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Construction Conditions and Applications of a Hilbert-Type Multiple Integral Inequality Involving Multivariable Upper Limit Functions and Higher-Order Partial Derivatives

Yong Hong ¹, Qian Zhao ¹ and Zhihong Zhao ^{2,*}

¹ Artificial Intelligence College, Guangzhou Huashang College, Guangzhou 511300, China; hongyong@gdufe.edu.cn (Y.H.); zhaoqian@gdhsc.edu.cn (Q.Z.)

² Graduate School, Beijing Institute of Technology, Zhuhai 519088, China

* Correspondence: zhaozh@bit.edu.cn

Abstract: Hilbert-type inequalities derive from the classical Hilbert inequality, and their theoretical work has key applications not only in operator theory but also in various analytic disciplines. In this paper, we have achieved the parametric conditions required for the construction of such inequalities, as well as the expressions for the optimal constant factors. Through the utilization of the construction theorem for Hilbert-type multiple integral inequalities with homogeneous kernels, our investigation centers on a Hilbert-type multiple integral inequality that involves multivariable upper limit functions and higher-order partial derivatives. Furthermore, we apply these results to discuss the boundedness and operator norms of integral operators with identical kernels.

Keywords: multivariable upper limit functions; higher-order partial derivatives; Hilbert-type multiple integral inequality; gamma function; bounded integral operator; operator norm

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1. Introduction and Preparatory Knowledge

Given that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. In 1934, ref. [1] obtained a Hilbert-type inequality for a homogeneous kernel $K(\xi, \eta)$ of the order -1 :

$$\int_0^{+\infty} \int_0^{+\infty} K(\xi, \eta) f(\xi) g(\eta) d\xi d\eta \leq K_p \|f\|_p \|g\|_q, \quad (1)$$

where the constant factor $K_p = \int_0^{+\infty} K(u, 1) u^{-\frac{1}{p}} du$ is the best value. When $K(\xi, \eta) = \frac{1}{\xi + \eta}$, (1) is transformed into the classical Hilbert integral inequality

$$\int_0^{+\infty} \int_0^{+\infty} \frac{f(\xi) g(\eta)}{\xi + \eta} d\xi d\eta \leq \frac{\pi}{\sin(\pi/p)} \|f\|_p \|g\|_q.$$

If we consider the integral operators with the same kernel as (1)

$$T_1(f)(\eta) = \int_0^{+\infty} K(\xi, \eta) f(\xi) d\xi, \quad T_2(g)(\xi) = \int_0^{+\infty} K(\xi, \eta) g(\eta) d\eta,$$

then (1) is equivalent to the operator inequalities

$$\|T_1(f)\|_p \leq K_p \|f\|_p, \quad \|T_2(g)\|_q \leq K_p \|g\|_q.$$

It can be seen that T_1 is a bounded operator in the Lebesgue space $L_p(0, +\infty)$, T_2 is a bounded operator in the Lebesgue space $L_q(0, +\infty)$, and the operator norms of T_1 and T_2 are both $\|T_1\| = \|T_2\| = K_p$. Since Hilbert-type inequalities have important applications in operator theory and analytical disciplines, they have received extensive attention. People have introduced various independent parameters and discussed Hilbert-type inequalities about homogeneous/quasi-homogeneous/non-homogeneous kernels such as $\frac{1}{(\xi+\eta)^\lambda}$, $\frac{1}{\xi^{\lambda_1+\eta^{\lambda_2}}}$, $\frac{\ln \xi^{\lambda_1} - \ln \eta^{\lambda_2}}{\xi^{\lambda_1} - \eta^{\lambda_2}}$, etc., and obtained many useful results [2–10]. These results show that in order to obtain an inequality of the Hilbert-type with the factor being the best constant, it is necessary to carefully match each parameter. In 2016, Hong and Wen [11] explored the laws of the best-matched parameters and subsequently achieved many important results [12–15]. Based on these studies, in order to solve the boundedness problem of related operators, people began to explore the construction problem of Hilbert-type inequalities, that is, what conditions should the parameters satisfy to construct a Hilbert-type inequality? And are such conditions necessary and sufficient? Obviously, the exploration of these issues is very important for the research on operator theory. Currently, important progress has been made in this area [16–18], and many results have been extended to the high-dimensional case [19–23]. In recent years, the exploration of Hilbert-type inequalities that feature variable upper limit functions and derivative functions has also become one of the hot topics in this field. They generalize the classical Hilbert-type inequalities from another perspective [21,24–26]. For example, Yang and Rassias [27] discussed a Hilbert-type inequality involving the multiple variable upper limit function associated with the function $f(\xi)$

$$F_m(\xi) = \int_0^\xi \int_0^{t_{m-1}} \cdots \int_0^{t_2} \int_0^{t_1} f(t_0) dt_0 dt_1 \cdots dt_{m-1}$$

and the higher-order derivative $g^{(n)}(\eta)$ of $g(\eta)$, and they obtained

$$\begin{aligned} & \int_0^{+\infty} \int_0^{+\infty} \frac{f(\xi)g(\eta)}{(\xi+\eta)^{\lambda+n}} d\xi d\eta \\ & \leq \frac{\Gamma(\lambda+m)}{\Gamma(\lambda+n)} B^{\frac{1}{p}}(\lambda_2, \lambda+m-\lambda_2) B^{\frac{1}{q}}(\lambda_1+m, \lambda-\lambda_1) \\ & \quad \times \left[\int_0^{+\infty} \xi^{p(1-\tilde{\lambda}_1-m)-1} F_m^p(\xi) d\xi \right]^{\frac{1}{p}} \left[\int_0^{+\infty} \eta^{q(1-\tilde{\lambda}_2)-1} (g^{(n)}(\eta))^q d\eta \right]^{\frac{1}{q}}, \end{aligned}$$

