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Bipolar Fuzzy Sheffer Stroke in BCK-Algebras

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Abstract: In this study, we examine bipolar fuzzy SBCK-subalgebras and their corresponding level sets of bipolar fuzzy sets in the setting of Sheffer stroke BCK-algebras. These concepts contribute significantly to the analysis of bipolar logical structures within this algebraic context. We demonstrate a bidirectional relationship between SBCK-subalgebras and their level sets, proving that each level set derived from a bipolar fuzzy SBCK-subalgebra constitutes a subalgebra, and, conversely, each such subalgebra defines an associated level set. This duality emphasizes the structural interplay between bipolar fuzzy logic and the Sheffer stroke operation in BCK-algebras.

Keywords: Sheffer stroke (BCK-algebra); BCK-ideal; bipolar fuzzy SBCK-subalgebra; bipolar fuzzy SBCK-ideal

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1. Introduction

The notion of BCK-algebras was initially formulated by Imai and Iséki in 1966 [1]. These algebraic structures, named after Blokh, Cohn, and Kisielewicz, have since become essential tools in the intersection of logic and algebra. In contrast with conventional Boolean algebras, BCK-algebras provide a more adaptable framework for logical operations, particularly by emphasizing implications and relationships in a manner that broadens the scope of classical algebraic systems. The exploration of BCK-algebras opens new possibilities in theoretical mathematics and computer science, offering valuable insights into logical systems and algorithmic processes.

The Sheffer operation, or Sheffer stroke (denoted as "|"), was introduced by H. M. Sheffer [2]. This operation is noteworthy because it can independently form a complete logical system, meaning that all axioms within such a system can be expressed solely through the Sheffer stroke. This self-sufficiency simplifies the management and manipulation of logical systems. Furthermore, Boolean algebra, which forms the algebraic foundation of classical propositional logic, can also be entirely represented through the Sheffer operation. Its functional completeness enables the construction of other logical operations such as AND, OR, and NOT, making it a fundamental element in both Boolean algebra and digital circuit design. The Sheffer operation is particularly recognized for streamlining algebraic structures by reducing the number of axioms, thus making the study of systems incorporating this operation more manageable. Prominent examples include Boolean algebras [3] and

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ortholattices [4]. More recently, the Sheffer operation has found applications in a variety of algebraic structures, such as basic algebras [5], Sheffer stroke Hilbert algebras [6], and UP-algebras [7], as outlined in the existing literature. This broad range of applications demonstrates the Sheffer operation's versatility and its capacity to simplify and advance the study of complex algebraic systems.

BCK-algebras, originally proposed by Imai and Iséki, draw inspiration from set theory and both classical and non-classical propositional calculi. These algebras have seen widespread use in areas such as group theory, functional analysis, probability theory, and topology. A comprehensive study in [8] investigates the combination of BCK-algebras with the Sheffer operation, providing an in-depth exploration of their properties and applications in these fields.

Bipolar fuzzy sets generalize classical fuzzy set theory by assigning to each element a pair of membership values, representing both its degree of positive and negative association. This duality allows for a more sophisticated approach to modeling uncertainty, particularly in cases where both positive and negative information are relevant. Unlike classical fuzzy set theory, where each element has a single degree of membership indicating truth or belonging, bipolar fuzzy sets recognize that real-world situations often require consideration of both favorable and unfavorable degrees simultaneously. This concept has been instrumental in advancing fields such as decision-making, pattern recognition, and information processing, where ambiguity and contradictions are prevalent. Studying bipolar fuzzy sets within various algebraic structures, like BCK-algebras, creates new opportunities for understanding their properties and applications, yielding deeper insights into the behavior of logic systems under uncertain conditions.

This study investigates bipolar fuzzy SBCK-subalgebras and the level sets of bipolar fuzzy sets within the framework of Sheffer stroke BCK-algebras. These constructs are instrumental in capturing the behavior of bipolar logic in the context of this algebraic structure. A central result of the paper is the establishment of a bidirectional correspondence between bipolar fuzzy SBCK-subalgebras and their associated level sets. It is demonstrated that each level set derived from a bipolar fuzzy SBCK-subalgebra forms a subalgebra, and, conversely, each such subalgebra determines a level set of an appropriate bipolar fuzzy set.

Moreover, the concept of a bipolar fuzzy SBCK-ideal is introduced and its algebraic properties are thoroughly examined. It is proven that any bipolar fuzzy SBCK-ideal inherently qualifies as a bipolar fuzzy SBCK-subalgebra. However, the converse does not generally hold, indicating that bipolar fuzzy SBCK-ideals possess distinctive structural features that differentiate them from more general subalgebras within Sheffer stroke BCK-algebras.

The list of acronyms is given in Table 1.

Table 1. List of acronyms.

Acronyms	Representation
SBCK-algebra	Sheffer stroke BCK-algebra
BF_{SBCK-S}	bipolar fuzzy Sheffer stroke BCK-subalgebra
BFS	bipolar fuzzy set
BVFS	bipolar-valued fuzzy set
BF_{SBCK-I}	bipolar fuzzy Sheffer stroke BCK-ideal
BFI	bipolar fuzzy ideal
$BF_{(\varrho,\nu)-t}$	bipolar fuzzy (ϱ, ν)-translation

2. Preliminaries

Definition 1 ([2]). *Let $(\mathcal{G}_S, |)$ be a groupoid. The binary operation $|$ is referred to as a Sheffer stroke if it fulfills the following axiomatic conditions:*

(S1) $\mathfrak{g}_{s_1} \mid \mathfrak{g}_{s_2} = \mathfrak{g}_{s_2} \mid \mathfrak{g}_{s_1}$,
(S2) $(\mathfrak{g}_{s_1} \mid \mathfrak{g}_{s_1}) | (\mathfrak{g}_{s_1} \mid \mathfrak{g}_{s_2}) = \mathfrak{g}_{s_1}$,
(S3) $\mathfrak{g}_{s_1} \mid ((\mathfrak{g}_{s_2} \mid \mathfrak{g}_{s_3}) \mid (\mathfrak{g}_{s_2} \mid \mathfrak{g}_{s_3})) = ((\mathfrak{g}_{s_1} \mid \mathfrak{g}_{s_2}) \mid (\mathfrak{g}_{s_1} \mid \mathfrak{g}_{s_2})) \mid \mathfrak{g}_{s_3}$,
(S4) $(\mathfrak{g}_{s_1} \mid ((\mathfrak{g}_{s_1} \mid \mathfrak{g}_{s_1}) \mid (\mathfrak{g}_{s_2} \mid \mathfrak{g}_{s_2}))) | (\mathfrak{g}_{s_1} \mid ((\mathfrak{g}_{s_1} \mid \mathfrak{g}_{s_1}) \mid (\mathfrak{g}_{s_2} \mid \mathfrak{g}_{s_2}))) = \mathfrak{g}_{s_1}$,

for all $\mathfrak{g}_{s_1}, \mathfrak{g}_{s_2}, \mathfrak{g}_{s_3} \in \mathcal{G}_S$.

As a notational convenience, we denote $\mathfrak{W}_{\mathfrak{g}_{s_1}}(\mathfrak{g}_{s_2}) = \mathfrak{g}_{s_1} \mid (\mathfrak{g}_{s_2} \mid \mathfrak{g}_{s_2})$.

Definition 2 ([8]). *A Sheffer stroke BCK-algebra (abbreviated as SBCK-algebra) is an algebraic system $(\mathfrak{B}_S, |)$ of arity two, where 0 denotes a distinguished constant element in $\mathfrak{B}_S$, and the structure satisfies the following axioms for all $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2}, \mathfrak{b}_{s_3} \in \mathfrak{B}_S$:*

$(SBCK_1)$ $(((\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_3})) \mid ((\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_3}))) \mid \mathfrak{W}_{\mathfrak{b}_{s_3}}(\mathfrak{b}_{s_2}) = 0$,
$(SBCK_2)$ $\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) = 0$ *and* $\mathfrak{W}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1}) \mid \mathfrak{W}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1}) = 0 \Rightarrow \mathfrak{b}_{s_1} = \mathfrak{b}_{s_2}$.

Proposition 1 ([8]). *Let $(\mathfrak{B}_S, |)$ be an SBCK-algebra. Then, the binary relation $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) = 0$ is a partial order on $\mathfrak{B}_S$.*

Definition 3 ([8]). *Let $(\mathfrak{B}_S, |)$ be an SBCK-algebra. A nonempty subset $\mathfrak{G}$ of $\mathfrak{B}_S$ is called an SBCK-subalgebra of $\mathfrak{B}_S$ if $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \in \mathfrak{G}$ for all $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{G}$.*

Definition 4 ([8]). *Let $(\mathfrak{B}_S, |)$ be an SBCK-algebra. A nonempty subset $\mathfrak{G}$ of $\mathfrak{B}_S$ is called an SBCK-ideal of $\mathfrak{B}_S$ if for all $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{G}$.*

1. $0 \in \mathfrak{G}$,
2. $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \in G$ *and* $\mathfrak{b}_{s_2} \in \mathfrak{G} \Rightarrow \mathfrak{b}_{s_1} \in \mathfrak{G}$.

Lemma 1. *Let μ be a fuzzy set in a nonempty set X and $\mathfrak{p}, \mathfrak{q} \in X$. Then,*

1. $1 - \max\{\mu(\mathfrak{p}), \mu(\mathfrak{q})\} = \min\{1 - \mu(\mathfrak{p}), 1 - \mu(\mathfrak{q})\}$,
2. $1 - \min\{\mu(\mathfrak{p}), \mu(\mathfrak{q})\} = \max\{1 - \mu(\mathfrak{p}), 1 - \mu(\mathfrak{q})\}$.

Definition 5 ([9]). *Let X be a nonempty set. A bipolar fuzzy set (briefly, BFS) $\mathcal{B}$ defined on X is a collection of ordered triples given by*

$$\mathcal{B} = \{(\mathfrak{b}_s, \hbar^-(\mathfrak{b}_s), \hbar^+(\mathfrak{b}_s)) \mid \mathfrak{b}_s \in X\},$$

where $\hbar^+ : X \to [0, 1]$ and $\hbar^- : X \to [-1, 0]$ are functions representing the positive and negative degrees of membership, respectively. Specifically, $\hbar^+(\mathfrak{b}_s)$ quantifies the extent to which an element $\mathfrak{b}_s$ satisfies a given property associated with $\mathcal{B}$, while $\hbar^-(\mathfrak{b}_s)$ indicates the degree to which $\mathfrak{b}_s$ conforms to a corresponding counter-property.

When $\hbar^+(\mathfrak{b}_s) \neq 0$ and $\hbar^-(\mathfrak{b}_s) = 0$, the element $\mathfrak{b}_s$ is interpreted as affirming the property but not its negation. Conversely, if $\hbar^+(\mathfrak{b}_s) = 0$ and $\hbar^-(\mathfrak{b}_s) \neq 0$, the element supports the counter-property while not satisfying the original one. In cases where both degrees are zero, i.e., $\hbar^+(\mathfrak{b}_s) = \hbar^-(\mathfrak{b}_s) = 0$, the element exhibits neutrality with respect to both the property and its counter-property, suggesting a potential overlap or ambiguity in their scopes.

For simplicity, we shall denote the bipolar fuzzy set $\mathcal{B} = \{(\mathfrak{b}_s, \hbar^-(\mathfrak{b}_s), \hbar^+(\mathfrak{b}_s) \mid \mathfrak{b}_s \in X\}$ as $\hbar = (X, \hbar^-, \hbar^+)$.

3. Bipolar Fuzzy Sets in SBCK-Algebras

This section presents the notion of bipolar fuzzy SBCK-subalgebras in the framework of SBCK-algebras. Throughout the remainder of this section, unless otherwise specified, the symbol $\mathfrak{B}_S$ will be assumed to denote an SBCK-algebra.

Definition 6. *A bipolar fuzzy set $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ in $\mathfrak{B}_S$ is called a bipolar fuzzy SBCK-subalgebra (briefly, BF_{SBCK-S}) of $\mathfrak{B}_S$ if*

A bipolar fuzzy set $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ in $\mathfrak{B}_S$ is called a bipolar fuzzy SBCK-subalgebra (briefly, BF_{SBCK-S}) of $\mathfrak{B}_S$ if

$$(\forall \mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S)\begin{pmatrix} \hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq \max\{\hbar^-(\mathfrak{b}_{s_1}), \hbar^-(\mathfrak{b}_{s_2})\} \\ \hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \min\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{b}_{s_2})\} \end{pmatrix}. \tag{1}$$

Example 1. *Let $\mathfrak{B}_S = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with the binary operation $|$ given in the following table:*

$\mid$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1
1	1	0	3	2	5	4	7	6
2	1	3	3	1	5	6	5	6
3	1	2	1	2	1	2	2	1
4	1	5	5	1	5	1	5	1
5	1	4	6	2	1	4	2	6
6	1	7	5	2	5	2	7	1
7	1	6	6	1	1	6	1	6

Then, $(\mathfrak{B}_S, |)$ is an SBCK-algebra. Define a BVFS $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ in $\mathfrak{B}_S$ by the table below:

$\mathfrak{B}_S$	0	1	2	3	4	5	6	7
$\hbar^-$	-0.9	-0.7	-0.9	-0.9	-0.9	-0.9	-0.7	-0.9
$\hbar^+$	0.95	0.7	0.8	0.85	0.8	0.8	0.8	0.85

It is routine to verify that the BVFS $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ in $\mathfrak{B}_S$ is a bipolar-valued fuzzy SBCK-subalgebra of $(\mathfrak{B}_S, |)$.

Definition 7. *Let μ be a fuzzy set defined on an SBCK-algebra $\mathfrak{B}_S$, and let $(\mathfrak{s}, \mathfrak{t}) \in [-1, 0] \times [0, 1]$. The following sets are defined as:*

$$\mathfrak{L}(\hbar^-, \mathfrak{s}) = \{\mathfrak{b}_s \in \mathfrak{B}_S : \mu(\mathfrak{b}_s) \leq \mathfrak{s}\},$$

and

$$\mathfrak{U}(\hbar^+, \mathfrak{t}) = \{\mathfrak{b}_s \in \mathfrak{B}_S : \mu(\mathfrak{b}_s) \geq \mathfrak{t}\}.$$

These sets are referred to as the negative $\mathfrak{s}$-cut and the positive $\mathfrak{t}$-cut of the bipolar fuzzy set $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$, respectively.

Example 2. *Let $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ be the BFS given in Example 1. For $\mathfrak{s} = -0.9$,*

$$\mathfrak{L}(\hbar^-, -0.9) = \{\mathfrak{b}_s \in \mathfrak{B}_S \mid \hbar^-(\mathfrak{b}_s) \leq -0.9\} = \{0, 2, 3, 4, 5, 7\}.$$

For $\mathfrak{t} = 0.85$,

$$\mathfrak{U}(\hbar^+, 0.85) = \{\mathfrak{b}_s \in \mathfrak{B}_S \mid \hbar^+(\mathfrak{b}_s) \geq 0.85\} = \{0, 3, 7\}.$$

Theorem 1. *A bipolar fuzzy set $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ on $\mathfrak{B}_S$ is a BF_{SBCK-S} of $\mathfrak{B}_S$ if and only if, for all pairs $(\mathfrak{s}, \mathfrak{t}) \in [-1, 0] \times [0, 1]$, its negative $\mathfrak{s}$-cut and positive $\mathfrak{t}$-cut are SBCK-subalgebras of $\mathfrak{B}_S$, provided they are nonempty.*

Proof. Assume that $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BF_{SBCK-S} of $\mathfrak{B}_S$ and $\mathfrak{L}(\hbar^-, \mathfrak{s}) \neq \emptyset \neq \mathfrak{U}(\hbar^+, \mathfrak{t})$ for all $(\mathfrak{s}, \mathfrak{t}) \in [-1, 0] \times [0, 1]$. Let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2}, \mathfrak{b}_{s_p}, \mathfrak{b}_{s_q} \in \mathfrak{B}_S$ be such that $(\mathfrak{b}_{s_1}, \mathfrak{b}_{s_p}) \in \mathfrak{B}_S(\hbar^-, \mathfrak{s}) \times \mathfrak{U}(\hbar^+, \mathfrak{t})$ and $(\mathfrak{b}_{s_2}, \mathfrak{b}_{s_q}) \in \mathfrak{B}_S(\hbar^-, \mathfrak{s}) \times \mathfrak{U}(\hbar^+, \mathfrak{t})$. Then $\hbar^-(\mathfrak{b}_{s_1}) \leq \mathfrak{s}$, $\hbar^-(\mathfrak{b}_{s_2}) \leq \mathfrak{s}$, $\hbar^+(\mathfrak{b}_{s_p}) \geq \mathfrak{t}$ and $\hbar^+(\mathfrak{b}_{s_q}) \geq \mathfrak{t}$. Using (1), we get

$$\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq \max\{\hbar^-(\mathfrak{b}_{s_1}), \hbar^-(\mathfrak{b}_{s_2})\} \leq \mathfrak{s}$$

and

$$\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q}) \mid \mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q})) \geq \min\{\hbar^+(\mathfrak{b}_{s_p}), \hbar^+(\mathfrak{b}_{s_q})\} \geq \mathfrak{t}$$

and so $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), (\mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q}) \mid \mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q})) \in \mathfrak{B}_S(\hbar^-, \mathfrak{s}) \times \mathfrak{U}(\hbar^+, \mathfrak{t})$. Therefore, $\mathfrak{L}(\hbar^-, \mathfrak{s})$ and $\mathfrak{U}(\hbar^+, \mathfrak{t})$ are SBCK-subalgebras of $\mathfrak{B}_S$.

