regional power supply network with five vertices and six edges. The blue circular vertices represent three key substations (target vertices): r (main supply to the urban core), s (main supply to the industrial area), and t (main supply to the outskirts of the city and important public utilities). The black triangular vertices represent auxiliary power supply facilities (non-target vertices): small-scale power generation stations and backup power facilities. The target vertices are responsible for converting high-voltage electrical energy into low-voltage electrical energy suitable for use in urban or industrial areas and distributing it to each user end. The non-target vertices can provide additional power support to the target vertices or serve as backup power sources.*

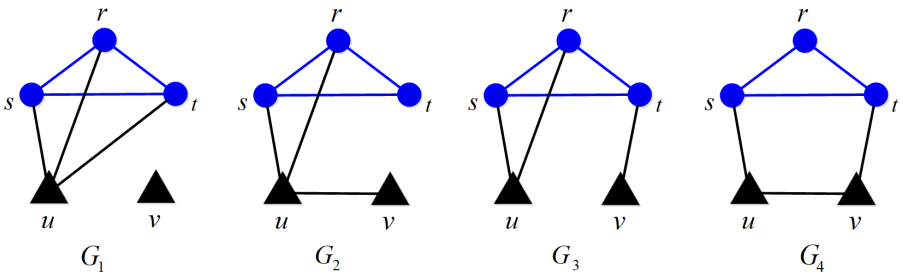


Figure 1. Four types of regional power supply networks with five vertices and six edges.

If the edge (supply line) failure probability is q, then for G_1, *rst*-subgraphs (S_0 and S_1) with zero edges and with one edge do not exist, so $n_0^{rst}(G_1) = n_1^{rst}(G_1) = 0$. The *rst*-subgraphs S_2 with two edges are $\{rs, rt\}$, $\{rs, st\}$, $\{rt, st\}$, so $n_2^{rst}(G_1) = 3$. The *rst*-subgraphs S_3 with three edges are $\{rs, rt, st\}$, $\{rs, rt, ru\}$, $\{rs, rt, su\}$, $\{rs, rt, tu\}$, $\{rs, st, ru\}$, $\{rs, st, su\}$, $\{rs, st, tu\}$, $\{rt, st, ru\}$, $\{rt, st, su\}$, $\{rt, st, tu\}$, $\{rs, su, ut\}$, $\{rs, ru, ut\}$, $\{rt, ut, su\}$, $\{rt, ru, su\}$, $\{st, su, ru\}$, $\{st, tu, ru\}$, $\{ru, su, tu\}$, so $n_3^{rst}(G_1) = 17$. The *rst*-subgraphs S_4 with four edges are $\{rs, rt, st, ru\}$, $\{rs, rt, st, su\}$, $\{rs, rt, st, tu\}$, $\{rs, rt, ru, su\}$, $\{rs, rt, ru, tu\}$, $\{rs, rt, su, tu\}$, $\{rs, st, ru, su\}$, $\{rs, st, ru, tu\}$, $\{rs, st, su, tu\}$, $\{rt, st, ru, su\}$,

$\{rt, st, ru, tu\}$, $\{rt, st, su, tu\}$, $\{ru, su, tu, rs\}$, $\{ru, su, tu, rt\}$, $\{ru, su, tu, st\}$, so $n_4^{rst}(G_1) = 15$. The rst-subgraphs S_5 with five edges are $\{rs, rt, st, ru, su\}$, $\{rs, rt, st, ru, tu\}$, $\{rs, rt, st, su, tu\}$, $\{rs, rt, ru, su, tu\}$, $\{rs, st, ru, su, tu\}$, $\{rt, st, ru, su, tu\}$, so $n_5^{rst}(G_1) = 6$. The rst-subgraph S_6 with six edges is $\{rs, rt, st, ru, su, tu\}$, so $n_6^{rst}(G_1) = 1$.

Therefore, the reliable polynomial of G_1 is $R_3(G_1; q) = 3(1-q)^2q^4 + 17(1-q)^3q^3 + 15(1-q)^4q^2 + 6(1-q)^5q + (1-q)^6$. Similarly, the reliable polynomials of G_2, G_3, and G_4 are $R_3(G_2; q) = 3(1-q)^2q^4 + 12(1-q)^3q^3 + 14(1-q)^4q^2 + 6(1-q)^5q + (1-q)^6$, $R_3(G_3; q) = 3(1-q)^2q^4 + 12(1-q)^3q^3 + 14(1-q)^4q^2 + 6(1-q)^5q + (1-q)^6$, and $R_3(G_4; q) = (1-q)^2q^4 + 4(1-q)^3q^3 + 7(1-q)^4q^2 + 5(1-q)^5q + (1-q)^6$, respectively. Obviously, $R_3(G_1; q) > R_3(G_2; q) = R_3(G_2; q) > R_3(G_2; q)$ for q close to 1. In fact, it has been proved in [11] that G_1 is the most reliable graph in $\mathcal{G}_{6,6}$. So, the critical packet certified regional power supply network maintains the stable transmission and distribution of electric energy, and this network can be constructed according to the structure of G_1. However, when more than one edge of the network fails, in order to reduce the loss as soon as possible, it is necessary to choose the line that is prioritized for repair, so the characterization of a more reliable network is necessary, and the next sections will give the related research.

3. Judgment Criteria of the Local Reliability Comparison with High Edge Failure Probability

In this section, we focus on the characterization of more reliable graphs by giving three locally more reliable comparison judgment theorems. Then, we construct several locally most reliable three-terminal graphs for q close to 1. We compare the results with existing literature results to further verify the validity and accuracy of the judgment theorems. In general, the calculation of the three-terminal reliability polynomial of the graph is NP-hard [15]. Therefore, we study the reliability of graphs by the following lemma.

Lemma 1. *Let the three-terminal reliability polynomials of G and H in $\mathcal{G}_{n,m}$ be $R_3(G; q) = \sum_{i=2}^{m} n_i^{rst}(G)(1-q)^i q^{m-i}$ and $R_3(H; q) = \sum_{i=2}^{m} n_i^{rst}(H)(1-q)^i q^{m-i}$, respectively.*

Suppose there exists a positive integer k such that $n_i^{rst}(G) = n_i^{rst}(H)$ and $n_k^{rst}(G) > n_k^{rst}(H)$ for $2 \le i < k$, then $R_3(G; q) > R_3(H; q)$ holds when $q \to 1$.