where $\tilde{\lambda}_1 = \frac{\lambda-\lambda_2}{p} + \frac{\lambda_1}{q}$ and $\tilde{\lambda}_2 = \frac{\lambda-\lambda_2}{q} + \frac{\lambda_2}{p}$. It was also pointed out that when $\lambda_1 + \lambda_2 = \lambda$ ($0 < \lambda_1, \lambda_2 < \lambda$), the optimal value is precisely what the constant factor embodies. This is a relatively complex result, but it has some deficiencies: (1) the parameter relationships are complex, which brings inconvenience to applications; (2) the existing results are not fully utilized, resulting in a very cumbersome proof process; (3) the conditions under which such a Hilbert-type inequality can be constructed are not discussed.

In this paper, we will consider an inequality of the Hilbert-type, a multiple integral one, which is applicable to multivariate functions and encompasses variable upper limit functions and higher-order partial derivatives. By using the construction theory of multiple integral inequalities conforming to the Hilbert type and featuring homogeneous kernels, we obtain the necessary and sufficient conditions for the tenability of the Hilbert-type multiple integral inequality involving variable upper limit functions and higher-order partial derivatives. The parameter relationships are concise, and the proof process is optimized. Finally, using the obtained results, we discuss the construction conditions of the corresponding bounded multiple integral operators and the calculation formula for the operator norm.

2. Preliminary Lemmas

To avoid unnecessary repetition, the following notations are introduced in this paper: $\mathbb{N}_+ = \{1, 2, \dots\}$,

$$\xi = (\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}_+^m, \quad \eta = (\eta_1, \eta_2, \dots, \eta_n) \in \mathbb{R}_+^n,$$

$$\|\xi\|_m = \sum_{i=1}^m \xi_i, \quad \|\eta\|_n = \sum_{j=1}^n \eta_j.$$

The variable upper limit function of $f(\xi) = f(\xi_1, \xi_2, \dots, \xi_m)$ is

$$\tilde{F}_i(f)(\xi) = \int_0^{\xi_i} \dots \int_0^{\xi_2} \int_0^{\xi_1} f(\xi_1, \dots, \xi_i, \dots, \xi_m) d\xi_1 d\xi_2 \dots d\xi_i.$$

It is stipulated that when $i = 0$, $\tilde{F}_i(f)(\xi) = f(\xi)$. The higher-order partial derivative of $g(\eta) = g(\eta_1, \eta_2, \dots, \eta_n)$ is

$$g_{\eta_1 \eta_2 \dots \eta_j}(\eta) = \frac{\partial^j g(\eta_1, \dots, \eta_j, \dots, \eta_n)}{\partial \eta_1 \partial \eta_2 \dots \partial \eta_j}.$$

It is stipulated that when $j = 0$, $g_{\eta_1 \dots \eta_j}(\eta) = g(\eta)$. The weighted high-dimensional Lebesgue space is

$$L_p^\alpha(\mathbb{R}_+^m) = \{f(\xi) : \|f\|_{p,\alpha} = \left(\int_{\mathbb{R}_+^m} \|\xi\|_m^\alpha |f(\xi)|^p d\xi \right)^{\frac{1}{p}} < +\infty\}.$$

Denote

$$W_0(p, \alpha, m) = \int_0^{+\infty} K(t, 1) t^{-\frac{\alpha+m}{p} + m-1} dt.$$

If $K(tu, tv) = t^\sigma K(u, v)$ ($t > 0$), then $K(u, v)$ is said to be a homogeneous function with an order of σ .

Lemma 1 ([20]). *Given that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, $\alpha, \beta \in \mathbb{R}$ and $m, n \in \mathbb{N}_+$. Let $K(u, v) > 0$ be a homogeneous measurable function of order σ , $\xi = (\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}_+^m$, $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in \mathbb{R}_+^n$, and $W_0(p, \alpha, m) < +\infty$. Then,*

(i) *There exists a constant $M > 0$ such that*

$$\int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^m} K(\|\xi\|_m, \|\eta\|_n) f(\xi) g(\eta) d\xi d\eta \leq M \|f\|_{p,\alpha} \|g\|_{q,\beta} \quad (2)$$

if and only if $\frac{\alpha+m}{p} + \frac{\beta+n}{q} = m + n + \sigma$.

(ii) *When $\frac{\alpha+m}{p} + \frac{\beta+n}{q} = m + n + \sigma$, that is, when (2) holds, it is the optimal constant factor that*

$$M_1 = \inf\{M\} = \frac{1}{\Gamma^{1/q}(m) \Gamma^{1/p}(n)} W_0(p, \alpha, m).$$

Lemma 2. *Let $\frac{1}{p} + \frac{1}{q} = 1$ ($p > 1$), $m, n \in \mathbb{N}_+$, $\alpha, \beta \in \mathbb{R}$, $\lambda > m - \frac{\alpha+m}{p} > 0$, $\xi = (\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}_+^m$, and $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in \mathbb{R}_+^n$. Then,*

(i) *There exists a constant $M > 0$ such that*

$$\int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^m} \frac{1}{(\|\xi\|_m + \|\eta\|_n)^\lambda} f(\xi) g(\eta) d\xi d\eta \leq M \|f\|_{p,\alpha} \|g\|_{q,\beta} \quad (3)$$

if and only if $\frac{\alpha+m}{p} + \frac{\beta+n}{q} = m + n - \lambda$.