Conversely, suppose that $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a bipolar fuzzy set on $\mathfrak{B}_S$, such that for all pairs $(\mathfrak{s}, \mathfrak{t}) \in [-1, 0] \times [0, 1]$, the negative $\mathfrak{s}$-cut and positive $\mathfrak{t}$-cut of $\hbar$ are SBCK-subalgebras of $\mathfrak{B}_S$, whenever these cuts are nonempty. Suppose that

$$\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q}) \mid \mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q}) > \max\{\hbar^-(\mathfrak{b}_{s_p}), \hbar^-(\mathfrak{b}_{s_q})\}$$

or

$$\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) < \min\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_2}})\}$$

for some $\mathfrak{b}_{s_p}, \mathfrak{b}_{s_q}, \mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$. Then, $\mathfrak{b}_{s_p}, \mathfrak{b}_{s_q} \in \mathfrak{B}_S(\hbar^-, \mathfrak{s})$ or $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{U}(\hbar^+, \mathfrak{t})$, where

$$\mathfrak{s} = \max\{\hbar^-(\mathfrak{b}_{s_p}), \hbar^-(\mathfrak{b}_{s_q})\}$$

and

$$\mathfrak{t} = \min\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{b}_{s_2})\}.$$

But $(\mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q})) \mid \mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q})) \notin \mathfrak{B}_S(\hbar^-, \mathfrak{s})$ or $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid (\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))) \notin \mathfrak{U}(\hbar^+, \mathfrak{t})$, a contradiction. Therefore,

$$\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q}) \mid \mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q})) \leq \max\{\hbar^-(\mathfrak{b}_{s_p}), \hbar^-(\mathfrak{b}_{s_q})\}$$

and

$$\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \min\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_2}})\}$$

for all $\mathfrak{b}_{s_p}, \mathfrak{b}_{s_q}, \mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$. Consequently, $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BF_{SBCK-S}, and thus a BF_{SBCK-S} of $\mathfrak{B}_S$. □

Theorem 2. *A BFS $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ in $\mathfrak{B}_S$ is a BF_{SBCK-S} of $\mathfrak{B}_S$ if and only if the fuzzy sets $\hbar^-_{\mathfrak{c}}$ and $\hbar^+$ are fuzzy SBCK-subalgebras of $\mathfrak{B}_S$, where $\hbar^-_{\mathfrak{c}} : L \to [0, 1]$ is defined by $\mathfrak{b}_s \mapsto 1 - \hbar^-(\mathfrak{b}_s)$.*

Proof. Assume that $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BF_{SBCK-S} of $\mathfrak{B}_S$. It is clear that $\hbar^+$ is a fuzzy subalgebra of $\mathfrak{B}_S$. For every $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$,

$$\begin{aligned}
\hbar^-_{\mathfrak{c}}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= 1 - \hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \\
&\geq 1 - \max\{\hbar^-(\mathfrak{b}_{s_1}), \hbar^-(\mathfrak{b}_{s_2})\} \\
&= \min\{1 - \hbar^-(\mathfrak{b}_{s_1}), 1 - \hbar^-(\mathfrak{b}_{s_2})\} \\
&= \min\{\hbar^-_{\mathfrak{c}}(\mathfrak{b}_{s_1}), \hbar^-_{\mathfrak{c}}(\mathfrak{b}_{s_2})\}.
\end{aligned}$$

Hence, $\hbar_{\mathfrak{c}}^{-}$ is a fuzzy SBCK-subalgebra of $\mathfrak{B}_S$.

Conversely, let $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ be a BFS of $\mathfrak{B}_S$ for which $\hbar_{\mathfrak{c}}^{-}$ and $\hbar^{+}$ are fuzzy SBCK-subalgebras of $\mathfrak{B}_S$. Let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$. Then,

$$\begin{aligned} 1 - \hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= \hbar_{\mathfrak{c}}^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \\ &\geq \min\{\hbar_{\mathfrak{c}}^{-}(\mathfrak{b}_s), \hbar_{\mathfrak{c}}^{-}(\mathfrak{b}_{s_2})\} \\ &= \min\{1 - \hbar^{-}(\mathfrak{b}_s), 1 - \hbar^{-}(\mathfrak{b}_{s_2})\} \\ &= 1 - \max\{\hbar^{-}(\mathfrak{b}_s), \hbar^{-}(\mathfrak{b}_{s_2})\} \end{aligned}$$

and then

$$\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq \max\{\hbar^{-}(\mathfrak{b}_{s_1}), \hbar^{-}(\mathfrak{b}_{s_2})\}.$$

Hence, $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ is a BF_{SBCK-S} of $\mathfrak{B}_S$. □

Theorem 3. *Given a nonempty subset $\mathfrak{H}$ of $\mathfrak{B}_S$, let $\hbar_{\mathfrak{H}} = (\mathfrak{B}_S, \hbar_{\mathfrak{H}}^{-}, \hbar_{\mathfrak{H}}^{+})$ be a function in $\mathfrak{B}_S$, defined as follows:*

$$\hbar_{\mathfrak{H}}^{-} : L \to [-1, 0], \quad \mathfrak{b}_{s_a} \mapsto \begin{cases} \mathfrak{s}^{-} & \text{if } \mathfrak{b}_{s_a} \in \mathfrak{H}, \\ \mathfrak{t}^{-} & \text{otherwise}, \end{cases}$$

$$\hbar_{\mathfrak{H}}^{+} : L \to [-1, 0], \quad \mathfrak{b}_{s_b} \mapsto \begin{cases} \mathfrak{s}^{+} & \text{if } \mathfrak{b}_{s_b} \in \mathfrak{H}, \\ \mathfrak{t}^{+} & \text{otherwise}, \end{cases}$$

where $\mathfrak{s}^{-} < \mathfrak{t}^{-}$ in $[-1, 0]$ and $\mathfrak{s}^{+} > \mathfrak{t}^{+}$ in $[0, 1]$. Then, $\hbar_{\mathfrak{H}} = (\mathfrak{B}_S, \hbar_{\mathfrak{H}}^{-}, \hbar_{\mathfrak{H}}^{+})$ is a BF_{SBCK-S} of $\mathfrak{B}_S$ if and only if $\mathfrak{H}$ is an SBCK-subalgebra of $\mathfrak{B}_S$. Moreover, we have

$$\mathfrak{H} = L_{\hbar_{\mathfrak{H}}} = \left\{ \mathfrak{b}_{s_x} \in \mathfrak{B}_S : \hbar_{\mathfrak{H}}^{-}(\mathfrak{b}_{s_x}) = \hbar_{\mathfrak{H}}^{-}(0),\ \hbar_{\mathfrak{H}}^{+}(\mathfrak{b}_{s_x}) = \hbar_{\mathfrak{H}}^{+}(0) \right\}.$$

Proof. Let $\hbar_{\mathfrak{H}} = (\mathfrak{B}_S, \hbar_{\mathfrak{H}}^{-}, \hbar_{\mathfrak{H}}^{+})$ be a BF_{SBCK-S} structure on $\mathfrak{B}_S$. Let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$ be such that $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{H}$. Using (1), we have

$$\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq \max\left\{\hbar^{-}(\mathfrak{b}_{s_1}), \hbar^{-}(\mathfrak{b}_{s_2})\right\} = \mathfrak{s}^{-},$$

$$\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \min\left\{\hbar^{+}(\mathfrak{b}_{s_1}), \hbar^{+}(\mathfrak{b}_{s_2})\right\} = \mathfrak{s}^{+}$$

Thus, we conclude that

$$\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) = \mathfrak{s}^{-} \quad \text{and} \quad \hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) = \mathfrak{s}^{+}.$$

This implies $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \in \mathfrak{H}$. Therefore, $\mathfrak{H}$ is a SBCK-subalgebra of $\mathfrak{B}_S$.

Conversely, assume that $\mathfrak{H}$ is a SBCK-subalgebra of $\mathfrak{B}_S$. For every $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$, if $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{H}$, then $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \in \mathfrak{H}$, which implies that

$$\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) = \mathfrak{s}^{-} = \max\left\{\hbar^{-}(\mathfrak{b}_{s_1}), \hbar^{-}(\mathfrak{b}_{s_2})\right\},$$

$$\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) = \mathfrak{s}^{+} = \min\left\{\hbar^{+}(\mathfrak{b}_{s_1}), \hbar^{+}(\mathfrak{b}_{s_2})\right\}$$

If either $\mathfrak{b}_{s_1} \notin \mathfrak{H}$ or $\mathfrak{b}_{s_2} \notin \mathfrak{H}$, we have

$$\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq \mathfrak{t}^{-} = \max\left\{\hbar^{-}(\mathfrak{b}_{s_1}), \hbar^{-}(\mathfrak{b}_{s_2})\right\},$$

$$\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \mathfrak{t}^{+} = \min\left\{\hbar^{+}(\mathfrak{b}_{s_1}), \hbar^{+}(\mathfrak{b}_{s_2})\right\}$$

Thus, the structure $\hbar_{\mathfrak{H}} = (\mathfrak{B}_S, \hbar^-_{\mathfrak{H}}, \hbar^+_{\mathfrak{H}})$ is a BF_{SBCK-S} structure on $\mathfrak{B}_S$. As $\mathfrak{H}$ is a SBCK-subalgebra of $\mathfrak{B}_S$, we deduce

$$\begin{aligned} L_{\hbar_{\mathfrak{H}}} &= \{\mathfrak{b}_{s_x} \in \mathfrak{B}_S : \hbar^-_{\mathfrak{H}}(\mathfrak{b}_{s_x}) = \hbar^-_{\mathfrak{H}}(0), \hbar^+_{\mathfrak{H}}(\mathfrak{b}_{s_x}) = \hbar^+_{\mathfrak{H}}(0)\} \\ &= \{\mathfrak{b}_{s_x} \in \mathfrak{B}_S : \hbar^-_{\mathfrak{H}}(\mathfrak{b}_{s_x}) = \mathfrak{s}^-, \hbar^+_{\mathfrak{H}}(\mathfrak{b}_{s_x}) = \mathfrak{s}^+\} \\ &= \{\mathfrak{b}_{s_x} \in \mathfrak{B}_S : \mathfrak{b}_{s_x} \in \mathfrak{H}\} \\ &= \mathfrak{H}. \end{aligned}$$

□

Lemma 2. *A BFS $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ in $\mathfrak{B}_S$ is a BF_{SBCK-S} of $\in \mathfrak{B}_S$, then $\hbar^+(0) \geq \hbar^+(\mathfrak{b}_s)$ and $\hbar^-(0) \leq \hbar^-(\mathfrak{b}_s)$ for all $\mathfrak{b}_s \in \mathfrak{B}_S$.*

Proof. For any $\mathfrak{b}_s \in \mathfrak{B}_S$,

$$\hbar^+(0) = \hbar^+(\mathfrak{W}_{\mathfrak{b}_s}(\mathfrak{b}_s) \mid \mathfrak{W}_{\mathfrak{b}_s}(\mathfrak{b}_s)) \geq \min\{\hbar^+(\mathfrak{b}_s), \hbar^+(\mathfrak{b}_s)\} = \hbar^+(\mathfrak{b}_s),$$

and

$$\hbar^-(0) = \hbar^-(\mathfrak{W}_{\mathfrak{b}_s}(\mathfrak{b}_s) \mid \mathfrak{W}_{\mathfrak{b}_s}(\mathfrak{b}_s)) \leq \max\{\hbar^-(\mathfrak{b}_s), \hbar^-(\mathfrak{b}_s)\} = \hbar^-(\mathfrak{b}_s).$$

□

Proposition 2. *For a BF_{SBCK-S} on $\in \mathfrak{B}_S$, the following condition holds:*

$$(\forall\, \mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S) \quad \begin{pmatrix} \hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \hbar^+(\mathfrak{b}_{s_2}), \\ \hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq \hbar^-(\mathfrak{b}_{s_2}) \end{pmatrix}$$

if and only if $\hbar^+(0) = \hbar^+(\mathfrak{b}_{s_1})$ and $\hbar^-(0) = \hbar^-(\mathfrak{b}_{s_1})$.

Proof. We have

$$\begin{aligned} \hbar^+(\mathfrak{b}_s) &= \hbar^+((\mathfrak{b}_s \mid \mathfrak{b}_s) \mid (\mathfrak{b}_s \mid \mathfrak{b}_s)) \\ &= \hbar^+(\mathfrak{W}_1(\mathfrak{b}_s \mid \mathfrak{b}_s) \mid \mathfrak{W}_1(\mathfrak{b}_s \mid \mathfrak{b}_s)) \\ &= \hbar^+((\mathfrak{b}_s \mid 1) \mid (\mathfrak{b}_s \mid 1)) \\ &= \hbar^+(\mathfrak{W}_{\mathfrak{b}_s}(0) \mid \mathfrak{W}_{\mathfrak{b}_s}(0)) \\ &\geq \hbar^+(0) \end{aligned}$$

and

$$\begin{aligned} \hbar^-(x) &= \hbar^-((\mathfrak{b}_s \mid \mathfrak{b}_s) \mid (\mathfrak{b}_s \mid \mathfrak{b}_s)) \\ &= \hbar^-(\mathfrak{W}_1(\mathfrak{b}_s \mid \mathfrak{b}_s) \mid \mathfrak{W}_1(\mathfrak{b}_s \mid \mathfrak{b}_s)) \\ &= \hbar^-((\mathfrak{b}_s \mid 1) \mid (\mathfrak{b}_s \mid 1)) \\ &= \hbar^-(\mathfrak{W}_{\mathfrak{b}_s}(0) \mid \mathfrak{W}_{\mathfrak{b}_s}(0)) \\ &\leq \hbar^-(0). \end{aligned}$$

Then, by Lemma 2, we have $\hbar^+(0) = \hbar^+(\mathfrak{b}_s)$ and $\hbar^-(0) = \hbar^-(\mathfrak{b}_s)$. The converse is clear. □

4. Bipolar Fuzzy Sets in SBCK-Ideals

This section investigates bipolar fuzzy SBCK-ideals within the context of SBCK-algebras. The theorems presented establish a formal framework for identifying the conditions under which these fuzzy sets qualify as SBCK-ideals, thereby enhancing their applicability in logical and algebraic contexts.

Definition 8. *A BFS $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ on $\mathfrak{B}_S$ is called a BF_{SBCK-I} of $\mathfrak{B}_S$ if*

$$(\forall \mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S)\left(\begin{array}{l} \hbar^+(0) \geq \hbar^+(\mathfrak{b}_{s_1}) \geq \min\{\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^+(\mathfrak{b}_{s_2})\} \\ \hbar^-(0) \leq \hbar^-(\mathfrak{b}_{s_1}) \leq \max\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_{s_2})\} \end{array} \right). \quad (2)$$

Example 3. *Let $(\mathfrak{B}_S, |)$ be an SBCK-algebra given in Example 1. Define a BVFS $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ in $\mathfrak{B}_S$ by the table below:*

$\mathfrak{B}_S$	0	1	2	3	4	5	6	7
$\hbar^-$	−0.85	−0.72	−0.72	−0.83	−0.83	−0.72	−0.83	−0.72
$\hbar^+$	0.9	0.2	0.2	0.2	0.2	0.2	0.2	0.9

It is routine to verify that the BFS $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ in $\mathfrak{B}_S$ is a BF_{SBCK-I} of $\mathfrak{B}_S$.

