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Borel Summation in a Martingale-Type Collar

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Abstract: Extrapolation of the asymptotic series with a cost functional imposed on iterated Borel summation is considered. The cost functionals are designed to determine the optimal control parameter, the role of which is performed by the number of iterations, which could be considered as fractional real or non-negative integers. New cost functional is inspired by a martingale with a penalty term written to penalize the solution to optimization with fractional number of iterations for deviations of the expected value of sought quantity from the results of a discrete iterated Borel summation. The optimization technique employed in the paper is unique since the penalty by itself is expressed through the sought quantity, such as critical amplitude dependent on the number of iterations. The solution to the extrapolation problem with the control parameter found by means of optimization with a new cost functional is accurate, robust and uniquely defined for a variety of extrapolation problems.

Keywords: iterative Borel summation; cost functional optimization; control parameters; collar penalty; critical amplitude

MSC: 34L30; 34A45; 40G99; 41A21

1. Introduction to Extrapolation with Cost Functional

The problem of non-uniqueness of solutions arises often while applying some general-type highly nonlinear optimal conditions. In particular, it is typical to various extrapolations, when the solutions to a various complicated problems are given in the form of a truncated asymptotic expansion of the real function of real variable $f(x)$ as $x \to 0$ [1,2]

$$f_k(x) = \sum_{n=0}^{k} a_n x^n \ . \tag{1}$$

From the truncation (1) we need to construct the behavior of $f(x)$ at $x \to \infty$ under assumption that as $x \to \infty$ the function behaves as a power-law

$$f(x) \simeq B x^{\beta}, \tag{2}$$

with unknown B and β. Assume also that from some additional arguments, one can find the (critical) index β. Thus, by extrapolating (1) to the form (2), we have to find "only" the critical amplitude B [3].

The latter problem can be attacked by means of the iterative Borel summation

$$b_k(x,p) = \sum_{n=0}^{k} \frac{a_n}{(\Gamma(1+n))^p} \ x^n \ , \tag{3}$$

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defined following to [1,4] to generalize celebrated Borel summation [5–8]. Instead of the original series (1) we are going to work with the series $b_k(x,p)$ with the control parameter p. The parameter p can be real or discrete in two different realizations of the Borel iterative summation suggested in [4].

The summation procedure for a Borel transform $b_k(x,p)$ is realized analytically by means of self-similar iterated root approximants [4,9],

$$C_k^*(x,p) = A_0\left(\left((1+A_1x)^2 + A_2x^2\right)^{3/2} + \ldots + A_kx^k\right)^{\beta/k}, \tag{4}$$

with all parameters $A_i = A_i(p)$ defined in a closed form by the accuracy-through order procedure.

Strikingly, the large-variable behaviour of the function (4) is a power-law with the index β at infinity. The sought amplitude at infinity

$$B_k(p) = c_k(p)\,(\Gamma(1+\beta))^p\,, \tag{5}$$

is found analytically; and the marginal amplitude

$$c_k(p) = A_0\left(\left(A_1(p)^2 + A_2(p)\right)^{3/2} + \ldots + A_k(p)\right)^{\beta/k}$$

derives from $C_k^*(x,p)$. It is given analytically as well, dependent on the parameter p.

In order to find p one can consider local optimization conditions in application to the large-variable amplitude $B_k(p)$ and real control parameter p. By analogy to [10,11] we can write down the minimal difference condition,

$$B_k(p) - B_{k-1}(p) = 0, \tag{6}$$

and the minimal derivative condition,

$$\frac{dB_k(p)}{dp} = 0\,. \tag{7}$$

Obviously, there are at least two possible solutions to optimization. In fact, there could be more than two solutions. However, due to a strictly one-parameter nature of the iterated Borel summation, the solutions can be relatively easily enumerated. Solving separately both these equations, we can obtain several solutions for the control parameters, which are enumerated as p_k^j, with $j = 0, 1, 2, \ldots$. The solutions to the two optimizations could be considered all together and on equal footing. Then, we calculate their average and variance of all solutions. In such a manner the non-uniqueness is getting addressed.

In addition to finding all solutions with a subsequent averaging, there is an alternative approach. As suggested in [10,12–15], a penalty term should be introduced into optimization problems to avoid a non-uniqueness of solutions altogether. In the latter approach, one can adapt classical Tikhonov regularization schemes to the extrapolation problems [10,14,15]. In such an approach, it is necessary to construct a cost functional to be minimized directly with respect to the control parameter considered as a variable. In addition, a group of lasso measures [12,13], can be adapted to the extrapolation problems [10].

Firstly, following [10], we introduce a cost functional $F(p, p_0)$ which combines the minimal-difference and minimal-derivative conditions, complemented with addition of a ridge-type penalty term. The cost functional $F(p, p_0)$ is written down by analogy to [10].

The penalty is introduced in such a way that it selects the unique solution which deviates minimally from the solutions found at different p_0, so that

$$F(p, p_0) = \lambda |B_k(p) - B_{k-1}(p)|^2 + (1-\lambda)\left|\frac{dB_{k-1}(p)}{dp}\right|^2 + \frac{1}{2}(p-p_0)^2 \,, \tag{8}$$

with $\lambda = \frac{1}{2}$. We set above in the expression (8), $p_0 = 0$ (Penalty$_0$, or *Penal*$_0$); $p_0 = 1/2$ (Penalty$_{1/2}$, or Penal$_{1/2}$); $p_0 = 1$ (Penalty$_1$, or Penal$_1$). Penalty$_0$ employs non-transformed iterated roots as a reference state, while Penalty$_1$ employs a pure Borel-transform as reference state.

Computations are performed variationally, for the Borel solutions with amplitudes dependent on p. The optimal control parameter minimizes the cost functional (8), so that

$$F\left(p_{opt}, p_0\right) = \min_p F(p, p_0) \,. \tag{9}$$

Secondly, we consider for sake of completeness the "No-Penalty" functional

$$F_0(p) = \lambda |B_k(p) - B_{k-1}(p)|^2 + (1-\lambda)\left|\frac{dB_{k-1}(p)}{dp}\right|^2. \tag{10}$$

Thirdly, in the discrete case of a non-negative integer p, we consider for extrapolation the simplest sequences introduced in [4],

$$b_k^* = \frac{B_k(1) + B_k(2)}{2} \,, \tag{11}$$

with $k = 1, 2, 3, \ldots$. It could be easily extended to larger integers p if necessary. Formula (11) will be called for brevity "Discr.".

Fourthly, let us demand that the results of a continuous iterative summation with arbitrary real p should deviate minimally from the results (11) of the discrete iterations. Technically such objective is achieved through adding a corresponding penalty term to $F_0(p)$ which takes such deviations into account, so that a new cost functional

$$F_C(p) = \lambda |B_k(p) - B_{k-1}(p)|^2 + (1-\lambda)\left|\frac{dB_{k-1}(p)}{dp}\right|^2 + \frac{1}{2}(B_k(p) - b_k^*)^2 \,, \tag{12}$$

is introduced. The cost-functional (12) will be called for brevity "Collar".