Proof. Since $G, H \in \mathcal{G}_{n,m}$, it is clear that

$$\begin{aligned} & R_3(G; q) - R_3(H; q) \\ = & \sum_{i=2}^{m} (n_i^{rst}(G) - n_i^{rst}(H))(1-q)^i q^{m-i} \\ = & \sum_{i=2}^{k-1} (n_i^{rst}(G) - n_i^{rst}(H))(1-q)^i q^{m-i} + \sum_{i=k}^{m} (n_i^{rst}(G) - n_i^{rst}(H))(1-q)^i q^{m-i}. \end{aligned}$$

Since $n_i^{rst}(G) = n_i^{rst}(H)$ $(2 \le i < k)$,

$$\begin{aligned} & R_3(G; q) - R_3(H; q) \\ = & \sum_{i=k}^{m} (n_i^{rst}(G) - n_i^{rst}(H))(1-q)^i q^{m-i} \\ = & (1-q)^k [(n_k^{rst}(G) - n_k^{rst}(H))q^{m-k} \\ + & (n_{k+1}^{rst}(G) - n_{k+1}^{rst}(H))(1-q)q^{m-k-1} + \cdots + (n_m^{rst}(G) - n_m^{rst}(H))(1-q)^{m-k}]. \end{aligned}$$

For $q \to 1$ and $1 \leq j \leq m-k$, it is easy to see that $(n_{k+j}^{rst}(G) - n_{k+j}^{rst}(H))(1-q)^j q^{m-k-j} = o((n_k^{rst}(G) - n_k^{rst}(H))q^{m-k})$. So, if $n_k^{rst}(G) > n_k^{rst}(H)$, then $R_3(G;q) - R_3(H;q) > 0$ as $q \to 1$; that is, $R_3(G;q) > R_3(H;q)$. □

From Lemma 1, we can derive the first round of the local reliability comparison criterion for a three-terminal graph with high edge failure probability. It is referred to as the First Judgment Criterion ($q \to 1$).

For a graph $G \in \mathcal{G}_{n,m}$, let $T^{rst}(G)$ be the number of edges among the target vertices r, s and t in G; that is, $T^{rst}(G) = |\{rs, rt, st\} \cap E(G)|$.

Theorem 1 (First Judgment Criterion ($q \to 1$))**.** *Let $G, H \in \mathcal{G}_{n,m}$, if $T^{rst}(G) > T^{rst}(H)$, then for $q \to 1$, $R_3(G;q) > R_3(H;q)$, which means that G is locally more reliable than H for $q \to 1$.*

Proof. The number of *rst*-subgraph with two edges of G and H can be expressed as $n_2^{rst}(G) = \binom{T^{rst}(G)}{2}$ and $n_2^{rst}(H) = \binom{T^{rst}(H)}{2}$.

If $T^{rst}(G) > T^{rst}(H)$, then $n_2^{rst}(G) > n_2^{rst}(H)$. By Lemma 1, we see that $R_3(G;q) > R_3(H;q)$ as q approaches 1, which indicates that G is more reliable than H for $q \to 1$. □

According to the First Judgment Criterion ($q \to 1$), in order to find the locally most reliable three-terminal graph as q approaches 1, one must first identify the class of all three-terminal graphs in $\mathcal{G}_{n,m}$ that contain the maximum number of edges in the triangle *rst* formed by three target vertices. For convenience, this class is denoted as $\mathcal{G}_{n,m}^1$.

Obviously, there are many graphs in $\mathcal{G}_{n,m}^1$. To search the locally most reliable three-terminal graph for $q \to 1$, we further construct a second round of the local reliability comparison criterion in $\mathcal{G}_{n,m}^1$.

The three-terminal complete graph with n vertices, denoted by K_n^{rst}, is a simple graph that has 3 target vertices r, s, t and $n-3$ non-target vertices, where any two vertices are adjacent. If an edge connecting one target vertex and one non-target vertex is deleted in the graph K_n^{rst}, the resulting graph is denoted by K_n^{rst-}.

Theorem 2 (Second Judgment Criterion ($q \to 1$))**.** *Let $G, H \in \mathcal{G}_{n,m}^1$ and $q \to 1$. If $2N_G[K_4^{rst-}] + 7N_G[K_4^{rst}] > 2N_H[K_4^{rst-}] + 7N_H[K_4^{rst}]$, then $R_3(G;q) > R_3(H;q)$, which means that G is locally more reliable than H for $q \to 1$.*

Proof. Since $G, H \in \mathcal{G}_{n,m}^1$, the triangle *rst* is contained in G and H, which means that $T^{rst}(G) = T^{rst}(H) = 3$. Thus, $n_2^{rst}(G) = \binom{T^{rst}(G)}{2} = 3 = \binom{T^{rst}(H)}{2} = n_2^{rst}(H)$.

Then, by Lemma 1, we need to compare the values of $n_3^{rst}(G)$ and $n_3^{rst}(H)$. Now, we consider the *rst*-subgraphs S_3 in G and H, as shown in Figure 2.

Case 1 S_3 is a non-minimal *rst*-subgraphs with three edges.

- Subcase 1.1 S_3 contains all the edges of the triangle *rst*, which means that $S_3 = \{rs, st, rt\}$. Obviously, the number of S_3 in this subcase is 1.
- Subcase 1.2 S_3 contains two edges of the triangle *rst*, which means that $S_3 \cong \{rs, st, v_iv_j\}$, where $1 \leq j \leq n$ and $4 \leq i \neq j \leq n$. The number of S_3 in this subcase is $3(m-3)$.

Case 2 S_3 is a minimal *rst*-subgraphs.

- Subcase 2.1 S_3 contains one edge of the triangle *rst*. Without loss of generality, assume that $S_3 \cong \{rs, sv_i, tv_i\}$, where $4 \leq i \leq n$. The number of S_3 in this subcase is $\binom{N_G[K_4^{rst-}]}{1}\binom{2}{1} + \binom{N_G[K_4^{rst}]}{1}\binom{3}{2}\binom{2}{1}$, which equals $2N_G[K_4^{rst-}] + 6N_G[K_4^{rst}]$.

Subcase 2.2 S_3 contains no edge of the triangle rst, which means that $S_3 \cong \{rv_i, sv_i, tv_i\}$, where $4 \leq i \leq n$. The number of S_3 in this subcase is $\binom{N_G[K_4^{rst}]}{1}$, which equals $N_G[K_4^{rst}]$.

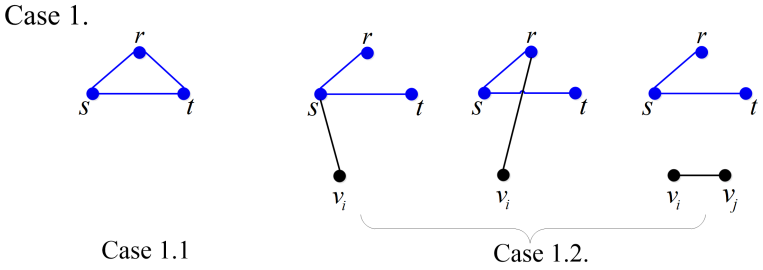


Figure 2. S_3, an rst-subgraph with 3 edges.

Combining the results of above cases, we have $n_3^{rst}(G) = 2N_G[K_4^{rst-}] + 7N_G[K_4^{rst}] + 3(m-3) + 1$.