Lemma 3. *If $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BFI of $\mathfrak{B}_S$, then*

$$(\forall\, \mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S)\left(\mathfrak{b}_{s_1} \preccurlyeq \mathfrak{b}_{s_2} \Rightarrow \left\{ \begin{array}{l} \hbar^+(\mathfrak{b}_{s_1}) \geq \hbar^+(\mathfrak{b}_{s_2}) \\ \hbar^-(\mathfrak{b}_{s_1}) \leq \hbar^-(\mathfrak{b}_{s_2}) \end{array} \right. \right). \quad (3)$$

Proof. Let $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ be a BFI of $\mathfrak{B}_S$ and $\mathfrak{b}_{s_1} \preccurlyeq \mathfrak{b}_{s_2}$. Then, by Theorem 2, we have

$$\hbar^+(\mathfrak{b}_s) \geq \min\{\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^+(\mathfrak{b}_{s_2})\} = \min\{\hbar^+(0), \hbar^+(\mathfrak{b}_{s_2})\} = \hbar^+(\mathfrak{b}_{s_2})$$

$$\hbar^-(\mathfrak{b}_s) \leq \max\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_{s_2})\} = \max\{\hbar^-(0), \hbar^-(\mathfrak{b}_{s_2})\} = \hbar^-(\mathfrak{b}_{s_2})$$

for all $\mathfrak{b}_s, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$. □

Theorem 4. *Let $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ be a BFI of X. Then, $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BFI of $\mathfrak{B}_S$, if and only if*

$$(\forall\, \mathfrak{b}_{s_1}, \mathfrak{b}_{s_2}, \mathfrak{b}_{s_3} \in \mathfrak{B}_S)\left((\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \preccurlyeq \mathfrak{b}_{s_3} \Rightarrow \left\{ \begin{array}{l} \hbar^+(\mathfrak{b}_s) \geq \min\{\hbar^+(\mathfrak{b}_{s_2}), \hbar^+(\mathfrak{b}_{s_3})\} \\ \hbar^-(\mathfrak{b}_s) \leq \max\{\hbar^-(\mathfrak{b}_{s_2}), \hbar^-(\mathfrak{b}_{s_3})\} \end{array} \right. \right). \quad (4)$$

Proof. Let $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ be a BFI of $\mathfrak{B}_S$, and assume that $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \preccurlyeq \mathfrak{b}_{s_3}$. Then,

$$((\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \mid (\mathfrak{b}_{s_3} \mid \mathfrak{b}_{s_3})) \mid ((\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \mid (\mathfrak{b}_{s_3} \mid \mathfrak{b}_{s_3})) = 0.$$

Furthermore, we have

$$\begin{aligned} \hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= \min\{\hbar^+\Big(((\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \mid (\mathfrak{b}_{s_3} \mid \mathfrak{b}_{s_3})) \\ &\quad \mid ((\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \mid (\mathfrak{b}_{s_3} \mid \mathfrak{b}_{s_3}))\Big), \hbar^+(\mathfrak{b}_{s_3})\} \\ &= \min\{\hbar^+(0), \hbar^+(\mathfrak{b}_{s_3})\} \\ &= \hbar^+(\mathfrak{b}_{s_3}), \end{aligned}$$

and similarly

$$\begin{aligned} \hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= \max\{\hbar^-\Big(((\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \mid (\mathfrak{b}_{s_3} \mid \mathfrak{b}_{s_3})) \\ &\quad \mid ((\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \mid (\mathfrak{b}_{s_3} \mid \mathfrak{b}_{s_3}))\Big), \hbar^-(\mathfrak{b}_{s_3})\} \\ &= \min\{\hbar^-(0), \hbar^-(\mathfrak{b}_{s_3})\} \\ &= \hbar^-(\mathfrak{b}_{s_3}). \end{aligned}$$

From this, we deduce that

$$\hbar^+(\mathfrak{b}_{\mathfrak{s}_1}) \geq \min\left\{\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^+(\mathfrak{b}_{s_2})\right\} \geq \min\{\hbar^+(\mathfrak{b}_{s_2}), \hbar^+(\mathfrak{b}_{s_3})\},$$

and

$$\hbar^-(\mathfrak{b}_{\mathfrak{s}_1}) \leq \max\left\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_{s_2})\right\} \leq \max\{\hbar^-(\mathfrak{b}_{s_2}), \hbar^-(\mathfrak{b}_{s_3})\}.$$

Conversely, suppose that $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BFS of $\mathfrak{B}_S$ satisfying the condition in (4). Since

$$\mathfrak{W}_0(\mathfrak{b}_{s_1}) \mid \mathfrak{W}_0(\mathfrak{b}_{s_1}) = \mathfrak{W}_{\mathfrak{b}_{s_1}|\mathfrak{b}_{s_1}}(1) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}|\mathfrak{b}_{s_1}}(1) = 1 \mid 1 = 0 \preccurlyeq \mathfrak{b}_{s_3},$$

we obtain

$$\hbar^+(0) \geq \hbar^+(\mathfrak{b}_{s_1}) \quad \text{and} \quad \hbar^-(0) \leq \hbar^-(\mathfrak{b}_{s_1}) \text{ for all } \mathfrak{b}_{s_1} \in \mathfrak{B}_S.$$

Next, as

$$((\mathfrak{b}_{s_1} \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \mid (\mathfrak{b}_{s_1} \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))) \mid (\mathfrak{b}_{s_2} \mid \mathfrak{b}_{s_2}) = \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid (\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) = 1,$$

we conclude that

$$(\mathfrak{b}_{s_1} \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \mid (\mathfrak{b}_{s_1} \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \preccurlyeq \mathfrak{b}_{s_2}.$$

Moreover, $(\mathfrak{b}_{s_1} \mid ((\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \mid (\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})))) \mid (\mathfrak{b}_{s_1} \mid ((\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \mid (\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})))) = (\mathfrak{b}_{s_1} \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \mid (\mathfrak{b}_{s_1} \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \preccurlyeq \mathfrak{b}_{s_2}$. Thus, we obtain

$$\hbar^+(\mathfrak{b}_{s_1}) \geq \min\{\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), {\mathfrak{b}_{s_1}}^+(\mathfrak{b}_{s_2})\},$$

and

$$\hbar^-(\mathfrak{b}_{s_1}) \leq \max\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_{s_2})\}.$$

Therefore, $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BFI of $\mathfrak{B}_S$. □

Theorem 5. *Every BF_{SBCK-I} of $\mathfrak{B}_S$ is a BF_{SBCK-S} of $\mathfrak{B}_S$.*

Proof. Let $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ be a BF_{SBCK-I} of $\mathfrak{B}_S$. Then

$$\begin{aligned}
&\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))\\
&\geq \min\{\hbar^+((\mathfrak{W}_{\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})|\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})}(\mathfrak{b}_{s_1})) \mid (\mathfrak{W}_{\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})|\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})}(\mathfrak{b}_{s_1})), \hbar^+(\mathfrak{b}_{s_1})\}\\
&= \min\{\hbar^+((\mathfrak{W}_{\mathfrak{b}_{s_2}|\mathfrak{b}_{s_2}}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))) \mid (\mathfrak{W}_{\mathfrak{b}_{s_2}|\mathfrak{b}_{s_2}}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))), \hbar^+(\mathfrak{b}_{s_1})\}\\
&= \min\{\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_2}|\mathfrak{b}_{s_2}}(1) \mid \mathfrak{W}_{\mathfrak{b}_{s_2}|\mathfrak{b}_{s_2}}(1), \hbar^+(\mathfrak{x})\}\\
&= \min\{\hbar^+(1|1), \hbar^+(\mathfrak{b}_{s_1})\}\\
&= \min\{\hbar^+(0), \hbar^+(\mathfrak{b}_{s_1})\}\\
&= \hbar^+(\mathfrak{b}_{s_1})\\
&\geq \min\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{b}_{s_2})\},
\end{aligned}$$

and

$$
\begin{aligned}
&\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))\\
&\leq \max\{f^{-}((\mathfrak{W}_{\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})|\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})}(\mathfrak{b}_{s_1})) \mid (\mathfrak{W}_{\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})|\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})}(\mathfrak{b}_{s_1})), \hbar^{-}(\mathfrak{b}_{s_1})\}\\
&= \max\{f^{-}((\mathfrak{W}_{\mathfrak{b}_{s_2}|\mathfrak{b}_{s_2}}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))) \mid (\mathfrak{W}_{\mathfrak{b}_{s_2}|\mathfrak{b}_{s_2}}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))), \hbar^{-}(\mathfrak{b}_{s_1})\}\\
&= \max\{f^{-}(\mathfrak{W}_{\mathfrak{b}_{s_2}|\mathfrak{b}_{s_2}}(1) \mid \mathfrak{W}_{\mathfrak{b}_{s_2}|\mathfrak{b}_{s_2}}(1)), \hbar^{-}(\mathfrak{b}_{s_1})\}\\
&= \max\{\hbar^{-}(1|1), \hbar^{-}(\mathfrak{b}_{s_1})\}\\
&= \max\{\hbar^{-}(0), \hbar^{-}(\mathfrak{b}_{s_1})\}\\
&= \hbar^{-}(\mathfrak{b}_{s_1})\\
&\leq \max\{\hbar^{-}(\mathfrak{b}_{s_1}), \hbar^{-}(\mathfrak{b}_{s_2})\}.
\end{aligned}
$$

Hence $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ is a BF_{SBCK-S} of X. □

Theorem 6. *A BFS $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ in $\mathfrak{B}_S$ is a BF_{SBCK-I} of $\mathfrak{B}_S$ if and only if its negative $\mathfrak{s}$-cut and positive $\mathfrak{t}$-cut are SBCK-ideals of $\mathfrak{B}_S$, whenever they are nonempty for all $(\mathfrak{s}, \mathfrak{t}) \in [-1, 0] \times [0, 1]$.*

Proof. Assume that $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ is a BF_{SBCK-I} of $\mathfrak{B}_S$ and $\mathfrak{L}(\hbar^{-}, \mathfrak{s}) \neq \emptyset \neq \mathfrak{U}(\hbar^{+}, \mathfrak{t})$ for all $(\mathfrak{s}, \mathfrak{t}) \in [-1, 0] \times [0, 1]$. Let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2}, \mathfrak{b}_{s_p}, \mathfrak{b}_{s_q} \in \mathfrak{B}_S$ be such that $(\mathfrak{b}_{s_2}, \mathfrak{b}_{s_q}) \in \mathfrak{B}_S(\hbar^{-}, \mathfrak{s}) \times \mathfrak{U}(\hbar^{+}, \mathfrak{t})$. Then, we have

$$\hbar^{-}(\mathfrak{b}_{s_2}) \leq \mathfrak{s} \quad \text{and} \quad \hbar^{+}(\mathfrak{b}_{s_q}) \geq \mathfrak{t}.$$

Using (1), we obtain

$$\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1}) \mid \mathfrak{W}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1})) \leq \hbar^{-}(\mathfrak{b}_{s_2}) \leq \mathfrak{s} \quad \text{and} \quad \hbar^{+}((\mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q}) \mid \mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q})) \geq \hbar^{+}(\mathfrak{b}_{s_q}) \geq \mathfrak{t}.$$

Thus, $(\mathfrak{W}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1}) \mid \mathfrak{W}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1})), (\mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q}) \mid \mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q})) \in \mathfrak{B}_S(\hbar^{-}, \mathfrak{s}) \times \mathfrak{U}(\hbar^{+}, \mathfrak{t})$. Let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2}, \mathfrak{b}_{s_p}, \mathfrak{b}_{s_q} \in \mathfrak{B}_S$ be such that $(\mathfrak{W}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1}) \mid \mathfrak{W}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1}), \mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q}) \mid \mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q})) \in \mathfrak{B}_S(\hbar^{-}, \mathfrak{s}) \times \mathfrak{U}(\hbar^{+}, \mathfrak{t})$ and $(\mathfrak{b}_s, \mathfrak{b}_{s_a}) \in \mathfrak{B}_S(\hbar^{-}, \mathfrak{s}) \times \mathfrak{U}(\hbar^{+}, \mathfrak{t})$. Then, we have

$$\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1}) \mid \mathfrak{W}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1})) \leq \mathfrak{s}, \quad \hbar^{-}(\mathfrak{b}_{s_1}) \leq \mathfrak{s}_1, \quad \hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_a}}(\mathfrak{b}_{s_b}) \mid \mathfrak{W}_{\mathfrak{b}_{s_a}}(\mathfrak{b}_{s_b})) \geq \mathfrak{t}, \quad \hbar^{+}(\mathfrak{b}_{s_a}) \geq \mathfrak{t}.$$

Using (1), we obtain

$$\hbar^{-}(\mathfrak{b}_{s_2}) \leq \max\left\{\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1}) \mid \mathfrak{W}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1})), \hbar^{-}(\mathfrak{b}_s)\right\} \leq \mathfrak{s}_1,$$

and

$$\hbar^{+}(\mathfrak{b}_{s_q}) \geq \min\left\{\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_a}}(\mathfrak{b}_{s_b}) \mid \mathfrak{W}_{\mathfrak{b}_{s_a}}(\mathfrak{b}_{s_b})), \hbar^{+}(\mathfrak{b}_{s_a})\right\} \geq \mathfrak{t}.$$

Thus, $(\mathfrak{b}_{s_2}, \mathfrak{b}_{s_q}) \in \mathfrak{B}_S(\hbar^{-}, \mathfrak{s}) \times \mathfrak{U}(\hbar^{+}, \mathfrak{t})$. Therefore, $\mathfrak{L}(\hbar^{-}, \mathfrak{s})$ and $\mathfrak{U}(\hbar^{+}, \mathfrak{t})$ are SBCK-ideals of $\mathfrak{B}_S$.

Conversely, let $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ be a BFS in $\mathfrak{B}_S$ for which its negative $\mathfrak{s}$-cut and positive $\mathfrak{t}$-cut are SBCK-ideals of $\mathfrak{B}_S$ whenever they are nonempty for all $(\mathfrak{s}, \mathfrak{t}) \in [-1, 0] \times [0, 1]$.

Suppose that $\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q}) \mid \mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q})) > \hbar^{-}(\mathfrak{b}_{s_q})$ for some $\mathfrak{b}_{s_p}, \mathfrak{b}_{s_q} \in \mathfrak{B}_S$. Then, $\mathfrak{b}_{s_q} \in \mathfrak{B}_S(\hbar^{-}, \hbar^{-}(\mathfrak{b}_{s_q}))$ but $(\mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q}) \mid \mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q})) \notin \mathfrak{B}_S(\hbar^{-}, \hbar^{-}(\mathfrak{b}_{s_q}))$, a contradiction. Hence, $\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1}) \mid \mathfrak{W}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1})) \leq \hbar^{-}(\mathfrak{b}_{s_2})$ for all $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$.

Suppose that $\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1}) \mid \mathfrak{W}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1})) < \hbar^{+}(\mathfrak{b}_{s_2})$ for some $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$. Then, $\mathfrak{b}_{s_2} \in \mathfrak{U}(\hbar^{+}, \hbar^{+}(\mathfrak{b}_{s_2}))$ but $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \notin \mathfrak{U}(\hbar^{+}, \hbar^{+}(\mathfrak{b}_{s_2}))$, a contradiction. Hence, $\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_a}}(\mathfrak{b}_{s_b}) \mid \mathfrak{W}_{\mathfrak{b}_{s_a}}(\mathfrak{b}_{s_b})) \geq \hbar^{-}(\mathfrak{b}_{s_q})$ for all $\mathfrak{b}_{s_a}, \mathfrak{b}_{s_q} \in \mathfrak{B}_S$.

Suppose that $\hbar^{-}(\mathfrak{b}_{s_q}) > \max\{\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q}) \mid \mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q})), \hbar^{-}(\mathfrak{b}_{s_p})\}$ or $\hbar^{+}(\mathfrak{b}_{s_2}) < \min\{\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^{+}(\mathfrak{b}_{s_1})\}$ for some $\mathfrak{b}_{s_p}, \mathfrak{b}_{s_q}, \mathfrak{b}_s, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$. Then, $(\mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q}) \mid$

$\mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q})), \mathfrak{b}_{s_p} \in \mathfrak{B}_S(\hbar^-, \mathfrak{s})$ or $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \mathfrak{b}_s \in \mathfrak{U}(\hbar^+, \mathfrak{t})$ where $\mathfrak{s} = \max\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q}) \mid \mathfrak{W}_{\mathfrak{b}_{s_p}}(\mathfrak{b}_{s_q}), \hbar^-(\mathfrak{b}_{s_p})\}$ and $\mathfrak{t} = \min\{\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^+(\mathfrak{b}_{s_1})\}$.

But $\mathfrak{b}_{s_q} \notin X(\hbar^-, \mathfrak{s})$ or $\mathfrak{b}_{s_2} \notin \mathfrak{U}(\hbar^+, \mathfrak{t})$, present a contradiction. Therefore, $\hbar^-(\mathfrak{b}_{s_2}) \leq \max\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_s)\}$ and $\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \min\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{b}_{s_2})\}$ for all $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$.

Consequently, $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BF_{SBCK-I} of $\mathfrak{B}_S$. □

Theorem 7. *A BFS $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ in $\mathfrak{B}_S$ is a BF_{SBCK-I} of $\mathfrak{B}_S$ if and only if the fuzzy sets $\hbar^-_{\mathfrak{c}}$ and $\hbar^+$ are fuzzy SBCK-ideals of $\mathfrak{B}_S$, where*

$$\hbar^-_{\mathfrak{c}} : L \to [0,1], \quad \mathfrak{b}_s \mapsto 1 - \hbar^-(\mathfrak{b}_s).$$

Proof. Let $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ be a BF_{SBCK-I} of $\mathfrak{B}_S$. It is clear that $\hbar^+$ is a fuzzy SBCK-ideal of $\mathfrak{B}_S$.