The optimal control parameter minimizes the cost functional (12), so that

$$F_C\left(p_{opt}\right) = \min_p F_C(p) \,. \tag{13}$$

The penalty term

$$H(p) = \frac{1}{2}(B_k(p) - b_k^*)^2, \tag{14}$$

included into the cost functional (12), is inspired by a martingale collar from the theory of random processes [16]. For martingales the expected values at future times could not be better than the last actually available value of the time series. Imposing martingale conditions on random processes significantly restricts the class of random processes considered in finance.

Simple methods respecting martingale constraints are even considered by respectful Makridakis Competitions as a naive benchmark for actual time series extrapolations; i.e., the last value in time known from the time series is used for extrapolation to the future.

Surprisingly, it is not easy to outperform the method of extrapolation based on such a naive assumption of martingale.

In our case we suggest an analog of such naive extrapolation with the penalty for any deviations from the value expected from the iterated roots constructed for original non-transformed truncation (1) as $p = 0$,

$$b_k^n = B_k(0). \tag{15}$$

To find a naive extrapolation it suffices to replace b_k^* in the cost functional (12) with b_k^n to arrive at a naive extrapolation with such defined cost functional $F^{naive}(p)$.

In our opinion it is desirable to have some general restrictions on the expected results of extrapolations considered in the current paper. Inspired by the martingale condition, we require the expected value of sought critical amplitude to be restricted by the results of a discrete Borel summation. To such an end the penalty term is written making the solution to optimization contingent on penalties for the deviations of the expected continuous value $B_k(p)$ from the result of a discrete iterated Borel summation given above as b_k^*. Indeed, the continuous approach to resummation is more general but more assumptive. While the discrete approach is more natural but more restrictive. In such a manner the discrete and continuous versions of iterations are getting reconciled.

The meaning of collaring achieved by imposing the novel penalty term becomes clear from the context, since the optimum is getting bounded from below and above by two competing terms arising from continuous no-penalty and penalty contributions.

Altogether, twenty different methods were advanced in the paper [10] to tackle the problem of non-uniqueness. Such a number originates from application of Borel–Leroy, Mittag-Leffler, Fractional Riemann–Liouville and Fractional Integral Borel summations in conjunction with various cost functionals. Clearly, it is highly desirable to advance a single method with a single penalty function which would allow for a unique solution to the extrapolation problem while also maintaining a good accuracy competitive with the results of [10].

Thus, the main objective of the current paper is to improve extrapolation accuracy and achieve uniqueness of the results using a new cost functional imposed on a single method of resummation.

2. Examples

When p_{opt} is found from any cost functional introduced above, the sought sequence of amplitudes

$$B_k^* = B_k(p_{opt}), \tag{16}$$

arises. Typically, when the sequence behaves monotonously the last value of the sequence with largest possible order k is chosen as the final answer. But sometimes monotonicity persists only to some smaller order. In such a case, the results calculated for such a number are going to be selected as the final answer.

We consider below various physical examples. The results are organized into tables as follows. Firstly, the simple averaging is applied to solutions of the optimization problems (6) and (7). Secondly, the cost functionals are considered without any penalty term. Thirdly, the ridge functional (8) is considered for three different values of the parameter p_0. Fourthly, the cost functional (12) is applied. Fifthly, the results of a discrete iterative Borel summation (11) are presented.

Analysis of minimal-difference and minimal derivative conditions gives an idea about the range where the solution could be located and the range of parameters where to look for it. Sometimes, it is useful to restrict the region of parameters to positive or negative half-axis, depending on where the solutions to minimal-difference and minimal derivative conditions are found.

As suggested in [4], when $\beta = -1, -2, ...$, we will work with inverse physical quantities, while taking an inverse again to return to the sought quantities.

2.1. Schwinger Model: Critical Amplitude

The ground-state energy E of the Schwinger model can be expanded in terms of the inverse coupling constant x [17–23], so that

$$E(x) \simeq 0.5642 - 0.219x + 0.1907x^2, \qquad x \to 0\,. \tag{17}$$

As $x \to \infty$ the ground-state energy behaves as a power-law of the type (2) with exact $B = 0.6418$, $\beta = -1/3$.

The results of calculations for the critical amplitude using different methods are shown in Table 1. The control parameters are found in the second order $(k = 2)$. However, the results for amplitude B are found in the third order $(k = 3)$, assuming that $a_3 = 0$.

Table 1. Amplitude B for Schwinger model. Second rrder $(k = 2)$.

Average	No Penalty	$\mathbf{Penal_0}$	$\mathbf{Penal_{1/2}}$	$\mathbf{Penal_1}$	Collar	Discr.
0.55552 ± 0.00838	0.547299	0.552069	0.636999	0.654163	0.655175	0.775767

Optimization according to (12) gives

$$B_1^*(p_{opt}) = 0.724306, \;\; B_2^*(p_{opt}) = 0.67106, \;\; B_3^*(p_{opt}) = 0.655175,$$

corresponding to $p_{opt} = 0.991466$. Therefore, we choose from the whole sequence the results the following estimate for the critical amplitude, $B \approx B_3^*(p_{opt}) \approx 0.6552$.

The behaviour of various cost functionals is illustrated in Figure 1. Complete cost functional, no-penalty cost functional and the collar-penalty terms $H(p)$ are presented. The penalty term plays a crucial role, since it contributes to formation of the solution at positive p. The latter is not present at all among all negative solutions to optimization conditions. Extrapolation with $F^{naive}(p)$ gives $B \approx 0.547315$.

It also makes sense to compare the results presented in Table 1 with the results obtained in [10], along the same lines and with similar goals in mind, but achieved by means of different summation techniques with "hidden" variables. Rather good result, $B \approx 0.64785$, comes from the ridge cost functional method developed for Borel–Leroy summation. Besides, all ridge cost functional methods are superior to the other optimization methods applied in [10].

2.2. Anomalous Dimension

The cusp anomalous dimension $\Omega(g)$ of a light-like Wilson loop in the $n = 4$ supersymmetric Yang-Mills theory, depends only on the coupling g [24]. The problem can be written in terms of the function $f(x) = \frac{\Omega(x)}{x}$, where $x = g^2$. The following weak-coupling expansion for $f(x)$ is known,

$$f(x) \simeq 4 - 13.1595x + 95.2444x^2 - 937.431x^3, \quad x \to 0.$$

In the strong-coupling limit $x \to \infty$, $f(x)$ takes the form of a power-law (2), with $B = 2$ and $\beta = -1/2$.

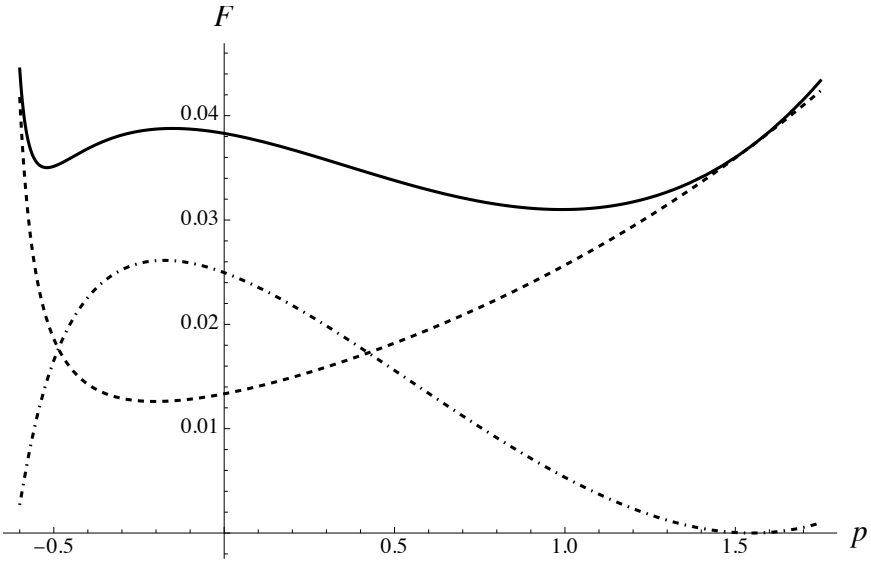


Figure 1. Amplitude for Schwinger model. Complete cost functional (12) (solid), no-penalty cost functional (10) (dashed); and the collar-penalty (14) (dotdashed), are compared dependent on the number of iterations p.

