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The Reverse Order Law for the $\{1, 3M, 4N\}$ —The Inverse of Two Matrix Products

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Abstract: By using the maximal and minimal ranks of some generalized Schur complement, the equivalent conditions for the reverse order law $(AB)\{1, 3M, 4K\} = B\{1, 3N, 4K\}A\{1, 3M, 4N\}$ are presented.

Keywords: reverse order law; generalized inverse; maximal and minimal ranks; generalized Schur complement; matrix product

MSC: 47A05; 15A09; 15A24

1. Introduction

At first, we will provide some definitions for the convenience of readers.

Definition 1 ([1,2]).

$C^{m \times n}$: the set of $m \times n$ complex matrices;

$R(A)$, $N(A)$: the range space and the null space of A , respectively;

$r(A)$, A^* : the rank and the conjugate transpose of A , respectively;

$A^{\otimes(M,N)} = A^{\#} = N^{-1}A^*M$: the weighted conjugate transpose matrices of A with positive-definite Hermitian matrices M and N .

Definition 2 ([3]). Let M be a positive-definite Hermitian matrix and let $A \in C^{m \times n}$, $b \in C^m$. The aim of the so-called weighted least squares problem [WLS] is to find $x \in C^n$ that satisfies the following:

$$\|Ax - b\|_M = \|M^{\frac{1}{2}}(Ax - b)\|_2^2 = \min_{x \in C^n} \|Ax - b\|_M,$$

where $M = M^{\frac{1}{2}}M^{\frac{1}{2}}$, $\|\cdot\|_2$ is a spectral norm, and $\|\cdot\|_M$ is a weighted M -norm.

Definition 3 ([2]). Let $A \in C^{m \times n}$, and let M and N be two positive-definite Hermitian matrices. X is the weighted Moore–Penrose inverse of A when it satisfies

$$(1) AXA = A, \quad (2) XAX = X, \quad (3M) (MAX)^* = MAX, \quad (4N) (NXA)^* = NXA, \quad (1)$$

where X is denoted by $X = A^{(1,2,3M,4N)} = A_{M,N}^{\dagger}$. Set's $A\{1, 3M, 4N\}$ -inverses of A are $A\{1, 3M, 4N\} = \{X \in C^{n \times m} \mid AXA = A, (MAX)^* = MAX, (NXA)^* = NXA\}$.



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The reverse order law for the weighted generalized inverse has been widely applied in the study of the WLS (see [4–6]), the weighted perturbation theory (see [7–12]), the optimization problems and other related topics (see [13–18]).

The reverse order laws for the weighted generalized inverses of matrix products have attracted considerable attentions (see [19–28]). It is well known that the core problem here concerns the reverse order law, which depends on what conditions the equations

$$A_n^{(i,j,\dots,k)} A_{n-1}^{(i,j,\dots,k)} \cdots A_1^{(i,j,\dots,k)} = (A_1 A_2 \cdots A_n)^{(i,j,\dots,k)}$$

hold, where $i, j, \dots, k \in \{1, 2, 3M, 4N\}$.

The purpose of this paper is to show some equivalent conditions for the following reverse order law:

$$B\{1, 3N, 4K\} A\{1, 3M, 4N\} = (AB)\{1, 3M, 4K\}. \quad (2)$$

Furthermore, the equivalent conditions for the inclusions

$$B\{1, 3N, 4K\} A\{1, 3M, 4N\} \subseteq (AB)\{1, 3M, 4K\} \quad (3)$$

and

$$B\{1, 3N, 4K\} A\{1, 3M, 4N\} \supseteq (AB)\{1, 3M, 4K\}, \quad (4)$$

are presented.

The following lemmas are essential in the rest of this paper.

Lemma 1 ([2,4]). *Let M and N be two positive-definite Hermitian matrices, and let $A \in \mathbb{C}^{m \times n}$. Then,*

$$X \in A\{1, 3M\} \iff A^* M A X = A^* M, \quad (5)$$

$$X \in A\{1, 4N\} \iff X A N^{-1} A^* = N^{-1} A^*, \quad (6)$$

$$X \in A\{1, 4N\} \iff X^* \in A^*\{1, 3N^{-1}\}. \quad (7)$$

Lemma 2 ([17]). *Suppose that A, B, C and D are four complex matrices, and suppose that M and N are two positive-definite Hermitian matrices. Then,*

$$r(D - C A_{M,N}^\dagger B) = r \begin{pmatrix} A^* M A N^{-1} A^* & A^* M B \\ C N^{-1} A^* & D \end{pmatrix} - r(A); \quad (8)$$

$$\max_{A(1,3M,4N)} r(D - C A^{(1,3M,4N)} B) = \min \left\{ r \begin{pmatrix} A^* M A & A^* M B \\ C & D \end{pmatrix} - r(A), \right. \\ \left. r \begin{pmatrix} A N^{-1} A^* & B \\ C N^{-1} A^* & D \end{pmatrix} - r(A) \right\}; \quad (9)$$

$$\min_{A^{(1,3M,4N)}} r(D - CA^{(1,3M,4N)}B) = r \begin{pmatrix} A^*MA & A^*MB \\ C & D \end{pmatrix} + r \begin{pmatrix} AN^{-1}A^* & B \\ CN^{-1}A^* & D \end{pmatrix} - r \begin{pmatrix} A & B \\ C & D \end{pmatrix} - r(A). \quad (10)$$

Lemma 3 ([29]). Let X and Y be two arbitrary matrices, and let $A_{ij} \in \mathbb{C}^{m_i \times n_j}$ ($1 \leq i, j \leq 3$). Then,

$$\min_{X,Y} r \begin{pmatrix} A_{11} & A_{12} & X \\ A_{21} & A_{22} & A_{23} \\ Y & A_{32} & A_{33} \end{pmatrix} = r \begin{pmatrix} A_{21} & A_{22} & A_{23} \end{pmatrix} + r \begin{pmatrix} A_{12} \\ A_{22} \\ A_{32} \end{pmatrix} + \max \left\{ r \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} - r \begin{pmatrix} A_{12} \\ A_{22} \end{pmatrix} - r \begin{pmatrix} A_{21} & A_{22} \end{pmatrix}, r \begin{pmatrix} A_{22} & A_{23} \\ A_{32} & A_{33} \end{pmatrix} - r \begin{pmatrix} A_{22} \\ A_{32} \end{pmatrix} - r \begin{pmatrix} A_{22} & A_{23} \end{pmatrix} \right\}. \quad (11)$$

The contents of this paper are organized as follows: In Section 2, we first present the equivalent conditions for Inclusions (3) and (4) using the maximal and minimal ranks of the generalized Schur complement and completions of partial matrices. Then, applying these results, we study the reverse order law (2). At last, in Theorem 3, we obtain necessary and sufficient conditions for the reverse order law (2) via rank conditions of known matrices.

2. Main Results

To obtain the main results (Theorem 3), we need present the minimal rank of the generalized Schur complement $D - B^{(1,3N,4K)}A^{(1,3M,4N)}$.

Lemma 4. Let M , N , and K be three positive-definite Hermitian matrices, and let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times k}$, $D \in \mathbb{C}^{k \times m}$. Then,

$$\begin{aligned} & \min_{A^{(1,3M,4N)}, B^{(1,3N,4K)}} r(D - B^{(1,3N,4K)}A^{(1,3M,4N)}) \\ &= r \begin{pmatrix} B^*NBD & B^*N \\ A^*M & A^*MA \end{pmatrix} + r \begin{pmatrix} N^{-1}A^* & BK^{-1}B^* \\ DAN^{-1}A^* & K^{-1}B^* \end{pmatrix} - r(A) - r(B) \\ &+ \max \left\{ r(B^*A^* - B^*NBDAN^{-1}A^*) - r(BDAN^{-1}A^* - N^{-1}A^*) - r(B^*NBDA - B^*N), \right. \\ & \left. r \begin{pmatrix} K^{-1}B^* & D \\ A^*MABK^{-1}B^* & A^*M \end{pmatrix} - r(DABK^{-1}B^* - K^{-1}B^*) - r(A^*MABD - A^*M) - n \right\}. \end{aligned} \quad (12)$$

