

Article

Existence Results for Some Classes of Weighted Equilibrium Problems [†]

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[†] This article is a revised and largely expanded version of a paper entitled Weighted Equilibrium Problems and Variational Inequalities, which was presented at ICNAAM 2024 Conference, Heraklion, Greece, 11–17 September 2024. Some of the results of this paper were presented at the ICNAAM 2024 Conference.

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Abstract: Starting from some systems of vector equilibrium problems, we obtain the existence of the solution for a class of weighted equilibrium problems, under different types of generalized pseudo-monotonicity assumptions. We present both new and previous results, making a connection between them and giving a few examples. Using the main theorem, we derive the solution existence for the initial systems and discuss a corresponding set-valued problem. Finally, we consider the case of a real normed space. We extend some previously obtained results from the literature about weighted variational inequalities, and we also give proofs for some results we previously announced. We give some relevant examples for our notions.

Keywords: weighted equilibrium problems; generalized monotonicity; generalized convexity

MSC: 90C26; 90C33; 90C48



Academic Editor: Simeon Reich

Received: 28 February 2025

Revised: 15 April 2025

Accepted: 18 April 2025

Published: 21 April 2025

Citation: Beldiman, M.-M.; Halanay, A.-D. Existence Results for Some Classes of Weighted Equilibrium Problems. *Axioms* **2025**, *14*, 316. <https://doi.org/10.3390/axioms14040316>

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1. Introduction

Equilibrium problems represent an important framework for many optimization [1], Nash equilibrium [2] or complementarity problems, arising in various fields such as finance [3], mechanics [4] and traffic management [5]. However, the study of abstract equilibrium problems is also highly relevant to mathematical research, both the existence of solutions [6,7] and solution methods [8,9].

The scalar equilibrium problems were introduced by Blum and Oettli [10] (and immediately after by Noor and Oettli [11]) as an extension of variational inequalities. Then, vector equilibrium problems and set-valued equilibrium problems were considered by Ansari, Oettli and Schlager [12]. In [13], Flores-Bazan introduced and investigated equilibrium problems on a Hilbert space. In [14], Noor defined the invex equilibrium problem. A modern and systematic presentation of equilibrium problems can be found in [15].

In [16], Ansari, Siddiqi and Khan introduced the notion of weighted variational inequalities, which has become a heavily studied subject since then. Their initial motivation was to obtain existence results for systems of general variational inequalities because the results are easier to obtain via the weighted case. Besides these theoretical considerations, there are numerous other applications in various areas, one of the most frequent being to traffic problems, such as traffic networks or time-dependent traffic equilibria (see, e.g., [17] and references therein). As well, in the last years weighted variational inequalities gained

importance as an independent object of study mainly in order to obtain the existence and unicity of solutions under generalized pseudo-monotonicity conditions [18]. A thorough discussion about pseudo-monotonicity in the sense of Karamardian can be found in [19].

In this paper, we extend the results of [16] about weighted variational inequalities and also those from [20] obtained on real normed spaces to the weighted equilibrium problems introduced by us in [21]. Our problems and systems of equilibrium problems are more general than those from [22] or [23]. For studying the existence of solutions, we needed to introduce notions of generalized pseudo-monotonicity more appropriate than those previously available. Until now, equilibrium problems under pseudo-monotonicity conditions have also been treated in [24]. Our results were announced at the ICNAAM2024 conference [25], and the main theorem was stated in [26].

The paper is divided into five sections, including the Introduction and Conclusions. After reviewing some preliminary notions, in Section 2 we introduce the systems that will be studied in the rest of the paper and give the equivalence conditions with the corresponding weighted systems. In the next section, we prove the main existence results, using two fixed-point theorems of Chowdhury and Tan [27,28]. Then, we use these results in order to obtain the existence of the solution for the weighted problems under generalized weighted monotonicity and generalized weighted B -pseudomonotonicity assumptions. In Section 4, we consider the particular case of real normed spaces (while in general we work over an arbitrary real topological vector space).

2. Statement of Problems

Denote by $\mathbb{R}_+^m$ (where $m \in \mathbb{N}$) the nonnegative orthant of $\mathbb{R}^m$. That is,

$$\mathbb{R}_+^m = \{x = (u_1, \dots, u_m) \in \mathbb{R}^m : x_j \geq 0, \text{ for all } j = 1, \dots, m\}$$

and let

$$T_+^m = \left\{ u = (u_1, \dots, u_m) \in \mathbb{R}_+^m : \sum_{j=1}^m u_j = 1 \right\}$$

be its simplex.

For each $i \in I$, take a real topological vector space X_i , K_i a convex subset (nonempty) of X_i and some set Y_i which can be anything. Let $X = \prod_{i \in I} X_i$ and $K = \prod_{i \in I} K_i$.

For an element $x \in X$, we denote by $x_i \in X_i$ its i th component, so $x = (x_i)_{i \in I}$.

We consider also two families of maps, both indexed by I . First let $f_i : K \rightarrow Y_i$, and then $\Phi_i : Y_i \times K_i \times K_i \rightarrow \mathbb{R}^{\ell_i}$ ($\ell_i \in \mathbb{N}$).