For every $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$,

$$\begin{aligned} \hbar^-_{\mathfrak{c}}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= 1 - \hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \\ &\geq 1 - \hbar^-(\mathfrak{b}_{s_2}) \\ &= 1 - \hbar^-(\mathfrak{b}_{s_2}) \\ &= \hbar^-_{\mathfrak{c}}(\mathfrak{b}_{s_2}), \end{aligned}$$

and

$$\begin{aligned} \hbar^-_{\mathfrak{c}}(\mathfrak{b}_{s_2}) &= 1 - \hbar^-(\mathfrak{b}_{s_2}) \\ &\geq 1 - \max\{\hbar^-(\mathfrak{b}_{s_1}), \hbar^-(\mathfrak{b}_{s_2})\} \\ &= \min\{1 - \hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), 1 - \hbar^-(\mathfrak{b}_{s_1})\} \\ &= \min\{\hbar^-_{\mathfrak{c}}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-_{\mathfrak{c}}(\mathfrak{b}_{s_1})\}. \end{aligned}$$

Hence, $\hbar^-_{\mathfrak{c}}$ is a fuzzy SBCK-ideal of X.

Conversely, let $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ be a BFS of $\mathfrak{B}_S$, for which $\hbar^-_{\mathfrak{c}}$ and $\hbar^+$ are fuzzy SBCK-ideals of $\mathfrak{B}_S$. Let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$. Then, as

$$\begin{aligned} 1 - \hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= \hbar^-_{\mathfrak{c}}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \\ &\geq \hbar^-_{\mathfrak{c}}(\mathfrak{b}_{s_2}) \\ &= 1 - \hbar^-(\mathfrak{b}_{s_2}), \end{aligned}$$

we obtain

$$\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq \hbar^-(\mathfrak{b}_{s_2}),$$

and as

$$\begin{aligned} 1 - \hbar^-(\mathfrak{b}_{s_2}) &= \hbar^-_{\mathfrak{c}}(\mathfrak{b}_{s_2}) \\ &\geq \min\{\hbar^-_{\mathfrak{c}}(\mathfrak{b}_s), \hbar^-_{\mathfrak{c}}(\mathfrak{b}_{s_2})\} \\ &= \min\{1 - \hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), 1 - \hbar^-(\mathfrak{b}_{s_1})\} \\ &= 1 - \max\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_{s_1})\}, \end{aligned}$$

we have

$$\hbar^-(\mathfrak{b}_{s_2}) \leq \max\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_{s_1})\}.$$

Hence, $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BF_{SBCK-I} of $\mathfrak{B}_S$. □

Theorem 8. *Given a nonempty subset $\mathfrak{H}$ of $\mathfrak{B}_S$, let $\hbar_{\mathfrak{H}} = (\mathfrak{B}_S, \hbar^-_{\mathfrak{H}}, \hbar^+_{\mathfrak{H}})$ be a BFS in $\mathfrak{B}_S$, defined as follows:*

$$\hbar^-_{\mathfrak{H}} : L \to [-1,0], \quad \mathfrak{b}_{s_a} \mapsto \begin{cases} \mathfrak{s}^- & \text{if } \mathfrak{b}_{s_a} \in \mathfrak{H}, \\ \mathfrak{t}^- & \text{otherwise}, \end{cases}$$

$$\hbar^+_{\mathfrak{H}} : L \to [-1,0], \quad \mathfrak{b}_{s_b} \mapsto \begin{cases} \mathfrak{s}^+ & \text{if } \mathfrak{b}_{s_b} \in \mathfrak{H}, \\ \mathfrak{t}^+ & \text{otherwise}, \end{cases}$$

where $\mathfrak{s}^- < \mathfrak{t}^-$ in $[-1,0]$ and $\mathfrak{s}^+ > \mathfrak{t}^+$ in $[0,1]$. Then, $\hbar_{\mathfrak{H}} = (\mathfrak{B}_S, \hbar^-_{\mathfrak{H}}, \hbar^+_{\mathfrak{H}})$ is a BF_{SBCK-I} of $\mathfrak{B}_S$, if and only if $\mathfrak{H}$ is an SBCK-ideal of $\mathfrak{B}_S$.

Proof. Assume that $\hbar_{\mathfrak{H}} = (\mathfrak{B}_S, \hbar^-_{\mathfrak{H}}, \hbar^+_{\mathfrak{H}})$ is a BFS of $\mathfrak{B}_S$. Let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$ be such that $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{H}$. Using (1), we have

$$\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq \hbar^-(\mathfrak{b}_{s_2}) = \mathfrak{s}^-,$$
$$\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \hbar^+(\mathfrak{b}_{s_2}) = \mathfrak{s}^+,$$

and so

$$\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) = \mathfrak{s}^- \quad \text{and} \quad \hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) = \mathfrak{s}^+.$$

This shows that $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \in \mathfrak{H}$. Next, we have

$$\hbar^-(\mathfrak{b}_{s_2}) \leq \max\left\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_s)\right\} = \mathfrak{s}^-,$$

$$\hbar^+(\mathfrak{b}_{s_2}) \geq \min\left\{\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^+(\mathfrak{b}_s)\right\} = \mathfrak{s}^+,$$

and so

$$\hbar^-(\mathfrak{b}_{s_2}) = \mathfrak{s}^- \quad \text{and} \quad \hbar^+(\mathfrak{b}_{s_2}) = \mathfrak{s}^+.$$

We have $\mathfrak{b}_{s_2} \in \mathfrak{H}$. Therefore, $\mathfrak{H}$ is an SBCK-ideal of $\mathfrak{B}_S$.

Conversely, let $\mathfrak{H}$ be an SBCK-ideal of $\mathfrak{B}_S$. For every $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$, if $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \in \mathfrak{H}$, then $\mathfrak{b}_{s_2} \in \mathfrak{H}$, which implies that

$$\hbar^-(\mathfrak{b}_{s_2}) = \mathfrak{s}^- = \hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),$$

$$\hbar^+(\mathfrak{b}_{s_2}) = \mathfrak{s}^+ = \hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})).$$

If $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \notin \mathfrak{H}$, then

$$\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) = \mathfrak{t}^- > \hbar^-(\mathfrak{b}_{s_2}),$$

$$\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) = \mathfrak{t}^+ < \hbar^+(\mathfrak{b}_{s_2}).$$

For every $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$, if $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{H}$, then $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \in \mathfrak{H}$, which implies that

$$\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) = \mathfrak{s}_1{}^- = \max\{\hbar^-(\mathfrak{b}_{s_1}), \hbar^-(\mathfrak{b}_{s_2})\}$$

and

$$\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) = \mathfrak{s}_1{}^+ = \min\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{b}_{s_2})\}.$$

If $\mathfrak{b}_{s_1} \notin \mathfrak{H}$ or $\mathfrak{b}_{s_2} \notin \mathfrak{H}$, then

$$\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq \mathfrak{t}^- = \max\{\hbar^-(\mathfrak{b}_{s_1}), \hbar^-(\mathfrak{b}_{s_2})\}$$

and

$$\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \mathfrak{t}^+ = \min\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{b}_{s_2})\}.$$

Therefore, $\hbar_{\mathfrak{H}} = (\mathfrak{B}_S, \hbar^-_{\mathfrak{H}F}, \hbar^+_{\mathfrak{H}})$ is a BF_{SBCK-I} of $\mathfrak{B}_S$. □

Proposition 3. *If $\hbar_\mathfrak{i} = (\mathfrak{B}_S, \hbar^-_\mathfrak{i}, \hbar^+_\mathfrak{i})$ is a family of bipolar fuzzy SBCK-ideals of $\mathfrak{B}_S$, then $\bigwedge\limits_{\mathfrak{i}\in\Delta} \hbar_\mathfrak{i}$ is a BF_{SBCK-I} of $\mathfrak{B}_S$.*

Proof. Let $\hbar_\mathfrak{i} = \{(\hbar^-_\mathfrak{i}, \hbar^+_\mathfrak{i}) : \mathfrak{i} \in \Delta\}$ be a family of bipolar fuzzy SBCK-ideals of a SBCK-algebra X. Let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$. We have

$$\begin{aligned}
(\bigwedge_{\mathfrak{i}\in\Delta} \hbar^+_\mathfrak{i})(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= \inf_{\mathfrak{i}\in\Delta}\{\hbar^+_\mathfrak{i}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))\} \\
&\geq \inf_{\mathfrak{i}\in\Delta}\{\hbar^+_\mathfrak{i}(\mathfrak{b}_{s_2})\} \\
&= (\bigwedge_{\mathfrak{i}\in\Delta} \hbar^+_\mathfrak{i})(\mathfrak{b}_{s_2})
\end{aligned}$$

and

$$\begin{aligned}
(\bigwedge_{\mathfrak{i}\in\Delta} \hbar^-_\mathfrak{i})(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= \sup_{\mathfrak{i}\in\Delta}\{\hbar^-_\mathfrak{i}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))\} \\
&\leq \sup_{\mathfrak{i}\in\Delta}\{\hbar^-_\mathfrak{i}(\mathfrak{b}_{s_2})\} \\
&= (\bigwedge_{\mathfrak{i}\in\Delta} \hbar^-_\mathfrak{i})(\mathfrak{b}_{s_2}).
\end{aligned}$$

For $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$, we have the following inequality:

$$\begin{aligned}
(\bigwedge_{\mathfrak{i}\in\Delta} \hbar^+_\mathfrak{i})(\mathfrak{b}_{s_2}) &= \inf_{\mathfrak{i}\in\Delta}\{\hbar^+_\mathfrak{i}(\mathfrak{b}_{s_2})\} \\
&\geq \inf_{\mathfrak{i}\in\Delta}\left\{\min\left(\hbar^+_\mathfrak{i}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^+_\mathfrak{i}(\mathfrak{b}_{s_1})\right)\right\} \\
&= \min\left\{\inf_{\mathfrak{i}\in\Delta} \hbar^+_\mathfrak{i}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \inf_{\mathfrak{i}\in\Delta} \hbar^+_\mathfrak{i}(\mathfrak{b}_{s_1})\right\} \\
&= \min\left\{(\bigwedge_{\mathfrak{i}\in\Delta} \hbar^+_\mathfrak{i})(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), (\bigwedge_{\mathfrak{i}\in\Delta} \hbar^+_\mathfrak{i})(\mathfrak{b}_{s_1})\right\}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
(\bigwedge_{\mathfrak{i}\in\Delta} \hbar^-_\mathfrak{i})(\mathfrak{b}_{s_2}) &= \sup_{\mathfrak{i}\in\Delta}\{\hbar^-_\mathfrak{i}(\mathfrak{b}_{s_2})\} \\
&\leq \sup_{\mathfrak{i}\in\Delta}\left\{\max\left(\hbar^-_\mathfrak{i}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-_\mathfrak{i}(\mathfrak{b}_{s_1})\right)\right\} \\
&= \max\left\{\sup_{\mathfrak{i}\in\Delta} \hbar^-_\mathfrak{i}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \sup_{\mathfrak{i}\in\Delta} \hbar^-_\mathfrak{i}(\mathfrak{b}_{s_1})\right\} \\
&= \max\left\{(\bigwedge_{\mathfrak{i}\in\Delta} \hbar^-_\mathfrak{i})(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), (\bigwedge_{\mathfrak{i}\in\Delta} \hbar^-_\mathfrak{i})(\mathfrak{b}_{s_1})\right\}.
\end{aligned}$$

Hence, $\bigwedge\limits_{\mathfrak{i}\in\Delta} \hbar_\mathfrak{i}$ is a BF_{SBCK-I} of $\mathfrak{B}_S$. □

5. Bipolar Fuzzy (ϱ, ν)-Translations in SBCK-Algebras

In this section, we introduce and explore the concept of bipolar fuzzy (ϱ, ν)-translations within SBCK-algebras.

Definition 9. *The inclusion $\subseteq$ is defined as follows: for any bipolar fuzzy sets $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ and $\psi = (\mathfrak{B}_S, \psi^-, \psi^+)$ on $\mathfrak{B}_S$,*

$$\hbar \subseteq \psi \quad \Leftrightarrow \quad \hbar^-(\mathfrak{b}_s) \geq \psi^-(\mathfrak{b}_s) \text{ and } \hbar^+(\mathfrak{b}_s) \leq \psi^+(\mathfrak{b}_s) \quad \forall \mathfrak{b}_s \in \mathfrak{B}_S.$$

We say that $\psi = (\mathfrak{B}_S, \psi^-, \psi^+)$ is a bipolar fuzzy extension of $\hbar = (X, \hbar^-, \hbar^+)$, and $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a bipolar fuzzy intension of $\psi = (\mathfrak{B}_S, \psi^-, \psi^+)$.

Example 4. *Let $(\mathfrak{B}_S, |)$ be a SBCK-algebra given in Example 1. Define bipolar-valued fuzzy sets $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ and $\psi = (\mathfrak{B}_S, \psi^-, \psi^+)$ in X by the table below:*

$\mathfrak{B}_S$	0	1	2	3	4	5	6	7
$\hbar^-$	-0.8	-0.7	-0.8	-0.8	-0.8	-0.8	-0.7	-0.8
$\hbar^+$	0.9	0.6	0.7	0.85	0.7	0.7	0.7	0.85
ψ^-	-0.9	-0.7	-0.9	-0.9	-0.9	-0.9	-0.7	-0.9
ψ^+	0.95	0.7	0.8	0.85	0.8	0.8	0.8	0.85

It is routine to verify that the BFS $\psi = (\mathfrak{B}_S, \psi^-, \psi^+)$ is a bipolar fuzzy extension of $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$, and $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a bipolar fuzzy intension of $\psi = (\mathfrak{B}_S, \psi^-, \psi^+)$.

Definition 10. *For any BFS $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ in $\mathfrak{B}_S$, we denote*

$$\perp = -1 - \inf\{\hbar^-(\mathfrak{b}_s) : \mathfrak{b}_s \in \mathfrak{B}_S\},$$

$$\top = 1 - \sup\{\hbar^+(\mathfrak{b}_s) : \mathfrak{b}_s \in \mathfrak{B}_S\}.$$

Let $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ be a BFS in $\mathfrak{B}_S$ and $(\varrho, \nu) \in [\perp, 0] \times [0, \top]$. By a $BF_{(\varrho,\nu)-t}$ of $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ of type I, we mean a BFS $\hbar^{\mathfrak{T}_1}_{(\varrho,\nu)} = (\mathfrak{B}_S, \hbar^-_{(\varrho,\mathfrak{T}_1)}, \hbar^+_{(\nu,\mathfrak{T}_1)})$, where

$$\hbar^-_{(\varrho,\mathfrak{T}_1)} :\in \mathfrak{B}_S \to [-1, 0],\ \mathfrak{b}_s \mapsto \hbar^-(\mathfrak{b}_s) + \varrho,$$

$$\hbar^+_{(\nu,\mathfrak{T}_1)} :\in \mathfrak{B}_S \to [0, 1],\ \mathfrak{b}_s \mapsto \hbar^+(\mathfrak{b}_s) + \nu.$$

Lemma 4. *If a BFS $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ in $\mathfrak{B}_S$ is a BF_{SBCK-S} of $\mathfrak{B}_S$, then for all $(\varrho, \nu) \in [\perp, 0] \times [0, \top]$, a $BF_{(\varrho,\nu)-t}$ $\hbar^{\mathfrak{T}_1}_{(\varrho,\nu)} = (\mathfrak{B}_S, \hbar^-_{(\varrho,\mathfrak{T}_1)}, \hbar^+_{(\nu,\mathfrak{T}_1)})$ of $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BF_{SBCK-S} of $\mathfrak{B}_S$.*

Proof. Assume that $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BF_{SBCK-S} of $\mathfrak{B}_S$. For any $(\varrho, \nu) \in [\perp, 0] \times [0, \top]$ and for all $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$, we have

$$\begin{aligned} \hbar^-_{(\varrho,\mathfrak{T}_1)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= \hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) + \varrho \\ &\leq \max\{\hbar^-(\mathfrak{b}_{s_1}), \hbar^-(\mathfrak{b}_{s_2})\} + \varrho \\ &= \max\{\hbar^-(\mathfrak{b}_{s_1}) + \varrho, \hbar^-(\mathfrak{b}_{s_2}) + \varrho\} \\ &= \max\{\hbar^-_{(\varrho,\mathfrak{T}_1)}(\mathfrak{b}_{s_1}), \hbar^-_{(\varrho,\mathfrak{T}_1)}(\mathfrak{b}_{s_2})\}, \end{aligned}$$

and

$$\begin{aligned} \hbar^+_{(\nu,\mathfrak{T}_1)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= \hbar\hbar + (\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) + \nu \\ &\geq \min\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{b}_{s_2})\} + \nu \\ &= \min\{\hbar^+(\mathfrak{b}_{s_1}) + \nu, \hbar^+(\mathfrak{b}_{s_2}) + \nu\} \\ &= \min\{\hbar^+_{(\nu,\mathfrak{T}_1)}(\mathfrak{b}_{s_1}), \hbar^+_{(\nu,\mathfrak{T}_1)}(\mathfrak{b}_{s_2})\}. \end{aligned}$$

Hence, $f^{T_1}_{(\varrho,\nu)} = (\mathfrak{B}_S, \hbar^-_{(\varrho,T_1)}, f^+_{(\nu,T_1)})$ is a BF_{SBCK-S} of $\mathfrak{B}_S$. □

Lemma 5. *If there exists* $(\varrho,\nu) \in [\bot,0]\times[0,\top]$, *such that the* $(\varrho,\nu)-$*translation* $\hbar^{T_1}_{(\varrho,\nu)} = (\mathfrak{B}_S, \hbar^-_{(\varrho,T_1)}, \hbar^+_{(\nu,T_1)})$ *of* $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ *is a* BF_{SBCK-S} *of* $\mathfrak{B}_S$, *then* $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ *is a* BF_{SBCK-S} *of* $\mathfrak{B}_S$.