Proof. It is well known that $A^{(1,3M,4N)} = A_{M,N}^\dagger + \tilde{F}_A V \tilde{E}_A$, where $\tilde{F}_A = I_n - A_{M,N}^\dagger A$, $\tilde{E}_A = I_m - A A_{M,N}^\dagger$ and V is an arbitrary matrix (see [17]). Combining this fact, we have

$$\begin{aligned} & r(D - B^{(1,3N,4K)}A^{(1,3M,4N)}) \\ &= r[(B_{N,K}^\dagger + \tilde{F}_B W \tilde{E}_B)(A_{M,N}^\dagger + \tilde{F}_A V \tilde{E}_A) - D] \\ &= r(B_{N,K}^\dagger A_{M,N}^\dagger + B_{N,K}^\dagger \tilde{F}_A V \tilde{E}_A + \tilde{F}_B W \tilde{E}_B A_{M,N}^\dagger + \tilde{F}_B W \tilde{E}_B \tilde{F}_A V \tilde{E}_A - D) \\ &= r \begin{pmatrix} O & O & O & O & I_n & V \\ O & O & -\tilde{E}_A & O & O & I_m \\ O & O & O & I_n & \tilde{F}_A & O \\ O & \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger & O & O \\ I_n & O & \tilde{E}_B A_{M,N}^\dagger & \tilde{E}_B & O & O \\ W & I_k & O & O & O & O \end{pmatrix} - k - m - 3n. \end{aligned} \quad (13)$$

where $\tilde{F}_A = I_n - A_{M,N}^\dagger A$, $\tilde{E}_A = I_m - A A_{M,N}^\dagger$, $\tilde{F}_B = I_k - B_{N,K}^\dagger B$, $\tilde{E}_B = I_n - B B_{N,K}^\dagger$ and the matrices V and W are arbitrary. Applying Lemma 3, we have

$$\begin{aligned} & \min_{A^{(1,3M,4N)}, B^{(1,3N,4K)}} r(D - B^{(1,3N,4K)} A^{(1,3M,4N)}) \\ &= r\left(\tilde{F}_B, B_{N,K}^\dagger A_{M,N}^\dagger - D, B_{N,K}^\dagger \tilde{F}_A\right) \\ &+ r\left(\begin{array}{c} -\tilde{E}_A \\ B_{N,K}^\dagger A_{M,N}^\dagger - D \\ \tilde{E}_B A_{M,N}^\dagger \end{array}\right) + \max \left\{ r\left(\begin{array}{cc} O & -\tilde{E}_A \\ \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D \end{array}\right) - r\left(\begin{array}{cc} O & -\tilde{E}_A \\ \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D \end{array}\right) \right. \\ &- r\left(\begin{array}{ccc} O & -\tilde{E}_A & O \\ \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger \tilde{F}_A \end{array}\right), r\left(\begin{array}{cc} B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger \tilde{F}_A \\ \tilde{E}_B A_{M,N}^\dagger & \tilde{E}_B \tilde{F}_A \end{array}\right) \\ &- \left. r\left(\begin{array}{cc} -\tilde{E}_A & O \\ B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger \tilde{F}_A \end{array}\right) - r\left(\begin{array}{ccc} \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger \tilde{F}_A \\ O & \tilde{E}_B A_{M,N}^\dagger & \tilde{E}_B \tilde{F}_A \end{array}\right) \right\} \end{aligned} \quad (14)$$

The next step is to use block matrix operations to simplify the rank of the right-hand side of (14). For the first one, combining $\tilde{F}_A = I_n - A_{M,N}^\dagger A$ and $\tilde{F}_B = I_k - B_{N,K}^\dagger B$, we have

$$\begin{aligned} & r\left(\tilde{F}_B, B_{N,K}^\dagger A_{M,N}^\dagger - D, B_{N,K}^\dagger \tilde{F}_A\right) \\ &= r\left(I_k - B_{N,K}^\dagger B, B_{N,K}^\dagger A_{M,N}^\dagger - D, B_{N,K}^\dagger - B_{N,K}^\dagger A_{M,N}^\dagger A\right) \\ &= r\left(I_k - B_{N,K}^\dagger B, B_{N,K}^\dagger A_{M,N}^\dagger - D, B_{N,K}^\dagger - DA\right) \\ &= r\left(\begin{array}{ccccc} B_{N,K}^\dagger & B_{N,K}^\dagger B & -B_{N,K}^\dagger A_{M,N}^\dagger & -B_{N,K}^\dagger & O \\ O & I_k - B_{N,K}^\dagger B & B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger - DA & O \\ O & O & A_{M,N}^\dagger & O & A_{M,N}^\dagger \end{array}\right) - r(A) - r(B) \\ &= r\left(\begin{array}{ccccc} B_{N,K}^\dagger & B_{N,K}^\dagger B & O & -B_{N,K}^\dagger & B_{N,K}^\dagger A_{M,N}^\dagger \\ B_{N,K}^\dagger & I_k & -D & -DA & O \\ O & O & A_{M,N}^\dagger & O & A_{M,N}^\dagger \end{array}\right) - r(A) - r(B) \\ &= r\left(\begin{array}{ccc} B_{N,K}^\dagger BD & B_{N,K}^\dagger BDA - B_{N,K}^\dagger & B_{N,K}^\dagger A_{M,N}^\dagger \\ A_{M,N}^\dagger & O & A_{M,N}^\dagger \end{array}\right) + k - r(A) - r(B) \\ &= r\left(\begin{array}{cc} B_{N,K}^\dagger BD & B_{N,K}^\dagger \\ A_{M,N}^\dagger & A_{M,N}^\dagger A \end{array}\right) + k - r(A) - r(B). \end{aligned} \quad (15)$$

From Definition 1, we have $A^\# = N^{-1} A^* M$, $B^\# = K^{-1} B^* N$ and

$$\left(\begin{array}{cc} B^\# B & O \\ O & A^\# A \end{array}\right) \left(\begin{array}{cc} B_{N,K}^\dagger BD & B_{N,K}^\dagger \\ A_{M,N}^\dagger & A_{M,N}^\dagger A \end{array}\right) = \left(\begin{array}{cc} B^\# BD & B^\# \\ A^\# & A^\# A \end{array}\right)$$

and

$$\left(\begin{array}{cc} B_{N,K}^\dagger (B_{N,K}^\dagger)^\# & O \\ O & A_{M,N}^\dagger (A_{M,N}^\dagger)^\# \end{array}\right) \left(\begin{array}{cc} B^\# BD & B^\# \\ A^\# & A^\# A \end{array}\right) = \left(\begin{array}{cc} B_{N,K}^\dagger BD & B_{N,K}^\dagger \\ A_{M,N}^\dagger & A_{M,N}^\dagger A \end{array}\right).$$

That is,

$$r\left(\begin{array}{cc} B_{N,K}^\dagger BD & B_{N,K}^\dagger \\ A_{M,N}^\dagger & A_{M,N}^\dagger A \end{array}\right) = r\left(\begin{array}{cc} B^\# BD & B^\# \\ A^\# & A^\# A \end{array}\right) = r\left(\begin{array}{cc} B^* NBD & B^* N \\ A^* M & A^* MA \end{array}\right). \quad (16)$$

Substituting Formula (16) into (15), we have

$$r\left(\tilde{F}_B, \quad B_{N,K}^\dagger A_{M,N}^\dagger - D, \quad B_{N,K}^\dagger \tilde{F}_A\right) = r\left(\begin{array}{cc} B^* N B D & B^* N \\ A^* M & A^* M A \end{array}\right) + k - r(A) - r(B). \quad (17)$$

For the second partitioned matrix of the right-hand of (14), we have

$$\begin{aligned} r\left(\begin{array}{cc} -\tilde{E}_A & \\ B_{N,K}^\dagger A_{M,N}^\dagger - D & \\ \tilde{E}_B A_{M,N}^\dagger & \end{array}\right) &= r\left(\begin{array}{cc} I_n & A_{M,N}^\dagger \\ O & -\tilde{E}_A \\ -B_{N,K}^\dagger & -D \\ -\tilde{E}_B & O \end{array}\right) - n \\ &= r\left(\begin{array}{cccc} A_{M,N}^\dagger & O & A_{M,N}^\dagger & O \\ O & I_n & A_{M,N}^\dagger & O \\ O & O & -\tilde{E}_A & O \\ O & -B_{N,K}^\dagger & -D & O \\ O & -\tilde{E}_B & O & O \\ O & B_{N,K}^\dagger & O & B_{N,K}^\dagger \end{array}\right) - n - r(A) - r(B) \\ &= r\left(\begin{array}{cccc} A_{M,N}^\dagger & O & A_{M,N}^\dagger & O \\ -A_{M,N}^\dagger & I_n & O & O \\ -A A_{M,N}^\dagger & O & -I_m & O \\ O & O & -D & O \\ O & -I_n & O & -B B_{N,K}^\dagger \\ O & B_{N,K}^\dagger & O & B_{N,K}^\dagger \end{array}\right) - n - r(A) - r(B) \\ &= r\left(\begin{array}{cc} A_{M,N}^\dagger & B B_{N,K}^\dagger \\ D A A_{M,N}^\dagger & B_{N,K}^\dagger \end{array}\right) + m - r(A) - r(B). \end{aligned} \quad (18)$$