Using them, we formulate the following systems of vector equilibrium problems:

(Φ -SVEP) Find $\tilde{x} \in K$ such that, for each $i \in I$

$$\Phi_i(f_i(\tilde{x}), \tilde{x}_i; y_i) \notin \mathbb{R}_+^{\ell_i} \setminus \{0\},$$

for all $y_i \in K_i$

and its weak form

(Φ -SVEP) $_w$ Find $\tilde{x} \in K$ such that, for each $i \in I$

$$\Phi_i(f_i(\tilde{x}), \tilde{x}_i; y_i) \notin \text{int } \mathbb{R}_+^{\ell_i},$$

for all $y_i \in K_i$.

The solution set of (Φ -SVEP) is denoted with K^S , while K^{WS} denotes the solution set of (Φ -SVEP) $_w$.

Remark 1. Our classes of equilibrium problems are in fact generalized equilibrium problems, because the three sets involved in the cartesian product are not the same, compared to the classical form, but for simplicity we will also call them equilibrium problems.

Related to the $(\Phi\text{-SVEP})$ and $(\Phi\text{-SVEP})_w$ problems, we consider the following weighted general equilibrium problems over product sets:

($\Phi\text{-WEP}$) Given the weight vector (in fact a family of vectors) $W = (W_1, \dots, W_n) \in \prod_{i=1}^n (\mathbb{R}_+^{\ell_i} \setminus \{0\})$, find $\tilde{x} \in K$ such that

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(\tilde{x}), \tilde{x}_i; y_i) \leq 0$$

for all $y_i \in K_i, i \in I$

and

($\Phi\text{-SWEP}$) Given the weight vector $W = (W_1, \dots, W_n)$ with $W_i \in \mathbb{R}_+^{\ell_i} \setminus \{0\}$, find $\tilde{x} \in K$ such that

$$W_i \cdot \Phi_i(f_i(\tilde{x}), \tilde{x}_i; y_i) \leq 0$$

for all $y_i \in K_i$, where “ $\cdot$ ” is the canonical scalar product on $\mathbb{R}_+^{\ell_i}$.

Definition 1. A solution of $(\Phi\text{-WEP})$ or of $(\Phi\text{-SWEP})$ is called normalized if for each $i \in I, W_i \in T_+^{\ell_i}$.

Denote by K^W (respectively K^{SW}) the solution set of $(\Phi\text{-WEP})$ (respectively $(\Phi\text{-SWEP})$) and by K_n^W (respectively K_n^{SW}) the normalised solution set of $(\Phi\text{-WEP})$ (respectively $(\Phi\text{-SWEP})$)

In what follows, we shall take the the weight vector to satisfy

$$W = (W_1, \dots, W_n) \in \prod_{i=1}^n (\mathbb{R}_+^{\ell_i} \setminus \{0\}).$$

The following result gives that $(\Phi\text{-WEP})$ and $(\Phi\text{-SWEP})$ have the same set of solutions and the same is true for normalized solutions.

Lemma 1. Let W be a given weight vector (respectively $W = (W_1, \dots, W_n) \in \prod_{i \in I} T_+^{\ell_i}$). We suppose that for any $i \in I, \Phi_i(f_i(x), x_i; x_i) = 0$ for all $x_i \in K_i$. Then, $K^W = K^{SW}$ (respectively $K_n^W = K_n^{SW}$).

Proof. Obviously, $K^{SW} \subseteq K^W$. Conversely, let $\tilde{x} \in K^W$. Then,

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(\tilde{x}), \tilde{x}_i; y_i) \leq 0$$

for all $y_i \in K_i, i \in I$.

For each $j \neq i$, take $y_j = \tilde{x}_j$. Since $\Phi_j(f_j(\tilde{x}), \tilde{x}_j; \tilde{x}_j) = 0$ for all $j \neq i, j \in I$, all the terms of the sum vanish except for the i term, giving us

$$W_i \cdot \Phi_i(f_i(\tilde{x}), \tilde{x}_i; y_i) \leq 0,$$

for all $y_i \in K_i$.

Hence, $\tilde{x} \in K^{SW}$ and, therefore, $K^W = K^{SW}$. $\square$

$(\Phi\text{-SVEP})_w$ or $(\Phi\text{-SVEP})$ can be solved using $(\Phi\text{-SWEP})$ by the following:

Lemma 2. Each normalized solution $\tilde{x} \in K_n^{SW}$ with vector $W \in \prod_{i=1}^n T_+^{\ell_i}$ (respectively $W \in \prod_{i=1}^n (\text{int} T_+^{\ell_i})$) of $(\Phi\text{-SWEP})$ is a solution of $(\Phi\text{-SVEP})_w$ (respectively $(\Phi\text{-SVEP})$).

Proof. Let $\tilde{x} \in K_n^{SW}$ be a normalized solution of $(\Phi\text{-SWEP})$ with weight vector $W \in \prod_{i=1}^n T_+^{\ell_i}$ (respectively $W \in \prod_{i=1}^n (\text{int} T_+^{\ell_i})$). Assume that there is a $\tilde{x} \in K$ which is not a solution of $(\Phi\text{-SVEP})_w$ (respectively $(\Phi\text{-SVEP})$). Then, there would exist some $i \in I$ and a $y_i \in K_i$ satisfying

$$\Phi_i(f_i(\tilde{x}), \tilde{x}_i; y_i) \in \text{int } \mathbb{R}_+^{\ell_i}, \text{ (respectively } \Phi_i(f_i(\tilde{x}), \tilde{x}_i; y_i) \in \mathbb{R}_+^{\ell_i} \setminus \{0\}).$$