Similarly, it can be obtained that $n_3^{rst}(H) = 2N_H[K_4^{rst-}] + 7N_H[K_4^{rst}] + 3(m-3) + 1$.

Thus, we have

$$n_3^{rst}(G) - n_3^{rst}(H) = (2N_G[K_4^{rst-}] + 7N_G[K_4^{rst}]) - (2N_H[K_4^{rst-}] + 7N_H[K_4^{rst}]).$$

Clearly, if $2N_G[K_4^{rst-}] + 7N_G[K_4^{rst}] > 2N_H[K_4^{rst-}] + 7N_H[K_4^{rst}]$, then $n_3^{rst}(G) > n_3^{rst}(H)$.

Therefore, by Lemma 1, it follows that $R_3(G;q) > R_3(H;q)$ for $q \to 1$, and the theorem holds. □

According to the Second Judgment Criterion ($q \to 1$), to find the locally most reliable three-terminal graph as $q \to 1$, it is necessary to identify the class of graphs within $\mathcal{G}_{n,m}^1$ that maximizes the sum of two times the number of induced subgraphs isomorphic to K_4^{rst-} and seven times the number of induced subgraphs isomorphic to K_4^{rst}. This class is denoted as $\mathcal{G}_{n,m}^2$.

Indeed, in a three-terminal graph G containing triangles rst, the number of induced subgraphs isomorphic to K_4^{rst} must be maximized in order to get the maximum value of $2N_G[K_4^{rst-}] + 7N_G[K_4^{rst}]$. In other words, each non-target vertex is adjacent to three target vertices as many as possible, as seen in Theorem 3.

Theorem 3. *Let $G \in \mathcal{G}_{n,m}^2$, then $N_G[K_4^{rst}] = min\{\lfloor \frac{m-3}{3} \rfloor, n-3\}$, and*

$$N_G[K_4^{rst-}] = \begin{cases} 1 & \text{if } n > \lfloor \frac{m-3}{3} \rfloor + 3 \text{ and } m \equiv 2 (\mathrm{mod} 3), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since $G \in \mathcal{G}_{n,m}^2$, it follows that $2N_G[K_4^{rst-}] + 7N_G[K_4^{rst}] \geq 2N_H[K_4^{rst-}] + 7N_H[K_4^{rst}]$ for any three-terminal graph H in $\mathcal{G}_{n,m}^1$. For convenience, let $N_G[K_4^{rst-}] = a$ and $N_G[K_4^{rst}] = b$. It is easy to see that $a \leq 1$. Otherwise, there must exist two non-target vertices u_1 and u_2 such that G contains two edges in $\{ru_1, su_1, tu_1\}$, and two edges in $\{ru_2, su_2, tu_2\}$. Without loss of generality, assume that these four edges are ru_1, su_1,

ru_2, and tu_2, then there must exist a graph $G' = G \cup \{tu_1\} - \{tu_2\} (G' \in \mathcal{G}^1_{n,m})$ satisfying $N_{G'}[K_4^{rst-}] = a - 2$, $N_{G'}[K_4^{rst}] = b + 1$. Thus, we have $2N_{G'}[K_4^{rst-}] + 7N_{G'}[K_4^{rst}] = 2a + 7b + 3 > 2a + 7b = 2N_G[K_4^{rst-}] + 7N_G[K_4^{rst}]$; that is, $2N_{G'}[K_4^{rst-}] + 7N_{G'}[K_4^{rst}] > 2N_G[K_4^{rst-}] + 7N_G[K_4^{rst}]$, a contradiction.

Clearly, $b \leq min\ \{\lfloor \frac{m-3}{3} \rfloor, n-3\}$. Since $a \leq 1$, $b = min\ \{\lfloor \frac{m-3}{3} \rfloor, n-3\}$. Otherwise, for any graph G'' in $\mathcal{G}^1_{n,m}$ containing $min\ \{\lfloor \frac{m-3}{3} \rfloor, n-3\}$ induced subgraphs K_4^{rst}, there is $2N_{G''}[K_4^{rst-}] + 7N_{G''}[K_4^{rst}] > 2N_G[K_4^{rst-}] + 7N_G[K_4^{rst}]$, a contradiction.

Moreover, when $n > \lfloor \frac{m-3}{3} \rfloor + 3$, $b = \lfloor \frac{m-3}{3} \rfloor$. At this point, if $m \equiv 0 (\mathrm{mod} 3)$ or $m \equiv 1 (\mathrm{mod} 3)$, then clearly, $a = 0$ holds; if $m \equiv 2 (\mathrm{mod} 3)$, then $a = 1$. When $n \leq \lfloor \frac{m-3}{3} \rfloor + 3$, $b = n - 3$, in this case $a = 0$.

In conclusion, we complete the proof. □

Theorem 3 determines the number of induced subgraphs K_4^{rst-} and induced subgraphs K_4^{rst} of the locally most reliable three-terminal graph as $q \to 1$. Consequently, we can directly determine the locally optimal structure for $9 \leq m \leq 3n - 6$ and $m \equiv 0 (\mathrm{mod} 3)$ or $m \equiv 2 (\mathrm{mod} 3)$. Please refer to the following corollary, which is consistent with the results of Theorem 3.1 and Theorem 3.2 in [11].

Corollary 1. *Let G be the locally most reliable three-terminal graph in $\mathcal{G}_{n,m}$ at $q \to 1$, then*

(1) for $9 \leq m \leq 3n - 6$ and $m \equiv 0 (\mathrm{mod} 3)$, $E(G) = \{rs, rt, st\} \cup \{v_i v_j | 1 \leq i \leq 3, 4 \leq j \leq \lfloor \frac{m-3}{3} \rfloor + 3\}$;

(2) for $9 \leq m \leq 3n - 6$ and $m \equiv 2 (\mathrm{mod} 3)$, $E(G) = \{rs, rt, st\} \cup \{v_i v_j | 1 \leq i \leq 3, 4 \leq j \leq \lfloor \frac{m-3}{3} \rfloor + 3\} \cup \{v_k v_{\lfloor \frac{m-3}{3} \rfloor + 4}, v_\ell v_{\lfloor \frac{m-3}{3} \rfloor + 4} | k \neq \ell \in \{1, 2, 3\}\}$.

However, when $9 \leq m \leq 3n - 6$ and $m \equiv 1 (\mathrm{mod} 3)$, or when $3n - 6 < m \leq \binom{n}{2}$, the locally most reliable three-terminal graph ($q \to 1$) cannot be determined only based on the Second Judgment Criterion. Therefore, we begin to construct the third criterion for local reliability comparison among three-terminal graphs with high edge failure probability. This will be referred to as the Third Judgment Criterion ($q \to 1$).