Similarly to (16), we have

$$\left(\begin{array}{cc} A_{M,N}^\dagger & B B_{N,K}^\dagger \\ D A A_{M,N}^\dagger & B_{N,K}^\dagger \end{array}\right) \left(\begin{array}{cc} A A^\# & O \\ O & B B^\# \end{array}\right) = \left(\begin{array}{cc} A^\# & B B^\# \\ D A A^\# & B^\# \end{array}\right)$$

and

$$\left(\begin{array}{cc} A^\# & B B^\# \\ D A A^\# & B^\# \end{array}\right) \left(\begin{array}{cc} (A_{M,N}^\dagger)^\# A_{M,N}^\dagger & O \\ O & (B_{N,K}^\dagger)^\# B_{N,K}^\dagger \end{array}\right) = \left(\begin{array}{cc} A_{M,N}^\dagger & B B_{N,K}^\dagger \\ D A A_{M,N}^\dagger & B_{N,K}^\dagger \end{array}\right).$$

That is,

$$r\left(\begin{array}{cc} A_{M,N}^\dagger & B B_{N,K}^\dagger \\ D A A_{M,N}^\dagger & B_{N,K}^\dagger \end{array}\right) = r\left(\begin{array}{cc} A^\# & B B^\# \\ D A A^\# & B^\# \end{array}\right) = r\left(\begin{array}{cc} N^{-1} A^* & B K^{-1} B^* \\ D A N^{-1} A^* & K^{-1} B^* \end{array}\right). \quad (19)$$

Substituting Formula (19) into (18), we have

$$r\left(\begin{array}{cc} -\tilde{E}_A & \\ B_{N,K}^\dagger A_{M,N}^\dagger - D & \\ \tilde{E}_B A_{M,N}^\dagger & \end{array}\right) = r\left(\begin{array}{cc} N^{-1} A^* & B K^{-1} B^* \\ D A N^{-1} A^* & K^{-1} B^* \end{array}\right) + m - r(A) - r(B). \quad (20)$$

Substituting $\tilde{E}_A = I_m - A A_{M,N}^\dagger$ and $\tilde{F}_B = I_k - B_{N,K}^\dagger B$ into the third block matrix in (14), we have

$$\begin{aligned}
 & r \begin{pmatrix} O & -\tilde{E}_A \\ \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D \end{pmatrix} = r \begin{pmatrix} A_{M,N}^\dagger & O & A_{M,N}^\dagger & O \\ O & O & -\tilde{E}_A & O \\ O & \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger \\ O & O & O & B_{N,K}^\dagger \end{pmatrix} - r(A) - r(B) \\
 & = r \begin{pmatrix} A_{M,N}^\dagger & O & A_{M,N}^\dagger & O \\ -AA_{M,N}^\dagger & O & -I_m & O \\ O & I_k & -D & B_{N,K}^\dagger \\ B_{N,K}^\dagger A_{M,N}^\dagger & B_{N,K}^\dagger B & O & B_{N,K}^\dagger \end{pmatrix} - r(A) - r(B) \\
 & = r(B_{N,K}^\dagger A_{M,N}^\dagger - B_{N,K}^\dagger BDAA_{M,N}^\dagger) + k + m - r(A) - r(B). \quad (21)
 \end{aligned}$$

This is because

$$\begin{aligned}
 r(B^\# A^\# - B^\# BDAA^\#) &= r[B^\# B(B_{N,K}^\dagger A_{M,N}^\dagger - B_{N,K}^\dagger BDAA_{M,N}^\dagger)AA^\#] \\
 &\leq r(B_{N,K}^\dagger A_{M,N}^\dagger - B_{N,K}^\dagger BDAA_{M,N}^\dagger)
 \end{aligned}$$

and

$$\begin{aligned}
 r(B_{N,K}^\dagger A_{M,N}^\dagger - B_{N,K}^\dagger BDAA_{M,N}^\dagger) &= r[B_{N,K}^\dagger (B_{N,K}^\dagger)^\# (B^\# A^\# - B^\# BDAA^\#) (A_{M,N}^\dagger)^\# A_{M,N}^\dagger] \\
 &\leq r(B^\# A^\# - B^\# BDAA^\#).
 \end{aligned}$$

Then,

$$\begin{aligned}
 r(B_{N,K}^\dagger A_{M,N}^\dagger - B_{N,K}^\dagger BDAA_{M,N}^\dagger) &= r(B^\# A^\# - B^\# BDAA^\#) \\
 &= r(B^* A^* - B^* NBDAN^{-1} A^*) \quad (22)
 \end{aligned}$$

That is,

$$r \begin{pmatrix} O & -\tilde{E}_A \\ \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D \end{pmatrix} = r(B^* A^* - B^* NBDAN^{-1} A^*) + k + m - r(A) - r(B). \quad (23)$$

For the fourth partitioned matrix on the right-hand of (14), we have

$$\begin{aligned}
 & r \begin{pmatrix} O & -\tilde{E}_A \\ \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D \end{pmatrix} = r \begin{pmatrix} O & -\tilde{E}_A \\ \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D \end{pmatrix} = r \begin{pmatrix} O & -\tilde{E}_A \\ \tilde{F}_B & B_{N,K}^\dagger BD - D \end{pmatrix} \\
 & = r \begin{pmatrix} O & -\tilde{E}_A \\ \tilde{F}_B & O \end{pmatrix} = r(\tilde{F}_B) + r \begin{pmatrix} A_{M,N}^\dagger & A_{M,N}^\dagger \\ O & \tilde{E}_A \end{pmatrix} - r(A) \\
 & = k - r(B) + r \begin{pmatrix} A_{M,N}^\dagger & A_{M,N}^\dagger \\ AA_{M,N}^\dagger & I_m \end{pmatrix} - r(A) \\
 & = r(BDAA_{M,N}^\dagger - A_{M,N}^\dagger) + k + m - r(B) - r(A). \quad (24)
 \end{aligned}$$

Using Definition 1, we obtain

$$r(BDAA_{M,N}^\dagger - A_{M,N}^\dagger) = r[(BDAA^\# - A^\#)(A_{M,N}^\dagger)^\# A_{M,N}^\dagger] \leq r(BDAA^\# - A^\#)$$

and

$$r(BDAA^\# - A^\#) = r[(BDAA_{M,N}^\dagger - A_{M,N}^\dagger)AA^\#] \leq r(BDAA_{M,N}^\dagger - A_{M,N}^\dagger).$$

That is,

$$r(BDAA_{M,N}^\dagger - A_{M,N}^\dagger) = r(BDAA^\# - A^\#) = r(BDAN^{-1}A^* - N^{-1}A^*). \quad (25)$$

Substituting Formula (25) into (24), we have

$$r \begin{pmatrix} O & -\tilde{E}_A \\ \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D \\ O & \tilde{E}_B A_{M,N}^\dagger \end{pmatrix} = r(BDAN^{-1}A^* - N^{-1}A^*) + k + m - r(B) - r(A). \quad (26)$$