Since $W_i \in T_+^{\ell_i}$ (respectively $W_i \in \text{int } T_+^{\ell_i}$), for each $i \in I$, i.e., each W_i is a vector all of whose components are positive (resp. strictly positive). Because each component of $\Phi_i(f_i(\tilde{x}), \tilde{x}_i; y_i)$ is also positive (resp. strictly positive), we have

$$W_i \cdot \Phi_i(f_i(\tilde{x}), \tilde{x}_i; y_i) > 0$$

for all $i \in I$, in contradiction of the fact that $\tilde{x} \in K_n^{SW}$. $\square$

Lemmas 1 and 2 imply the next lemma:

Lemma 3. Each normalized solution $\tilde{x} \in K_n^W$ with weight vector $W \in \prod_{i=1}^n T_+^{\ell_i}$ (respectively $W \in \prod_{i=1}^n (\text{int} T_+^{\ell_i})$) is a solution of $(\Phi\text{-SVEP})_w$ (respectively of $(\Phi\text{-SVEP})$).

We also consider a Minty type weighted equilibrium problem:
 $(\Phi\text{-MWEP})$ Find $\tilde{x} \in K$ such that

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(y), \tilde{x}_i; y_i) \leq 0,$$

for all $y_i \in K_i, i \in I$. Let K^{WM} be the solution set of the problem $(\Phi\text{-MWEP})$.

For the reader's convenience, all these notions are summarized in Table 1.

Table 1. Notations.

Notation	Problems	Solution Sets
$(\Phi\text{-SVEP})$	System of vector equilibrium problems defined by Φ	K^S
$(\Phi\text{-SVEP})_w$	System of weak vector equilibrium problems defined by Φ	K_w^S
$(\Phi\text{-WEP})_w$	Weighted equilibrium problem defined by Φ	K^W and K_n^W (normalized)
$(\Phi\text{-SWEP})_w$	System of weighted equilibrium problems defined by Φ	K^{SW} and K_n^{SW} (normalized)
$(\Phi\text{-MWEP})_w$	Weighted equilibrium problem of Minty type defined by Φ	K^{MW}

Notice that all the problems except for the last one are of Stampacchia type. The link with the Minty type problem will appear in the last section after the introductions of pseudo-monotonicity.

3. Existence Results for $(\Phi\text{-WEP})$

Now, we consider in this section some notions of generalized weighted monotonicity, and in order to simplify the exposition we shall employ the term “weighted monotonicity” instead of “generalized weighted monotonicity”. Next, we will prove some existence results for $(\Phi\text{-WEP})$.

In this section the families $(f_i)_{i \in I}$, $(\Phi_i)_{i \in I}$, $(W_i)_{i \in I}$ as well as the equilibrium problems are the same as those in Section 2.

Definition 2. The family $(f_i)_{i \in I}$ is said to be

(i) *weighted pseudo-monotone with respect to (W, Φ)* , if, for all $x, y \in K$, we have

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; y_i) \leq 0 \implies \sum_{i \in I} W_i \cdot \Phi_i(f_i(y), x_i; y_i) \leq 0$$

and *weighted strictly pseudo-monotone with respect to (W, Φ)* if the second inequality is strict for all $x \neq y$;

(ii) *weighted maximal pseudo-monotone with respect to (W, Φ)* if it is *weighted pseudo-monotone with respect to (W, Φ)* and, for all $x, y \in K$, we have

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(z), x_i; z_i) \leq 0, \forall z \in (x, y] \implies \sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; y_i) \leq 0 \quad (1)$$

where $(x, y] = \prod_{i \in I} (x_i, y_i]$ and *weighted maximal strictly pseudo-monotone with respect to (W, Φ)* if it is *weighted strictly pseudo-monotone with respect to (W, Φ)* and (1) holds.

If the family $(f_i)_{i \in I}$ satisfies the stronger condition

$$\sum_{i \in I} W_i \cdot (\Phi_i(f_i(y), x_i; y_i) - \Phi_i(f_i(x), y_i; x_i)) \leq 0$$

then it is called *weighted monotone with respect to (W, Φ)* and *weighted strictly monotone with respect to (W, ψ)* if the inequality is strict for all $x \neq y$.

Definition 3. The family $(f_i)_{i \in I}$ is *weighted hemicontinuous with respect to (W, Φ)* if, for any $x, y \in K$ and any $\lambda \in [0, 1]$, the mapping $\lambda \mapsto \sum_{i \in I} W_i \cdot \Phi_i(f_i(x + \lambda(y - x)), x_i; y_i)$ is continuous.

Proposition 1. We suppose that the family $(f_i)_{i \in I}$ of functions satisfies the following conditions:

- (i₁) it is *weighted hemicontinuous and weighted pseudo-monotone with respect to (W, Φ)* ;
- (i₂) for any $i \in I$,

$$\begin{aligned} & \Phi_i(f_i(x + \lambda(y - x)), x_i; x_i + \lambda(y_i - x_i)) \\ &= \lambda^\tau \Phi_i(f_i(x + \lambda(y - x)), x_i; y_i), \text{ for any } \lambda \in (0, 1], \end{aligned}$$

where $\tau > 0$.

Then, $(f_i)_{i \in I}$ is *weighted maximal pseudo-monotone with respect to (W, Φ)* .