For $G \in \mathcal{G}^2_{n,m}$, let $\widehat{G} = G - \{r, s, t\} - \{v_i | N_G[K_4^{rst-}] + N_G[K_4^{rst}] + 4 \leq i \leq n\}$. Obviously, $|V(\widehat{G})| = N_G[K_4^{rst-}] + N_G[K_4^{rst}]$, $|E(\widehat{G})| = \widehat{m} = m - 3 - 2N_G[K_4^{rst-}] - 3N_G[K_4^{rst}]$.

Theorem 4 (Third Judgment Criterion ($q \to 1$))**.** *Let $G, H \in \mathcal{G}^2_{n,m}$, $\widehat{G}$ and $\widehat{H}$ satisfy the above establishment. If one of the following conditions holds when $9 \leq m \leq 3n - 6$ and $m \equiv 1 (\mathrm{mod} 3)$, or $3n - 6 < m \leq \binom{n}{2}$, then $R_3(G; q) > R_3(H; q)$ as $q \to 1$; that is, G is locally more reliable than H when q approaches 1.*

(1) $N_{\widehat{G}}(P_2) > N_{\widehat{H}}(P_2)$;

(2) $N_{\widehat{G}}(P_2) = N_{\widehat{H}}(P_2)$, but $N_{\widehat{G}}(P_3) > N_{\widehat{H}}(P_3)$.

Proof. Since $G, H \in \mathcal{G}^2_{n,m}$, $n_2^{rst}(G) = n_2^{rst}(H) = 3$. For $9 \leq m \leq 3n - 6$ and $m \equiv 1 (\mathrm{mod} 3)$, or $3n - 6 < m \leq \binom{n}{2}$, by Theorem 3, it is easy to see that $N_G[K_4^{rst-}] = N_H[K_4^{rst-}] = 0$ and

$$N_G[K_4^{rst}] = N_H[K_4^{rst}] = \begin{cases} \lfloor \frac{m-3}{3} \rfloor & 9 \leq m \leq 3n - 6 \text{ and } m \equiv 1 (\mathrm{mod} 3), \\ n - 3 & 3n - 6 < m \leq \binom{n}{2}. \end{cases}$$

Thus, combined with the proof of Theorem 2, we obtain

$$n_3^{rst}(G) = n_3^{rst}(H) = \begin{cases} 7\lfloor \frac{m-3}{3} \rfloor + 3m - 8 & 9 \leq m \leq 3n - 6 \text{ and } m \equiv 1 (\mathrm{mod} 3), \\ 7n + 3m - 29 & 3n - 6 < m \leq \binom{n}{2}. \end{cases}$$

By Lemma 1, in order to compare the reliability of G and H, we need to compare $n_4^{rst}(G)$ and $n_4^{rst}(H)$. Consider the cases of S_4 in G and H.

Case 1 S_4 is a non-minimal *rst*-subgraph with four edges.

Subcase 1.1 The minimal *rst*-subgraph contained in S_4 has two edges, which means that $S_4 \cong \{rs, st, v_iv_j, v_kv_l\}$, where $1 \le i, j, k, l \le n$. The number of S_4 in this subcase is $3\binom{m-3}{2} + \binom{m-3}{1}$.

Subcase 1.2 The minimal *rst*-subgraph contained in S_4 has three edges, which means that $S_4 \cong \{rs, sv_i, v_it, v_jv_k\}$ or $S_4 \cong \{rv_i, sv_i, v_it, v_jv_k\}$, where $1 \le j \le n, 4 \le i, k \le n$. The number of S_4 in this subcase is $\binom{N_G[K_4^{rst}]}{1}\binom{m-3}{1} + \binom{N_G[K_4^{rst}]}{1}\binom{3}{2}\binom{2}{1}\binom{m-6}{1}$, which equals $N_G[K_4^{rst}](7m - 39)$.

Case 2 S_4 is a minimal *rst*-subgraph, as shown in Figure 3, which means that $S_4 \cong \{rs, sv_i, v_iv_j, tv_j\}$, $S_4 \cong \{rv_i, rv_j, sv_i, tv_j\}$ or $S_4 \cong \{rv_i, sv_j, v_iv_j, tv_j\}$, where $4 \le i, j \le n$. The number of S_4 in this case is $\binom{N_{\widehat{G}}(P_2)}{1}\binom{2}{1}\binom{3}{1}\binom{2}{1} + \binom{N_{\widehat{G}}(P_2)}{1}\binom{2}{1}\binom{3}{1} + \binom{N_G[K_4^{rst}]}{2}\binom{3}{2}\binom{2}{1}$, which equals $6\binom{N_G[K_4^{rst}]}{2} + 18N_{\widehat{G}}(P_2)$.

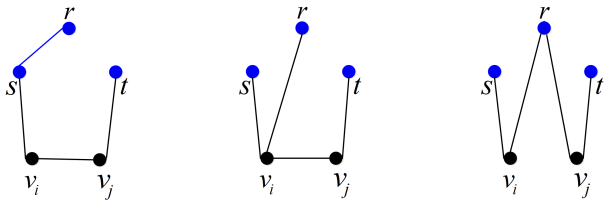


Figure 3. Minimal *rst*-subgraphs with 4 edges.

Thus, we have

$$n_4^{rst}(G) = 3\binom{m-3}{2} + (m-3) + 6\binom{N_G[K_4^{rst}]}{2} + N_G[K_4^{rst}](7m - 39) + 18N_{\widehat{G}}(P_2).$$

Similarly,

$$n_4^{rst}(H) = 3\binom{m-3}{2} + (m-3) + 6\binom{N_H[K_4^{rst}]}{2} + N_H[K_4^{rst}](7m - 39) + 18N_{\widehat{H}}(P_2).$$

Therefore, $n_4^{rst}(G) - n_4^{rst}(H) = 18[N_{\widehat{G}}(P_2) - N_{\widehat{H}}(P_2)]$.

(1) If $N_{\widehat{G}}(P_2) > N_{\widehat{H}}(P_2)$, then $n_4^{rst}(G) - n_4^{rst}(H) > 0$; that is $n_4^{rst}(G) > n_4^{rst}(H)$. So, by Lemma 1, $R_3(G; q) > R_3(H; q)$ for $q \to 1$.

(2) If $N_{\widehat{G}}(P_2) = N_{\widehat{H}}(P_2)$, then $n_4^{rst}(G) = n_4^{rst}(H)$. So, we need to further compare the values of $n_5^{rst}(G)$ and $n_5^{rst}(H)$, and the *rst*-subgraph S_5 of a three-terminal graph has the following cases.

Case 1 S_5 is a non-minimal *rst*-subgraph with five edges.

Subcase 1.1 The minimal *rst*-subgraph contained in S_5 has two edges, which means that $S_5 \cong \{rs, st, v_{i_1}v_{i_2}, v_{i_3}v_{i_4}, v_{i_5}v_{i_6}\}$, where $1 \le i_1, i_2, i_3, i_4, i_5, i_6 \le n$. The number of S_5 in this subcase is $\binom{3}{2}\binom{m-3}{2} + \binom{m-3}{2}$, which equals $3\binom{m-3}{3} + \binom{m-3}{2}$.