For the fifth block matrix in (14), via the row or column elementary block matrix operations and $\tilde{E}_A A = O$, $\tilde{E}_A = I_m - AA_{M,N}^\dagger$, $\tilde{F}_A = I_n - A_{M,N}^\dagger A$ and $\tilde{F}_B = I_k - B_{N,K}^\dagger B$, we have

$$\begin{aligned} & r \begin{pmatrix} O & -\tilde{E}_A & O \\ \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger \tilde{F}_A \end{pmatrix} = r \begin{pmatrix} O & -\tilde{E}_A & O \\ \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger - DA \\ O & \tilde{E}_B A_{M,N}^\dagger & O \end{pmatrix} \\ & = r \begin{pmatrix} A_{M,N}^\dagger & O & A_{M,N}^\dagger & O & O \\ O & O & \tilde{E}_A & O & O \\ O & \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger - DA & B_{N,K}^\dagger \\ O & O & O & O & B_{N,K}^\dagger \end{pmatrix} - r(A) - r(B) \\ & = r \begin{pmatrix} A_{M,N}^\dagger & O & A_{M,N}^\dagger & O & O \\ AA_{M,N}^\dagger & O & I_m & O & O \\ O & I_k & -D & -DA & B_{N,K}^\dagger \\ B_{N,K}^\dagger A_{M,N}^\dagger & B_{N,K}^\dagger B & O & -B_{N,K}^\dagger & B_{N,K}^\dagger \end{pmatrix} - r(A) - r(B) \\ & = r(B_{N,K}^\dagger BDA - B_{N,K}^\dagger) + m + k - r(A) - r(B). \end{aligned} \quad (27)$$

From Definition 1, we have

$$\begin{aligned} r(B_{N,K}^\dagger BDA - B_{N,K}^\dagger) &= r[B_{N,K}^\dagger (B_{N,K}^\dagger)^\# (B^\# BDA - B^\#)] \leq r(B^\# BDA - B^\#) \\ &= r[B^\# B (B_{N,K}^\dagger BDA - B_{N,K}^\dagger)] \leq r(B_{N,K}^\dagger BDA - B_{N,K}^\dagger). \end{aligned}$$

That is,

$$r(B_{N,K}^\dagger BDA - B_{N,K}^\dagger) = r(B^\# BDA - B^\#) = r(B^* NBDA - B^* N). \quad (28)$$

Substituting Formula (28) into (27), we have

$$r \begin{pmatrix} O & -\tilde{E}_A & O \\ \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger \tilde{F}_A \end{pmatrix} = r(B^* NBDA - B^* N) + m + k - r(A) - r(B). \quad (29)$$

For the sixth block matrix in (14), using the same method as above, we have

$$\begin{aligned}
& r \begin{pmatrix} B_{N,K}^+ A_{M,N}^+ - D & B_{N,K}^+ \tilde{F}_A \\ \tilde{E}_B A_{M,N}^+ & \tilde{E}_B \tilde{F}_A \end{pmatrix} = r \begin{pmatrix} B_{N,K}^+ A_{M,N}^+ - D & B_{N,K}^+ \tilde{F}_A \\ A_{M,N}^+ - BD & \tilde{F}_A \end{pmatrix} \\
& = r \begin{pmatrix} B_{N,K}^+ BD - D & O \\ A_{M,N}^+ - BD & \tilde{F}_A \end{pmatrix} = r \begin{pmatrix} B_{N,K}^+ & O & O & O \\ B_{N,K}^+ & B_{N,K}^+ BD - D & O & O \\ O & A_{M,N}^+ - BD & \tilde{F}_A & A_{M,N}^+ \\ O & O & O & A_{M,N}^+ \end{pmatrix} - r(A) - r(B) \\
& = r \begin{pmatrix} B_{N,K}^+ & -B_{N,K}^+ BD & O & O \\ B_{N,K}^+ & -D & O & O \\ O & -BD & I_n & A_{M,N}^+ \\ O & -A_{M,N}^+ & A_{M,N}^+ A & A_{M,N}^+ \end{pmatrix} - r(A) - r(B) \\
& = r \begin{pmatrix} B_{N,K}^+ & D \\ A_{M,N}^+ A B B_{N,K}^+ & A_{M,N}^+ \end{pmatrix} + n - r(A) - r(B). \tag{30}
\end{aligned}$$

From Definition 1, we have

$$\begin{aligned}
& r \begin{pmatrix} B_{N,K}^+ & D \\ A_{M,N}^+ A B B_{N,K}^+ & A_{M,N}^+ \end{pmatrix} \\
& = r \left\{ \begin{pmatrix} I & O \\ O & A_{M,N}^+ (A_{M,N}^+)^{\#} \end{pmatrix} \begin{pmatrix} B^{\#} & D \\ A^{\#} A B B^{\#} & A^{\#} \end{pmatrix} \begin{pmatrix} (B_{N,K}^+)^{\#} B_{N,K}^+ & O \\ O & I \end{pmatrix} \right\} \\
& \leq r \begin{pmatrix} B^{\#} & D \\ A^{\#} A B B^{\#} & A^{\#} \end{pmatrix}
\end{aligned}$$

and

$$\begin{aligned}
& r \begin{pmatrix} B^{\#} & D \\ A^{\#} A B B^{\#} & A^{\#} \end{pmatrix} \\
& = r \left\{ \begin{pmatrix} I & O \\ O & A^{\#} A \end{pmatrix} \begin{pmatrix} B_{N,K}^+ & D \\ A_{M,N}^+ A B B_{N,K}^+ & A_{M,N}^+ \end{pmatrix} \begin{pmatrix} B B^{\#} & O \\ O & I \end{pmatrix} \right\} \\
& \leq r \begin{pmatrix} B_{N,K}^+ & D \\ A_{M,N}^+ A B B_{N,K}^+ & A_{M,N}^+ \end{pmatrix}.
\end{aligned}$$

That is,

$$r \begin{pmatrix} B_{N,K}^+ & D \\ A_{M,N}^+ A B B_{N,K}^+ & A_{M,N}^+ \end{pmatrix} = r \begin{pmatrix} B^{\#} & D \\ A^{\#} A B B^{\#} & A^{\#} \end{pmatrix} = r \begin{pmatrix} K^{-1} B^* & D \\ A^* M A B K^{-1} B^* & A^* M \end{pmatrix} \tag{31}$$

Substituting Formula (31) into (30), we have

$$r \begin{pmatrix} B_{N,K}^+ A_{M,N}^+ - D & B_{N,K}^+ \tilde{F}_A \\ \tilde{E}_B A_{M,N}^+ & \tilde{E}_B \tilde{F}_A \end{pmatrix} = r \begin{pmatrix} K^{-1} B^* & D \\ A^* M A B K^{-1} B^* & A^* M \end{pmatrix} + n - r(A) - r(B). \tag{32}$$

For the seventh partitioned matrix in (14), using the same method as above with $\tilde{E}_A A = O$, $\tilde{E}_A = I_m - A A_{M,N}^\dagger$, $\tilde{F}_A = I_n - A_{M,N}^\dagger A$ and $\tilde{E}_B = I_n - B B_{N,K}^\dagger$, we have

$$\begin{aligned} & r \begin{pmatrix} -\tilde{E}_A & O \\ B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger \tilde{F}_A \\ \tilde{E}_B A_{M,N}^\dagger & \tilde{E}_B \tilde{F}_A \end{pmatrix} = r \begin{pmatrix} -\tilde{E}_A & O \\ B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger - D A \\ \tilde{E}_B A_{M,N}^\dagger & \tilde{E}_B \end{pmatrix} \\ & = r \begin{pmatrix} -\tilde{E}_A & O \\ D A A_{M,N}^\dagger - D & B_{N,K}^\dagger - D A \\ O & \tilde{E}_B \end{pmatrix} \\ & = r \begin{pmatrix} A_{M,N}^\dagger & A_{M,N}^\dagger & O & O \\ O & -\tilde{E}_A & O & O \\ O & D A A_{M,N}^\dagger - D & B_{N,K}^\dagger - D A & O \\ O & O & \tilde{E}_B & O \\ O & O & B_{N,K}^\dagger & B_{N,K}^\dagger \end{pmatrix} - r(A) - r(B) \\ & = r \begin{pmatrix} A_{M,N}^\dagger & A_{M,N}^\dagger & O & O \\ -A A_{M,N}^\dagger & -I_m & O & O \\ -D A A_{M,N}^\dagger & -D & -D A & -B_{N,K}^\dagger \\ O & O & I_n & B B_{N,K}^\dagger \\ O & O & B_{N,K}^\dagger & B_{N,K}^\dagger \end{pmatrix} - r(A) - r(B) \\ & = r(D A B B_{N,K}^\dagger - B_{N,K}^\dagger) + m + n - r(A) - r(B). \end{aligned} \quad (33)$$

From Definition 1, we have

$$r(D A B B_{N,K}^\dagger - B_{N,K}^\dagger) = r[(D A B B^\# - B^\#)(B_{N,K}^\dagger)^\# B_{N,K}^\dagger] \leq r(D A B B^\# - B^\#)$$

and

$$r(D A B B^\# - B^\#) = r[(D A B B_{N,K}^\dagger - B_{N,K}^\dagger) B B^\#] \leq r(D A B B_{N,K}^\dagger - B_{N,K}^\dagger).$$