Proof. Assume that, for any $x, y \in K$,

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(z), x_i; z_i) \leq 0, \text{ for all } z \in (x, y].$$

Then,

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(x + \lambda(y - x)), x_i; x_i + \lambda(y_i - x_i)) \leq 0, \text{ for all } \lambda \in (0, 1].$$

Now, using (i_2) we get

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(x + \lambda(y - x)), x_i; y_i) \leq 0,$$

for all $\lambda \in (0, 1]$.

Because the family $(f_i)_{i \in I}$ is weighted hemicontinuity with respect to (W, Φ) ,

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; y_i) \leq 0.$$

Hence, the family $(f_i)_{i \in I}$ is weighted maximal pseudo-monotone with respect to (W, Φ) . $\square$

We now give two examples of functionals for which the hypothesis (i_2) of Proposition 1 is satisfied.

Example 1. For $i \in I$, let $Y_i = X_i^*$, $\Phi_i(f_i(z), x_i; y_i) = (\langle f_i(z), y_i - x_i \rangle)^{2s+1}$, where s is a positive integer. Condition (i_2) from Proposition 1 is satisfied.

In the real case i.e., when $Y_i = X_i = X_i^* = \mathbb{R}$, we see that $\Phi_i(f_i(z), x_i; y_i) = f_i(z) \cdot (y_i - x_i)^T \exp(x_i)$, $i \in I$, also verifies (i_2) .

If $(f_i)_{i \in I}$ is a weighted pseudo-monotone and weighted hemicontinuous family of functions, then it is weighted maximal pseudo-monotone.

This means that if the family $(f_i)_{i \in I}$ also verifies the condition (i_1) , then it will also be weighted maximal pseudo-monotone. For an example of a family of functions $(f_i)_{i \in I}$ and family of weight vectors $W = (W_i)_{i \in I}$ such that $(f_i)_{i \in I}$ is maximal weighted pseudo-monotone with respect to W and a certain family of functionals, see Example 2.

Lemma 4. If the family $(f_i)_{i \in I}$ of functions is weighted maximal pseudo-monotone with respect to (W, Φ) , then $K^W = K^{WM}$.

Proof. Because of the weighted pseudo-monotonicity of the family $(f_i)_{i \in I}$ with respect to (W, Φ) , $K^W \subseteq K^{WM}$. Conversely, let $\tilde{x} \in K^{WM}$. Then,

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(y), \tilde{x}_i; y_i) \leq 0,$$

for all $y_i \in K_i$, $i \in I$. Since for each $i \in I$, K_i is convex, we have $(\tilde{x}_i, y_i] \subseteq K_i$; therefore,

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(z), \tilde{x}_i; z_i) \leq 0,$$

for all $z_i \in (\tilde{x}_i, y_i]$, $i \in I$. By the weighted maximal pseudo-monotonicity of the family $(f_i)_{i \in I}$ of functions with respect to (W, Φ) , we have

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(\tilde{x}), \tilde{x}_i; y_i) \leq 0,$$

for all $y_i \in K_i$, $i \in I$.

This shows that $\tilde{x} \in K^W$ and hence $K^W = K^{WM}$. $\square$

Remark 2. By Proposition 1 and Lemma 4, if the family $(f_i)_{i \in I}$ of functions is weighted hemicontinuous and weighted pseudo-monotone with respect to (W, Φ) and satisfies the condition (i_2) of Proposition 1, then $K^W = K^{WM}$.

Remark 3. In the following, we will need Theorem 2.2 from [27] due to Chowdhury and Tan. This is a result on the existence of fixed-points for set-valued maps. More precisely, if we have two set-valued maps defined on a convex non-empty subset K of a topological space with values in 2^K , one having a convex image and both satisfying four technical conditions, the one with a convex image will have a fixed-point.

Using this result to our problems, assuming some weighted pseudo-monotonicity and convexity conditions, we can prove our main existence result, as follows.

Theorem 1. Assume the following conditions:

- (i_1) the family $(f_i)_{i \in I}$ is weighted maximal pseudo-monotone with respect to (W, Φ) ;
- (i_2) there exists a closed compact and nonempty subset D of K and $\tilde{y} \in D$ such that, for any $x \in K \setminus D$,

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; \tilde{y}_i) > 0;$$

- (i_3) the mapping $y \mapsto \sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; y_i)$ is convex on K ;

- (i_4) $\sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; x_i) = 0$ for all $x \in K$;

- (i_5) for all $x, y \in K$ and every generalized sequence $\{x^\alpha\}_{\alpha \in \Gamma}$ in K converging to x , we have

$$\liminf_{\alpha \in \Gamma} \sum_{i \in I} W_i \cdot \Phi_i(f_i(x^\alpha), x_i^\alpha; y_i) = \sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; y_i).$$

Then, there exists a solution $\tilde{x} \in K^W$ and hence $\tilde{x} \in K^{SW}$. Additionally,

If $W \in \prod_{i \in I} T_+^{\ell_i}$, then there exists a normalized solution $\tilde{x} \in K_\eta^W$ and hence $\tilde{x} \in K_w^S$.

If $W \in \prod_{i=1}^n \left(\text{int } T_+^{\ell_i} \right)$, then $\tilde{x} \in K^S$.