(ii) When $\frac{\alpha+m}{p} + \frac{\beta+n}{q} = m+n-\lambda$, that is, when (3) holds, the best constant factor is

$$\overline{M} = \frac{1}{\Gamma(\lambda)\Gamma^{1/q}(m)\Gamma^{1/p}(n)}\Gamma\left(m - \frac{\alpha+m}{p}\right)\Gamma\left(n - \frac{\beta+n}{q}\right).$$

Proof. Let $K(u, v) = \frac{1}{(u+v)^\lambda}$ with $u > 0, v > 0$. Then, $K(u, v)$, with the property of $K(u, v) > 0$, is a homogeneous function of order $-\lambda$. Since $\lambda > m - \frac{\alpha+m}{p} > 0$, we have $\lambda + \frac{\alpha+m}{p} - m > 0$. Thus,

$$\begin{aligned} W_0(p, \alpha, m) &= \int_0^{+\infty} \frac{1}{(1+t)^\lambda} t^{-\frac{\alpha+m}{p}+m-1} dt \\ &= B\left(m - \frac{\alpha+m}{p}, \lambda + \frac{\alpha+m}{p} - m\right) < +\infty. \end{aligned}$$

When $\frac{\alpha+m}{p} + \frac{\beta+n}{q} = m+n-\lambda$, we have $\lambda + \frac{\alpha+m}{p} - m = n - \frac{\beta+n}{q}$. Then,

$$\begin{aligned} W_0(p, \alpha, m) &= B\left(m - \frac{\alpha+m}{p}, n - \frac{\beta+n}{q}\right) \\ &= \frac{1}{\Gamma(\lambda)}\Gamma\left(m - \frac{\alpha+m}{p}\right)\Gamma\left(n - \frac{\beta+n}{q}\right). \end{aligned}$$

In conclusion, Lemma 2 holds according to Lemma 1. $\square$

3. Multiple Integral Inequality of the Hilbert-Type Incorporating Multivariable Upper Limit Functions and Higher-Order Partial Derivatives

Theorem 1. Given that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, $m, n \in \mathbb{N}_+$, $\alpha, \beta \in \mathbb{R}$, $0 \leq k_1 \leq m, 0 \leq k_2 \leq n$, $\lambda + k_1 - k_2 > m - \frac{\alpha+m}{p} > 0$, $\xi = (\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}_+^m$, and $\eta = (\eta_1, \eta_2, \dots, \eta_n) \in \mathbb{R}_+^n$. When $i = 1, 2, \dots, k_1$, $\tilde{F}_i(f)(\xi) = o(e^{t\xi_i})$ ($\xi_i \rightarrow +\infty, t > 0$); when $j = 1, 2, \dots, k_2$, $g_{\eta_1\eta_2\cdots\eta_{j-1}}(\eta) = o(e^{t\eta_j})$ ($\eta_j \rightarrow +\infty, t > 0$), and $\lim_{\eta_j \rightarrow +\infty} g_{\eta_1\eta_2\cdots\eta_{j-1}}(\eta) = 0$.

(i) There exists a constant $M > 0$ independent of f and g , such that

$$\int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^m} \frac{f(\xi)g(\eta)}{(\|\xi\|_m + \|\eta\|_n)^\lambda} d\xi d\eta \leq M \|\tilde{F}_{k_1}(f)\|_{p,\alpha} \|g_{\eta_1\eta_2\cdots\eta_{k_2}}\|_{q,\beta} \quad (4)$$

if and only if $\frac{\alpha+m}{p} + \frac{\beta+n}{q} = m+n-k_1+k_2-\lambda$.

(ii) When $\frac{\alpha+m}{p} + \frac{\beta+n}{q} = m+n-k_1+k_2-\lambda$, the constant factor of (4) that attains the optimal value is

$$M_0 = \inf\{M\} = \frac{1}{\Gamma(\lambda)\Gamma^{1/q}(m)\Gamma^{1/p}(n)}\Gamma\left(m - \frac{\alpha+m}{p}\right)\Gamma\left(n - \frac{\beta+n}{q}\right).$$

Proof. First, when $a > 0$, according to the property of the Gamma function, one has

$$\int_0^{+\infty} t^{\lambda-1} e^{-at} dt = \frac{1}{a^\lambda} \int_0^{+\infty} u^{\lambda-1} e^{-u} du = \frac{1}{a^\lambda} \Gamma(\lambda).$$

Then,

$$\frac{1}{(\|\xi\|_m + \|\eta\|_n)^\lambda} = \frac{1}{\Gamma(\lambda)} \int_0^{+\infty} t^{\lambda-1} e^{-t(\|\xi\|_m + \|\eta\|_n)} dt. \quad (5)$$

(i) According to (5), we have

$$\begin{aligned} A(f, g) &:= \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^m} \frac{f(\xi)g(\eta)}{(\|\xi\|_m + \|\eta\|_n)^\lambda} d\xi d\eta \\ &= \frac{1}{\Gamma(\lambda)} \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^m} f(\xi)g(\eta) \left(\int_0^{+\infty} t^{\lambda-1} e^{-t(\|\xi\|_m + \|\eta\|_n)} dt \right) d\xi d\eta \\ &= \frac{1}{\Gamma(\lambda)} \int_0^{+\infty} t^{\lambda-1} \left(\int_{\mathbb{R}_+^m} e^{-t \sum_{i=1}^m \xi_i} f(\xi) d\xi \right) \left(\int_{\mathbb{R}_+^n} e^{-t \sum_{j=1}^n \eta_j} g(\eta) d\eta \right) dt. \end{aligned}$$