The results of calculations for the critical amplitude by different methods are shown in Table 2. Extrapolation with $F^{naive}(p)$ gives $B \approx 1.96677$.

Optimization according to (12) gives

$$B_1^*(p_{opt}) = 1.83868, \;\; B_2^*(p_{opt}) = 2.01166, \;\; B_3^*(p_{opt}) = 2.04874,$$

corresponding to $p_{opt} = 0.991466$. Therefore, we choose from the whole sequence the resulting estimate, $B \approx B_3^*(p_{opt}) \approx 2.0487$.

Table 2. Amplitude B for anomalous dimension ($k = 3$).

Average	No Penalty	$\mathbf{Penal_0}$	$\mathbf{Penal_{1/2}}$	$\mathbf{Penal_1}$	Collar	Discr.
2.09614 ± 0.07409	1.9851	1.97238	2.0084	2.06978	2.04874	3.12385

The behaviour of various cost functionals is illustrated in Figure 2. Complete cost functional (solid), no-penalty cost functional (dashed); and the collar-penalty terms $H(p)$ (dotdashed) are presented. In the case of cusp anomalous dimensions the optimum is uniquely defined. One can see a compromise emerging here as the result of interplay between the no-penalty and penalty terms. Yet, imposition of the collar-penalty only slightly shifts the position of optimum. Such an effect is to be expected since the no-penalty cost functional works already works quite well. As to the results of the paper [10], one can see that Borel–Leroy summation gives $B \approx 2.0118$. The best result, $B \approx 1.9938$, comes from the ridge cost functional method applied together with Mittag-Leffler summation.

2.3. Bose Temperature Shift

The ideal Bose gas is unstable below the condensation temperature [25]. Atomic interactions stabilize the system and cause shift $\Delta T_c \equiv T_c - T_0$, of the Bose–Einstein condensation temperature T_c of a non-ideal Bose system compared with the Bose–Einstein condensation temperature T_0 of the ideal uniform Bose gas [26]. The shift is quantified

using the parameter c_1, so that $\frac{\Delta T_c}{T_0} \simeq c_1\gamma$, for $\gamma \to 0$, where γ is a gas parameter dependent on atomic scattering length and on gas density.

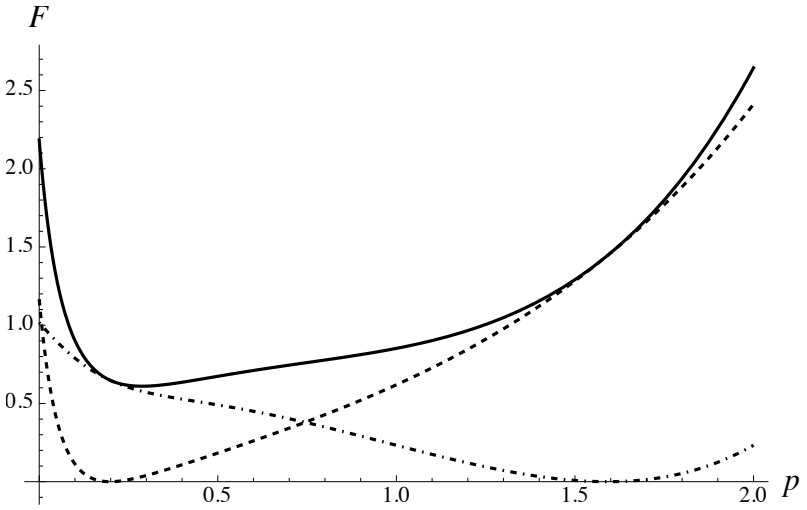


Figure 2. Cusp anomalous dimension. Complete cost functional (12) (solid), no-penalty cost functional (10) (dashed); and the collar-penalty (14) (dotdashed), are compared dependent on the number of iterations p.

In order to calculate c_1 it was suggested to calculate an auxiliary function $c_1(g)$ [27–29]. Eventually, c_1 corresponds to the limit

$$B \equiv c_1 = \lim_{g\to\infty} c_1(g) \tag{18}$$

of the auxiliary function. The latter function can be expanded as $g \to 0$, so that

$$c_1(g) \simeq 0.223286g - 0.0661032g^2 + 0.026446g^3 - 0.0129177g^4 + 0.00729073g^5, \tag{19}$$

and the index $\beta = -1$ here, as well as in the two analogous examples presented below. Monte Carlo simulations produce $B = 1.3 \pm 0.05$ [30–32].

The results of calculations by different methods are shown in Table 3. The original iterated roots producing complex numbers as $k = 4$. Using the value of $b_3^n = B_3(0)$, we find that extrapolation with $F^{naive}(p)$ gives $B \approx 1.21311$.

Optimization according to (12) gives

$$B_1^*(p_{opt}) = 0.754224,\ B_2^*(p_{opt}) = 0.996264,\ B_3^*(p_{opt}) = 1.23226,\ B_4^*(p_{opt}) = 1.26857,$$

corresponding to $p_{opt} = 0.719219$. Therefore, we choose from the whole sequence the results the following estimate, $B \approx B_4^*(p_{opt}) \approx 1.269$, corresponding to the last member of the monotonously increasing sequence.

Table 3. Amplitude B for Bose temperature shift. Fourth order ($k = 4$).

Average	No Penalty	$\mathbf{Penal_0}$	$\mathbf{Penal_{1/2}}$	$\mathbf{Penal_1}$	Collar	Discr.
1.30245 ± 0.224497	1.26375	1.15251	1.21322	1.38294	1.26857	1.30288

The behaviour of various cost functionals is illustrated in Figure 3. Complete cost functional (solid), no-penalty cost functional (dashed); and the collar-penalty terms $H(p)$ (dotdashed) are presented. In the case of Bose temperature shift the optimum is uniquely defined. One can see a compromise emerging here between the no-penalty and penalty

terms. In this case imposition of the collar-penalty only slightly changes the position of optimum. Such a small effect is to be expected since the no-penalty cost functional works already quite well.