That is,

$$r(D A B B_{N,K}^\dagger - B_{N,K}^\dagger) = r(D A B B^\# - B^\#) = r(D A B K^{-1} B^* - K^{-1} B^*). \quad (34)$$

Substituting Formula (34) into (33), we have

$$r \begin{pmatrix} -\tilde{E}_A & O \\ B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger \tilde{F}_A \\ \tilde{E}_B A_{M,N}^\dagger & \tilde{E}_B \tilde{F}_A \end{pmatrix} = r(D A B K^{-1} B^* - K^{-1} B^*) + n + m - r(A) - r(B). \quad (35)$$

For the last partitioned matrix in (14), using the basic block matrix operations for rows or columns and $B \tilde{F}_B = O$, $\tilde{F}_A = I_n - A_{M,N}^\dagger A$, $\tilde{F}_B = I_k - B_{N,K}^\dagger B$ and $\tilde{E}_B = I_n - B B_{N,K}^\dagger$, we have

$$\begin{aligned}
& r \begin{pmatrix} \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger \tilde{F}_A \\ O & \tilde{E}_B A_{M,N}^\dagger & \tilde{E}_B \tilde{F}_A \end{pmatrix} = r \begin{pmatrix} \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger \tilde{F}_A \\ O & A_{M,N}^\dagger - BD & \tilde{F}_A \end{pmatrix} \\
& = r \begin{pmatrix} \tilde{F}_B & B_{N,K}^\dagger BD - D & O \\ O & A_{M,N}^\dagger - BD & \tilde{F}_A \end{pmatrix} \\
& = r \begin{pmatrix} B_{N,K}^\dagger & O & O & O & O \\ B_{N,K}^\dagger & \tilde{F}_B & B_{N,K}^\dagger BD - D & O & O \\ O & O & A_{M,N}^\dagger - BD & \tilde{F}_A & A_{M,N}^\dagger \\ O & O & O & O & A_{M,N}^\dagger \end{pmatrix} - r(A) - r(B) \\
& = r \begin{pmatrix} B_{N,K}^\dagger & B_{N,K}^\dagger B & -B_{N,K}^\dagger BD & O & O \\ B_{N,K}^\dagger & I_k & -D & O & O \\ O & O & -BD & I_n & A_{M,N}^\dagger \\ O & O & -A_{M,N}^\dagger & A_{M,N}^\dagger A & A_{M,N}^\dagger \end{pmatrix} - r(A) - r(B) \\
& = r(A_{M,N}^\dagger ABD - A_{M,N}^\dagger) + n + k - r(A) - r(B). \tag{36}
\end{aligned}$$

From Definition 1, we have

$$r(A_{M,N}^\dagger ABD - A_{M,N}^\dagger) = r[A_{M,N}^\dagger (A_{M,N}^\dagger)^\# (A^\# ABD - A^\#)] \leq r(A^\# ABD - A^\#)$$

and

$$r(A^\# ABD - A^\#) = r[A^\# A (A_{M,N}^\dagger ABD - A_{M,N}^\dagger)] \leq r(A_{M,N}^\dagger ABD - A_{M,N}^\dagger).$$

That is,

$$r(A_{M,N}^\dagger ABD - A_{M,N}^\dagger) = r(A^\# ABD - A^\#) = r(A^* M ABD - A^* M). \tag{37}$$

Substituting Formula (37) into (36), we have

$$r \begin{pmatrix} \tilde{F}_B & B_{N,K}^\dagger A_{M,N}^\dagger - D & B_{N,K}^\dagger \tilde{F}_A \\ O & \tilde{E}_B A_{M,N}^\dagger & \tilde{E}_B \tilde{F}_A \end{pmatrix} = r(A^* M ABD - A^* M) + n + k - r(A) - r(B). \tag{38}$$

According to the above proof process, by substituting Formulas (17), (20), (23), (26), (29), (32), (35) and (38) into (14), we have Formula (12). Thus, Lemma 4 is proved. $\square$

Theorem 1. Let M, N , and K be three positive-definite Hermitian matrices, and let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times k}$. Then,

$$\begin{aligned}
& (AB)\{1, 3M, 4K\} \subseteq B\{1, 3N, 4K\}A\{1, 3M, 4N\} \\
& \iff r(AB) = \max \left\{ \min\{a_1, a_2\}, \min\{k - r(B), m - r(A)\} + a_3 + a_4 - n \right\}. \tag{39}
\end{aligned}$$

where

$$a_1 = r \begin{pmatrix} B^* NB & B^* A^* \\ B^* A^* MAB & B^* A^* M A N^{-1} A^* \end{pmatrix}, \quad a_3 = r \begin{pmatrix} B^* N \\ B^* A^* M A \end{pmatrix} \tag{40}$$

and

$$a_2 = r \begin{pmatrix} AN^{-1}A^* & ABK^{-1}B^*A^* \\ B^*A^* & B^*NBK^{-1}B^*A^* \end{pmatrix}, \quad a_4 = r \begin{pmatrix} AN^{-1} \\ ABK^{-1}B^* \end{pmatrix}. \quad (41)$$

Proof. We can see that there are some $(AB)^{(1,3M,4K)} \in (AB)\{1, 3M, 4K\}$, $A^{(1,3M,4N)} \in A\{1, 3M, 4N\}$ and $B^{(1,3N,4K)} \in B\{1, 3N, 4K\}$, such that the following three formulas are equivalent:

$$(AB)\{1, 3M, 4K\} \subseteq B\{1, 3N, 4K\}A\{1, 3M, 4N\}, \quad (42)$$

$$B^{(1,3N,4K)}A^{(1,3M,4N)} = (AB)^{(1,3M,4K)} \quad (43)$$

and

$$\max_{(AB)^{(1,3M,4K)}} \min_{A^{(1,3M,4N)}, B^{(1,3N,4K)}} r((AB)^{(1,3M,4K)} - B^{(1,3N,4K)}A^{(1,3M,4N)}) = 0. \quad (44)$$

It is well known that $AB(AB)_{M,K}^\dagger = AB(AB)^{(1,3M,4K)} = P_{R(AB), N(B^\#A^\#)}$ and $(AB)_{M,K}^\dagger AB = (AB)^{(1,3M,4K)}AB = P_{R(B^\#A^\#), N(AB)}$ (see [4]). Combining this fact with Formula (12) in Lemma 4, we have

$$\begin{aligned} & \min_{A^{(1,3M,4N)}, B^{(1,3N,4K)}} r((AB)^{(1,3M,4K)} - B^{(1,3N,4K)}A^{(1,3M,4N)}) \\ &= r \begin{pmatrix} O & B^*N \\ A^*M - A^*MAB(AB)_{M,K}^\dagger & A^*MA \end{pmatrix} + r \begin{pmatrix} N^{-1}A^* & BK^{-1}B^* \\ O & K^{-1}B^* - (AB)_{M,K}^\dagger ABK^{-1}B^* \end{pmatrix} \\ &- r(A) - r(B) + \max \left\{ r(B^*A^* - B^*NB(AB)^{(1,3M,4K)}AN^{-1}A^*) \right. \\ &- r(N^{-1}A^* - B(AB)^{(1,3M,4K)}AN^{-1}A^*) - r(B^*NB(AB)^{(1,3M,4K)}A - B^*N), \\ &r \begin{pmatrix} A^*M & A^*MABK^{-1}B^* \\ (AB)^{(1,3M,4K)} & K^{-1}B^* \end{pmatrix} - r((AB)_{M,K}^\dagger ABK^{-1}B^* - K^{-1}B^*) \\ &- \left. r(A^*MAB(AB)_{M,K}^\dagger - A^*M) - n \right\}. \quad (45) \end{aligned}$$