Using integration by parts and noting that when $i = 1, 2, \dots, k_1$, $\tilde{F}_i(f)(\xi) = o(e^{t\xi_i})$ ($\xi_i \rightarrow +\infty, t > 0$), we have

$$\begin{aligned} &\int_{\mathbb{R}_+^m} e^{-t \sum_{i=1}^m \xi_i} f(\xi) d\xi \\ &= \int_{\mathbb{R}_+^{m-1}} e^{-t \sum_{i \neq 1}^m \xi_i} \left(\int_0^{+\infty} e^{-t\xi_1} \tilde{F}_0(f)(\xi_1, \xi_2, \dots, \xi_m) d\xi_1 \right) d\xi_2 \dots d\xi_m \\ &= \int_{\mathbb{R}_+^{m-1}} e^{-t \sum_{i \neq 1}^m \xi_i} \left(\int_0^{+\infty} e^{-t\xi_1} d\tilde{F}_1(f)(\xi_1, \xi_2, \dots, \xi_m) \right) d\xi_2 \dots d\xi_m \\ &= \int_{\mathbb{R}_+^{m-1}} e^{-t \sum_{i \neq 1}^m \xi_i} \left(\lim_{\xi_1 \rightarrow +\infty} e^{-t\xi_1} \tilde{F}_1(f)(\xi_1, \xi_2, \dots, \xi_m) \right. \\ &\quad \left. + t \int_0^{+\infty} e^{-t\xi_1} \tilde{F}_1(f)(\xi_1, \xi_2, \dots, \xi_m) d\xi_1 \right) d\xi_2 \dots d\xi_m \\ &= t \int_{\mathbb{R}_+^m} e^{-t \sum_{i=1}^m \xi_i} \tilde{F}_1(f)(\xi_1, \xi_2, \dots, \xi_m) d\xi_1 d\xi_2 \dots d\xi_m \\ &= \dots \\ &= t^{k_1} \int_{\mathbb{R}_+^m} e^{-t \sum_{i=1}^m \xi_i} \tilde{F}_{k_1}(f)(\xi_1, \xi_2, \dots, \xi_m) d\xi_1 d\xi_2 \dots d\xi_m \\ &= t^{k_1} \int_{\mathbb{R}_+^m} e^{-t\|\xi\|_m} \tilde{F}_{k_1}(f)(\xi) d\xi. \end{aligned}$$

Since when $j = 1, 2, \dots, k_2$, $g_{\eta_1 \eta_2 \dots \eta_{j-1}}(\eta) = o(e^{t\eta_j})$ ($\eta_j \rightarrow +\infty, t > 0$) and $\lim_{\eta_j \rightarrow 0^+} g_{\eta_1 \eta_2 \dots \eta_{j-1}}(\eta) = 0$, then

$$e^{-t\eta_j} g_{\eta_1 \eta_2 \dots \eta_{j-1}}(\eta) \Big|_0^{+\infty} = \lim_{\eta_j \rightarrow +\infty} e^{-t\eta_j} g_{\eta_1 \eta_2 \dots \eta_{j-1}}(\eta) - \lim_{\eta_j \rightarrow 0^+} g_{\eta_1 \eta_2 \dots \eta_{j-1}}(\eta) = 0.$$

From this and using integration by parts, we have

$$\begin{aligned} &\int_{\mathbb{R}_+^n} e^{-t \sum_{j=1}^n \eta_j} g(\eta) d\eta \\ &= \int_{\mathbb{R}_+^{n-1}} e^{-t \sum_{j \neq 1}^n \eta_j} \left(\int_0^{+\infty} e^{-t\eta_1} g(\eta_1, \eta_2, \dots, \eta_n) d\eta_1 \right) d\eta_2 \dots d\eta_n \\ &= \int_{\mathbb{R}_+^{n-1}} e^{-t \sum_{j \neq 1}^n \eta_j} \left(-\frac{1}{t} \int_0^{+\infty} g(\eta_1, \eta_2, \dots, \eta_n) d(e^{-t\eta_1}) \right) d\eta_2 \dots d\eta_n \\ &= \int_{\mathbb{R}_+^{n-1}} e^{-t \sum_{j \neq 1}^n \eta_j} \left(-\frac{1}{t} e^{-t\eta_1} g(\eta_1, \eta_2, \dots, \eta_n) \Big|_0^{+\infty} \right. \\ &\quad \left. + \frac{1}{t} \int_0^{+\infty} e^{-t\eta_1} g_{\eta_1}(\eta_1, \eta_2, \dots, \eta_n) d\eta_1 \right) d\eta_2 \dots d\eta_n \\ &= \frac{1}{t} \int_{\mathbb{R}_+^{n-1}} e^{-t \sum_{j=1}^n \eta_j} g_{\eta_1}(\eta_1, \eta_2, \dots, \eta_n) d\eta_1 d\eta_2 \dots d\eta_n \\ &= \dots \\ &= \frac{1}{t^{k_2}} \int_{\mathbb{R}_+^n} e^{-t \sum_{j=1}^n \eta_j} g_{\eta_1 \eta_2 \dots \eta_{k_2}}(\eta_1, \eta_2, \dots, \eta_n) d\eta_1 d\eta_2 \dots d\eta_n \\ &= \frac{1}{t^{k_2}} \int_{\mathbb{R}_+^n} e^{-t\|\eta\|_n} g_{\eta_1 \eta_2 \dots \eta_{k_2}}(\eta) d\eta. \end{aligned}$$