Proof. Assume that $\hbar^{\mathfrak{T}_1}_{(\varrho,\nu)} = (\mathfrak{B}_S, \hbar^-_{(\varrho,\mathfrak{T}_1)}, \hbar^+_{(\nu,\mathfrak{T}_1)})$ is a BF_{SBCK-S} of $\mathfrak{B}_S$ for $(\varrho,\nu)\in[\bot,0]\times[0,\top]$ and for all $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$, we have

$$\begin{aligned}
\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) + \varrho &= \hbar^-_{(\varrho,\mathfrak{T}_1)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \\
&\leq \max\Big\{\hbar^-_{(\varrho,\mathfrak{T}_1)}(\mathfrak{b}_{s_1}), \hbar^-_{(\varrho,\mathfrak{T}_1)}(\mathfrak{b}_{s_2})\Big\} \\
&= \max\big\{\hbar^-(\mathfrak{b}_{s_1}) + \varrho, \hbar^-(\mathfrak{b}_{s_2}) + \varrho\big\} \\
&= \max\big\{\hbar^-(\mathfrak{b}_s), \hbar^-(\mathfrak{b}_{s_2})\big\} + \varrho.
\end{aligned}$$

and

$$\begin{aligned}
\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) + \nu &= \hbar^+_{(\nu,\mathfrak{T}_1)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \\
&\geq \min\Big\{\hbar^+_{(\nu,\mathfrak{T}_1)}(\mathfrak{b}_{s_1}), \hbar^+_{(\nu,\mathfrak{T}_1)}(\mathfrak{b}_{s_2})\Big\} \\
&= \min\big\{\hbar^+(\mathfrak{b}_{s_1}) + \nu, \hbar^+(\mathfrak{b}_{s_2}) + \nu\big\} \\
&= \min\big\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{b}_{s_2})\big\} + \nu.
\end{aligned}$$

Hence, $\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq \max\{\hbar^-(\mathfrak{b}_{s_1}), \hbar^-(\mathfrak{b}_{s_2})\}$ and $\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \min\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{b}_{s_2})\}$. Hence, $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BF_{SBCK-S} of $\mathfrak{B}_S$. □

Lemma 6. *If a BFS* $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ *in* $\mathfrak{B}_S$ *is a* BF_{SBCK-I} *of* $\mathfrak{B}_S$, *then for all* $(\varrho,\nu)\in[\bot,0]\times[0,\top]$, *a* $BF_{(\varrho,\nu)-\mathfrak{t}}$ $\hbar^{\mathfrak{T}_1}_{(\varrho,\nu)} = (\mathfrak{B}_S, \hbar^-_{(\varrho,\mathfrak{T}_1)}, \hbar^+_{(\nu,\mathfrak{T}_1)})$ *of* $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ *is a* BF_{SBCK-I} *of* $\mathfrak{B}_S$.

Proof. Assume that $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BF_{SBCK-I} of $\mathfrak{B}_S$. For any $(\varrho,\nu)\in[\bot,0]\times[0,\top]$ and for all $\mathfrak{b}_s, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$, we have

$$\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq \hbar^-(\mathfrak{b}_{s_2}) \quad \text{and} \quad \hbar^+(\mathfrak{b}_{s_1} \mid \mathfrak{b}_{s_2}) \geq \hbar^+(\mathfrak{b}_{s_2}).$$

Then,

$$\begin{aligned}
\hbar^-_{(\varrho,\mathfrak{T}_1)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= \hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) + \varrho \\
&\leq \hbar^-(\mathfrak{b}_{s_2}) + \varrho \\
&= \hbar^-_{(\varrho,\mathfrak{T}_1)}(\mathfrak{b}_{s_2}), \\
\hbar^+_{(\nu,\mathfrak{T}_1)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= \hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) + \nu \\
&\geq \hbar^+(\mathfrak{b}_{s_2}) + \nu \\
&= \hbar^+_{(\nu,\mathfrak{T}_1)}(\mathfrak{b}_{s_2}).
\end{aligned}$$

Next, let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$. Then,

$$\hbar^-(\mathfrak{b}_{s_2}) \leq \max\Big\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_{s_1})\Big\}$$

and

$$\hbar^+(\mathfrak{b}_{s_2}) \geq \min\Big\{\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^+(\mathfrak{b}_{s_1})\Big\}.$$

Then, we have

$$\begin{aligned}
\hbar^-_{(\varrho,\mathfrak{T}_1)}(\mathfrak{b}_{s_2}) &= \hbar^-(\mathfrak{b}_{s_2}) + \varrho \\
&\leq \max\Big\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_{s_1})\Big\} + \varrho \\
&= \max\Big\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) + \varrho, \hbar^-(\mathfrak{b}_{s_1}) + \varrho\Big\} \\
&= \max\Big\{\hbar^-_{(\varrho,\mathfrak{T}_1)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-_{(\varrho,\mathfrak{T}_1)}(\mathfrak{b}_{s_1})\Big\},
\end{aligned}$$

and

$$\begin{aligned}\hbar^{+}_{(\nu,\mathfrak{T}_1)}(\mathfrak{b}_{s_2}) &= \hbar^{+}(\mathfrak{b}_{s_2})+\nu\\ &\geq \min\Big\{\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),\hbar^{+}(\mathfrak{b}_{s_1})\Big\}+\nu\\ &= \min\Big\{\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))+\nu,\hbar^{+}(\mathfrak{b}_{s_1})+\nu\Big\}\\ &= \min\Big\{\hbar^{+}_{(\nu,\mathfrak{T}_1)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),\hbar^{+}_{(\nu,\mathfrak{T}_1)}(\mathfrak{b}_{s_1})\Big\}.\end{aligned}$$

Hence, $\hbar^{\mathfrak{T}_1}_{(\varrho,\nu)}=(\mathfrak{B}_S,\hbar^{-}_{(\varrho,\mathfrak{T}_1)},\hbar^{+}_{(\nu,\mathfrak{T}_1)})$ is a BF_{SBCK-I} of $\mathfrak{B}_S$. □

Theorem 9. *If there exists* $(\varrho,\nu)\in[\bot,0]\times[0,\top]$, *such that the BF* $(\varrho,\nu)-$*translation* $\hbar^{\mathfrak{T}_1}_{(\varrho,\nu)}=(\mathfrak{B}_S,\hbar^{-}_{(\varrho,\mathfrak{T}_1)},\hbar^{+}_{(\nu,\mathfrak{T}_1)})$ *of* $\hbar=(\mathfrak{B}_S,\hbar^{-},\hbar^{+})$ *is a* BF_{SBCK-I} *of* $\mathfrak{B}_S$, *then* $\hbar=(\mathfrak{B}_S,\hbar^{-},\hbar^{+})$ *is a* BF_{SBCK-I} *of* $\mathfrak{B}_S$.

Proof. Assume that $\hbar^{\mathfrak{T}_1}_{(\varrho,\nu)}=(\mathfrak{B}_S,\hbar^{-}_{(\varrho,\mathfrak{T}_1)},\hbar^{+}_{(\nu,\mathfrak{T}_1)})$ is a BF_{SBCK-I} of $\mathfrak{B}_S$ for $(\varrho,\nu)\in[\bot,0]\times[0,\top]$, and for all $\mathfrak{b}_{s_1},\mathfrak{b}_{s_2}\in\mathfrak{B}_S$, we have

$$\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))\leq\hbar^{-}(\mathfrak{b}_{s_2})\quad\text{and}\quad\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))\geq\hbar^{+}(\mathfrak{b}_{s_2}).$$

Then, we have the following:

$$\begin{aligned}\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))+\varrho &= \hbar^{-}_{(\varrho,\mathfrak{T}_1)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))\\ &\leq \hbar^{-}_{(\varrho,\mathfrak{T}_1)}(\mathfrak{b}_{s_2})\\ &= \hbar^{-}(\mathfrak{b}_{s_2})+\varrho,\end{aligned}$$

and

$$\begin{aligned}\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))+\nu &= \hbar^{+}_{(\nu,\mathfrak{T}_1)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))\\ &\geq \hbar^{+}_{(\nu,\mathfrak{T}_1)}(\mathfrak{b}_{s_2})\\ &= \hbar^{+}(\mathfrak{b}_{s_2})+\nu.\end{aligned}$$

Now, let $\mathfrak{b}_{s_1},\mathfrak{b}_{s_2}\in\mathfrak{B}_S$. Then, we have the following:

$$\begin{aligned}\hbar^{-}(\mathfrak{b}_{s_2})+\varrho &= \hbar^{-}_{(\varrho,\mathfrak{T}_1)}(\mathfrak{b}_{s_2})\\ &\leq \max\{\hbar^{-}_{(\varrho,\mathfrak{T}_1)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),\hbar^{-}_{(\varrho,\mathfrak{T}_1)}(\mathfrak{b}_{s_1})\}\\ &= \max\{\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))+\varrho,\hbar^{-}(\mathfrak{b}_{s_1})+\varrho\}\\ &= \max\{\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),\hbar^{-}(\mathfrak{b}_{s_1})\}+\varrho\end{aligned}$$

and

$$\begin{aligned}\hbar^{+}(\mathfrak{b}_{s_2})+\nu &= \hbar^{+}_{(\nu,\mathfrak{T}_1)}(\mathfrak{b}_{s_2})\\ &\geq \min\{\hbar^{+}_{(\nu,\mathfrak{T}_1)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),\hbar^{+}_{(\nu,\mathfrak{T}_1)}(\mathfrak{b}_{s_1})\}\\ &= \min\{\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))+\nu,\hbar^{+}(\mathfrak{b}_{s_1})+\nu\}\\ &= \min\{\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),\hbar^{+}(\mathfrak{b}_{s_1})\}+\nu.\end{aligned}$$

Hence, we obtain the following inequalities:

$$\hbar^{-}(\mathfrak{b}_{s_2})\leq\max\{\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),\hbar^{-}(\mathfrak{b}_{s_1})\},$$

and

$$\hbar^{+}(\mathfrak{b}_{s_2})\geq\min\{\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),\hbar^{+}(\mathfrak{b}_{s_1})\}.$$

Therefore, $\hbar=(\mathfrak{B}_S,\hbar^{-},\hbar^{+})$ is a BF_{SBCK-I} of $\mathfrak{B}_S$. □

Remark 1. *If* $\hbar=(\mathfrak{B}_S,\hbar^{-},\hbar^{+})$ *is a BFS of* $\mathfrak{B}_S$, *then for all* $(\varrho,\nu)\in[\bot,0]\times[0,\top]$, *we have*

$$\hbar^{-}_{(\varrho,\mathfrak{T}_1)}(\mathfrak{b}_s)=\hbar^{-}(\mathfrak{b}_s)+\varrho\leq\hbar^{-}(\mathfrak{b}_s),$$

and

$$\hbar^{+}_{(\nu,\mathfrak{T}_1)}(\mathfrak{b}_s) = \hbar^{+}(\mathfrak{b}_s) + \nu \geq \hbar^{+}(\mathfrak{b}_s),$$

for all $\mathfrak{b}_s \in \mathfrak{B}_S$. *Hence, the* $BF_{(\varrho,\nu)-\mathfrak{t}}$ $\hbar^{\mathfrak{T}_1}_{(\varrho,\nu)} = (\mathfrak{B}_S, \hbar^{-}_{(\varrho,\mathfrak{T}_1)}, \hbar^{+}_{(\nu,\mathfrak{T}_1)})$ *of* $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ *is a bipolar fuzzy extension of* $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ *for all* $(\varrho, \nu) \in [\perp, 0] \times [0, \top]$.

Definition 11. *For any BFS,* $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ *in* $\mathfrak{B}_S$, *we denote*

$$\pm = \sup\{\hbar^{-}(\mathfrak{b}_s) : \mathfrak{b}_s \in \mathfrak{B}_S\},$$

$$\mp = \inf\{\hbar^{+}(\mathfrak{b}_s) : \mathfrak{b}_s \in \mathfrak{B}_S\}.$$

Let $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ *be a BFS in* $\mathfrak{B}_S$ *and* $(\varrho, \nu) \in [\pm, 0] \times [0, \mp]$. *By a* $BF_{(\varrho,\nu)-\mathfrak{t}}$ *of* $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ *of Type II, we mean a BFS* $\hbar^{\mathfrak{T}_2}_{(\varrho,\nu)} = (\mathfrak{B}_S, \hbar^{-}_{(\varrho,\mathfrak{T}_2)}, \hbar^{+}_{(\nu,\mathfrak{T}_2)})$, *where*

$$\hbar^{-}_{(\varrho,\mathfrak{T}_2)} : L \to [-1, 0],\ \mathfrak{b}_s \mapsto \hbar^{-}(\mathfrak{b}_s) - \varrho,$$

$$\hbar^{+}_{(\nu,\mathfrak{T}_2)} : L \to [0, 1],\ \mathfrak{b}_s \mapsto \hbar^{+}(\mathfrak{b}_s) - \nu.$$

Theorem 10. *If a BFS* $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ *in* $\mathfrak{B}_S$ *is a* BF_{SBCK-S} *of* $\mathfrak{B}_S$, *then for all* $(\varrho, \nu) \in [\pm, 0] \times [0, \mp]$, *a BF* $(\varrho, \nu)-$*translation* $\hbar^{\mathfrak{T}_2}_{(\varrho,\nu)} = (\mathfrak{B}_S, \hbar^{-}_{(\varrho,\mathfrak{T}_2)}, \hbar^{+}_{(\nu,\mathfrak{T}_2)})$ *of* $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ *is a* BF_{SBCK-S} *of* $\mathfrak{B}_S$.

Proof. Assume that $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ is a BF_{SBCK-S} of $\mathfrak{B}_S$. For any $(\varrho, \nu) \in [\pm, 0] \times [0, \mp]$ and for all $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$, we have

$$\begin{aligned}
\hbar^{-}_{(\varrho,T_2)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= \hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) - \varrho \\
&\leq \max\{\hbar^{-}(\mathfrak{b}_{s_1}), \hbar^{-}(\mathfrak{b}_{s_2})\} - \varrho \\
&= \max\{\hbar^{-}(\mathfrak{b}_{s_1}) - \varrho, \hbar^{-}(\mathfrak{b}_{s_2}) - \varrho\} \\
&= \max\{\hbar^{-}_{(\varrho,\mathfrak{T}_2)}(\mathfrak{b}_{s_1}), \hbar^{-}_{(\varrho,\mathfrak{T}_2)}(\mathfrak{b}_{s_2})\},
\end{aligned}$$

$$\begin{aligned}
\hbar^{+}_{(\nu,T_2)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= \hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) - \nu \\
&\geq \min\{\hbar^{+}(\mathfrak{b}_{s_1}), \hbar^{+}(\mathfrak{b}_{s_2})\} - \nu \\
&= \min\{\hbar^{+}(\mathfrak{b}_{s_1}) - \nu, \hbar^{+}(\mathfrak{b}_{s_2}) - \nu\} \\
&= \min\{\hbar^{+}_{(\nu,\mathfrak{T}_2)}(\mathfrak{b}_{s_1}), \hbar^{+}_{(\nu,\mathfrak{T}_2)}(\mathfrak{b}_{s_2})\}.
\end{aligned}$$

Hence, $\hbar^{\mathfrak{T}_2}_{(\varrho,\nu)} = (\mathfrak{B}_S, \hbar^{-}_{(\varrho,\mathfrak{T}_2)}, \hbar^{+}_{(\nu,\mathfrak{T}_2)})$ is a BF_{SBCK-S} of $\mathfrak{B}_S$. □

Theorem 11. *If there exists* $(\varrho, \nu) \in [\pm, 0] \times [0, \mp]$, *such that the BF* $(\varrho, \nu)-$*translation* $\hbar^{\mathfrak{T}_2}_{(\varrho,\nu)} = (\mathfrak{B}_S, \hbar^{-}_{(\varrho,\mathfrak{T}_2)}, \hbar^{+}_{(\nu,\mathfrak{T}_2)})$ *of* $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ *is a* BF_{SBCK-S} *of* $\mathfrak{B}_S$, *then* $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ *is a* BF_{SBCK-I} *of* $\mathfrak{B}_S$.