Applying (8) of Lemma 2, we have

$$\begin{aligned} & r \begin{pmatrix} O & B^*N \\ A^*M - A^*MAB(AB)_{M,K}^\dagger & A^*MA \end{pmatrix} \\ &= r \left[\begin{pmatrix} O & B^*N \\ A^*M & A^*MA \end{pmatrix} - \begin{pmatrix} O \\ A^*MAB \end{pmatrix} (AB)_{M,K}^\dagger \begin{pmatrix} I_m & O \end{pmatrix} \right] \\ &= r \begin{pmatrix} (AB)^*MABK^{-1}(AB)^* & (AB)^*M & O \\ O & O & B^*N \\ A^*MABK^{-1}(AB)^* & A^*M & A^*MA \end{pmatrix} - r(AB) \\ &= a_3 + r(A) - r(AB) \quad (46) \end{aligned}$$

and

$$\begin{aligned}
 & r \begin{pmatrix} N^{-1}A^* & BK^{-1}B^* \\ O & K^{-1}B^* - (AB)_{M,K}^\dagger ABK^{-1}B^* \end{pmatrix} \\
 = & r \left[\begin{pmatrix} N^{-1}A^* & BK^{-1}B^* \\ O & K^{-1}B^* \end{pmatrix} - \begin{pmatrix} O \\ I_k \end{pmatrix} (AB)_{M,K}^\dagger \begin{pmatrix} O & ABK^{-1}B^* \end{pmatrix} \right] \\
 = & r \begin{pmatrix} (AB)^* MABK^{-1}(AB)^* & O & (AB)^* MABK^{-1}B^* \\ O & N^{-1}A^* & BK^{-1}B^* \\ K^{-1}(AB)^* & O & K^{-1}B^* \end{pmatrix} - r(AB) \\
 = & a_4 + r(B) - r(AB)
 \end{aligned} \tag{47}$$

and

$$\begin{aligned}
 & r((AB)_{M,K}^\dagger ABK^{-1}B^* - K^{-1}B^*) = r(K^{-1}B^* - (AB)_{M,K}^\dagger ABK^{-1}B^*) \\
 = & r \begin{pmatrix} (AB)^* MABK^{-1}(AB)^* & (AB)^* MABK^{-1}B^* \\ K^{-1}(AB)^* & K^{-1}B^* \end{pmatrix} - r(AB) \\
 = & r(B) - r(AB)
 \end{aligned} \tag{48}$$

and

$$\begin{aligned}
 & r(A^* MAB(AB)_{M,K}^\dagger - A^* M) = r(A^* M - A^* MAB(AB)_{M,K}^\dagger) \\
 = & r \begin{pmatrix} (AB)^* MABK^{-1}(AB)^* & (AB)^* M \\ A^* MABK^{-1}(AB)^* & A^* M \end{pmatrix} - r(AB) \\
 = & r(A) - r(AB).
 \end{aligned} \tag{49}$$

Applying (9) and (10) of Lemma 2, we have

$$\begin{aligned}
 & \min_{(AB)^{(1,3M,4K)}} r(N^{-1}A^* - B(AB)^{(1,3M,4K)} AN^{-1}A^*) \\
 = & r \begin{pmatrix} (AB)^* MAB & (AB)^* MAN^{-1}A^* \\ B & N^{-1}A^* \end{pmatrix} + r \begin{pmatrix} ABK^{-1}(AB)^* & AN^{-1}A^* \\ BK^{-1}(AB)^* & N^{-1}A^* \end{pmatrix} \\
 - & r \begin{pmatrix} AB & AN^{-1}A^* \\ B & N^{-1}A^* \end{pmatrix} - r(AB) \\
 = & a_4 - r(AB) \\
 = & \max_{(AB)^{(1,3M,4K)}} r(N^{-1}A^* - B(AB)^{(1,3M,4K)} AN^{-1}A^*)
 \end{aligned}$$

and

$$\begin{aligned}
 & \min_{(AB)^{(1,3M,4K)}} r(B^* NB(AB)^{(1,3M,4K)} A - B^* N) \\
 = & \min_{(AB)^{(1,3M,4K)}} r(B^* N - B^* NB(AB)^{(1,3M,4K)} A) \\
 = & r \begin{pmatrix} (AB)^* MAB & (AB)^* MA \\ B^* NB & B^* N \end{pmatrix} + r \begin{pmatrix} ABK^{-1}(AB)^* & A \\ B^* NBK^{-1}(AB)^* & B^* N \end{pmatrix} \\
 - & r \begin{pmatrix} AB & A \\ B^* NB & B^* N \end{pmatrix} - r(AB) \\
 = & a_3 - r(AB) \\
 = & \max_{(AB)^{(1,3M,4K)}} r(B^* NB(AB)^{(1,3M,4K)} A - B^* N).
 \end{aligned}$$

For any $(AB)^{(1,3M,4K)} \in (AB)\{1, 3M, 4K\}$, we have

$$r(N^{-1}A^* - B(AB)^{(1,3M,4K)}AN^{-1}A^*) = a_4 - r(AB) \quad (50)$$

and

$$r(B^*NB(AB)^{(1,3M,4K)}A - B^*N) = a_3 - r(AB). \quad (51)$$

Combining Formulas (45)–(51), we have

$$\begin{aligned} & \max_{(AB)^{(1,3M,4K)}} \min_{A^{(1,3M,4N)}, B^{(1,3N,4K)}} r((AB)^{(1,3M,4K)} - B^{(1,3N,4K)}A^{(1,3M,4N)}) \\ &= \max \left\{ \max_{(AB)^{(1,3M,4K)}} r(B^*A^* - B^*NB(AB)^{(1,3M,4K)}AN^{-1}A^*), \right. \\ & \quad \left. \max_{(AB)^{(1,3M,4K)}} r \begin{pmatrix} A^*M & A^*MABK^{-1}B^* \\ (AB)^{(1,3M,4K)} & K^{-1}B^* \end{pmatrix} + a_3 + a_4 - r(A) - r(B) - n \right\}. \quad (52) \end{aligned}$$

Using (9) of Lemma 2 with $A = AB$, $B = AN^{-1}A^*$, $C = B^*NB$ and $D = B^*A^*$, we obtain

$$\begin{aligned} & \max_{(AB)^{(1,3M,4K)}} r(B^*A^* - B^*NB(AB)^{(1,3M,4K)}AN^{-1}A^*) \\ &= \min \left\{ r \begin{pmatrix} (AB)^*MAB & (AB)^*MAN^{-1}A^* \\ B^*NB & B^*A^* \end{pmatrix} - r(AB), \right. \\ & \quad \left. r \begin{pmatrix} ABK^{-1}(AB)^* & AN^{-1}A^* \\ B^*NBK^{-1}(AB)^* & B^*A^* \end{pmatrix} - r(AB) \right\} \\ &= \min \left\{ a_1 - r(AB), a_2 - r(AB) \right\}. \quad (53) \end{aligned}$$

Using (9) of Lemma 2 with $\tilde{A} = AB$, $\tilde{X} = (AB)^{(1,3M,4K)}$, $\tilde{B} = \begin{pmatrix} I_m & O \end{pmatrix}$, $\tilde{C} = \begin{pmatrix} O \\ -I_k \end{pmatrix}$ and $\tilde{D} = \begin{pmatrix} A^*M & A^*MABK^{-1}B^* \\ O & K^{-1}B^* \end{pmatrix}$, we obtain

$$\begin{aligned} & \max_{(AB)^{(1,3M,4K)}} r \begin{pmatrix} A^*M & A^*MABK^{-1}B^* \\ (AB)^{(1,3M,4K)} & K^{-1}B^* \end{pmatrix} = \max_{(AB)^{(1,3M,4K)}} r(\tilde{D} - \tilde{C}\tilde{X}\tilde{B}) \\ &= \min \left\{ r \begin{pmatrix} (AB)^*MAB & (AB)^*M & O \\ O & A^*M & A^*MABK^{-1}B^* \\ -I_k & O & K^{-1}B^* \end{pmatrix}, \right. \\ & \quad \left. r \begin{pmatrix} ABK^{-1}(AB)^* & I_m & O \\ O & A^*M & A^*MABK^{-1}B^* \\ -K^{-1}(AB)^* & O & K^{-1}B^* \end{pmatrix} \right\} - r(AB) \\ &= \min \left\{ r(A) - r(AB) + k, r(B) - r(AB) + m \right\}. \quad (54) \end{aligned}$$

Combining Formulas (52), (53) and (54), we have

$$\begin{aligned} & \max_{(AB)^{(1,3M,4K)}} \min_{A^{(1,3M,4N)}, B^{(1,3N,4K)}} r((AB)^{(1,3M,4K)} - B^{(1,3N,4K)} A^{(1,3M,4N)}) \\ &= \max \left\{ \min\{a_1, a_2\}, \min\{k - r(B), m - r(A)\} + a_3 + a_4 - n \right\} - r(AB). \end{aligned} \quad (55)$$

According to Formulas (42), (43) and (44) (i.e., (55)), we obtain Formula (39). $\square$

Theorem 2. Let M , N , and K be three positive-definite Hermitian matrices, and let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{n \times k}$. Then,

$$\begin{aligned} & B\{1, 3N, 4K\}A\{1, 3M, 4N\} \subseteq (AB)\{1, 3M, 4K\} \\ \iff & \min \left\{ a_1 + m - r(A), a_3 \right\} = r(B) \text{ and } \min \left\{ a_2 + k - r(B), a_4 \right\} = r(A), \end{aligned} \quad (56)$$

where a_1, a_2, a_3 and a_4 are the same as in Theorem 1.