In conclusion, we find

$$\begin{aligned}
 & A(f, g) \\
 &= \frac{1}{\Gamma(\lambda)} \int_0^{+\infty} t^{\lambda+k_1-k_2-1} \left(\int_{\mathbb{R}_+^m} e^{-t\|\xi\|_m} \tilde{F}_{k_1}(f)(\xi) d\xi \right) \left(\int_{\mathbb{R}_+^n} e^{-t\|\eta\|_n} g_{\eta_1\eta_2\cdots\eta_{k_2}}(\eta) d\eta \right) dt \\
 &= \frac{1}{\Gamma(\lambda)} \int_0^{+\infty} t^{\lambda+k_1-k_2-1} \left(\int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^m} e^{-t(\|\xi\|_m+\|\eta\|_n)} \tilde{F}_{k_1}(f)(\xi) g_{\eta_1\eta_2\cdots\eta_{k_2}}(\eta) d\xi d\eta \right) dt \\
 &= \frac{1}{\Gamma(\lambda)} \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^m} \left(\int_0^{+\infty} t^{\lambda+k_1-k_2-1} e^{-t(\|\xi\|_m+\|\eta\|_n)} dt \right) \tilde{F}_{k_1}(f)(\xi) g_{\eta_1\eta_2\cdots\eta_{k_2}}(\eta) d\xi d\eta \\
 &= \frac{\Gamma(\lambda+k_1-k_2)}{\Gamma(\lambda)} \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^m} \frac{\tilde{F}_{k_1}(f)(\xi) g_{\eta_1\eta_2\cdots\eta_{k_2}}(\eta)}{(\|\xi\|_m+\|\eta\|_n)^{\lambda+k_1-k_2}} d\xi d\eta.
 \end{aligned}$$

By Lemma 2(i), there exists a constant $\overline{M} > 0$ such that

$$\int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^m} \frac{\tilde{F}_{k_1}(f)(\xi) g_{\eta_1\eta_2\cdots\eta_{k_2}}(\eta)}{(\|\xi\|_m+\|\eta\|_n)^{\lambda+k_1-k_2}} d\xi d\eta \leq \overline{M} \|\tilde{F}_{k_1}(f)\|_{p,\alpha} \|g_{\eta_1\eta_2\cdots\eta_{k_2}}\|_{q,\beta} \quad (6)$$

if and only if $\frac{\alpha+m}{p} + \frac{\beta+n}{q} = m+n-(\lambda+k_1-k_2) = m+n-k_1+k_2-\lambda$. Let $M = \frac{\Gamma(\lambda+k_1-k_2)}{\Gamma(\lambda)} \overline{M}$; then, we obtain (4).

(ii) When $\frac{\alpha+m}{p} + \frac{\beta+n}{q} = m+n-k_1+k_2-\lambda = m+n-(\lambda+k_1-k_2)$, according to Lemma 2(ii), the best constant factor of (6) is

$$\overline{M}_0 = \frac{1}{\Gamma(\lambda+k_1-k_2)\Gamma^{1/q}(m)\Gamma^{1/p}(n)} \Gamma\left(m-\frac{\alpha+m}{p}\right) \Gamma\left(n-\frac{\beta+n}{q}\right).$$

Then, the best constant factor M_0 of (4) is

$$M_0 = \frac{\Gamma(\lambda+k_1-k_2)}{\Gamma(\lambda)} \overline{M}_0 = \frac{1}{\Gamma(\lambda)\Gamma^{1/q}(m)\Gamma^{1/p}(n)} \Gamma\left(m-\frac{\alpha+m}{p}\right) \Gamma\left(n-\frac{\beta+n}{q}\right).$$

□

4. Applications in Operator Theory

Let $p > 1, q > 1, \alpha, \beta \in \mathbb{R}, \xi = (\xi_1, \xi_2, \dots, \xi_m) \in \mathbb{R}_+^m, \eta = (\eta_1, \eta_2, \dots, \eta_n) \in \mathbb{R}_+^n, 0 \leq k_1 \leq m$, and $0 \leq k_2 \leq n$. Define

$$\begin{aligned}
 I_+^{k_1} L_p^\alpha(\mathbb{R}_+^m) &= \left\{ f(\xi) \geq 0 : \|\tilde{F}_{k_1}(f)\|_{p,\alpha} < +\infty, \tilde{F}_i(f)(\xi) = o(e^{t\xi_i}) \right. \\
 &\quad \left. (\xi_i \rightarrow +\infty, t > 0), i = 1, 2, \dots, k_1 \right\}, \\
 \partial^{k_2} L_q^\beta(\mathbb{R}_+^n) &= \left\{ g(\eta) : \|g_{\eta_1\eta_2\cdots\eta_{k_2}}\|_{q,\beta} < +\infty, g_{\eta_1\eta_2\cdots\eta_{k_2}}(\eta) = o(e^{t\eta_j}) \right. \\
 &\quad \left. (\eta_j \rightarrow +\infty, t > 0), \lim_{\eta_j \rightarrow 0^+} g_{\eta_1\eta_2\cdots\eta_{j-1}}(\eta) = 0, j = 1, 2, \dots, k_2 \right\}.
 \end{aligned}$$

In the following, we use Theorem 1 to discuss the boundedness and norms of multiple integral operators defined on $I_+^{k_1} L_p^\alpha(\mathbb{R}_+^m)$ and $\partial^{k_2} L_q^\beta(\mathbb{R}_+^n)$.