Proof. For each $x \in K$, define the multivalued maps $\mathcal{S}, \mathcal{T} : K \rightarrow 2^K$ by

$$\mathcal{S}(x) = \left\{ y \in K : \sum_{i \in I} W_i \cdot \Phi_i(f_i(y), x_i; y_i) > 0 \right\}$$

and

$$\mathcal{T}(x) = \left\{ y \in K : \sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; y_i) > 0 \right\}.$$

Using (i_3) we get that $\mathcal{T}(x)$ is convex for all $x \in K$. (i_1) implies that $\mathcal{S}(x) \subseteq \mathcal{T}(x)$.

Define now the set

$$\left[\mathcal{S}^{-1}(y) \right]^c = \left\{ x \in K : \sum_{i \in I} W_i \cdot \Phi_i(f_i(y), x_i; y_i) \leq 0 \right\}.$$

This set, called the complement of $\mathcal{S}^{-1}(y)$ in K , is closed in K (by (i_5)) and hence $\mathcal{S}^{-1}(y)$ is open in K . Therefore, $\mathcal{S}^{-1}(y)$ is compactly open (see [27]) for any $y \in K$.

We will prove that there exists $\tilde{x} \in K$ such that $\mathcal{S}(\tilde{x}) = \emptyset$. In order to do this, we assume that, for all $x \in K$, $\mathcal{S}(x)$ is nonempty. Then, relative to $\mathcal{T}$, all the conditions of Theorem 2.2 from [27] are satisfied so there exists $\hat{x} \in K$ such that $\hat{x} \in \mathcal{T}(\hat{x})$. It follows that

$$0 = \sum_{i \in I} W_i \cdot \Phi_i(f_i(\hat{x}), \hat{x}_i; \hat{x}_i) > 0,$$

a contradiction. Hence, since there exists $\tilde{x} \in K$ such that $\mathcal{S}(\tilde{x}) = \emptyset$, we have that, for all $i \in I$ and all $y_i \in K_i$,

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(y), \tilde{x}_i; y_i) \leq 0,$$

with $y = (y_i)_{i \in I}$. By Lemma 4, $\tilde{x} \in K$ is a solution of $(\Phi\text{-WEP})$, and so by Lemma 1 it is a solution of $(\Phi\text{-SWE})$.

If $W \in \prod_{i \in I} T_+^{\ell_i}$, then $\tilde{x} \in K$ is a normalized solution of $(\Phi\text{-SWE})$, and hence by Lemma 2 it is a solution of $(\Phi\text{-SVEP})_w$. If $W \in \prod_{i \in I} (\text{int } T_+^{\ell_i})$, using again Lemma 2, it follows that $\tilde{x} \in K$ is a solution of $(\Phi - \text{SVEP})$ $\square$

As a consequence, for the hemicontinuous case we obtain

Corollary 1. Assume that (i_2) – (i_5) from Theorem 1 are satisfied and also that (i'_1) holds, where (i'_1) the family $(f_i)_{i \in I}$ is weighted hemicontinuous and pseudo-monotone with respect to (W, Φ) .

Then, there exists an element $\tilde{x} \in K^W$ and hence $\tilde{x} \in K^{SW}$. Moreover, if $W \in \prod_{i \in I} T_+^{\ell_i}$, then there exists a normalised solution $\tilde{x} \in K_n^W$ and hence $\tilde{x} \in K_w^S$. Also, if $W \in \prod_{i=1}^n (\text{int } T_+^{\ell_i})$, then $\tilde{x} \in K^S$.

Theorem 2. We suppose the following:

- (j_1) The family $(f_i)_{i \in I}$ is weighted maximal strictly pseudo-monotone with respect to (W, Φ) .
- (j_2) There exists a closed compact and nonempty subset D of K and an $\tilde{y} \in D$ such that, for all $x \in K \setminus D$,

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; \tilde{y}_i) > 0.$$

- (j_3) $\sum_{i \in I} W_i \cdot (\Phi_i(f_i(x), x_i; y_i) + \Phi_i(f_i(x), y_i; x_i)) \geq 0$ for all $x, y \in K$.

Then, $(\Phi\text{-WEP})$ has a unique solution which is also a unique solution of $(\Phi\text{-SWE})$.

Moreover, if $W \in \prod_{i \in I} T_+^{\ell_i}$, then there exists a unique normalized solution $\tilde{x} \in K_n^W$ which is also a unique solution of $(\Phi\text{-SVEP})_w$ and for $W \in \prod_{i=1}^n (\text{int } T_+^{\ell_i})$, $\tilde{x} \in K^S$ is unique.

Proof. From Theorem 1, the problem $(\Phi\text{-WEP})$ has a solution, so we only need to prove unicity. Suppose we have to solutions x' and x'' of $(\Phi\text{-WEP})$, such that $x' \neq x''$. Then,

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(x''), x''_i; x'_i) \leq 0.$$

Because the family $(f_i)_{i \in I}$ is weighted strictly pseudo-monotone with respect to (W, Φ) , we obtain that

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(x''), x''_i; x'_i) \leq 0 \xrightarrow{(j_3)} \sum_{i \in I} W_i \cdot \Phi_i(f_i(x'), x'_i; x''_i) > 0,$$

so x' is not a solution of $(\Phi - \text{WEP})$, a contradiction. $\square$

From Corollary 1 and Theorem 2, we obtain

Corollary 2. Assume conditions (j_2) and (j_3) from Theorem 2, and also assume that (j'_1) holds, where

(j'_1) The family $(f_i)_{i \in I}$ is weighted hemicontinuous and weighted strictly pseudo-monotone with respect to (W, Φ) .