Proof. For the convenience of future research, we simply provide an idea for proving this theorem. For any $A^{(1,3M,4N)} \in A\{1, 3M, 4N\}$ and $B^{(1,3N,4K)} \in B\{1, 3N, 4K\}$, according to Lemma 1, Formulas (57)–(59) are equivalent, as follows:

$$B\{1, 3N, 4K\}A\{1, 3M, 4N\} \subseteq (AB)\{1, 3M, 4K\}. \quad (57)$$

$$\begin{aligned} & (AB)^* M A B B^{(1,3N,4K)} A^{(1,3M,4N)} = (AB)^* M \\ \text{and} \\ & B^{(1,3N,4K)} A^{(1,3M,4N)} A B K^{-1} (AB)^* = K^{-1} (AB)^*. \end{aligned} \quad (58)$$

and

$$\begin{aligned} & \max_{A^{(1,3M,4N)}, B^{(1,3N,4K)}} r(B^* A^* M - B^* A^* M A B B^{(1,3N,4K)} A^{(1,3M,4N)}) = 0 \\ \text{and} \\ & \max_{B^{(1,3N,4K)}, A^{(1,3M,4N)}} r(K^{-1} B^* A^* - B^{(1,3N,4K)} A^{(1,3M,4N)} A B K^{-1} B^* A^*) = 0. \end{aligned} \quad (59)$$

From (9) of Lemma 2 with $A = A$, $B = I_m$, $C = B^* A^* M A B B^{(1,3N,4K)}$ and $D = B^* A^* M$, we have

$$\begin{aligned} & \max_{A^{(1,3M,4N)}} r(B^* A^* M - B^* A^* M A B B^{(1,3N,4K)} A^{(1,3M,4N)}) \\ &= \min \left\{ r \begin{pmatrix} A^* M A & A^* M \\ B^* A^* M A B B^{(1,3N,4K)} & B^* A^* M \end{pmatrix} - r(A), \right. \\ & \quad \left. r \begin{pmatrix} A N^{-1} A^* & I_m \\ B^* A^* M A B B^{(1,3N,4K)} N^{-1} A^* & B^* A^* M \end{pmatrix} - r(A) \right\} \\ &= \min \left\{ r(B^* A^* M A - B^* A^* M A B B^{(1,3N,4K)}), \right. \\ & \quad \left. r(B^* A^* M A N^{-1} A^* - B^* A^* M A B B^{(1,3N,4K)} N^{-1} A^*) + m - r(A) \right\}. \end{aligned} \quad (60)$$

Using (10) of Lemma 2 with $A = B$, $B = I_n$, $C = B^*A^*MAB$ and $D = B^*A^*MA$, we have

$$\begin{aligned} & \min_{B(1,3N,4K)} r(B^*A^*MA - B^*A^*MABB^{(1,3N,4K)}) \\ &= r \begin{pmatrix} B^*NB & B^*N \\ B^*A^*MAB & B^*A^*MA \end{pmatrix} + r \begin{pmatrix} BK^{-1}B^* & I_n \\ B^*A^*MABK^{-1}B^* & B^*A^*MA \end{pmatrix} \\ & - r \begin{pmatrix} B & I_n \\ B^*A^*MAB & B^*A^*MA \end{pmatrix} - r(B) = a_3 - r(B). \end{aligned} \quad (61)$$

Using (10) of Lemma 2 with $A = B$, $B = N^{-1}A^*$, $C = B^*A^*MAB$ and $D = B^*A^*MAN^{-1}A^*$, we have

$$\begin{aligned} & \min_{B(1,3N,4K)} r(B^*A^*MAN^{-1}A^* - B^*A^*MABB^{(1,3N,4K)}N^{-1}A^*) \\ &= r \begin{pmatrix} B^*NB & B^*A^* \\ B^*A^*MAB & B^*A^*MAN^{-1}A^* \end{pmatrix} + r \begin{pmatrix} BK^{-1}B^* & N^{-1}A^* \\ B^*A^*MABK^{-1}B^* & B^*A^*MAN^{-1}A^* \end{pmatrix} \\ & - r \begin{pmatrix} B & N^{-1}A^* \\ B^*A^*MAB & B^*A^*MAN^{-1}A^* \end{pmatrix} - r(B). \end{aligned} \quad (62)$$

This is because

$$\begin{aligned} & r \begin{pmatrix} BK^{-1}B^* & N^{-1}A^* \\ B^*A^*MABK^{-1}B^* & B^*A^*MAN^{-1}A^* \end{pmatrix} \\ &= r \left[\begin{pmatrix} BK^{-1/2} & N^{-1}A^* \\ B^*A^*MABK^{-1/2} & B^*A^*MAN^{-1}A^* \end{pmatrix} \begin{pmatrix} (BK^{-1/2})^* & O \\ O & I \end{pmatrix} \right] \\ &\leq r \begin{pmatrix} BK^{-1/2} & N^{-1}A^* \\ B^*A^*MABK^{-1/2} & B^*A^*MAN^{-1}A^* \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} & r \begin{pmatrix} BK^{-1/2} & N^{-1}A^* \\ B^*A^*MABK^{-1/2} & B^*A^*MAN^{-1}A^* \end{pmatrix} \\ &= r \left[\begin{pmatrix} BK^{-1}B^* & N^{-1}A^* \\ B^*A^*MABK^{-1}B^* & B^*A^*MAN^{-1}A^* \end{pmatrix} \begin{pmatrix} (BK^{-1}B^*)^\dagger BK^{-1/2} & O \\ O & I \end{pmatrix} \right] \\ &\leq r \begin{pmatrix} BK^{-1}B^* & N^{-1}A^* \\ B^*A^*MABK^{-1}B^* & B^*A^*MAN^{-1}A^* \end{pmatrix}. \end{aligned} \quad (63)$$

Then, we have

$$\begin{aligned} r \begin{pmatrix} BK^{-1}B^* & N^{-1}A^* \\ B^*A^*MABK^{-1}B^* & B^*A^*MAN^{-1}A^* \end{pmatrix} &= r \begin{pmatrix} BK^{-1/2} & N^{-1}A^* \\ B^*A^*MABK^{-1/2} & B^*A^*MAN^{-1}A^* \end{pmatrix} \\ &= r \begin{pmatrix} B & N^{-1}A^* \\ B^*A^*MAB & B^*A^*MAN^{-1}A^* \end{pmatrix}. \end{aligned} \quad (64)$$

Combining Formulas (62)–(64), we have

$$\begin{aligned}
 & \min_{B^{(1,3N,4K)}} r(B^* A^* M A N^{-1} A^* - B^* A^* M A B B^{(1,3N,4K)} N^{-1} A^*) \\
 &= r \begin{pmatrix} B^* N B & B^* A^* \\ B^* A^* M A B & B^* A^* M A N^{-1} A^* \end{pmatrix} - r(B) \\
 &= a_1 - r(B).
 \end{aligned} \tag{65}$$

From Formulas (60), (61) and (65), we have

$$\begin{aligned}
 & \max_{A^{(1,3M,4N)}, B^{(1,3N,4K)}} r(B^* A^* M - B^* A^* M A B B^{(1,3N,4K)} A^{(1,3M,4N)}) \\
 &= \min \left\{ \min_{B^{(1,3N,4K)}} r(B^* A^* M A - B^* A^* M A B B^{(1,3N,4K)}), \right. \\
 & \quad \left. \left[\min_{B^{(1,3N,4K)}} r(B^* A^* M A N^{-1} A^* - B^* A^* M A B B^{(1,3N,4K)} N^{-1} A^*) \right] + m - r(A) \right\} \\
 &= \min \left\{ a_1 - r(A) - r(B) + m, a_3 - r(B) \right\}.
 \end{aligned} \tag{66}$$

On the other hand, using Lemma 2 again and similarly to the method in (66), we have

$$\begin{aligned}
 & \max_{B^{(1,3N,4K)}, A^{(1,3M,4N)}} r(K^{-1} B^* A^* - B^{(1,3N,4K)} A^{(1,3M,4N)} A B K^{-1} B^* A^*) \\
 &= \min \left\{ a_2 + k - r(A) - r(B), a_4 - r(A) \right\}.
 \end{aligned} \tag{67}$$

Finally, by combining Formulas (57), (58), (59), (66) and (67), we have Formula (56). $\square$

The reverse order law (2) holds if and only if the two inclusions (3) and (4) hold. Thus, combining Theorems 1 and 2, we immediately obtain the main result of this paper.