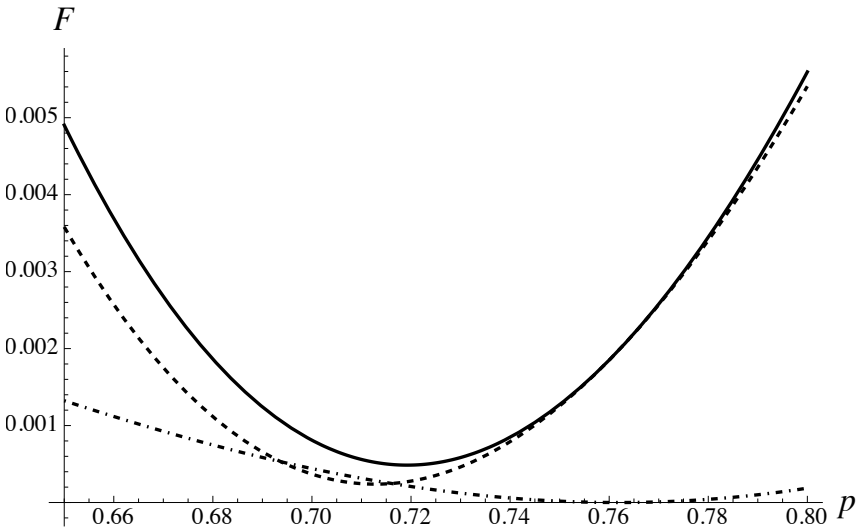


Figure 3. Bose temperature shift. Complete cost-functional (12) (solid), no-penalty cost-functional (10) (dashed); and the collar-penalty (14) (dotdashed), are compared dependent on the number of iterations p.

The approaches developed in the paper [10], also work very well. In particular, Mittag-Leffler summation gives $B \approx 1.3397$, and Borel–Leroy summation gives $B \approx 1.2868$. Fractional Riemann–Liouville summation brings $B \approx 1.2895$, while fractional integral summation results in $B \approx 1.2641$. The best result from various ridge cost functionals is $B \approx 1.244$, and slightly out of the expected range.

2.4. Three-Dimensional Harmonic Trap

Wave functions and energy of trapped Bose-condensed atoms in a spherically-symmetric harmonic trap are modelled by the stationary nonlinear Schrödinger equation [25]. The problem is reduced to studying only the radial part of the condensate wave function. The ground state energy E of the Bose-condensate in such a trap is approximated as $g \to 0$, by the fourth-order truncation

$$E(g) \simeq \frac{3}{2} + \frac{1}{2}g - \frac{3}{16}g^2 + \frac{9}{64}g^3 - \frac{35}{256}g^4. \tag{20}$$

where g plays the role of an effective coupling parameter. It takes trapping into account quantitatively. For very strong trapping $g \to \infty$, the energy behaves as the power-law (2), with the amplitude at infinity $B = \frac{5}{4}$, and the index $\beta = 2/5$ [25].

The results of calculations using different methods are shown in Table 4. Extrapolation with $F^{naive}(p)$ gives $B \approx 1.27936$.

Optimization according to (12) gives

$$B_1^*(p_{opt}) = 1.34529,\ B_2^*(p_{opt}) = 1.30808,\ B_3^*(p_{opt}) = 1.28897,\ B_4^*(p_{opt}) = 1.27932,$$

corresponding to $p_{opt} = 0.300336$. Therefore, we choose from the whole sequence the resultant numbers, the following estimate, $B \approx B_4^*(p_{opt}) \approx 1.2793$, corresponding to the last member of the given sequence.

Table 4. Amplitude B for three dimensional quantum trap ($k = 4$).

Average	No Penalty	$\mathbf{Penal_0}$	$\mathbf{Penal_{1/2}}$	$\mathbf{Penal_1}$	Collar	Discr.
1.28248 ± 0.0027477	1.28202	1.2786	1.27996	1.28354	1.27932	1.25848

The behaviour of various cost functionals is illustrated in Figure 4. Complete cost functional, no-penalty cost functional and the collar-penalty $H(p)$ are presented. In the case of three-dimensional quantum harmonic trap the role of penalty is to make a choice among the two alternative solutions. The cost-penalty term does affect the choice of the solutions, but only marginally influences the results of optimization. Mittag-Leffler summation leads to $B \approx 1.282$, while ridge cost functionals give close results, $B \approx 1.285$ [10].

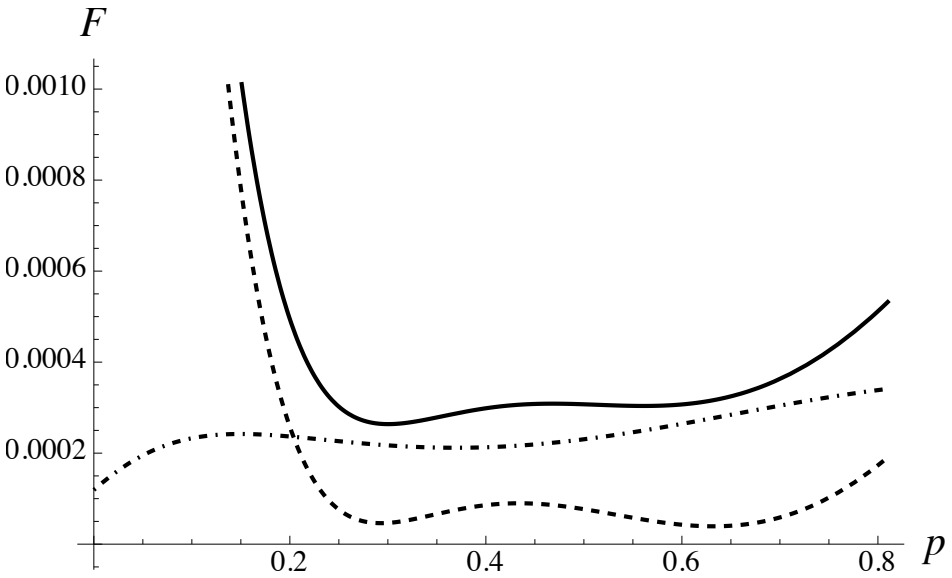


Figure 4. Three-dimensional quantum trap. Complete cost-functional (12) (solid), no-penalty cost-functional (10) (dashed); and the collar-penalty (14) (dotdashed), are compared dependent on the number of iterations p.

2.5. Three-Dimensional Random Polymer

The expansion factor $\mathrm{Y}(g)$ of a three-dimensional polymer can be represented as a truncated series in a single dimensionless interaction parameter g [33–35]. The expansion factor can be presented as the sixth-order truncation of the type of (1),

$$\mathrm{Y}_6(g) \simeq 1 + \frac{4}{3}g - 2.075385396g^2 + 6.296879676g^3 - 25.05725072g^4 + 116.134785g^5 - 594.71663g^6, \quad (21)$$

as $g \to 0$. The strong-coupling behaviour of the expansion factor, as $g \to \infty$, is a power-law of the type (2), with $B \approx 1.5309$, $\beta \approx 0.3544$ [34]. The results of calculations using different methods are shown in Table 5. Extrapolation with $F^{naive}(p)$ gives $B \approx 1.53637$.

Optimization according to (12) gives

$$B_1^*(p_{opt}) = 1.48206,\ B_2^*(p_{opt}) = 1.51241,\ B_3^*(p_{opt}) = 1.52351,$$

$$B_4^*(p_{opt}) = 1.528,\ B_5^*(p_{opt}) = 1.5303,\ B_6^*(p_{opt}) = 1.5316,$$

corresponding to $p_{opt} = 0.658003$. Therefore, we choose from the whole sequence the results the following estimate, $B \approx B_6^*(p_{opt}) \approx 1.5316$, corresponding to the last member of the monotonously increasing sequence.