Theorem 2. Given that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, $m, n \in \mathbb{N}_+$, $\alpha, \beta \in \mathbb{R}$, $0 \leq k_1 \leq m$, and $\lambda + k_1 > m - \frac{\alpha+m}{p} > 0$. The integral operator T_1 is defined as follows:

$$T_1(f)(\eta) = \int_{\mathbb{R}_+^m} \frac{f(\xi)}{(\|\xi\|_m + \|\eta\|_n)^\lambda} d\xi, \quad f(\xi) \in I_+^{k_1} L_p^\alpha(\mathbb{R}_+^m).$$

(i) T_1 is a bounded operator from $I_+^{k_1} L_p^\alpha(\mathbb{R}_+^m)$ to $L_p^{\beta(1-p)}(\mathbb{R}_+^n)$ if and only if $\frac{\alpha+m}{p} + \frac{\beta+n}{q} = m + n - k_1 - \lambda$. That is, there exists a constant $M_1 > 0$ such that

$$\|T_1(f)\|_{p, \beta(1-p)} \leq M_1 \|\tilde{F}_{k_1}(f)\|_{p, \alpha}, \quad (7)$$

(ii) When T_1 is bounded, its operator norm is

$$\|T_1\| = \frac{1}{\Gamma(\lambda) \Gamma^{1/q}(m) \Gamma^{1/p}(n)} \Gamma\left(m - \frac{\alpha+m}{p}\right) \Gamma\left(n - \frac{\beta+n}{q}\right).$$

Proof. First, we prove that (7) is equivalent to

$$\int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^m} \frac{f(\xi)g(\eta)}{(\|\xi\|_m + \|\eta\|_n)^\lambda} d\xi d\eta \leq M_1 \|\tilde{F}_{k_1}(f)\|_{p, \alpha} \|g\|_{q, \beta}. \quad (8)$$

If (8) holds, let $g(\eta) = \|\eta\|_n^{\beta(1-p)} |T_1(f)(\eta)|^{p-1}$. Since $f(\xi) \in I_+^{k_1} L_p^\alpha(\mathbb{R}_+^m)$, then $f(\xi) \geq 0$. Thus, $\tilde{F}_{k_1}(f)(\xi) \geq 0$. Then,

$$\begin{aligned} \|T_1(f)\|_{p, \beta(1-p)}^p &= \int_{\mathbb{R}_+^n} \|\eta\|_n^{\beta(1-p)} |T_1(f)(\eta)|^p d\eta \\ &= \int_{\mathbb{R}_+^n} g(\eta) T_1(f)(\eta) d\eta = \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^m} \frac{f(\xi)g(\eta)}{(\|\xi\|_m + \|\eta\|_n)^\lambda} d\xi d\eta \\ &\leq M_1 \|\tilde{F}_{k_1}(f)\|_{p, \alpha} \|g\|_{q, \beta} \\ &= M_1 \|\tilde{F}_{k_1}(f)\|_{p, \alpha} \left(\int_{\mathbb{R}_+^n} \|\eta\|_n^\beta \left(\|\eta\|_n^{\beta(1-p)} |T_1(f)(\eta)|^{p-1} \right)^q d\eta \right)^{\frac{1}{q}} \\ &= M_1 \|\tilde{F}_{k_1}(f)\|_{p, \alpha} \left(\int_{\mathbb{R}_+^n} \|\eta\|_n^{\beta(1-p)} |T_1(f)(\eta)|^p d\eta \right)^{\frac{1}{q}} \\ &= M_1 \|\tilde{F}_{k_1}(f)\|_{p, \alpha} \|T_1(f)\|_{p, \beta(1-p)}^{p-1}. \end{aligned}$$

From this, we obtain $\|T_1(f)\|_{p, \beta(1-p)} \leq M_1 \|\tilde{F}_{k_1}(f)\|_{p, \alpha}$, that is, (7) holds.

Conversely, if (7) holds, according to the Hölder integral inequality and noting that $-\frac{\beta p}{q} = \beta(1-p)$, we have

$$\begin{aligned} \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^m} \frac{f(\xi)g(\eta)}{(\|\xi\|_m + \|\eta\|_n)^\lambda} d\xi d\eta &= \int_{\mathbb{R}_+^n} g(\eta) T_1(f)(\eta) d\eta \\ &= \int_{\mathbb{R}_+^n} \left(\|\eta\|_n^{-\frac{\beta}{q}} T_1(f)(\eta) \right) \left(\|\eta\|_n^{\frac{\beta}{q}} g(\eta) \right) d\eta \\ &\leq \left(\int_{\mathbb{R}_+^n} \|\eta\|_n^{-\frac{\beta p}{q}} |T_1(f)(\eta)|^p d\eta \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}_+^n} \|\eta\|_n^\beta |g(\eta)|^q d\eta \right)^{\frac{1}{q}} \\ &= \|T_1(f)\|_{p, \beta(1-p)} \|g\|_{q, \beta} \leq M_1 \|\tilde{F}_{k_1}(f)\|_{p, \alpha} \|g\|_{q, \beta}. \end{aligned}$$

So, (8) holds.

In conclusion, (7) is equivalent to (8).