Then $(\Phi\text{-WEP})$ has a unique solution which is the unique solution of $(\Phi\text{-SWEP})$. Moreover, if $W \in \prod_{i \in I} T_+^{\ell_i}$, then there is a unique $\tilde{x} \in K_n^W$ which is also the unique solution of $(\Phi\text{-SVEP})_w$ and for $W \in \prod_{i=1}^n \left(\text{int } T_+^{\ell_i} \right)$, $\tilde{x} \in K^S$ is unique.

Remark 4. The statement of Theorem 1 was first published in [21].

Remark 5. We used Theorem 1 in [26] to obtain an existence result for the solution of weighted set-valued equilibrium problems which we introduced there.

Next, we will prove a result similar to Theorem 1 under B -pseudomonotonicity assumptions.

Definition 4. The family $(f_i)_{i \in I}$ is said to be weighted B -pseudo-monotone with respect to (W, Φ) , if for every $x \in K$ and every net $\{x^\alpha\}_{\alpha \in \Gamma}$ in K converging to x with

$$\limsup_{\alpha} \left[\sum_{i \in I} W_i \cdot \Phi_i(f_i(x^\alpha), x_i^\alpha; x_i) \right] \leq 0,$$

we have

$$\limsup_{\alpha} \left[\sum_{i \in I} W_i \cdot \Phi_i(f_i(x^\alpha), x_i^\alpha; y_i) \right] \geq \sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; y_i),$$

for all $y \in K$.

Remark 6. For the following, we shall need Theorem 6 from [28], also due to Chowdhury and Tan. As in Remark 3, this 1996 result also gives conditions for the existence of fixed-points for set-valued maps. In it, contrary to the case of Remark 3, the conditions for the existence of a fixed-point do not depend on a second map.

Using this result, we obtain the existence of solutions for our problems, this time under weighted B -pseudo-monotonicity.

Theorem 3. Assume the following conditions:

(k_1) The family $(f_i)_{i \in I}$ is weighted B -pseudo-monotone with respect to (W, Φ) and for every $A \in \mathbf{F}(K)$ the map $x \mapsto \sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; y_i)$ is lower semi-continuous on $\text{co} A$.

(k_2) There exists a closed compact and nonempty subset D of K and also some $\tilde{y} \in D$ such that, for all $x \in K \setminus D$,

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; \tilde{y}_i) > 0.$$

(k_3) For every $X \in K$ $\sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; x_i) = 0$.

Then, there exists a solution $\tilde{x} \in K^W$ and hence $\tilde{x} \in K^{SW}$.

Additionally,

If $W \in \prod_{i \in I} T_+^{\ell_i}$ there exists a normalized solution $\tilde{x} \in K_n^W$ and hence $\tilde{x} \in K_w^S$.

If $W \in \prod_{i=1}^n \left(\text{int } T_+^{\ell_i} \right)$, then $\tilde{x} \in K^S$.

Proof. For any $x \in K$, let $\mathcal{T} : K \rightarrow 2^K$ be as in the proof of Theorem 1. Then, for all $x \in K$, $\mathcal{T}(x)$ is convex (i.e., condition 1 from Theorem 6 from [28] holds). Let $A \in \mathbf{F}(K)$. Then,

$$\left[\mathcal{T}^{-1}(y)\right]^c \cap coA = \left\{x \in coA : \sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; y_i) \leq 0\right\}$$

is closed in coA by the lower semi-continuity of the map $x \mapsto \sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; y_i)$ on coA . Hence, $\mathcal{T}^{-1}(y) \cap coA$ is open in coA , that is, condition 2 from Theorem 6 from [28] holds.

Now, we prove that condition 3 from the same theorem holds. In order to do this, let $x, y \in coA$ and the net $\{x^\alpha\}_{\alpha \in \Gamma}$ in K convergent to x , such that

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(x^\alpha), x_i^\alpha; ty_i + (1-t)x_i) \leq 0,$$

for all $\alpha \in \Gamma, t \in [0, 1]$.

In the above inequality, we consider first that $t = 0$ and then $t = 1$. For $t = 0$ we have

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(x^\alpha), x_i^\alpha; x_i) \leq 0,$$

for all $\alpha \in \Gamma$. Hence,

$$\limsup_{\alpha} \sum_{i \in I} W_i \cdot \Phi_i(f_i(x^\alpha), x_i^\alpha; x_i) \leq 0$$

and then, by (k_1) , we get

$$\limsup_{\alpha} \sum_{i \in I} W_i \cdot \Phi_i(f_i(x^\alpha), x_i^\alpha; y_i) \geq \sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; y_i), \text{ for all } y \in K.$$

Combining this inequality with the first one obtained for $t = 1$, we get

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(x), x_i; y_i) \leq 0,$$

that is, $y \notin \mathcal{T}(x)$. Thus, condition 3 from Theorem 6 from [28] is satisfied.

Also, according to (k_2) and the definition of $\mathcal{T}$, for all $x \in D$, the set $\mathcal{T}(x)$ is nonempty. Thus, condition 5 from Theorem 6 from [28] holds. Moreover, we see that (k_2) corresponds to condition (4) from this theorem.