Table 5. Amplitude B for three-dimensional random polymer ($k = 6$).

Average	No Penalty	$\mathbf{Penal_0}$	$\mathbf{Penal_{1/2}}$	$\mathbf{Penal_1}$	Collar	Discr.
1.5361 ± 0.00087	1.53239	1.53666	1.53302	1.52724	1.5316	1.49426

The behaviour of various cost functionals is illustrated in Figure 5. Complete cost functional, no-penalty cost functional and the collar-penalty terms are presented. In the case of three-dimensional polymer the role of collar-penalty amounts to selecting the optimum from the two alternatives, while also slightly correcting the results obtained from the no-penalty cost functional. In comparison, Borel–Leroy summation gives slightly better $B \approx 1.5323$, while ridge functional methods give at best case $B \approx 1.5283$ [10].

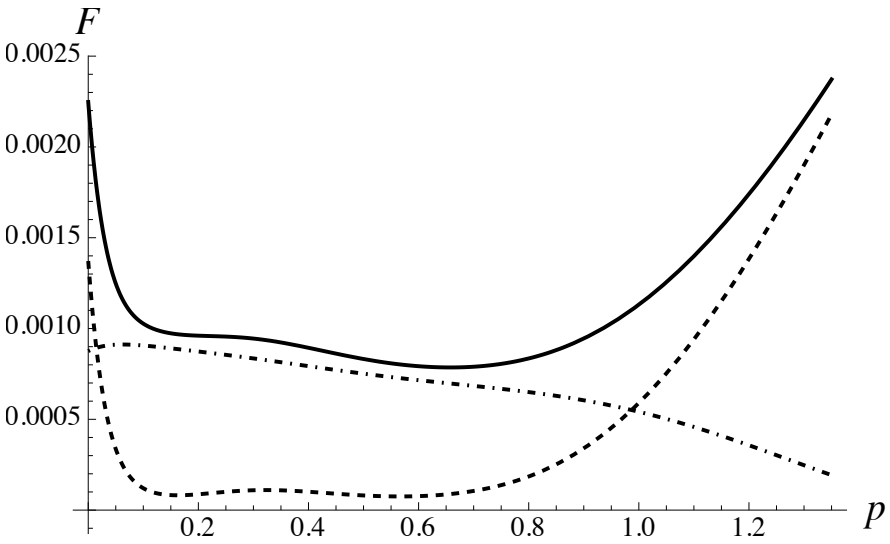


Figure 5. Three-dimensional polymer. Complete cost-functional (12) (solid), no-penalty cost-functional (10) (dashed); and the collar-penalty (14) (dotdashed), are compared dependent on the number of iterations p.

2.6. Pressure of a Two-Dimensional Membrane

For a two-dimensional membrane, its pressure can be calculated by perturbation theory. The expansion parameter is called the wall stiffness. It is quantified by the dimensionless parameter g [36]. By means of perturbation theory the following expansion for the pressure,

$$p(g) = \frac{\pi^2}{8g^2}\left(1 + \sum_{n=1}^{6} a_n g^n\right), \tag{22}$$

was found [36]; where

$$a_1 = \frac{1}{4}, \quad a_2 = \frac{1}{32}, \quad a_3 = 2.176347 \times 10^{-3},$$

$$a_4 = 0.552721 \times 10^{-4}, \quad a_5 = -0.721482 \times 10^{-5}, \quad a_6 = -1.777848 \times 10^{-6}.$$

At infinite g we arrive at the so-called rigid-wall limit. The Monte Carlo simulations of [37] give the best estimate for the amplitude,

$$B = p(\infty) = 0.0798 \pm 0.0003, \tag{23}$$

while $\beta = 2$, since the resummation is conventionally applied to $f_6(g) = g^2 p(g)$. The results of calculations by different methods are shown in Table 6. Extrapolation with $F^{naive}(p)$ gives $B \approx 0.0744091$.

Optimization according to (12) gives

$$B_1^*(p_{opt}) = 0.0120857,\ B_2^*(p_{opt}) = 0.0385531,\ B_3^*(p_{opt}) = 0.0608046,$$

$$B_4^*(p_{opt}) = 0.0731286,\ B_5^*(p_{opt}) = 0.0775616,\ B_6^*(p_{opt}) = 0.0668629,$$

corresponding to $p_{opt} = -0.673544$. Therefore, we choose from the whole sequence the following estimate, $B \approx B_5^*(p_{opt}) \approx 0.077562$, corresponding to the last member of the monotonously increasing subsequence.

Table 6. Amplitude B for membrane pressure ($k = 6$).

Average	No Penalty	$\mathbf{Penal_0}$	$\mathbf{Penal_{1/2}}$	$\mathbf{Penal_1}$	Collar	Discr.
0.0732382 ± 0.00001786	0.0735549	0.070683	0.0648976	0.0598286	0.0775616	0.0562949

The behaviour of various cost functionals is illustrated in Figure 6. Complete cost functional, no-penalty cost functional and the collar-penalty $H(p)$ terms are shown. In the case of two-dimensional membrane the final results for the critical amplitude are better than the results found with a no-penalty cost functional optimum. The uniquely defined optimum is found in between the no-penalty and collar-penalty contributions.

Fractional integral summation brings $B \approx 0.07665$, while fractional Riemann-Liouville summation gives $B \approx 0.8076$ [10]. Ridge cost functionals produce by far inferior results, with $B \approx 0.059829$. The results obtained from the cost functional with collar-penalty are also in agreement with the results obtained by various methods in our previous works [38].

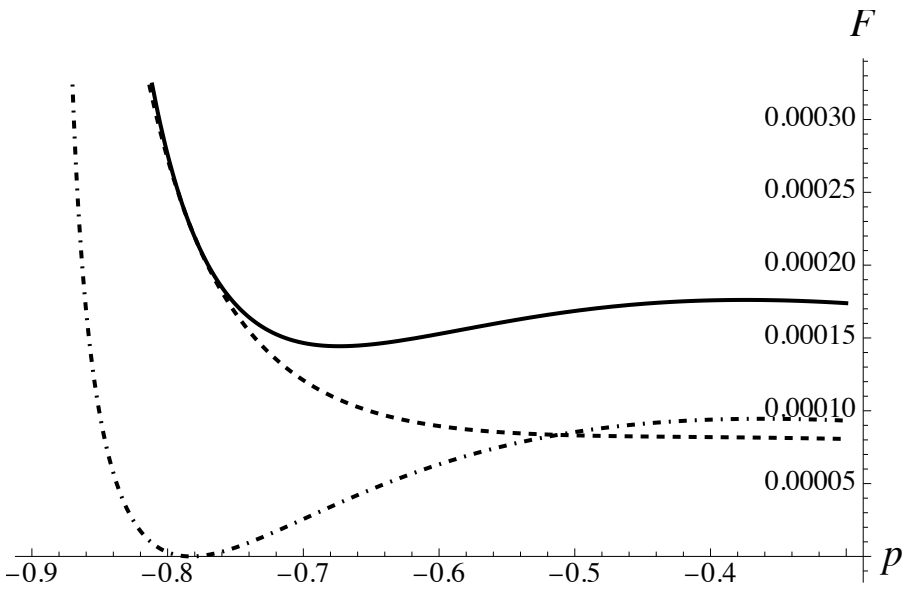


Figure 6. Membrane pressure. Complete cost-functional (12) (solid), no-penalty cost-functional (10) (dashed); and the collar-penalty (14) (dotdashed), are compared dependent on the number of iterations p.