Subcase 1.2 The minimal *rst*-subgraph contained in S_5 has three edges, which means that $S_5 \cong \{rs, sv_i, v_it, v_{j_1}v_{j_2}, v_{k_1}v_{k_2}\}$ or $S_5 \cong \{rv_i, sv_i, v_it, v_{j_1}v_{j_2}, v_{k_1}v_{k_2}\}$, where $1 \le j_1, k_1 \le n, 4 \le i, j_2, k_2 \le n$. The number of S_5 in this subcase is $\binom{N_G[K_4^{rst}]}{1}\binom{3}{2}\binom{2}{1}\binom{m-6}{2} - \binom{N_G[K_4^{rst}]}{2}\binom{3}{2}\binom{2}{1} - \binom{N_G[K_4^{rst}]}{2}\binom{3}{2}\binom{2}{1} + \binom{N_G[K_4^{rst}]}{1}\binom{3}{1}\binom{m-6}{1} + \binom{N_G[K_4^{rst}]}{1}\binom{m-6}{2}$, which equals $7N_G[K_4^{rst}]\ \binom{m-6}{2} + 3(m-6)N_G[K_4^{rst}] - 12\binom{N_G[K_4^{rst}]}{2}$.

Subcase 1.3 The minimal *rst*-subgraph contained in S_5 has four edges, which means that $S_5 \cong \{rs, sv_i, v_iv_j, tv_j, v_kv_\ell\}$ $(v_kv_\ell \notin \{tv_i, rv_j, sv_j\})$ or $S_5 \cong$

$\{rv_i, rv_j, sv_i, v_jt, v_kv_\ell\}$ $(v_kv_\ell \notin \{tv_i, sv_j\})$ or $S_5 \cong \{rv_i, sv_i, v_iv_j, tv_j, v_kv_\ell\}$ $(v_kv_\ell \neq tv_i)$, where $1 \leq k \leq n$, $4 \leq i, j, \ell \leq n$. The number of S_5 in this subcase is $\binom{N_{\widehat{G}}(P_2)}{1}\binom{3}{1}\binom{2}{1} + \binom{N_{\widehat{G}}(P_2)}{1}\binom{3}{1}\binom{2}{1}\binom{2}{1}\binom{m-10}{1} + \binom{N_G[K_4^{rst}]}{2}\binom{3}{1}\binom{2}{1}\binom{m-9}{1} + \binom{N_{\widehat{G}}(P_2)}{1}\binom{2}{1}\binom{3}{2}\binom{m-10}{1}$, which equals $6N_{\widehat{G}}(P_2) + 18(m-10)N_{\widehat{G}}(P_2) + 6(m-9)\binom{N_G[K_4^{rst}]}{2}$.

Case 2 S_5 is a minimal *rst*-subgraph, as shown in Figure 4, which means that $S_5 \cong \{rs, sv_i, v_iv_j, v_jv_k, tv_k\}$, $S_5 \cong \{rv_i, rv_j, sv_i, v_jv_k, tv_k\}$, $S_5 \cong \{rv_i, sv_i, v_iv_j, v_jv_k, tv_j\}$ or $S_5 \cong \{v_iv_j, v_jv_k, rv_j, sv_i, tv_k\}$, where $4 \leq i, j, k \leq n$. The number of S_5 in this case is $\binom{N_{\widehat{G}}(P_3)}{1}\binom{2}{1}\binom{3}{1}\binom{2}{1} + \binom{N_{\widehat{G}}(P_3)}{1}\binom{2}{1}\binom{3}{1} + \binom{N_{\widehat{G}}(P_2)}{1}\binom{2}{1}\binom{3}{1}\binom{2}{1}\binom{N_G[K_4^{rst}]-2}{1} + \binom{N_{\widehat{G}}(P_3)}{1}\binom{3}{1}\binom{2}{1}$, which equals $24N_{\widehat{G}}(P_3) + 12N_{\widehat{G}}(P_2)(N_G[K_4^{rst}] - 2)$.

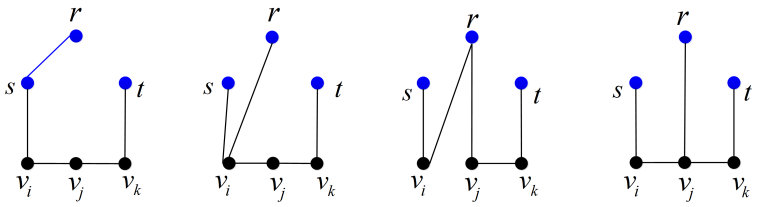


Figure 4. Minimal *rst*-subgraphs with 5 edges.

Thus, we have

$$\begin{aligned} n_5^{rst}(G) = {} & 3\binom{m-3}{3} + \binom{m-3}{2} + [7\binom{m-6}{2}(m-3) + 3(m-6)]N_G[K_4^{rst}] \\ & + 6(m-11)\binom{N_G[K_4^{rst}]}{2} + 18[(m-10)+6]N_{\widehat{G}}(P_2) \\ & + 12N_{\widehat{G}}(P_2)(N_G[K_4^{rst}] - 2) + 24N_{\widehat{G}}(P_3). \end{aligned}$$

Similarly,

$$\begin{aligned} n_5^{rst}(H) = {} & 3\binom{m-3}{3} + \binom{m-3}{2} + [7\binom{m-6}{2}(m-3) + 3(m-6)]N_H[K_4^{rst}] \\ & + 6(m-11)\binom{N_H[K_4^{rst}]}{2} + 18[(m-10)+6]N_{\widehat{H}}(P_2) \\ & + 12N_{\widehat{H}}(P_2)(N_H[K_4^{rst}] - 2) + 24N_{\widehat{H}}(P_3). \end{aligned}$$

It is easy to see that $n_5^{rst}(G) - n_5^{rst}(H) = 24[N_{\widehat{G}}(P_3) - N_{\widehat{H}}(P_3)]$.

Clearly, if $N_{\widehat{G}}(P_3) > N_{\widehat{H}}(P_3)$, then $n_5^{rst}(G) - n_5^{rst}(H) > 0$; that is, $n_5^{rst}(G) > n_5^{rst}(H)$. So, by Lemma 1, $R_3(G;q) > R_3(H;q)$ for $q \to 1$.

In conclusion, we complete the proof. □

According to the Third Judgment Criterion ($q \to 1$), it is easy to see that for $9 \leq m \leq 3n-6$ and $m \equiv 1(\mathrm{mod}3)$, there is only one graph that satisfies the condition, and hence, it is the locally most reliable three-terminal graph ($q \to 1$), leading to the following corollary, which is consistent with Theorem 3.1 in [11].