We obtained that all hypothesis of Theorem 6 from [28] are satisfied. So, there is an $\hat{x} \in K$ such that $\hat{x} \in \mathcal{T}(\hat{x})$; in other words,

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(\hat{x}), \hat{x}_i; \hat{x}_i) > 0$$

which contradicts (k_3) . Thus, there exists $\tilde{x} \in K$ such that $\mathcal{T}(\tilde{x}) = \emptyset$, that is

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(\tilde{x}), \tilde{x}_i; y_i) \leq 0,$$

for all $y_i \in K_i, i \in I$. Hence, $\tilde{x}$ is a solution of $(\Phi\text{-WEP})$ and so, by Lemma 3, it is a solution of $(\Phi\text{-SWEP})$.

If $W \in \prod_{i \in I} T_+^{\ell_i}$, then $\tilde{x} \in K$ is a normalized solution of $(\Phi\text{-SWEP})$ and hence, by Lemma 2, it is a solution of $(\Phi\text{-SVEP})_w$. Further, if $W \in \prod_{i=1}^n (\text{int } T_+^{\ell_i})$, then again by Lemma 2, $\tilde{x} \in K$ is a solution of $(\Phi\text{-SVEP})$. $\square$

Example 2. Let $K = K_1 \times K_2 \subset \mathbb{R}^2, K_1 = [-2, 2], K_2 = [-5, -3] \cup [3, 5]$.

Let $f_1, f_2 : K \rightarrow \mathbb{R}$, $f_1(x) = |x_1 - x_2|$, $f_2(x) = 0$.

Take also the functionals $\Psi_i : \mathbb{R} \times K_i \times K_i \rightarrow \mathbb{R}$, $\Psi_i(f(x), x_i; y_i) = f(x)(y_i^2 - x_i^2)$ with weights $W = \left(\frac{1}{2}, \frac{1}{2}\right)$.

Then, the family $(\Psi_i)_{i=1,2}$ verify the assumptions of Theorems 1.

We have that

$$\sum_{i \in I} W_i \cdot \Psi_i(f_i(x), x_i; y_i) = \frac{1}{2} |x_2 - x_1| (y_1^2 - x_1^2).$$

Let us check that this indeed satisfies the hypothesis of Theorem 1:

- (i₁) It is clear that $|x_2 - x_1| (y_1^2 - x_1^2) \leq 0$ implies that $|y_2 - y_1| (y_1^2 - x_1^2) \leq 0$;
- (i₂) Take $\tilde{y} = 2$. Then $D = \{-2, 2\} \times K_2$;
- (i₃) The map $y \mapsto |x_2 - x_1| (y_1^2 - x_1^2)$ is convex as the composition of two convex maps (a projection and a quadratic one);
- (i₄) Clear by construction;
- (i₅) All the maps which appear are continuous, so this condition is also satisfied.

Also, the same family satisfies the hypothesis of Theorem 3. If we take $K_1 = [-2, 0]$, $(\Psi_i)_{i=1,2}$ satisfies also the conditions of Theorem 2.

4. The Case of a Real Normed Space

Finally, we consider new results for the case when $X_i = X$ is a real normed space.

Assume that for any $i \in I$, D_i are open such that $D_i \supset K_i$ and $\Phi_i : Y_i \times K_i \times D_i$, $\Phi_i(f_i(x), x_i, x_i) = 0$, for all $x_i \in K_i, i \in I$.

Proposition 2. We suppose that for any $i \in I$, $\Phi_i(f_i(x), x_i, \cdot)$ is convex for all $x \in K$ and there exists the directional derivative Φ_i^0 in the third variable. If $\tilde{x}$ is a solution of $(\Phi - WEP)$, then

$$\sum_{i \in I} W_i \cdot \Phi_i^0(f_i(\tilde{x}), \tilde{x}_i, \tilde{x}_i)(h_i) \leq 0 \text{ for all } h_i \in K_i - \tilde{x}_i, i \in I.$$

Proof. We have

$$\sum_{i \in I} W_i \frac{1}{t} [\Phi_i(f_i(\tilde{x}_i), \tilde{x}_i, \tilde{x}_i + t(y_i - \tilde{x}_i)) - \Phi_i(f_i(\tilde{x}), \tilde{x}_i, \tilde{x}_i)] \leq 0,$$

for all $t \in (0, 1], y_i \in K_i, i \in I$.

Taking the limit $t \rightarrow 0^+$, we get

$$\sum_{i \in I} W_i \psi_i^0(f_i(\tilde{x}), \tilde{x}_i, \tilde{x}_i)(h_i) \leq 0, \text{ for all } h_i \in K_i - \tilde{x}_i, h_i \neq 0, i \in I.$$

□

Theorem 4. Assume that $\Phi_i(f_i(x), x_i, \cdot)$ is quasiconcave for all $x \in K$, $\Phi_i(f_i(x), x_i, x_i) = 0$ for any $x_i \in K_i$ and there exist $\tilde{x}, \tilde{y} \in K$ such that

$$\sum_{i \in I} W_i \Phi_i(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i) \leq 0$$

and

$$\sum_{i \in I} W_i \Phi_i^0(f_i(\tilde{x}_i), \tilde{x}_i, \tilde{y}_i)(h_i) < 0$$

for any $h_i \in K_i - \tilde{y}_i, h_i \neq 0, i \in I$.

Then, $\tilde{x} \in K^W$.