2.7. One-Dimensional Bose Gas

In terms of the dimensionless coupling parameter g, the ground-state energy $E(g)$ of the one-dimensional Bose gas with contact interactions, was found perturbatively in [39,40]. In the variables

$$e(x) \equiv E\left(x^2\right)/x^2\,, \qquad g \equiv x^2\,,$$

the weak-coupling expansion for $e(x)$, as $x \to 0$ is

$$\begin{aligned} e(x) \simeq 1 - 0.4244131815783876\, x + 0.06534548302432888\, x^2 - \\ 0.001587699865505945\, x^3 - 0.00016846018782773904\, x^4 - \\ 0.00002086497335840174\, x^5 - 3.1632142185373668\, 10^{-6}\, x^6 - \\ 6.106860595675022\, 10^{-7}\, x^7 - 1.4840346726187777\, 10^{-7}\, x^8\, . \end{aligned} \tag{24}$$

In the limit of strong coupling, as $x \to \infty$, the exact result

$$E(\infty) = B = \frac{\pi^2}{3} \approx 3.289868, \tag{25}$$

is known, called the Tonks–Girardeau limit. Formally, we set $\beta = -2$, while dealing with the expression for $e(x)$ at $x \to \infty$.

The results of calculations by different techniques are shown in Table 7. Extrapolation with $F^{naive}(p)$ gives $B \approx 3.31924$.

Optimization according to (12) gives

$$B_1^*(p_{opt}) = 31.2996,\ B_2^*(p_{opt}) = 8.71224,\ B_3^*(p_{opt}) = 5.05396,\ B_4^*(p_{opt}) = 3.92106,$$

$$B_5^*(p_{opt}) = 3.43773,\ B_6^*(p_{opt}) = 3.27213,\ B_7^*(p_{opt}) = 3.32055,\ B_8^*(p_{opt}) = 3.74645,$$

corresponding to $p_{opt} = -0.4951568$. Therefore, we choose from the whole sequence the following estimate, $B \approx B_6^*(p_{opt}) \approx 3.2721$, corresponding to the last member of the monotonously decreasing subsequence.

Table 7. Amplitude B for one-dimensional Bose gas ($k = 8$).

Average	No Penalty	$\mathbf{Penal_0}$	$\mathbf{Penal_{1/2}}$	$\mathbf{Penal_1}$	Collar	Discr.
3.38948 ± 0.0003415	3.31759	3.52747	4.00746	4.50733	3.27213	4.90572

The behaviour of various cost functionals is illustrated in Figure 7. Complete cost functional, no-penalty cost functional and the collar-penalty terms are presented. In the case of one-dimensional Bose gas, the final result of optimization is uniquely defined. It is achieved due to a compromise achieved between the no-penalty and collar-penalty dependencies.

Fractional integral summation brings rather reasonable numbers, $B \approx 3.08574$; Mittag-Leffler summation gives $B \approx 3.51951$, while various ridge cost functional methods appear to be inferior, giving $B \approx 4.5$ [10].

2.8. Schwinger Model: Energy Gap

The energy gaps for the bound states for the Schwinger model are presented as asymptotic expansions at small $z = (1/ga)^4$. Here a is the lattice spacing and g is a coupling parameter and [17]. The energy gap $\Delta(z)$ for the scalar state at small z is expressed as a truncated series of the type of (1), with the coefficients a_n are rapidly increasing by absolute value. They are known up to the 13th order. For instance,

$$a_0 = 1, \quad a_1 = 6, \quad a_2 = -26, \quad a_3 = 190.6666666667,$$

$$a_4 = -1756.666666667, \quad a_5 = 18048.33650794.$$

The series derived in [17] for the Schwinger model are notoriously bad for resummation up to the point of their abandonment. However, they can still be very instructive.

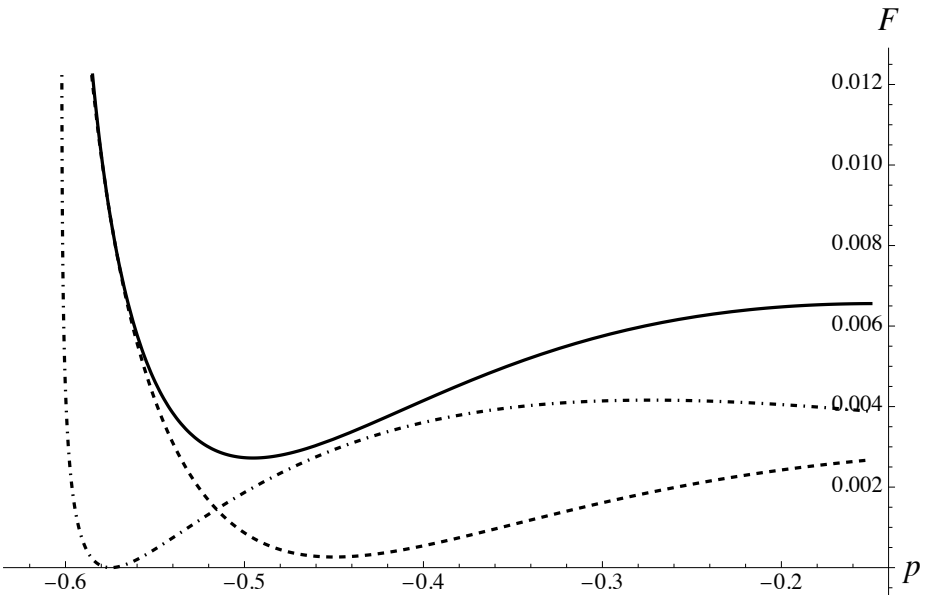


Figure 7. Bose gas. Complete cost functional (12) (solid), no-penalty cost functional (10) (dashed); and the collar-penalty (14) (dotdashed), are compared dependent on the number of iterations p.

In the continuous limit of the $z \to \infty$ the gap behaves as a power-law of the type of (2) [17], with $B = 1.1284$, $\beta = 1/4$.

The results of calculations are shown in Table 8. Extrapolation with $F^{naive}(p)$ gives only $B \approx 1.28933$.

Optimization according to (12) gives

$$B_1^*(p_{opt}) = 0.961103,\ B_2^*(p_{opt}) = 1.02057,\ B_3^*(p_{opt}) = 1.05411,\ B_4^*(p_{opt}) = 1.07414,$$

$$B_5^*(p_{opt}) = 1.08814,\ B_6^*(p_{opt}) = 1.09807,\ B_7^*(p_{opt}) = 1.10572,\ B_8^*(p_{opt}) = 1.11169,$$

$$B_9^*(p_{opt}) = 1.11654,\ B_{10}^*(p_{opt}) = 1.12051,\ B_{11}^*(p_{opt}) = 1.12387,$$

corresponding to $p_{opt} = 1.4352$. Therefore, we choose from the whole sequence the results, the following estimate for the critical amplitude, $B \approx B_{11}^*(p_{opt}) \approx 1.1239$, corresponding to the last member of the monotonously increasing sequence.

Table 8. Amplitude B for energy gap of Schwinger model ($k = 11$).