Proof. Assume that $\hbar^{\mathfrak{T}_2}_{(\varrho,\nu)} = (\mathfrak{B}_S, \hbar^{-}_{(\varrho,\mathfrak{T}_2)}, \hbar^{+}_{(\nu,\mathfrak{T}_2)})$ is a BF_{SBCK-S} of $\mathfrak{B}_S$ for $(\varrho, \nu) \in [\pm, 0] \times [0, \mp]$ and for all $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$, we have

$$\begin{aligned}
\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) - \varrho &= \hbar^{-}_{(\varrho,\mathfrak{T}_2)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \\
&\leq \max\{\hbar^{-}_{(\varrho,\mathfrak{T}_2)}(\mathfrak{b}_{s_1}), \hbar^{-}_{(\varrho,\mathfrak{T}_2)}(\mathfrak{b}_{s_2})\} \\
&= \max\{\hbar^{-}(\mathfrak{b}_{s_1}) - \varrho, \hbar^{-}(\mathfrak{b}_{s_2}) - \varrho\} \\
&= \max\{\hbar^{-}(\mathfrak{b}_{s_1}), \hbar^{-}(\mathfrak{b}_{s_2})\} - \varrho,
\end{aligned}$$

and

$$\begin{aligned}\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) - \nu &= \hbar^{+}_{(\nu,\mathfrak{T}_2)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \\ &\geq \min\{\hbar^{+}_{(\nu,\mathfrak{T}_2)}(\mathfrak{b}_{s_1}), \hbar^{+}_{(\nu,\mathfrak{T}_2)}(\mathfrak{b}_{s_2})\} \\ &= \min\{\hbar^{+}(\mathfrak{b}_{s_1}) - \nu, \hbar^{+}(\mathfrak{b}_{s_2}) - \nu\} \\ &= \min\{\hbar^{+}(\mathfrak{b}_{s_1}), \hbar^{+}(\mathfrak{b}_{s_2})\} - \nu.\end{aligned}$$

Hence, we obtain $\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq \max\{\hbar^{-}(\mathfrak{b}_{s_1}), \hbar^{-}(\mathfrak{b}_{s_2})\}$ and $\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \min\{\hbar^{+}(\mathfrak{b}_{s_1}), \hbar^{+}(\mathfrak{b}_{s_2})\}$. Therefore, $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ is a BF_{SBCK-I} of $\mathfrak{B}_S$. □

Theorem 12. *If a BFS $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ in $\mathfrak{B}_S$ is a BF_{SBCK-I} of $\mathfrak{B}_S$, then for all $(\varrho, \nu) \in [\pm, 0] \times [0, \mp]$, a BF $(\varrho, \nu)-$translation $\hbar^{\mathfrak{T}_2}_{(\varrho,\nu)} = (\mathfrak{B}_S, \hbar^{-}_{(\varrho,\mathfrak{T}_2)}, \hbar^{+}_{(\nu,\mathfrak{T}_2)})$ of $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ is a BF_{SBCK-I} of $\mathfrak{B}_S$.*

Proof. Assume that $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ is a BF_{SBCK-I} on $\mathfrak{B}_S$. For any $(\varrho, \nu) \in [\pm, 0] \times [0, \mp]$ and for all $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$, we have the following relations:

$$\begin{aligned}\hbar^{-}_{(\varrho,\mathfrak{T}_2)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= \hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) - \varrho \\ &\leq \hbar^{-}(\mathfrak{b}_{s_2}) - \varrho \\ &= \hbar^{-}_{(\varrho,\mathfrak{T}_2)}(\mathfrak{b}_{s_2}),\end{aligned}$$

and

$$\begin{aligned}\hbar^{+}_{(\nu,\mathfrak{T}_2)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= \hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) - \nu \\ &\geq \hbar^{+}(\mathfrak{b}_{s_2}) - \nu \\ &= \hbar^{+}_{(\nu,\mathfrak{T}_2)}(\mathfrak{b}_{s_2}).\end{aligned}$$

Next, let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$. Then, it follows that

$$\hbar^{-}(\mathfrak{b}_{s_2}) \leq \max\{\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^{-}(\mathfrak{b}_{s_1})\}$$

$$\hbar^{+}(\mathfrak{b}_{s_2}) \geq \min\{\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^{+}(\mathfrak{b}_{s_1})\}$$

Thus, we have

$$\begin{aligned}\hbar^{-}_{(\varrho,\mathfrak{T}_2)}(\mathfrak{b}_{s_2}) &= \hbar^{-}(\mathfrak{b}_{s_2}) - \varrho \\ &\leq \max\{\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^{-}(\mathfrak{b}_{s_1})\} - \varrho \\ &= \max\{\hbar^{-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) - \varrho, \hbar^{-}(\mathfrak{b}_{s_1}) - \varrho\} \\ &= \max\{\hbar^{-}_{(\varrho,\mathfrak{T}_2)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^{-}_{(\varrho,\mathfrak{T}_2)}(\mathfrak{b}_{s_1})\},\end{aligned}$$

and

$$\begin{aligned}\hbar^{+}_{(\nu,\mathfrak{T}_2)}(\mathfrak{b}_{s_2}) &= \hbar^{+}(\mathfrak{b}_{s_2}) - \nu \\ &\geq \min\{\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^{+}(\mathfrak{b}_{s_1})\} - \nu \\ &= \min\{\hbar^{+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) - \nu, \hbar^{+}(\mathfrak{b}_{s_1}) - \nu\} \\ &= \min\{\hbar^{+}_{(\nu,\mathfrak{T}_2)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^{+}_{(\nu,\mathfrak{T}_2)}(\mathfrak{b}_{s_1})\}.\end{aligned}$$

Therefore, $f^{T_2}_{(\varrho,\nu)} = (\mathfrak{B}_S, f^{-}_{(\varrho,T_2)}, f^{+}_{(\nu,T_2)})$ is a BF_{SBCK-I} of $\mathfrak{B}_S$. □

Theorem 13. *If there exists $(\varrho, \nu) \in [\pm, 0] \times [0, \mp]$, such that the BF $(\varrho, \nu)-$translation*

$$\hbar^{\mathfrak{T}_2}_{(\varrho,\nu)} = (\mathfrak{B}_S, \hbar^{-}_{(\varrho,\mathfrak{T}_2)}, \hbar^{+}_{(\nu,\mathfrak{T}_2)}),$$

of $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ is a BF_{SBCK-I} of $\mathfrak{B}_S$, then $\hbar = (\mathfrak{B}_S, \hbar^{-}, \hbar^{+})$ is also a BF_{SBCK-I} of $\mathfrak{B}_S$.

Proof. Assume that $\hbar^{\mathfrak{T}_2}_{(\varrho,\nu)} = (\mathfrak{B}_S, \hbar^-_{(\varrho,\mathfrak{T}_2)}, \hbar^+_{(\nu,\mathfrak{T}_2)})$ is a BF_{SBCK-I} of $\mathfrak{B}_S$ for $(\varrho,\nu) \in [\pm,0] \times [0,\mp]$ and for all $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$, we have

$$\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq \hbar^-(\mathfrak{b}_{s_2}) \quad \text{and} \quad \hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \hbar^+(\mathfrak{b}_{s_2}).$$

Then,

$$\begin{aligned} \hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) - \varrho &= \hbar^-_{(\varrho,\mathfrak{T}_2)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \\ &\leq \hbar^-_{(\varrho,\mathfrak{T}_2)}(\mathfrak{b}_{s_2}) \\ &= \hbar^-(\mathfrak{b}_{s_2}) - \varrho, \end{aligned}$$

and

$$\begin{aligned} \hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) - \nu &= \hbar^+_{(\nu,\mathfrak{T}_2)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \\ &\geq \hbar^+_{(\nu,\mathfrak{T}_2)}(\mathfrak{b}_{s_2}) \\ &= \hbar^+(\mathfrak{b}_{s_2}) - \nu. \end{aligned}$$

Now, let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{B}_S$. Then

$$\begin{aligned} \hbar^-(\mathfrak{b}_{s_2}) - \varrho &= \hbar^-_{(\varrho,\mathfrak{T}_2)}(\mathfrak{b}_{s_2}) \\ &\leq \max\left\{\hbar^-_{(\varrho,\mathfrak{T}_2)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-_{(\varrho,\mathfrak{T}_2)}(\mathfrak{b}_{s_1})\right\} \\ &= \max\left\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) - \varrho, \hbar^-(\mathfrak{b}_{s_1}) - \varrho\right\} \\ &= \max\left\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_{s_1})\right\} - \varrho, \end{aligned}$$

and

$$\begin{aligned} \hbar^+(\mathfrak{b}_{s_2}) - \nu &= \hbar^+_{(\nu,\mathfrak{T}_2)}(\mathfrak{b}_{s_2}) \\ &\geq \min\left\{\hbar^+_{(\nu,\mathfrak{T}_2)}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^+_{(\nu,\mathfrak{T}_2)}(\mathfrak{b}_{s_1})\right\} \\ &= \min\left\{\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) - \nu, \hbar^+(\mathfrak{b}_{s_1}) - \nu\right\} \\ &= \min\left\{\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^+(\mathfrak{b}_{s_1})\right\} - \nu. \end{aligned}$$

Hence,

$$\hbar^-(\mathfrak{b}_{s_2}) \leq \max\left\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_{s_1})\right\}$$

and

$$\hbar^+(\mathfrak{b}_{s_2}) \geq \min\left\{\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^+(\mathfrak{b}_{s_1})\right\}.$$

Therefore, $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BF_{SBCK-I} of $\mathfrak{B}_S$. □

Remark 2. *If $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BFS of $\mathfrak{B}_S$, then for all $(\varrho,\nu) \in [\pm,0] \times [0,\mp]$,*

$$\hbar^-_{(\varrho,\mathfrak{T}_2)}(\mathfrak{b}_s) = \hbar^-(\mathfrak{b}_s) - \varrho \leq \hbar^-(\mathfrak{b}_s),$$

and

$$\hbar^+_{(\nu,\mathfrak{T}_2)}(\mathfrak{b}_s) = \hbar^+(\mathfrak{b}_s) - \nu \geq \hbar^+(\mathfrak{b}_s),$$

for all $\mathfrak{b}_s \in \mathfrak{B}_S$. Hence, the $BF_{(\varrho,\nu)-\mathfrak{t}}$,

$$\hbar^{\mathfrak{T}_2}_{(\varrho,\nu)} = (\mathfrak{B}_S, \hbar^-_{(\varrho,\mathfrak{T}_2)}, \hbar^+_{(\nu,\mathfrak{T}_2)}),$$

is a bipolar fuzzy extension of $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ for all $(\varrho,\nu) \in [\pm,0] \times [0,\mp]$.

Definition 12. *Let $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ be a BFS of $\mathfrak{B}_S$. The BFS $\overline{\hbar} = (\mathfrak{L}, \overline{\hbar^-}, \overline{\hbar^+})$ is defined for all $\mathfrak{b}_s \in \mathfrak{B}_S$ as*

$$\overline{\hbar^-}(\mathfrak{b}_s) = -1 - \hbar^-(\mathfrak{b}_s),$$

and

$$\overline{\hbar^+}(\mathfrak{b}_s) = 1 - \hbar^+(\mathfrak{b}_s).$$

This BFS $\overline{\hbar} = (\mathfrak{L}, \overline{\hbar^-}, \overline{\hbar^+})$ *is called the complement of* $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ *in* $\mathfrak{B}_S$.

Definition 13. *Let* $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ *be a BFS in* $\mathfrak{B}_S$. *For* $(\mathfrak{t}^-, \mathfrak{t}^+) \in [-1, 0] \times [0, 1]$, *the sets*

$$\mathfrak{N}_{\mathfrak{A}}(\hbar; \mathfrak{t}^-) = \{\mathfrak{b}_s \in \mathfrak{B}_S \mid \hbar^-(\mathfrak{b}_s) \leq \mathfrak{t}^-\}$$

and

$$\mathfrak{N}_{\mathfrak{U}}(\hbar; \mathfrak{t}^-) = \{\mathfrak{b}_s \in \mathfrak{B}_S \mid \hbar^-(\mathfrak{b}_s) \geq \mathfrak{t}^-\}$$

are called the negative lower $\mathfrak{t}^-$*-cut and the negative upper* $\mathfrak{t}^-$*-cut of* $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$, *respectively. The sets*

$$\mathfrak{P}_{\mathfrak{A}}(\hbar; \mathfrak{t}^+) = \{\mathfrak{b}_s \in \mathfrak{B}_S \mid \hbar^+(\mathfrak{b}_s) \leq \mathfrak{t}^+\}$$

and

$$\mathfrak{P}_{\mathfrak{U}}(\hbar; \mathfrak{t}^+) = \{\mathfrak{b}_s \in \mathfrak{B}_S \mid \hbar^+(\mathfrak{b}_s) \geq \mathfrak{t}^+\}$$

are called the positive lower $\mathfrak{t}^+$*-cut and the positive upper* $\mathfrak{t}^+$*-cut of* $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$, *respectively.*

Theorem 14. *Let* $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ *be a BFS in* $\mathfrak{B}_S$. *Then,* $\overline{\hbar} = (\mathfrak{L}, \overline{\hbar^-}, \overline{\hbar^+})$ *is a* BF_{SBCK-S} *of* $\mathfrak{B}_S$ *if and only if for all* $(\mathfrak{t}^-, \mathfrak{t}^+) \in [-1, 0] \times [0, 1]$, $\mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$ *and* $\mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$ *are SBCK-subalgebras of* $\mathfrak{B}_S$ *if* $\mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$ *and* $\mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$ *are nonempty.*

Proof. Assume that $\overline{\hbar} = (\mathfrak{L}, \overline{\hbar^-}, \overline{\hbar^+})$ is a BF_{SBCK-S} of $\mathfrak{B}_S$. Let $(\mathfrak{t}^-, \mathfrak{t}^+) \in [-1, 0] \times [0, 1]$ be such that $\mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$ and $\mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$ are nonempty.

(i) Let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$. Then, $\hbar^-(\mathfrak{b}_{s_1}) \geq \mathfrak{t}^-$ and $\hbar^-(\mathfrak{b}_{s_2}) \geq \mathfrak{t}^-$, so $\mathfrak{t}^-$ is a lower bound of $\{\hbar^-(\mathfrak{b}_{s_1}), \hbar^-(\mathfrak{b}_{s_2})\}$. As $\overline{\hbar} = (\mathfrak{L}, \overline{\hbar^-}, \overline{\hbar^+})$ is a BF_{SBCK-I} of $\mathfrak{B}_S$, we have

$$\overline{\hbar^-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq \max\{\overline{\hbar^-}(\mathfrak{b}_{s_1}), \overline{\hbar^-}(\mathfrak{b}_{s_2})\}.$$

By Lemma 1 (1), we have

$$\begin{aligned} -1 - \hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &\leq \max\{-1 - \hbar^-(\mathfrak{b}_{s_1}), -1 - \hbar^-(\mathfrak{b}_{s_2})\} \\ &= -1 - \min\{\hbar^-(\mathfrak{b}_{s_1}), \hbar^-(\mathfrak{b}_{s_2})\}. \end{aligned}$$

Thus,

$$\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \min\{\hbar^-(\mathfrak{b}_{s_1}), \hbar^-(\mathfrak{b}_{s_2})\} \geq \mathfrak{t}^-.$$

Therefore, $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \in \mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$, and hence $\mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$ is an SBCK-subalgebra of $\mathfrak{B}_S$.

(ii) Let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$. Then $\hbar^+(\mathfrak{b}_{s_1}) \leq \mathfrak{t}^+$ and $\hbar^+(\mathfrak{b}_{s_2}) \leq \mathfrak{t}^+$, so $\mathfrak{t}^+$ is an upper bound of $\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{b}_{s_2})\}$. As $\overline{\hbar} = (\mathfrak{L}, \overline{\hbar^-}, \overline{\hbar^+})$ is a BF_{SBCK-S} of $\mathfrak{B}_S$, we obtain

$$\overline{\hbar^+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \min\{\overline{\hbar^+}(\mathfrak{b}_{s_1}), \overline{\hbar^+}(\mathfrak{b}_{s_2})\}.$$

By Lemma 1 (2), we have

$$\begin{aligned} 1 - \hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &\geq \min\{1 - \hbar^+(\mathfrak{b}_{s_1}), 1 - \hbar^+(\mathfrak{b}_{s_2})\} \\ &= -1 - \max\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{b}_{s_2})\}. \end{aligned}$$

Thus,

$$\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq \max\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{b}_{s_2})\} \leq \mathfrak{t}^+,$$

and so $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \in \mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$. Therefore, $\mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$ is an SBCK-subalgebra of $\mathfrak{B}_S$.