Theorem 3. Let M , N , and K be three positive-definite Hermitian matrices. Let $A \in C^{m \times n}$ and $B \in C^{n \times k}$. Let a_1 , a_2 , a_3 and a_4 be the same as in Theorem 1. Then, the following statements are equivalent:

- (1) $B\{1, 3N, 4K\}A\{1, 3M, 4N\} = (AB)\{1, 3M, 4K\}$
- (2) $r(AB) = \max \left\{ \min\{a_1, a_2\}, \min\{k - r(B), m - r(A)\} + a_3 + a_4 - n \right\}$ and $r(B) = \min \left\{ a_1 + m - r(A), a_3 \right\}$ and $r(A) = \min \left\{ a_2 + k - r(B), a_4 \right\}$.

Corollary 1. Let $A \in C^{m \times n}$ and $B \in C^{n \times k}$. Then, the following statements are equivalent:

- (1) $B\{1, 3, 4\}A\{1, 3, 4\} = (AB)\{1, 3, 4\}$
- (2) $r(AB) = \max \left\{ \min\{b_1, b_2\}, \min\{k - r(B), m - r(A)\} + b_3 + b_4 - n \right\}$ and $r(B) = \min \left\{ b_1 + m - r(A), b_3 \right\}$ and $r(A) = \min \left\{ b_2 + k - r(B), b_4 \right\}$,

where

$$b_1 = r \begin{pmatrix} B^*B & B^*A^* \\ B^*A^*AB & B^*A^*AA^* \end{pmatrix}, \quad b_3 = r \begin{pmatrix} B^* \\ B^*A^*A \end{pmatrix}$$

and

$$b_2 = r \begin{pmatrix} AA^* & ABB^*A^* \\ B^*A^* & B^*BB^*A^* \end{pmatrix}, \quad b_4 = r \begin{pmatrix} A \\ ABB^* \end{pmatrix}.$$

3. Conclusions

The reverse order law for a matrix product's weighted generalized inverses was studied in this article by using the ranks of generalized Schur complements. The work in this paper can be utilized as a useful tool in many algorithms for the computation of matrix equations' weighted least squares technique.

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References

- Hernandez, A.; Lattanz, M.; Thome, N. On a partial order defined by the weighted Moore–Penrose inverse. *Appl. Math. Comput.* **2013**, *219*, 7310–7318. [\[CrossRef\]](#)
- Wang, G.R.; Wei, Y.M.; Qiao, S.Z. *Generalized Inverse: Theory and Computations*; Science Press: Beijing, China, 2004.
- Wei, M. *Theory and Calculation of Generalized Least Squares Problem*; Science Press: Beijing, China, 2006.
- Ben-Israel, A.; Greville, T.N.E. *Generalized Inverse: Theory and Applications*. In *Wiley-Interscience*, 1974, 2nd ed.; Springer: New York, NY, USA, 2002.
- Campbell, S.L.; Meyer, C.D. *Generalized Inverse of Linear Transformations*; Dover: New York, NY, USA, 1979.
- Cvetković-Ilić, D.S.; Milosevic, J. Reverse order laws for $\{1, 3\}$ -generalized inverses. *Linear Multilinear Algebra* **2018**, *234*, 114–117.
- Pierro, A.R.D.; Wei, M. Reverse order law for reflexive generalized inverses of products of matrices. *Linear Algebra Appl.* **1998**, *277*, 299–311. [\[CrossRef\]](#)
- Djordjević, D.S. Further results on the reverse order law for generalized inverses. *SIAM J. Matrix Anal. Appl.* **2007**, *29*, 1242–1246.
- Greville, T.N.E. Note on the generalized inverses of a matrix products. *SIAM Rev.* **1966**, *8*, 518–521. [\[CrossRef\]](#)
- Hartwig, R.E. The reverse order law revisited. *Linear Algebra Appl.* **1986**, *76*, 241–246. [\[CrossRef\]](#)
- Liu, Y.; Tian, Y. A mixed-type reverse order law for generalized inverse of a triple matrix product. *Acta Math. Sin. Chin. Ser.* **2009**, *52*, 197–204.
- Liu, Q.; Wei, M. Reverse order law for least squares g -inverses of multiple matrix products. *Linear Mult. Algebra* **2008**, *56*, 491–506. [\[CrossRef\]](#)
- Penrose, R. A generalized inverse for matrix. *Proc. Camb. Philos. Soc.* **1955**, *51*, 406–413. [\[CrossRef\]](#)
- Sampazo, R.J.B.; Sun, W.; Yuan, J. On trust region algorithm for nonsmooth optimization. *Appl. Math. Comput.* **1997**, *85*, 109–116.
- Rao, C.R.; Mitra, S.K. *Generalized Inverse of Matrices and Its Applications*; Wiley: New York, NY, USA, 1971.
- Tian, Y. Reverse order laws for the generalized inverse of multiple matrix products. *Linear Algebra Appl.* **1994**, *211*, 85–100. [\[CrossRef\]](#)

17. Tian, Y. More on maximal and minimal ranks of Schur complements with applications. *Appl. Math. Comput.* **2004**, *152*, 675–692. [\[CrossRef\]](#)
18. Wei, M. Reverse order laws for generalized inverse of multiple matrix products. *Linear Algebra Appl.* **1999**, *293*, 273–288. [\[CrossRef\]](#)
19. Sitha, B.; Behera, R.; Sahoo, J.K.; Mohapatra, R.N.; Stanimirović, P.; Stupina, A. Characterizations of weighted generalized inverses. *Aequat. Math.* **2025**, *1*–36. [\[CrossRef\]](#)
20. Kyrchei, I. Weighted singular value decomposition and determinantal representations of the quaternion weighted Moore-Penrose inverse. *Appl. Math. Comput.* **2017**, *309*, 1–16. [\[CrossRef\]](#)
21. Nikolov, J.; Cvetković-Ilić, D.S. Reverse order laws for the weighted generalized inverses. *Appl. Math. Lett.* **2011**, *24*, 2140–2145. [\[CrossRef\]](#)
22. Sun, W.; Wei, Y. Inverse order rule for weighted generalized inverse. *SIAM J. Matrix Anal. Appl.* **1998**, *19*, 772–775. [\[CrossRef\]](#)
23. Sun, W.; Wei, Y. Triple reverse-order rule for weighted generalized inverses. *Appl. Math. Comput.* **2002**, *125*, 221–229.
24. Tian, Y. A family of 512 reverse order laws for generalized inverse of a matrix product: A review. *Heliyon* **2020**, *6*, e04924. [\[CrossRef\]](#)
25. Tian, Y. Characterizations of matrix equalities involving the sums and products of multiple matrices and their generalized inverse. *Electron. Res. Arch.* **2023**, *31*, 5866–5893. [\[CrossRef\]](#)
26. Xiong, Z.P.; Liu, Z.S. Applications of completions of operator matrices to some properties of operator products on Hilbert spaces. *Complex Anal. Oper. Theory* **2018**, *12*, 123–140. [\[CrossRef\]](#)
27. Xiong, Z.P.; Zheng, B. The reverse order laws for $\{1, 2, 3\}$ - and $\{1, 2, 4\}$ -inverses of two matrix product. *Appl. Math. Lett.* **2008**, *21*, 649–655. [\[CrossRef\]](#)
28. Zheng, B.; Xiong, Z.P. On reverse order laws for the weighted generalized inverse. *Arab. J. Sci. Eng.* **2008**, *34*, 195–203.
29. Cohen, N.; Johnson, C.R.; Rodman, L.; Woerdeman, H.J. Ranks of completions of partial matrices. *Oper. Theory Adv. Appl.* **1989**, *40*, 165–185.

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