Corollary 2. *Let* $9 \leq m \leq 3n-6$ *and* $m \equiv 1(\mathrm{mod}3)$, *if G is the locally most reliable three-terminal graph in* $\mathcal{G}_{n,m}$ *for* $q \to 1$, *then* $E(G) = \{rs, rt, st\} \cup \{v_iv_j | 1 \leq i \leq 3, 4 \leq j \leq \lfloor \frac{m-3}{3} \rfloor + 3\} \cup \{v_4v_5\}$.

In addition, when $3n-6 < m \leq \binom{n}{2}$, all non-target vertices in all graphs within $\mathcal{G}_{n,m}^2$ are adjacent to three target vertices, and the number of subgraphs induced by these non-target vertices that contain P_2 is consistently $m-3n+6$. According to Lemma 1, to find

the locally most reliable three-terminal graph for $3n-6 < m \leq \binom{n}{2}$ and $q \to 1$, we must find the structure within $\mathcal{G}^2_{n,m}$ such that the subgraph induced by $n-3$ non-target vertices with the maximum number of P_3. This graph class is denoted as $\mathcal{G}^3_{n,m}$. In fact, finding $\mathcal{G}^3_{n,m}$ can be translated into finding the all-terminal graph in $\mathcal{A}_{n-3,m-3n+6}$ with the maximum number of P_3.

Up to now, numerous studies have focused on the problem of maximizing the number of P_3 in $\mathcal{A}_{n,m}$. In 1999, Byer [16] characterized six types of graphs in $\mathcal{A}_{n,m}$ and proved that graphs with the most P_3 belong to this set. In 2009, Ábrego et al. [17] classified six types of graph divisions based on edge counts of graphs and established that the partition of graphs containing at most P_3 in $\mathcal{A}_{n,m}$ must belong to one of these six classifications. Consequently, by combining and extending the relevant results in [16,17], we conclude that $\mathcal{G}^3_{n,m}$ is composed of several graphs from among these six types, as stated in Lemma 2.

Lemma 2. *Let $0 < m \leq \binom{n}{2}$, and k, k', j, j' be the unique integers satisfying the following condition,*

$$m = \binom{k+1}{2} - j = \binom{n}{2} - \binom{k'+1}{2} + j', 1 \leq j \leq k, 1 \leq j' \leq k'.$$

If H is a graph with the maximum number $N_H(P_3)$ in $\mathcal{A}_{n,m}$, then H belongs to at least one of the following six classes.

$S^1_{n,m} = K_{n-k'}((n-k') \vee j')((n-k'-1) \vee (k'-j'))$*-Quasi-Star graph;*
$S^2_{n,m} = K_{n-k'}((n-k') \vee (k'-1))((n-2k'+j') \vee 1)$, *where* $k'+1 \leq 2k'-j'-1 \leq n-1$;
$S^3_{n,m} = K_{n-k'+2}((n-k'-1) \vee (k'-2))$, *where* $j' = 3, n \geq 10$;
$C^1_{n,m} = K_k((k-j) \vee 1) + (n-k-1) \cdot K_1$*-Quasi-Complete graph;*
$C^2_{n,m} = K_k(1 \vee (k-j)) + (n-2k+j) \cdot K_1$, *where* $k+1 \leq 2k-j-1 \leq n-1$;
$C^3_{n,m} = K_{k-2}((k-2) \vee 3) + (n-k-1) \cdot K_1$, *where* $j = 3, n \geq 10$.
In particular,

(1) *when* $0 \leq m \leq n-1$,
   if $m = 3$, *then* $H \cong K_3 + (n-3) \cdot K_1$ *or* $H \cong K_1(1 \vee 3) + (n-4) \cdot K_1$;
   if $m \neq 3$, *then* $H \cong K_1(1 \vee m) + (n-m-1) \cdot K_1$.

(2) *when* $\binom{n-1}{2} \leq m \leq \binom{n}{2}$,
   if $m = \binom{n}{2} - 3$, *then* $H \cong K_{n-3}((n-3) \vee 3)$ *or* $H \cong K_{n-1}((n-4) \vee 1)$;
   if $m \neq \binom{n}{2} - 3$, *then* $H \cong K_{n-1}((n-\binom{n}{2}+m-1) \vee 1)$.

Lemma 3. *Let $n \geq 5$, $3n-6 < m \leq \binom{n}{2}$ and $\widehat{m} = m-3n+6$, if G is the locally most reliable three-terminal graph for $q \to 1$, then $G \in G^3_{n,m}$, and $\widehat{G}$ must belong to $\{S^1_{n-3,\widehat{m}}, S^2_{n-3,\widehat{m}}, S^3_{n-3,\widehat{m}}, C^1_{n-3,\widehat{m}}, C^2_{n-3,\widehat{m}}, C^3_{n-3,\widehat{m}}\}$, which are shown as Figure 5.*
In particular,

(1) *when* $3n-6 < m \leq 4n-10$,
   if $m = 3n-3$, *then* $G \cong K_3 \vee (K_3 + (n-6) \cdot K_1)$ *or* $G \cong K_3 \vee (K_1(1 \vee 3) + (n-7) \cdot K_1)$;
   if $m \neq 3n-3$, *then* $G \cong K_3 \vee (K_1(1 \vee \widehat{m}) + (n-\widehat{m}-4) \cdot K_1)$.

(2) *when* $\binom{n}{2} - n + 4 \leq m \leq \binom{n}{2}$,
   if $m = \binom{n}{2} - 3$, *then* $G \cong K_3 \vee (K_{n-6}((n-6) \vee 3))$ *or* $G \cong K_3 \vee (K_{n-4}((n-7) \vee 1))$;
   if $m \neq \binom{n}{2} - 3$, *then* $G \cong K_3 \vee (K_{n-4}((n-\binom{n-3}{2}+\widehat{m}-4) \vee 1))$.

Proof. Let G be the locally most reliable three-terminal graph for $q \to 1$, then by Theorem 4, we have $G \in \mathcal{G}^3_{n,m}$ and $\widehat{G} \in \mathcal{A}_{n-3,\widehat{m}}$.

Since $\widehat{G}$ has the maximum number P_3 in $\mathcal{A}_{n-3,\widehat{m}}$, then by Lemma 2, we see that $\widehat{G}$ belongs to $\{S^1_{n-3,\widehat{m}}, S^2_{n-3,\widehat{m}}, S^3_{n-3,\widehat{m}}, C^1_{n-3,\widehat{m}}, C^2_{n-3,\widehat{m}}, C^3_{n-3,\widehat{m}}\}$.

In particular, if $3n-6 < m \leq 4n-10$, then $0 < \widehat{m} \leq n-4$, and if $\binom{n}{2} - n + 4 \leq m \leq \binom{n}{2}$, then $\binom{n-4}{2} \leq \widehat{m} \leq \binom{n-3}{2}$. Therefore, by Lemma 2, the corresponding conclusion in this lemma holds.