Conversely, assume that for all $(\mathfrak{t}^-, \mathfrak{t}^+) \in [-1, 0] \times [0, 1]$, $\mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$ and $\mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$ are SBCK-subalgebras of $\mathfrak{B}_S$ if they are nonempty.

(i) Let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{L}$. Then, $\hbar^-(\mathfrak{b}_{s_1}), \hbar^-(\mathfrak{b}_{s_2}) \in [-1, 0]$. Choose $\mathfrak{t}^- = \min\{\hbar^-(\mathfrak{b}_{s_1}), \hbar^-(\mathfrak{b}_{s_2})\}$. Thus, $\hbar^-(\mathfrak{b}_{s_1}) \geq \mathfrak{t}^-$ and $\hbar^-(\mathfrak{b}_{s_2}) \geq \mathfrak{t}^-$, so $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-) \neq \emptyset$. By assumption, we have $\mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$ is an SBCK-subalgebra of $\mathfrak{B}_S$ and so $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \in \mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$. Thus,

$$\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \mathfrak{t}^- = \min\{\hbar^-(\mathfrak{b}_{s_1}), \hbar^-(\mathfrak{b}_{s_2})\}.$$

By Lemma 1 (1), we have

$$\begin{aligned} \overline{\hbar^-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= -1 - \hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \\ &\leq -1 - \min\{\hbar^-(\mathfrak{b}_{s_1}), \hbar^-(\mathfrak{b}_{s_2})\} \\ &= \max\{-1 - \hbar^-(\mathfrak{b}_{s_1}), -1 - \hbar^-(\mathfrak{b}_{s_2})\} \\ &= \max\{\overline{\hbar^-}(\mathfrak{b}_{s_1}), \overline{\hbar^-}(\mathfrak{b}_{s_2}), \end{aligned}$$

(ii) Let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{L}$. Then $\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{b}_{s_2}) \in [0, 1]$. Choose $\mathfrak{t}^+ = \max\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{b}_{s_2})\}$. Thus $\hbar^+(\mathfrak{b}_s) \leq \mathfrak{t}^+$ and $\hbar^+(\mathfrak{b}_{s_2}) \leq \mathfrak{t}^+$, so $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+) \neq \emptyset$. By assumption, we have $\mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$ is an SBCK-subalgebra of $\mathfrak{B}_S$ and so $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \in \mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$. Thus,

$$\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq \mathfrak{t}^+ = \max\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{b}_{s_2})\}.$$

By Lemma 1 (2), we have

$$\begin{aligned} \overline{\hbar^+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &= 1 - \hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \\ &\geq 1 - \max\{\hbar^+(\mathfrak{b}_{s_1}), \hbar^+(\mathfrak{b}_{s_2})\} \\ &= \min\{1 - \hbar^+(\mathfrak{b}_{s_1}), 1 - \hbar^+(\mathfrak{b}_{s_2})\} \\ &= \min\{\overline{\hbar^+}(\mathfrak{b}_{s_1}), \overline{\hbar^+}(\mathfrak{b}_{s_2})\}. \end{aligned}$$

Thus, $\overline{\hbar} = (\mathfrak{L}, \overline{\hbar^-}, \overline{\hbar^+})$ is a BF_{SBCK-S} of $\mathfrak{B}_S$. □

Theorem 15. *Let $\overline{\hbar} = (\mathfrak{L}, \overline{\hbar^-}, \overline{\hbar^+})$ represent a BFS in $\mathfrak{B}_S$. Then, $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ qualifies as a BF_{SBCK-I} of $\mathfrak{B}_S$, if and only if, for every pair $(\mathfrak{t}^-, \mathfrak{t}^+) \in [-1, 0] \times [0, 1]$, the sets $\mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$ and $\mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$ are SBCK-ideals of $\mathfrak{B}_S$, provided that these sets are nonempty.*

Proof. Assume that $\overline{\hbar} = (\mathfrak{L}, \overline{\hbar^-}, \overline{\hbar^+})$ is a BF_{SBCK-I} of $\mathfrak{B}_S$. Let $(\mathfrak{t}^-, \mathfrak{t}^+) \in [-1, 0] \times [0, 1]$ be such that $\mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$ and $\mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$ are nonempty. Let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$ be such that $\mathfrak{b}_{s_2} \in \mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$. Then, $\hbar^-(\mathfrak{b}_{s_2}) \geq \mathfrak{t}^-$. As $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BF_{SBCK-I} of $\mathfrak{B}_S$, we have

$$\begin{aligned} \overline{\hbar^-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &\leq \overline{\hbar^-}(\mathfrak{b}_{s_2}) \\ 1 - \hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &\leq 1 - \hbar^+(\mathfrak{b}_{s_2}) \\ \hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) &\geq \hbar^+(\mathfrak{b}_{s_2}) \\ &\geq \mathfrak{t}^-. \end{aligned}$$

Hence, $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \in \mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$.

Next, let $\mathfrak{b}_s, \mathfrak{b}_{s_2} \in \mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$ be such that $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), x \in \mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$. Then $\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \mathfrak{t}^-$ and $\hbar^-(\mathfrak{b}_s) \geq \mathfrak{t}^-$. As $\overline{\hbar} = (L, \overline{\hbar^-}, \overline{\hbar^+})$ is a BF_{SBCK-I} of $\mathfrak{B}_S$, we have

$$\begin{array}{rcl}
\overline{\hbar^-}(\mathfrak{b}_{s_2}) & \leq & \max\{\overline{\hbar^-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \overline{\hbar^-}(\mathfrak{b}_{s_1})\} \\
1 - \hbar^-(\mathfrak{b}_{s_2}) & \leq & \max\{1 - \hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), 1 - \hbar^-(\mathfrak{b}_{s_1})\} \\
1 - \hbar^-(\mathfrak{b}_{s_2}) & \leq & 1 - \min\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_{s_1})\} \\
\hbar^-(\mathfrak{b}_{s_2}) & \geq & \min\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_{s_1})\} \\
& \geq & \mathfrak{t}^-.
\end{array}$$

Hence, $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \in \mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$. Therefore, $\mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$ is an SBCK-ideal of $\mathfrak{B}_S$.

Let $\mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$ be such that $\mathfrak{b}_{s_2} \in \mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$. Then, $\hbar^+(\mathfrak{b}_{s_2}) \geq \mathfrak{t}^+$. As $\hbar = (\mathfrak{B}_S, \hbar^-, \hbar^+)$ is a BF_{SBCK-I} of $\mathfrak{B}_S$, we have

$$\begin{array}{rcl}
\overline{\hbar^+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) & \geq & \overline{\hbar^+}(\mathfrak{b}_{s_2}) \\
1 - \hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) & \geq & 1 - \hbar^+(\mathfrak{b}_{s_2}) \\
\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) & \leq & \hbar^+(\mathfrak{b}_{s_2}) \\
& \leq & \mathfrak{t}^+.
\end{array}$$

Hence, $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \in \mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$.

Now, let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$ be such that $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \mathfrak{b}_{s_1} \in \mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$. Then, $\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \mathfrak{t}^+$ and $\hbar^+(\mathfrak{b}_{s_1}) \geq \mathfrak{t}^+$. As $\overline{\hbar} = (L, \overline{\hbar^-}, \overline{\hbar^+})$ is a BF_{SBCK-I} of $\mathfrak{B}_S$, we have

$$\begin{array}{rcl}
\overline{\hbar^+}(\mathfrak{b}_{s_2}) & \leq & \max\{\overline{\hbar^+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \overline{\hbar^+}(\mathfrak{b}_{s_1})\} \\
1 - \hbar^+(\mathfrak{b}_{s_2}) & \leq & \max\{1 - \hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), 1 - \hbar^+(\mathfrak{b}_s)\} \\
1 - \hbar^+(\mathfrak{b}_{s_2}) & \leq & 1 - \min\{\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^+(\mathfrak{b}_{s_1})\} \\
\hbar^+(\mathfrak{b}_{s_2}) & \geq & \min\{\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^+(\mathfrak{b}_{s_1})\} \\
& \geq & \mathfrak{t}^+.
\end{array}$$

Hence, $\mathfrak{b}_{s_2} \in \mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$. Therefore, $\mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$ is an SBCK-ideal of $\mathfrak{B}_S$.

Conversely, assume that for all $(\mathfrak{t}^-, \mathfrak{t}^+) \in [-1, 0] \times [0, 1]$, the sets $\mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$ and $\mathfrak{P}_{\mathfrak{L}}(\hbar, \mathfrak{t}^+)$ are SBCK-ideals of $\mathfrak{B}_S$, provided that these sets are non-empty. Let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{L}$. Then, we have $\hbar^-(\mathfrak{b}_{s_2}) \in [-1, 0]$. Choose $\mathfrak{t}^- = \hbar^-(\mathfrak{b}_{s_2})$. Consequently, $\hbar^-(\mathfrak{b}_{s_2}) \geq \mathfrak{t}^-$, which implies $\mathfrak{b}_{s_2} \in \mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-) \neq \emptyset$. By assumption, $\mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$ is an SBCK-ideal of $\mathfrak{B}_S$, so $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \in \mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$. Thus, we obtain $\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \hbar^-(\mathfrak{b}_{s_2})$, and consequently,

$$\overline{\hbar^-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) = -1 - \hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \leq -1 - \hbar^-(\mathfrak{b}_{s_2}) = \overline{\hbar^-}(\mathfrak{b}_{s_2}).$$

Now, let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{L}$. Then, we have $\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_{s_1}) \in [-1, 0]$. Choose $\hbar^- = \min\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_{s_1})\}$. Therefore, $\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})) \geq \mathfrak{t}^-$ and $\hbar^-(\mathfrak{b}_{s_1}) \geq \mathfrak{t}^-$, implying that $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \mathfrak{b}_{s_1} \in \mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-) \neq \emptyset$. By assumption, $\mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$ is an SBCK-ideal of $\mathfrak{B}_S$, so $\mathfrak{b}_{s_2} \in \mathfrak{N}_{\mathfrak{U}}(\hbar, \mathfrak{t}^-)$. Thus,

$$\hbar^-(\mathfrak{b}_{s_2}) \geq \mathfrak{t}^- = \min\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}) \mid \mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})), \hbar^-(\mathfrak{b}_{s_1})\}.$$

Now,

$$\begin{aligned}\overline{\hbar^-}(\mathfrak{b}_{s_2}) &= -1-\hbar^-(\mathfrak{b}_{s_2})\\ &\leq -1-\min\{\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),\hbar^-(\mathfrak{b}_{s_1})\}\\ &= \max\{-1-\hbar^-(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),-1-\hbar^-(\mathfrak{b}_{s_1})\}\\ &= \max\{\overline{\hbar^-}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),\overline{\hbar^-}(\mathfrak{b}_{s_1})\}.\end{aligned}$$

Let $\mathfrak{b}_s,\mathfrak{b}_{s_2}\in\mathfrak{L}$. Then, we have $\hbar^+(\mathfrak{b}_{s_2})\in[0,1]$. Choose $\mathfrak{t}^+=\hbar^+(\mathfrak{b}_{s_2})$. Hence, $\hbar^+(\mathfrak{b}_{s_2})\leq\mathfrak{t}^+$, implying $\mathfrak{b}_{s_2}\in\mathfrak{P}_{\mathfrak{L}}(\hbar,\mathfrak{t}^+)\neq\emptyset$. By assumption, $\mathfrak{P}_{\mathfrak{L}}(\hbar,\mathfrak{t}^+)$ is an SBCK-ideal of $\mathfrak{B}_S$, so $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))\in\mathfrak{P}_{\mathfrak{L}}(\hbar,\mathfrak{t}^+)$. Thus, we have $\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))\geq\mathfrak{t}^+=\hbar^+(\mathfrak{b}_{s_2})$, and consequently,

$$\overline{\hbar^+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))=1-\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))\geq 1-\hbar^+(\mathfrak{b}_{s_2})=\overline{\hbar^+}(\mathfrak{b}_{s_2}).$$

Let $\mathfrak{b}_s,\mathfrak{b}_{s_2}\in\mathfrak{L}$. Then, we have $\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),\hbar^+(\mathfrak{b}_{s_1})\in[0,1]$. Choose $\hbar^+=\max\{\hbar\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),\hbar^+(\mathfrak{b}_s)\}$. Therefore, $\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2}))\leq\mathfrak{t}^+$ and $\hbar^+(\mathfrak{b}_{s_1})\leq\mathfrak{t}^+$, implying that $(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),\mathfrak{b}_{s_1}\in\mathfrak{P}_{\mathfrak{L}}(\hbar,\mathfrak{t}^+)\neq\emptyset$. By assumption, $\mathfrak{P}_{\mathfrak{L}}(\hbar,\mathfrak{t}^+)$ is an SBCK-ideal of $\mathfrak{B}_S$, so $\mathfrak{b}_{s_2}\in\mathfrak{P}_{\mathfrak{L}}(\hbar,\mathfrak{t}^+)$. Thus,

$$\hbar^+(\mathfrak{b}_{s_2})\geq\mathfrak{t}^+=\max\{\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),\hbar^+(\mathfrak{b}_{s_1})\}.$$

Now,

$$\begin{aligned}\overline{\hbar^+}(\mathfrak{b}_{s_2}) &= 1-\hbar^+(\mathfrak{b}_{s_2})\\ &\leq 1-\max\{\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),\hbar^+(\mathfrak{b}_{s_1})\}\\ &= \min\{1-\hbar^+(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),1-\hbar^+(\mathfrak{b}_{s_1})\}\\ &= \min\{\overline{\hbar^+}(\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})\mid\mathfrak{W}_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})),\overline{\hbar^+}(\mathfrak{b}_{s_1})\}.\end{aligned}$$

Hence, $\overline{\hbar}=(L,\overline{\hbar^-},\overline{\hbar^+})$ is a BF_{SBCK-I} of $\mathfrak{B}_S$. □

Definition 14 ([8]). *Let $(\mathfrak{P},|_{\mathfrak{P}},0_{\mathfrak{P}})$ and $(\mathfrak{Q},|_{\mathfrak{Q}},0_{\mathfrak{Q}})$ denote SBCK-algebras. A mapping $\mathfrak{h}:\mathfrak{P}\to\mathfrak{Q}$ is called a homomorphism if it satisfies the condition*

$$\mathfrak{h}(\mathfrak{b}_{s_1}\mid_{\mathfrak{P}}\mathfrak{b}_{s_2})=\mathfrak{h}(\mathfrak{b}_{s_1})\mid_{\mathfrak{Q}}\mathfrak{h}(\mathfrak{b}_{s_2})$$

for all $\mathfrak{b}_{s_1},\mathfrak{b}_{s_2}\in\mathfrak{B}_S$, and additionally, $\mathfrak{h}(0_{\mathfrak{P}})=0_{\mathfrak{Q}}$.

As a notational convenience, we denote $\mathfrak{W}^A_{\mathfrak{b}_{s_1}}(\mathfrak{b}_{s_2})=\mathfrak{b}_{s_1}\mid_A(\mathfrak{b}_{s_2}\mid_A\mathfrak{b}_{s_2})$.