In Theorem 1, take $k_2 = 0$ and rename M as M_1 . Then, (4) becomes (8), and the condition $\frac{\alpha+m}{p} + \frac{\beta+n}{q} = m+n-k_1+k_2-\lambda$ is reduced to $\frac{\alpha+m}{p} + \frac{\beta+n}{q} = m+n-k_1-\lambda$. So, according to Theorem 1, the conclusions of Theorem 2 hold. $\square$

Theorem 3. Given that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, $m, n \in \mathbb{N}_+$, $\alpha, \beta \in \mathbb{R}$, $0 \leq k_2 \leq n$, and $\lambda - k_2 > m - \frac{\alpha+m}{p} > 0$. The integral operator is defined as follows:

$$T_2(g)(\xi) = \int_{\mathbb{R}_+^n} \frac{g(\eta)}{(\|\xi\|_m + \|\eta\|_n)^\lambda} d\eta, \quad g(\eta) \in \partial^{k_2} L_q^\beta(\mathbb{R}_+^n).$$

(i) T_2 is a bounded operator from $\partial^{k_2} L_q^\beta(\mathbb{R}_+^n)$ to $L_q^{\alpha(1-q)}(\mathbb{R}_+^m)$ if and only if $\frac{\alpha+m}{p} + \frac{\beta+n}{q} = m+n-k_2-\lambda$. That is, there exists a constant $M_2 > 0$ such that

$$\|T_2(g)\|_{q, \alpha(1-q)} \leq M_2 \|g_{\eta_1 \eta_2 \dots \eta_{k_2}}\|_{q, \beta}. \quad (9)$$

(ii) When T_2 is bounded, its operator norm is

$$\|T_2\| = \frac{1}{\Gamma(\lambda) \Gamma^{1/q}(m) \Gamma^{1/p}(n)} \Gamma\left(m - \frac{\alpha+m}{p}\right) \Gamma\left(n - \frac{\beta+n}{q}\right).$$

Proof. Similarly, we first prove that (9) is equivalent to

$$\int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^m} \frac{f(\xi)g(\eta)}{(\|\xi\|_m + \|\eta\|_n)^\lambda} d\xi d\eta \leq M_2 \|f\|_{p, \alpha} \|g_{\eta_1 \eta_2 \dots \eta_{k_2}}\|_{q, \beta}. \quad (10)$$

If (10) holds, let $f(\xi) = \|\xi\|_m^{\alpha(1-q)} |T_2(g)(\xi)|^{q-1}$. Since $|g_{\eta_1 \eta_2 \dots \eta_{k_2}}(\eta)|$ and $|g|_{\eta_1 \eta_2 \dots \eta_{k_2}}(\eta)$ are equal almost everywhere, then $\|g_{\eta_1 \eta_2 \dots \eta_{k_2}}\|_{q, \beta} = \|g|_{\eta_1 \eta_2 \dots \eta_{k_2}}\|_{q, \beta}$. It follows that

$$\begin{aligned} \|T_2(g)\|_{q, \alpha(1-q)}^q &= \int_{\mathbb{R}_+^m} \|\xi\|_m^{\alpha(1-q)} |T_2(g)(\xi)|^q d\xi \\ &= \int_{\mathbb{R}_+^m} f(\xi) \left| \int_{\mathbb{R}_+^n} \frac{g(\eta)}{(\|\xi\|_m + \|\eta\|_n)^\lambda} d\eta \right| d\xi \\ &\leq \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^m} \frac{f(\xi) |g(\eta)|}{(\|\xi\|_m + \|\eta\|_n)^\lambda} d\xi d\eta \\ &\leq M_2 \|f\|_{p, \alpha} \|g|_{\eta_1 \eta_2 \dots \eta_{k_2}}\|_{q, \beta} = M_2 \|f\|_{p, \alpha} \|g_{\eta_1 \eta_2 \dots \eta_{k_2}}\|_{q, \beta} \\ &= M_2 \|g_{\eta_1 \eta_2 \dots \eta_{k_2}}\|_{q, \beta} \left(\int_{\mathbb{R}_+^m} \|\xi\|_m^\alpha \left(\|\xi\|_m^{\alpha(1-q)} |T_2(g)(\xi)|^{q-1} \right)^p d\xi \right)^{\frac{1}{p}} \\ &= M_2 \|g_{\eta_1 \eta_2 \dots \eta_{k_2}}\|_{q, \beta} \left(\int_{\mathbb{R}_+^m} \|\xi\|_m^{\alpha(1-q)} |T_2(g)(\xi)|^q d\xi \right)^{\frac{1}{p}} \\ &= M_2 \|g_{\eta_1 \eta_2 \dots \eta_{k_2}}\|_{q, \beta} \|T_2(g)\|_{q, \alpha(1-q)}^{q-1}. \end{aligned}$$

From this, we obtain $\|T_2(g)\|_{q, \alpha(1-q)} \leq M_2 \|g_{\eta_1 \eta_2 \dots \eta_{k_2}}\|_{q, \beta}$, that is, (9) holds.

Conversely, if (9) holds, according to the Hölder integral inequality and noting that $-\frac{\alpha q}{p} = \alpha(1-q)$, we have

$$\begin{aligned}
& \int_{\mathbb{R}_+^n} \int_{\mathbb{R}_+^m} \frac{f(\xi)g(\eta)}{(\|\xi\|_m + \|\eta\|_n)^\lambda} d\xi d\eta \\
&= \int_{\mathbb{R}_+^m} f(\xi) T_2(g)(\xi) d\xi \\
&= \int_{\mathbb{R}_+^m} \left(\|\xi\|_m^{\frac{\alpha}{p}} f(\xi) \right) \left(\|\xi\|_m^{-\frac{\alpha}{p}} T_2(g)(\xi) \right) d\xi \\
&\leq \left(\int_{\mathbb{R}_+^m} \|\xi\|_m^\alpha |f(\xi)|^p d\xi \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}_+^m} \|\xi\|_m^{-\frac{\alpha q}{p}} |T_2(g)(\xi)|^q d\xi \right)^{\frac{1}{q}} \\
&= \|f\|_{p,\alpha} \|T_2(g)\|_{q,\alpha(1-q)} \leq M_2 \|f\|_{p,\alpha} \|g_{\eta_1 \eta_2 \dots \eta_{k_2}}\|_{q,\beta}.
\end{aligned}$$

So, (10) holds.