In conclusion, we complete the proof. □

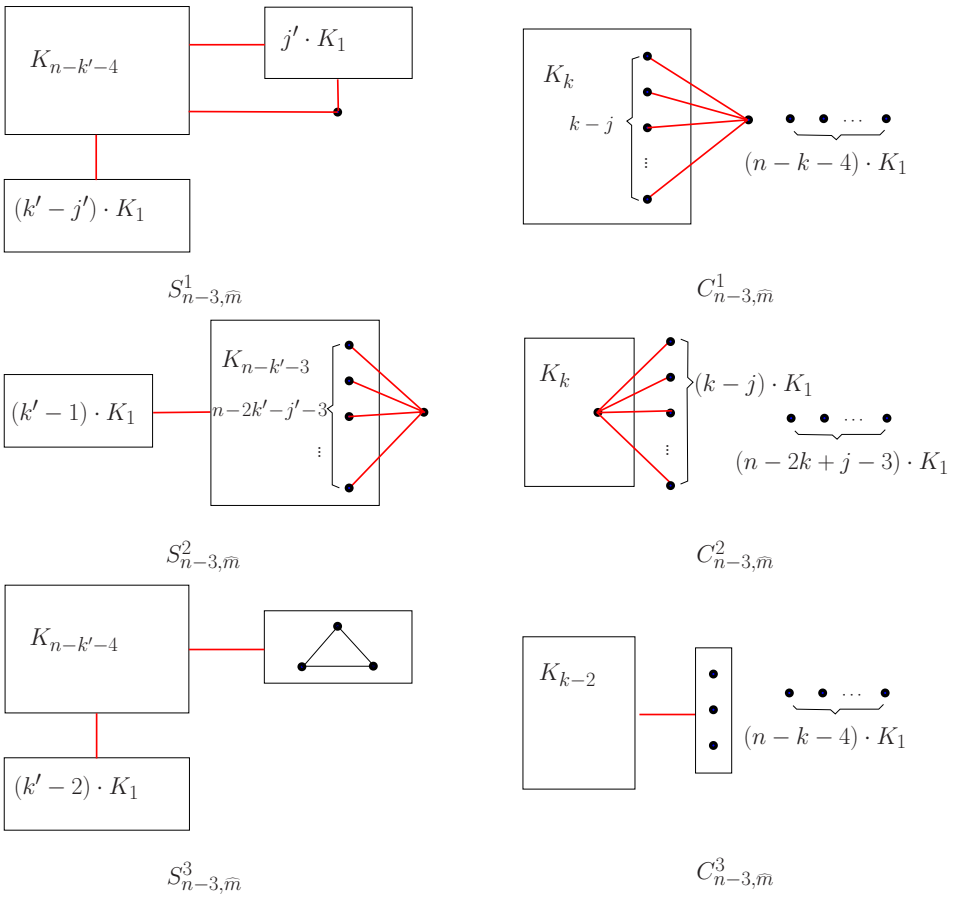


Figure 5. Six types of special graphs in $\mathcal{A}_{n-3,\widehat{m}}$. (A solid red line between two disjoint graphs indicates that all vertices in one graph are adjacent to all vertices in the other graph).

Lemma 3 gives locally most reliable three-terminal graphs for $3n-6 < m \leq 4n-10$ ($m \neq 3n-3$) and $\binom{n}{2} - n + 4 \leq m \leq \binom{n}{2}$ ($m \neq \binom{n}{2} - 3$), and the conclusions agree with the result of Theorem 3.7 in [11] and the result of Theorem 1 in [12], respectively.

Moreover, for $m = 3n-3$ and $m = \binom{n}{2} - 3$, Lemma 3 restricts the locally most reliable three-terminal graphs to two classes. Furthermore, in [11], $K_3 \vee (K_1(1 \vee 3) + (n-7) \cdot K_1)$ is the locally most reliable three-terminal graphs for $m = 3n-3$ and in [12], $K_3 \vee (K_{n-6}((n-6) \vee 3))$ is the locally most reliable three-terminal graphs for $m = \binom{n}{2} - 3$.

4. Flowchart for Searching Locally Most Reliable Structures with High Edge Failure Probability

In the preceding section, we deduced three theorems related to comparative judgments of local reliability in the limiting case where the edge failure probability tends towards 1 (i.e., as q approaches 1).

Based on these judgment criteria, this section proposes a search method for identifying locally most reliable three-terminal graphs with a high probability of edge failure, as detailed in the following flowchart, which is shown as Figure 6.