Proof. For $t \in (0, 1]$ and $y \in K, y \neq \tilde{y} = (\tilde{y}_i)_{i \in I}$, we get by quasiconcavity

$$\sum_{i \in I} W_i \Phi_i(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i + t(y_i - \tilde{y}_i)) \geq \sum_{i \in I} W_i \cdot \min\{\Phi_i(f_i(\tilde{x}), \tilde{x}_i, y_i), \Phi_i(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i)\},$$

the minimum being taken component-wise. We obtain

$$\begin{aligned} & \sum_{i \in I} W_i \cdot \frac{1}{t} [\Phi_i(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i + t(y_i - \tilde{y}_i)) - \Phi_i(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i)] \\ & \geq \min \left\{ \sum_{i \in I} W_i \frac{1}{t} [\Phi_i(f_i(\tilde{x}), \tilde{x}_i, y_i) - \Phi_i(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i)], 0 \right\}. \end{aligned}$$

Using the fact that for any $i \in I, y_i \in K_i$,

$$\begin{aligned} \lim_{t \rightarrow 0^+} \frac{1}{t} [\Phi_i(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i + t(y_i - \tilde{y}_i)) - \Phi_i(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i)] \\ = \Phi_i^0(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i)(y_i - \tilde{y}_i), \end{aligned}$$

then for any $\epsilon > 0$ such that

$$\sum_{i \in I} W_i \cdot \Phi_i^0(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i)(y_i - \tilde{y}_i) < -2\epsilon,$$

there exists $t_0 > 0$ such that for any $0 < t < t_0$ we have

$$\begin{aligned} & \sum_{i \in I} W_i \cdot \frac{1}{t} [\Phi_i(f_i(\tilde{x}), \tilde{x}_i, y_i + t(y_i - \tilde{x}_i)) - \Phi_i(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i)] \\ & < \sum_{i \in I} W_i \Phi_i^0(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i)(y_i - \tilde{y}_i) + \epsilon < -2\epsilon + \epsilon = -\epsilon. \end{aligned}$$

Hence, for $0 < t < t_0$ we have

$$\sum_{i \in I} W_i \frac{1}{t} \min\{\Phi_i(f_i(\tilde{x}), \tilde{x}_i, y_i) - \Phi_i(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i), 0\} < -\epsilon.$$

Thus, we obtain

$$\begin{aligned} & \sum_{i \in I} W_i \frac{1}{t} \min\{\Phi_i(f_i(\tilde{x}), \tilde{x}_i, y_i) - \Phi_i(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i), 0\} \\ & < t \left[\sum_{i \in I} W_i \cdot \Phi_i^0(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i)(y_i - \tilde{y}_i) + \epsilon \right] < 0 \end{aligned}$$

i.e.,

$$\begin{aligned} & \sum_{i \in I} W_i \cdot [\Phi_i(f_i(\tilde{x}), \tilde{x}_i, y_i) - \Phi_i(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i)] \\ & < t \left[\sum_{i \in I} W_i \cdot \Phi_i^0(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i)(y_i - \tilde{y}_i) + \epsilon \right] < 0 \end{aligned}$$

and therefore

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(\tilde{x}), \tilde{x}_i, y_i) < \sum_{i \in I} W_i \cdot \Phi_i(f_i(\tilde{x}), \tilde{x}_i, \tilde{y}_i) \leq 0 \quad \text{for all } y \in K, y \neq \tilde{y}.$$

By $\sum_{i \in I} W_i = 1$, $W_i \geq 0$, $i \in I$ and $\Phi_i(f_i(\tilde{x}), \tilde{x}_i, \tilde{x}_i) = 0$ we get

$$\sum_{i \in I} W_i \cdot \Phi_i(f_i(\tilde{x}), \tilde{x}_i, \tilde{x}_i) = 0.$$

Hence, by the last strict inequality we obtain $\tilde{y} = \tilde{x}$ and

$$\sum_{i \in I} W_i \Phi_i(f_i(\tilde{x}), \tilde{x}_i, y_i) \leq 0$$

for all $y_i \in K_i$, $i \in I$. Thus, we obtained that $\tilde{x} \in K^W$. $\square$

5. Conclusions and Future Work

We obtained the existence of solutions for some classes of weighted equilibrium problems starting from some systems of vector equilibrium problems. In order to do this, we assumed some pseudo-monotonicity-type conditions for a family of argument functions of our problem-defining functionals. These hypotheses allow us to apply some fixed-point theorems leading to the conclusion that our main problems have a solution. From here follows the existence of solutions for some other equivalent weighted problems, but also for the initial systems of equilibrium problems. Some previous results were extended or put in a new context here, the main ones using the previously mentioned modern fixed-point theorems, the others using our main theorems. As an example, our results were applied to the case of normed real vector spaces.

We extended some previously obtained results from the literature from weighted variational inequalities to weighted equilibrium problems, and we also gave proofs for some results we previously announced.

In further works, we will study algorithmic methods for solution finding, and we will continue to study the existence of solutions both under weaker assumptions and on different function spaces.

In addition, we hope to be able to study this type of problem on Hilbert spaces, which will allow them to be applied to traffic equilibria by studying more realistic transportation models and developing numerical methods.

Author Contributions: All the authors contributed equally. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: During the elaboration of this paper no new data was generated or analyzed. Data sharing is not applicable for this paper.

Conflicts of Interest: The authors declare no conflict of interest.

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