Average	No Penalty	$\mathbf{Penal_0}$	$\mathbf{Penal_{1/2}}$	$\mathbf{Penal_1}$	Collar	Discr.
1.27442 ± 0.001489	1.28974	1.24247	1.20479	1.16304	1.12387	1.11717

The behaviour of various cost functionals is illustrated in Figure 8. Complete cost functional, no-penalty cost functional and the collar-penalty terms are presented. In the case of the energy gap of the Schwinger model the penalty term plays a crucial role, since it contributes to formation of the solution at positive p. The latter solution is not present at all among all negative solutions to local optimization conditions and no-penalty cost functional.

As to the results obtained in [10], we observe that fractional integral summation brings much inferior result, $B \approx 1.3525$, while the ridge cost-functional method with Fractional Riemann–Liouville summation gives much better numbers, $B \approx 1.1576$.

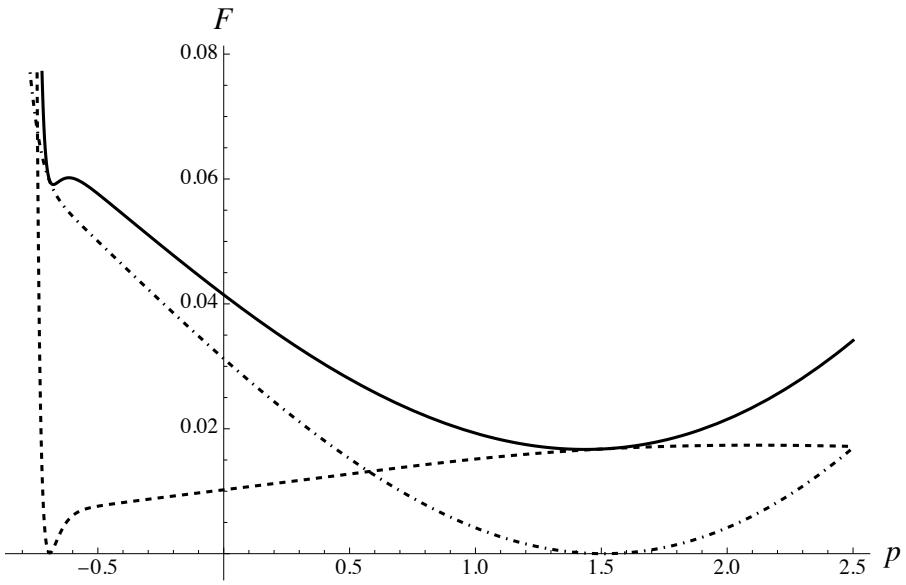


Figure 8. Energy gap. Complete cost functional (12) (solid), no-penalty cost functional (10) (dashed); and the collar-penalty (14) (dotdashed), are compared dependent on the number of iterations p.

2.9. Gaussian Polymer: Debye–Hukel Function

The Debye–Hukel function,

$$\Phi(x) = \frac{2}{x} - \frac{2(1-\exp(-x))}{x^2}, \tag{26}$$

describes the correlation function of the Gaussian polymer [35]. For small $x > 0$ it can be expanded into Taylor series with rapidly decaying by magnitude coefficients,

$$\Phi(x) = 1 - \frac{x}{3} + \frac{x^2}{12} - \frac{x^3}{60} + \frac{x^4}{360} + ..., \quad x \to 0\,. \tag{27}$$

As $x \to +\infty$ $\Phi(x)$ behaves as a power-law of the type (2), with $B = 2$, $\beta = -1$.

The results of calculations are shown in Table 9. Extrapolation with $F^{naive}(p)$ gives $B \approx 1.98779$. Optimization according to (12) gives

$$B_1^*(p_{opt}) = 3,\ B_2^*(p_{opt}) = 2.28961,\ B_3^*(p_{opt}) = 2.09317,\ B_4^*(p_{opt}) = 2.04466,$$

$$B_5^*(p_{opt}) = 2.02832,\ B_6^*(p_{opt}) = 2.02326,\ B_7^*(p_{opt}) = 2.01723,\ B_8^*(p_{opt}) = 2.01238,$$

$$B_9^*(p_{opt}) = 2.00981,\ B_{10}^*(p_{opt}) = 2.00858,\ B_{11}^*(p_{opt}) = 2.00729,$$

corresponding to $p_{opt} = -0.519639$. Therefore, we choose from the whole sequence the results the following estimate, $B \approx B_{11}^*(p_{opt}) \approx 2.0073$, corresponding to the last member of the monotonously decreasing sequence.

Table 9. Amplitude B for correlation function of Gaussian polymer ($k = 11$).

Average	No Penalty	$\mathbf{Penal_0}$	$\mathbf{Penal_{1/2}}$	$\mathbf{Penal_1}$	Collar	Discr.
1.95793 ± 0.044	1.9996	1.97146	1.99516	2.07892	2.00729	2.18329

The behaviour of various cost functionals is illustrated in Figure 9. Complete cost functional, no-penalty cost functional and the collar-penalty terms $H(p)$ are presented. In the case of Gaussian polymer, the collar-penalty term helps to select proper solution among two alternatives existing within the no-penalty functional. Collar-penalty just

slightly shifts the value of p_{opt} compared to the results found from the no-penalty cost functional. The latter result is to be expected, since the no-penalty cost functional optimization solution is already quite good. As shown in [10], Mittag-Leffler summation summation gives $B \approx 1.9715$, while ridge cost-functional applied in conjunction with the Mittag-Leffler summation brings $B \approx 1.9672$.

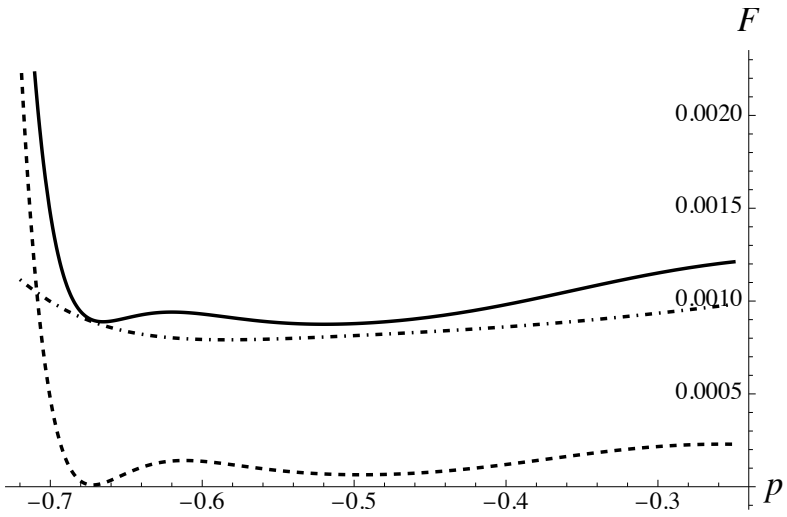


Figure 9. Gaussian polymer. Complete cost functional (12) (solid), no-penalty cost functional (10) (dashed); and the collar-penalty (14) (dotdashed), are compared dependent on the number of iterations p.