Theorem 16. *Let $(\mathfrak{P},|_{\mathfrak{P}},0_{\mathfrak{P}})$ and $(\mathfrak{Q},|_{\mathfrak{Q}},0_{\mathfrak{Q}})$ be SBCK-algebras, and let $\varrho:\mathfrak{P}\to\mathfrak{Q}$ be a surjective homomorphism. Suppose $\hbar$ is a BFS on $\mathfrak{Q}$. Then, $\hbar$ is a BF_{SBCK-I} of $\mathfrak{B}_S$ if and only if $\hbar^\varrho$ is a BF_{SBCK-I} of $\mathfrak{P}$, where $\hbar^\varrho:\mathfrak{P}\to[-1,0]$ on $\mathfrak{B}_S$ is defined by $\hbar^\varrho(\mathfrak{b}_s)=\hbar(\varrho(\mathfrak{b}_s))$ for all $\mathfrak{b}_s\in\mathfrak{P}$.*

Proof. Let $(\mathfrak{P},|_{\mathfrak{P}},0_{\mathfrak{P}})$ and $(\mathfrak{Q},|_{\mathfrak{Q}},0_{\mathfrak{Q}})$ be SBCK-algebras, with $\hbar:\mathfrak{P}\to\mathfrak{Q}$ as a surjective homomorphism and $\hbar$ being a BF_{SBCK-I} of $\mathfrak{Q}$. Let $\mathfrak{b}_{s_1},\mathfrak{b}_{s_2}\in\mathfrak{P}$. Then

$$\begin{aligned}\hbar^{-\varrho}(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1})\mid\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1})) &= \hbar^-\Big(\varrho\Big(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1})\mid\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}(\mathfrak{b}_{s_1})\Big)\Big)\\ &= \hbar^-\Big(\mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1})\mid\mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1})\Big)\\ &\leq \hbar^-(\varrho(\mathfrak{b}_{s_2}))\\ &= \hbar^{-\varrho}(\mathfrak{b}_{s_2}),\end{aligned}$$

and

$$\begin{aligned}\hbar^{-\varrho}(\mathfrak{b}_{s_2}) &= \hbar^{-}(\varrho(\mathfrak{b}_{s_2}))\\ &\leq \max\Big\{\hbar^{-}(\varrho(\mathfrak{b}_{s_2})),\hbar^{-}\Big(\mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1}) \mid \mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1})\Big)\Big\}\\ &= \max\Big\{\hbar^{-}(\varrho(\mathfrak{b}_{s_2})),\hbar^{+}\Big(\varrho\Big(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1}\Big)\Big)\Big\}\\ &= \max\Big\{\hbar^{-\varrho}(\mathfrak{b}_{s_1}),\hbar^{-\varrho}\Big(\varrho\Big(\mathfrak{W}^{P}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{P}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1}\Big)\Big)\Big\}.\end{aligned}$$

Besides, we have

$$\begin{aligned}\hbar^{+\varrho}(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1}) &= \hbar^{+}\Big(\varrho\Big(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1}\Big)\Big)\\ &= \hbar^{+}\Big(\mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1}) \mid \mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1})\Big)\\ &\geq \hbar^{+}(\varrho(\mathfrak{b}_{s_2}))\\ &= \hbar^{+\varrho}(\mathfrak{b}_{s_2}),\end{aligned}$$

and

$$\begin{aligned}\hbar^{+\varrho}(\mathfrak{b}_{s_2}) &= \hbar^{+}(\varrho(\mathfrak{b}_{s_2}))\\ &\geq \min\Big\{\hbar^{+}(\varrho(\mathfrak{b}_{s_1})),\hbar^{+}\Big(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1}\Big)\Big\}\\ &= \min\Big\{\hbar^{+}(\varrho(\mathfrak{b}_{s_1})),\hbar^{+}\Big(\varrho\Big(\mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1}) \mid \mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1})\Big)\Big)\Big\}\\ &= \min\Big\{\hbar^{+\varrho}(\mathfrak{b}_{s_1}),\hbar^{+\varrho}\Big(\mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1}) \mid \mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1})\Big)\Big\}.\end{aligned}$$

Thus, $\hbar^{\varrho}$ is a BF_{SBCK-I} of $\mathfrak{P}$.

Conversely, suppose $\hbar^{\varrho}$ is a BF_{SBCK-I} of P. Let $\mathfrak{b}_{s_2}, \mathfrak{b}_{s_{22}} \in Q$, such that $\varrho(\mathfrak{b}_{s_1}) = \mathfrak{b}_{s_{21}}$ and $\varrho(\mathfrak{b}_{s_2}) = \mathfrak{b}_{s_{22}}$ for $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{P}$. Then, we have

$$\begin{aligned}\hbar^{-}(\mathfrak{W}^{\mathfrak{Q}}_{\mathfrak{b}_{s_2}\mathfrak{b}_{s_{22}}}\mathfrak{b}_{s_{21}} \mid \mathfrak{W}^{\mathfrak{Q}}_{\mathfrak{b}_{s_2}\mathfrak{b}_{s_{22}}}\mathfrak{b}_{s_{21}}) &= \hbar^{-}(\mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1}) \mid \mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1}))\\ &= \hbar^{-\varrho}(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1})\\ &\leq \hbar^{-\varrho}(\mathfrak{b}_{s_2})\\ &= \hbar^{-}(\varrho(\mathfrak{b}_{s_2}))\\ &= \hbar^{-}(\mathfrak{b}_{s_2}),\end{aligned}$$

and

$$\begin{aligned}\hbar^{-}(\mathfrak{b}_{s_{22}}) &= \hbar^{-}(\varrho(\mathfrak{b}_{s_2}))\\ &= \hbar^{-\varrho}(\mathfrak{b}_{s_2})\\ &\leq \max\Big\{\hbar^{-\varrho}(\mathfrak{b}_{s_1}),\hbar^{-\varrho}(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s1})\Big\}\\ &= \max\Big\{\hbar^{-}(\varrho(\mathfrak{b}_{s_1})),\hbar^{-}(\varrho(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1}))\Big\}\\ &= \max\Big\{\hbar^{-}(\varrho(\mathfrak{b}_{s_1})),\hbar^{-}(\mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1}) \mid \mathfrak{W}^{\mathfrak{Q}}Q_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1}))\Big\}\\ &= \max\Big\{\hbar^{-}(\mathfrak{b}_{s_{21}}),\hbar^{-}(\mathfrak{W}^{\mathfrak{Q}}_{\mathfrak{b}_{s_{22}}}\mathfrak{b}_{s_{21}} \mid \mathfrak{W}^{\mathfrak{Q}}_{\mathfrak{b}_{s_{22}}}\mathfrak{b}_{s_{21}})\Big\}.\end{aligned}$$

Morever, we obtain

$$\begin{aligned}\hbar^{+}(\mathfrak{W}^{\mathfrak{Q}}_{\mathfrak{b}_{s_{22}}}\mathfrak{b}_{s_{21}} \mid \mathfrak{W}^{\mathfrak{Q}}_{\mathfrak{b}_{s_{22}}}\mathfrak{b}_{s_{21}}) &= \hbar^{+}(\mathfrak{W}^{\mathfrak{Q}}_{\mathfrak{b}_{s_{22}}}\mathfrak{b}_{s_{21}} \mid \mathfrak{W}^{\mathfrak{Q}}_{\mathfrak{b}_{s_{22}}}\mathfrak{b}_{s_{21}})\\ &= \hbar^{+\varrho}(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1})\\ &\geq \hbar^{+\varrho}(\mathfrak{b}_{s_2})\\ &= \hbar^{+}(\varrho(\mathfrak{b}_{s_2}))\\ &= \hbar^{+}(\mathfrak{b}_{s_{22}}),\end{aligned}$$

and

$$\begin{aligned}
\hbar^{+}(\mathfrak{b}_{s_{22}}) &= \hbar^{+}(\varrho(\mathfrak{b}_{s_2})) \\
&= \hbar^{+\varrho}(\mathfrak{b}_{s_2}) \\
&\geq \min\left\{\hbar^{+}(\mathfrak{b}_{s_1}), \hbar^{+\varrho}(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1})\right\} \\
&= \min\left\{\hbar^{+}(\varrho(\mathfrak{b}_{s_1})), \hbar^{+}(\varrho(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1}))\right\} \\
&= \min\left\{\hbar^{+}(\varrho(\mathfrak{b}_{s_1})), \hbar^{+}(\mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1}) \mid \mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1}))\right\} \\
&= \min\left\{\hbar^{+}(\mathfrak{b}_{s_{21}}), \hbar^{+}(\mathfrak{W}^{\mathfrak{Q}}_{\mathfrak{b}_{s_{22}}}\mathfrak{b}_{s_{21}} \mid \mathfrak{W}^{\mathfrak{Q}}_{\mathfrak{b}_{s_{22}}}\mathfrak{b}_{s_{21}})\right\}.
\end{aligned}$$

Hence, $\hbar$ is a BF_{SBCK-I} of $\mathfrak{Q}$. □

Theorem 17. *Let $(\mathfrak{P}, |_{\mathfrak{P}}, 0_{\mathfrak{P}})$ and $(\mathfrak{Q}, |_{\mathfrak{Q}}, 0_{\mathfrak{Q}})$ be SBCK-algebras, and let $\hbar : \mathfrak{P} \to \mathfrak{Q}$ be a surjective homomorphism. Assume that $\hbar$ is a BFS in $\mathfrak{Q}$. Then, $\hbar$ is a bipolar fuzzy BF_{SBCK-S} of $\mathfrak{Q}$, if and only if $\hbar^{\varrho}$ is a BF_{SBCK-S} of $\mathfrak{B}_S$, where $\hbar^{\varrho} : \mathfrak{P} \to [-1, 0]$ on $\mathfrak{B}_S$ is defined by $\hbar^{\varrho} = \hbar(\varrho(\mathfrak{b}_s))$ for all $\mathfrak{b}_s \in \mathfrak{P}$.*

Proof. Let $(\mathfrak{P}, |_{\mathfrak{P}}, 0_{\mathfrak{P}})$ and $(\mathfrak{Q}, |_{\mathfrak{Q}}, 0_{\mathfrak{Q}})$ be SBCK-algebras, and let $\hbar : \mathfrak{P} \to \mathfrak{Q}$ be a surjective homomorphism. Assume that $\hbar$ is a BF_{SBCK-S} of $\mathfrak{Q}$. Let $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{P}$. Then

$$\begin{aligned}
\hbar^{-\varrho}(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1}) &= \hbar^{-}(\varrho(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1})) \\
&= \hbar^{-}(\mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1}) \mid \mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1})) \\
&\leq \max\{\hbar^{-}(\varrho(\mathfrak{b}_{s_1})), \hbar^{-}(\varrho(\mathfrak{b}_{s_2}))\} \\
&= \max\{\hbar^{-\varrho}(\mathfrak{b}_{s_1}), \hbar^{-\varrho}(\mathfrak{b}_{s_2})\},
\end{aligned}$$

and

$$\begin{aligned}
\hbar^{+\varrho}(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1}) &= \hbar^{+}(\varrho(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1})) \\
&= \hbar^{+}(\mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1}) \mid \mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1})) \\
&\geq \min\{\hbar^{+}(\varrho(\mathfrak{b}_{s_1})), \hbar^{+}(\varrho(\mathfrak{b}_{s_2}))\} \\
&= \min\{\hbar^{+\varrho}(\mathfrak{b}_{s_1}), \hbar^{+\varrho}(\mathfrak{b}_{s_2})\}.
\end{aligned}$$

Hence, $\hbar^{\varrho}$ is a BF_{SBCK-S} of $\mathfrak{B}_S$.

Conversely, let $\hbar^{\varrho}$ be a BF_{SBCK-S} of $\mathfrak{P}$. Let $\mathfrak{b}_{s_{21}}, \mathfrak{b}_{s_{22}} \in \mathfrak{Q}$, such that $\varrho(\mathfrak{b}_{s_1}) = \mathfrak{b}_{s_{21}}$ and $\varrho(\mathfrak{b}_{s_2}) = \mathfrak{b}_{s_{22}}$ for $\mathfrak{b}_{s_1}, \mathfrak{b}_{s_2} \in \mathfrak{P}$. Then

$$\begin{aligned}
\hbar^{+}(\mathfrak{W}^{\mathfrak{Q}}_{\mathfrak{b}_{s_{22}}}\mathfrak{b}_{s_{21}} \mid \mathfrak{W}^{\mathfrak{Q}}_{\mathfrak{b}_{s_{22}}}\mathfrak{b}_{s_{21}}) &= \hbar^{+}(\mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1}) \mid \mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1})) \\
&= \hbar^{+}(\varrho(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1})) \\
&= \hbar^{+\varrho}(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1}) \\
&\geq \min\{\hbar^{+\varrho}(\mathfrak{b}_{s_1}), \hbar^{+\varrho}(\mathfrak{b}_{s_2})\} \\
&= \min\{\hbar^{+}(\varrho(\mathfrak{b}_{s_1})), \hbar^{+}(\varrho(\mathfrak{b}_{s_2}))\} \\
&= \min\{\hbar^{+}(\mathfrak{b}_{s_{21}}), \hbar^{+}(\mathfrak{b}_{s_{22}})\},
\end{aligned}$$

and

$$\begin{aligned}
\hbar^{-}(\mathfrak{W}^{\mathfrak{Q}}_{\mathfrak{b}_{s_{22}}}\mathfrak{b}_{s_{21}} \mid \mathfrak{W}^{\mathfrak{Q}}_{\mathfrak{b}_{s_{22}}}\mathfrak{b}_{s_{21}}) &= \hbar^{-}(\mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1}) \mid \mathfrak{W}^{\mathfrak{Q}}_{\varrho(\mathfrak{b}_{s_2})}\varrho(\mathfrak{b}_{s_1})) \\
&= \hbar^{-}(\varrho(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1})) \\
&= \hbar^{-\varrho}(\mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1} \mid \mathfrak{W}^{\mathfrak{P}}_{\mathfrak{b}_{s_2}}\mathfrak{b}_{s_1}) \\
&\leq \max\{\hbar^{-\varrho}(\mathfrak{b}_{s_1}), \hbar^{-\varrho}(\mathfrak{b}_{s_2})\} \\
&= \max\{\hbar^{-}(\varrho(\mathfrak{b}_{s_1})), \hbar^{-}(\varrho(\mathfrak{b}_{s_2}))\} \\
&= \max\{\hbar^{-}(\mathfrak{b}_{s_{21}}), \hbar^{-}(\mathfrak{b}_{s_{22}})\}.
\end{aligned}$$

Hence, $\hbar$ is a BF_{SBCK-S} of $\mathfrak{B}_S$. □

6. Conclusions

In this paper, we have thoroughly examined the concept of bipolar fuzzy SBCK-subalgebras and their level sets within the framework of Sheffer stroke BCK-algebras. Our primary contribution is the identification of a key relationship between these subalgebras and the corresponding level sets in Sheffer stroke BCK-algebras, specifically proving that the level set of a bipolar fuzzy SBCK-subalgebra corresponds to a subalgebra, and the reverse holds as well. This finding enriches the algebraic structure of Sheffer stroke BCK-algebras and lays a foundational framework that may support further exploration of bipolar fuzzy logic within the context of Sheffer stroke BCK-algebras.

Furthermore, we introduced the notion of a bipolar fuzzy SBCK-ideal, which represents a significant extension of the concept of SBCK-subalgebras in Sheffer stroke BCK-algebras. We demonstrated that while every bipolar fuzzy SBCK-ideal is also a bipolar fuzzy SBCK-subalgebra, the converse does not necessarily hold. This distinction highlights the unique characteristics of bipolar fuzzy SBCK-ideals and their behavior within the broader algebraic structure. These results provide a clearer understanding of the role of fuzzy logic in the theory of BCK-algebras and expand upon previous work in this area.

Despite the valuable contributions of this study, several directions for future research remain. These areas promise to extend the scope and applicability of the concepts discussed in this paper.

Extension to Other Algebraic Structures: While this paper focused on Sheffer stroke BCK-algebras, future work could investigate the applicability of the identified relationships between subalgebras and level sets in other types of algebraic structures, such as general BCK-algebras or other non-classical logical algebras. Understanding whether similar results hold in these broader contexts could provide a more generalized framework for bipolar fuzzy logic.

Interaction with Other Logical Operations: Another natural extension of this work is to study the interaction between bipolar fuzzy SBCK-subalgebras and other logical operations within the context of non-classical logics, such as fuzzy logic, intuitionistic logic, and paraconsistent logic. Investigating how these operations behave in the presence of bipolar fuzzy sets and how they can be integrated with the Sheffer stroke operation could lead to new insights into logical systems and their applications.

Practical Applications in Computational Systems: While this study has been theoretical, the practical applications of bipolar fuzzy SBCK-subalgebras and SBCK-ideals in computational systems are vast. Future research could explore how these algebraic structures can be applied in areas such as fuzzy decision-making, multi-criteria decision analysis, and information retrieval systems, where bipolar fuzzy sets often provide a useful model for uncertain or ambiguous information. Additionally, the study of how these concepts can improve algorithms for machine learning or artificial intelligence, particularly in areas that require the processing of imprecise or conflicting data, could lead to valuable advancements in computational theory.

Algorithmic Approaches: Future research could also focus on the development of efficient algorithms for computing and manipulating bipolar fuzzy SBCK-subalgebras and SBCK-ideals. Designing algorithms that can automatically identify and work with these structures could facilitate their application in real-world systems, particularly in contexts where large amounts of data are involved, such as data mining or complex decision support systems.

Further Investigation of Properties of SBCK-Ideals: A more in-depth study of the properties and behavior of bipolar fuzzy SBCK-ideals within Sheffer stroke BCK-algebras is another promising direction. Specifically, investigating the conditions under which the reverse of the relationship between bipolar fuzzy SBCK-ideals and SBCK-subalgebras

might hold could lead to new insights and contribute to a more complete characterization of these structures.

Cross-disciplinary Applications: The theoretical framework established in this paper may also find applications in fields outside of pure mathematics, such as computer science, economics, and cognitive science. By applying bipolar fuzzy logic to model decision-making processes, economic behaviors under uncertainty, or even cognitive reasoning, future research could explore the real-world impact of these algebraic concepts in interdisciplinary domains.

In conclusion, while the results presented in this paper represent a significant step forward in the study of bipolar fuzzy SBCK-subalgebras and their associated level sets in Sheffer stroke BCK-algebras, there are many exciting avenues for further research that could expand upon these findings and open up new directions for both theoretical exploration and practical application.

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