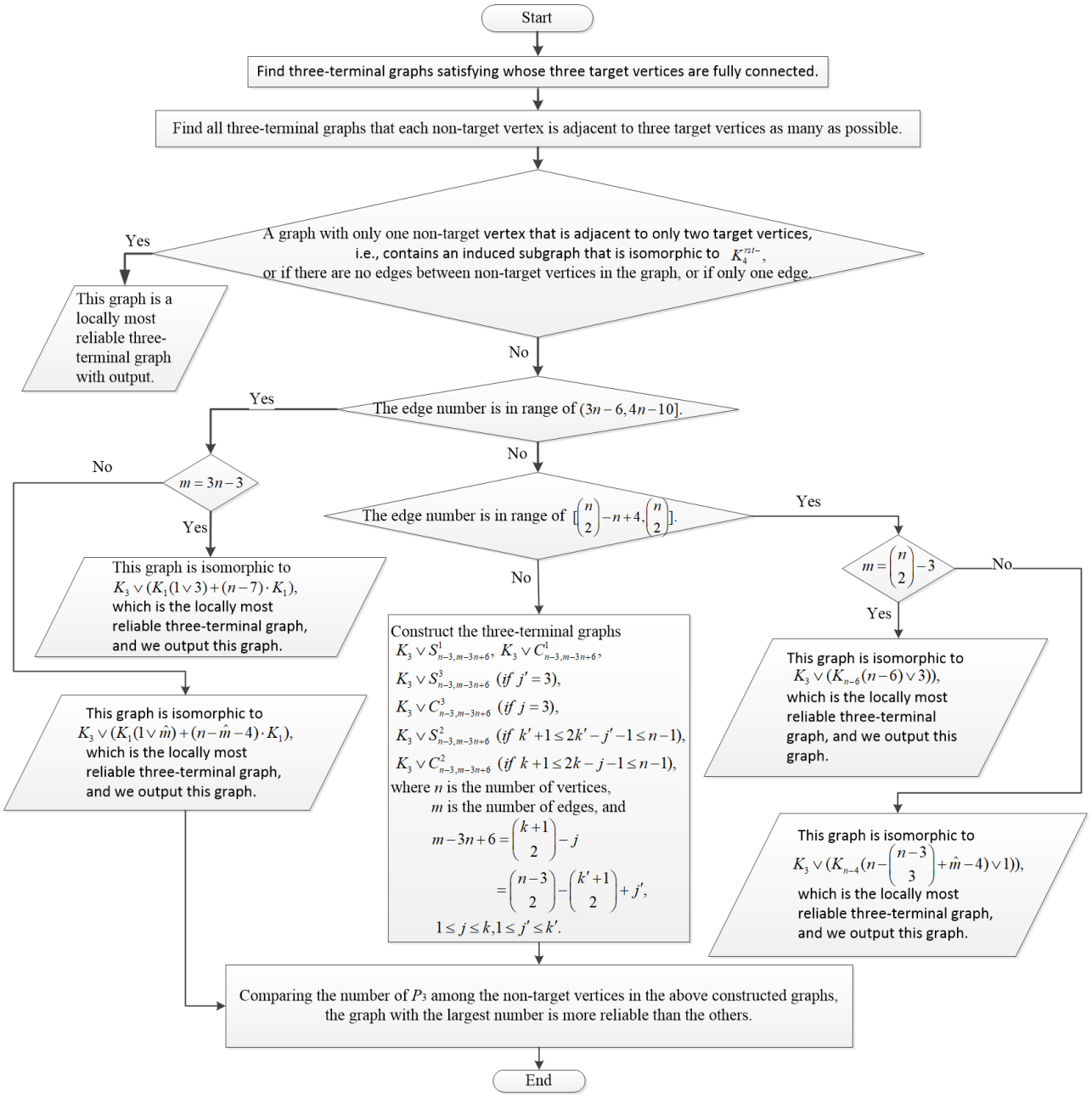


Figure 6. Flowchart for searching locally most reliable three-terminal graphs under high edge failure probability.

5. Conclusions

The focus of this study is to provide three local reliability comparison criteria for $q \to 1$ by comparing the reliability polynomial coefficients of the three-terminal network in a particular order and maximizing these coefficients in turn. As a result, we identify a number of properties satisfied by locally more reliable graphs when the probability of edge failure is high.

We characterize the locally more reliable three-terminal graph step by step according to three criteria. First, connect edges between target vertices, then connect edges between target and non-target vertices with each non-target vertex joining as many target vertices as possible, and finally, connect edges between non-target vertices to obtain as many P_3 subgraphs as possible. With these features, we characterize the locally most reliable three-terminal graph in the range $5 \le m \le 4n-10$ except when $m = 3n-3$, and the results

are consistent with the corresponding results in [11]. We also characterize the locally most reliable three-terminal graph in the range $\binom{n}{2} - n + 4 \leq m \leq \binom{n}{2}$ except for $m = \binom{n}{2} - 3$, and the results agree with the corresponding results in [12]. For other ranges of three-terminal graph families, we have greatly restricted the search for locally optimal structures to six classes of graphs for comparison, thus significantly reducing the complexity of this study. It is undoubtedly a worthwhile and interesting research topic to find the locally most reliable three-terminal graph among these six classes of graphs.

Furthermore, based on three comparison criteria of $q \to 1$, we propose a method to search for the most locally reliable three-terminal structure in a flowchat. By applying this method, we can design highly reliable graphs with three target vertices under the condition that edges fail with high probability. This result not only enriches the research in the field of graph theory, but also provides a theoretical basis for fault repair strategies in related networks.

After research, we found that the construction of the locally most reliable three-terminal graph was more complex and the classification scenarios were more diverse than the two-terminal graph (as identified in [11]) under high edge failure probability conditions. However, the methods for constructing the locally optimal structures of these two networks shared commonalities. Therefore, we speculate that such construction methods may be extendable to the analysis of locally most reliable k-terminal graphs ($k \geq 4$), which would be an interesting direction of research.

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References

1. Brown, J.; DeGagné, C.D.C. Roots of two-terminal reliability polynomials. *Networks* **2021**, *78*, 153–163. [CrossRef]
2. Gong, S.C.; Lin, L. On the construction of locally most reliable two-terminal graphs. *Discret. Appl. Math.* **2024**, *356*, 393–402. [CrossRef]
3. Da, G.F.; Zhang, X.; He, Z.W.; Ding, W.Y. Estimating the all-terminal signatures for networks by using deep neural network. *Reliab. Eng. Syst. Saf.* **2025**, *253*, 110496. [CrossRef]
4. Brown, J.I.; Cox, D. Nonexistence of optimal graphs for all terminal reliability. *Networks* **2014**, *63*, 146–153. [CrossRef]
5. Brown, J.; Mol, L. On the roots of all-terminal reliability polynomials. *Discret. Math.* **2017**, *340*, 1287–1299. [CrossRef]
6. Liang, J.; Zhao, H.X.; Xie, S. A Method for calculating the reliability of 2-Separable networks and its applications. *Axioms* **2024**, *7*, 459–470. [CrossRef]
7. Niu, Y.F.; Wang, Y.H.; Xu, X.Z. New decomposition algorithm for computing two-terminal network reliability. *Comput. Eng. Appl.* **2011**, *47*, 79–82.
8. Archer, K.; Graves, C.; Milan, D. Classes of uniformly most reliable graphs for all-terminal reliability. *Discret. Appl. Math.* **2019**, *267*, 12–29. [CrossRef]
9. Bertrand, H.; Goff, O.; Graves, C.; Sun, M. On uniformly most reliable two-terminal graphs. *Networks* **2018**, *72*, 200–216. [CrossRef]
10. Xie, S.; Zhao, H.X.; Yin, J. Nonexistence of uniformly most reliable two-terminal graphs. *Theor. Comput. Sci.* **2021**, *892*, 279–288 [CrossRef]
11. Xie, S.; Zhao, H.X.; Dai, L.M.; Yin, J. On locally most reliable three-terminal graphs of sparse graphs. *AIMS Math.* **2021**, *6*, 7518–7531. [CrossRef]
12. Xie, S.; Zhao, H.X.; Yin, J. Uniformly most reliable three-terminal Graph of dense graphs. *Math. Probl. Eng.* **2021**, *2021*, 1–10. [CrossRef]

13. Simon, F. Splitting the *K*-Terminal Reliability. *arXiv* **2011**, arXiv:1104.3444.
14. Bondy, J.A.; Murty, U.S.R. *Graph Theory*; Springer: Berlin/Heidelberg, Germany, 2008.
15. Ball, M.O. Complexity of network reliability computation. *Networks* **1980**, *10*, 153–165. [CrossRef]
16. Byer, O.D. Two path extremal graphs and an application to a Ramsey-type problem. *Discret. Math.* **1999**, *196*, 51–64. [CrossRef]
17. Ábrego, B.M.; Fernndez-Merchant, S.; Neubauer, M.G.; Watkins, W. Sum of squares of degrees in a graph. *J. Inequalities Pure Appl. Math.* **2009**, *10*, 1–34.

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