In conclusion, (9) is equivalent to (10).

In Theorem 1, take $k_1 = 0$. Then, the condition $\frac{\alpha+m}{p} + \frac{\beta+n}{q} = m+n-k_1+k_2-\lambda$ is reduced to $\frac{\alpha+m}{p} + \frac{\beta+n}{q} = m+n+k_2-\lambda$. Rename M as M_2 ; then, (4) is reduced to (10). So, according to Theorem 1, the conclusion of Theorem 3 holds. $\square$

In Theorem 2, taking $\alpha = \beta = 0$, we can obtain the following Corollary 1.

Corollary 1. Given that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, $m, n \in \mathbb{N}_+$, $0 \leq k_1 \leq m$, and $\lambda + k_1 > \frac{m}{q}$. The integral operator T_1 is defined as follows:

$$T_1(f)(\eta) = \int_{\mathbb{R}_+^m} \frac{f(\xi)}{(\|\xi\|_m + \|\eta\|_n)^\lambda} d\xi, \quad f(\xi) \in I_+^{k_1} L_p(\mathbb{R}_+^m).$$

- (i) T_1 is a bounded operator from $I_+^{k_1} L_p(\mathbb{R}_+^m)$ to $L_p(\mathbb{R}_+^n)$ if and only if $\frac{m}{q} + \frac{n}{p} = \lambda + k_1$.
- (ii) When T_1 is bounded, its operator norm is

$$\|T_1\| = \frac{1}{\Gamma(\lambda) \Gamma^{1/q}(m) \Gamma^{1/p}(n)} \Gamma\left(\frac{m}{q}\right) \Gamma\left(\frac{n}{p}\right).$$

In Theorem 3, taking $\alpha = \beta = 0$, we can obtain the following Corollary 3.

Corollary 2. Given that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, $m, n \in \mathbb{N}_+$, $0 \leq k_2 \leq n$, and $\lambda - k_2 > \frac{m}{q}$. The integral operator T_2 is defined as follows:

$$T_2(g)(\xi) = \int_{\mathbb{R}_+^n} \frac{g(\eta)}{(\|\xi\|_m + \|\eta\|_n)^\lambda} d\eta, \quad g(\eta) \in \partial^{k_2} L_q(\mathbb{R}_+^n).$$

- (i) T_2 is a bounded operator from $\partial^{k_2} L_q(\mathbb{R}_+^n)$ to $L_q(\mathbb{R}_+^m)$ if and only if $\frac{m}{q} + \frac{n}{p} = \lambda - k_2$.
- (ii) When T_2 is bounded, its operator norm is

$$\|T_2\| = \frac{1}{\Gamma(\lambda) \Gamma^{1/q}(m) \Gamma^{1/p}(n)} \Gamma\left(\frac{m}{q}\right) \Gamma\left(\frac{n}{p}\right).$$

If we further take $n = m$ in Corollary 1, then we can obtain the following Corollary 3.

Corollary 3. Given that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, $m \in \mathbb{N}_+$, $0 \leq k_1 \leq m$, and $\lambda + k_1 > \frac{m}{q}$. The integral operator T_1 is defined as follows:

$$T_1(f)(\eta) = \int_{\mathbb{R}_+^m} \frac{f(\xi)}{(\|\xi\|_m + \|\eta\|_m)^\lambda} d\xi, \quad f(\xi) \in I_+^{k_1} L_p(\mathbb{R}_+^m).$$

- (i) T_1 is a bounded operator from $I_+^{k_1} L_p(\mathbb{R}_+^m)$ to $L_p(\mathbb{R}_+^m)$ if and only if $\lambda = m - k_1$.

(ii) When T_1 is bounded, its operator norm is

$$\|T_1\| = \frac{1}{\Gamma(m - k_1)\Gamma(m)} \frac{\pi}{\sin(m\pi/p)}.$$

If we further take $m = n$ in Corollary 2, then we can obtain the following Corollary 4.

Corollary 4. Given that $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$, $n \in \mathbb{N}_+$, $0 \leq k_2 \leq n$, and $\lambda - k_2 > \frac{n}{q}$. The integral operator T_2 is defined as follows:

$$T_2(g)(\xi) = \int_{\mathbb{R}_+^n} \frac{g(\eta)}{(\|\xi\|_n + \|\eta\|_n)^\lambda} d\eta, \quad g(\eta) \in \partial^{k_2} L_q(\mathbb{R}_+^n).$$

(i) T_2 is a bounded operator from $\partial^{k_2} L_q(\mathbb{R}_+^n)$ to $L_q(\mathbb{R}_+^n)$ if and only if $\lambda = n + k_2$.

(ii) When T_2 is bounded, its operator norm is

$$\|T_2\| = \frac{1}{\Gamma(n + k_2)\Gamma(n)} \frac{\pi}{\sin(n\pi/p)}.$$

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