2.10. Quantum Quartic Oscillator

The anharmonic oscillator is described by the model Hamiltonian with non-linearity quantified through a positive coupling (anharmonicity) parameter g [41]. Perturbation theory for the ground-state energy $E(g)$ has the form of a long expansion of the type (1) [41], with the coefficients a_n rapidly growing in magnitude. For instance, the starting coefficients

$$a_0 = \frac{1}{2}\,, \qquad a_1 = \frac{3}{4}\,, \qquad a_2 = -\frac{21}{8} \qquad a_3 = \frac{333}{16}\,, \qquad a_4 = -\frac{30885}{128}\,.$$

are shown here.

The strong-coupling limit-case of $E(g)$ as $g \to \infty$, is given by the power-law of the type of (2), with the critical amplitude $B = 0.667986$, and index $\beta = 1/3$.

The results of calculations by different methods are shown in Table 10. The original iterated roots produced complex numbers in all orders higher than $k = 3$. Using the value of $b_3^n = B_3(0)$, we find that extrapolation with $F^{naive}(p)$ gives $B \approx 0.683744$.

Optimization according to (12) gives

$$B_1^*(p_{opt}) = 0.724777,\ B_2^*(p_{opt}) = 0.720537,\ B_3^*(p_{opt}) = 0.71,\ B_4^*(p_{opt}) = 0.698752,$$

$$B_5^*(p_{opt}) = 0.688949,\ B_6^*(p_{opt}) = 0.681999,\ B_7^*(p_{opt}) = 0.679277,\ B_8^*(p_{opt}) = 0.679949,$$

$$B_9^*(p_{opt}) = 0.682148,\ B_{10}^*(p_{opt}) = 0.683519,\ B_{11}^*(p_{opt}) = 0.683146,$$

corresponding to $p_{opt} = 1.14941$. Therefore, we choose from the whole sequence the following estimate, $B \approx B_7^*(p_{opt}) \approx 0.67928$, corresponding to the last member of the monotonously decreasing subsequence.

Table 10. Amplitude B for ground state energy of quartic oscillator ($k = 11$).

Average	No Penalty	$\mathbf{Penal_0}$	$\mathbf{Penal_{1/2}}$	$\mathbf{Penal_1}$	Collar	Discr.
0.684332 ± 0.01811	0.70825	0.679135	0.67847	0.676165	0.679277	0.685669

The behaviour of various cost functionals is illustrated in Figure 10. Complete cost functional, no-penalty cost functional and the collar-penalty terms $H(p)$ are presented. In the case of a quartic quantum oscillator the collar-penalty term helps to achieve better selection among several alternatives existing within the no-penalty functional.

Overall, the results for various cost functionals presented in the Table 10, appear to be rather close, and are not too sensitive to the type of penalty introduced. Fractional Riemann–Liouville summation gives reasonable $B \approx 0.674121$, while ridge cost functional methods give $B \approx 0.6771$ [10]. The best results, $B = 0.667638$, can be found by applying the minimal difference optimization in the 11th order of the fractional Borel summation discussed in [38]. In the 10th order, the estimate for critical amplitude, $B \approx 0.66936$, is already quite good, as demonstrated in [38].

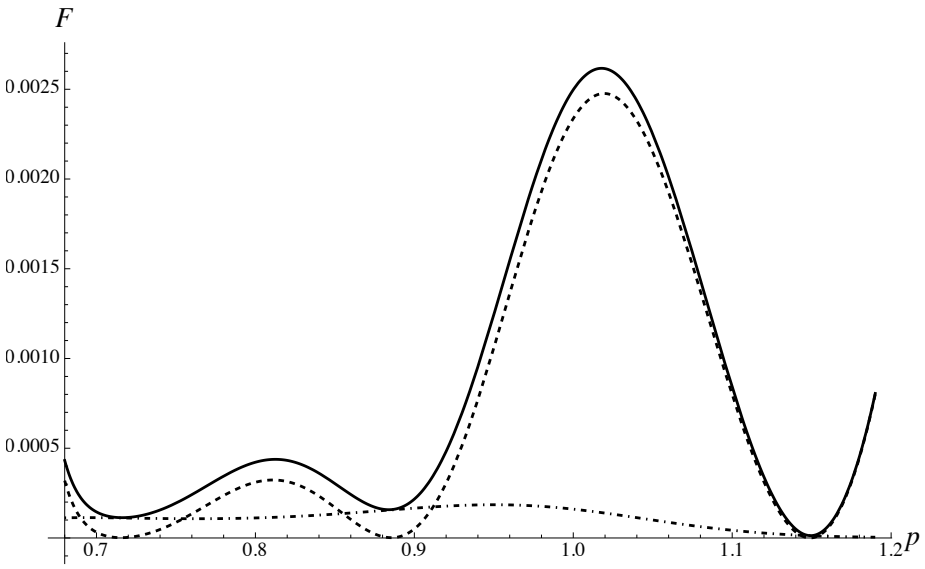


Figure 10. Quartic quantum oscillator. Complete cost-functional (12) (solid), no-penalty cost-functional (10) (dashed); and the collar-penalty (14) (dotdashed), are compared dependent on the number of iterations p.

3. Concluding Remarks

We consider iterated Borel-type summation with the number of iterations employed as control parameters associated with some optimization problem. For the optimization problem a new cost functional is suggested inspired by a martingale, with a penalty term written to penalize the optimal number of iterations for the deviations of expected value of critical amplitude from the results of a discrete iterated Borel summation. The optimization technique employs the penalty which by itself is expressed through the critical amplitude. The variational solution to the problem is accurate, robust and uniquely defined for a variety of extrapolation problems. Making the penalty dependent on expected value is novel, to the best of our knowledge.

The cost functional (12) systematically outperforms, or stays close, to the naive cost functional with the penalty conditioned on deviations of the critical amplitude from the naive value (15). Current approach is also much simpler and direct than also accurate

two-parameter fractional summation techniques of [38]. By applying the complementarity principle, it is possible to make the solution uniquely defined as well [38].

The penalty term in the formula (12) is written with the aim to penalize for any deviations of the expected value $B_k(p)$ with fractional p from the result of a discrete iterated Borel summation b_k^*. In such a manner the discrete and continuous versions of iterated Borel summations are getting reconciled. The fractional approach is more general, since it employs real control parameters, but also is more assumptive in making the number of iterations to be fractional. While the discrete approach is more natural but restricted to a non-negative number of iterations.

The influence of the collar-penalty term (14) could be very weak and rather strong, or else it can lead to a compromise solution depending on the situation. The goals here are not to spoil the results when a no-penalty cost-functional (10) works well, and to produce a plausible solution emerging from the discrete version of Borel summation (11) when no-penalty cost functionals fail.

The collar cost functionals lead to a good quality, robust extrapolants. They produce reasonable results for a wide variety of problems with rapidly growing, rapidly decaying or even irregular coefficients a_k. In such a sense the method based on a collar cost functionals works better than all other ridge cost-functionals based on Tikhonov's ideas, advanced in the current paper and in [10]. It is also more robust than a direct averaging over all solutions to the minimal-derivative and minimal-difference optimization problems.

It is possible to envisage several extensions to the cost functional (12). For instance, in place of the simplest expected value given by the critical amplitude for given k by itself, one can employ some average approximations arising from averaging over approximants with varying k.

It would be also of a general interest to lift the property of differentiability pertinent to functional of the type (12). Considering a non-differentiable cost functionals could lead to interesting properties of the optimum to be found. The optimum could become a catastrophe. Of course, the accuracy of the ensuing cost functionals should be checked against its differentiable counterparts.

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