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Three Kinds of Denting Points and Their Further Applications in Banach Spaces

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Abstract: In this paper, we study three kinds of dentabilities and present their important applications in the geometric theory of Banach spaces. First, the relations between weak-denting points and extreme points are established. Moreover, we characterize strong convex spaces and weakly locally uniformly smooth spaces by using weakly locally uniform dentability. Finally, we provide some important applications of *strong** dentability in Fréchet-smooth Banach spaces while investigating the density of *strong** points in dual Banach spaces.

Keywords: weak-denting point; weakly locally uniformly dentability; *strong** dentability; Fréchet smooth; density

MSC: 46B10; 46B20; 46B22

1. Introduction

Let X be a Banach space and X^* be its dual space. $U(X)$ and $S(X)$ are denoted as the closed unit ball and the unit sphere of Banach space X, respectively.

$U(X^*)$ and $S(X^*)$ are denoted as the closed unit ball and the closed unit sphere of the dual space of X^*, respectively. w, w^* and $\|\cdot\|$ represent the w, w^* and norm topologies, respectively. By $co(A)$, $\overline{co}(A)$ and $\overline{co}^w(A)$, we denote the convex hull, closed convex hull and weak closed convex hull of set A, respectively. Furthermore $\overline{\text{co}}^{w^*}(A^*)$ denotes the *weak** closed convex hull of set A^*. $x_n \xrightarrow{w} x$ denotes $\{x_n\}$ converging to x and $x_n^* \xrightarrow{w^*} x^*$ denotes $\{x_n^*\}$ *weakly** converging to x^*. Let us define the following:

$$S_x = \{f \in S(X^*) : f(x) = 1 = \|x\|\}$$

$$A_f = \{x \in S(X) : f(x) = 1 = \|f\|\}$$

$$F(f, \delta) = \{x \in B(X) : f(x) \geq 1 - \delta\}$$

In early 1936, the concept of uniformly convex space was introduced by J. A. Clarkson. He also proved that uniformly convex space has the Radon–Nikodým property. In 1966, M.A. Riefel introduced the notion dentability of a set and linked the Radon–Nikodým property to the geometry of Banach space via the notion of dentability [1]. In 1940, M. Krein and D. Milman asserted that if K is a convex compact subset of an l. c. t. space, then K is the closed convex hull of its extreme points, i.e., $K = \overline{\text{cov}}(Ext(K))$; the result is known as the Krein–Milman theorem [2]. Milman theorem [3], also known as a partial converse of the Krein–Milman theorem, states that if A is a subset of A and the closed

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convex hull of A is all of K, then every extreme point of K belongs to the closure of A, i.e., $(A \subset K; K = \overline{\text{cov}}(A)) \Rightarrow \text{Ext}(K) \subset \overline{A})$. In 1961, Bishop and Phelps derived the Bishop–Phelp theorem [4], which demonstrates that every Banach space is subreflexive; in other words, they proved that in any Banach space X, given a functional $x^* \in X^*$ and an arbitrary positive number $\varepsilon > 0$, it is always possible to find a new functional $y^* \in X^*$ that attains its norm and satisfies $\| y^* - x^* \| < \varepsilon$. This means that, for every Banach space X, the set $NA(X)$ (the set of all norm-attaining functionals on X) is norm-dense in the dual space X^*. In 1961, N. V. Efimov and S. B. Stechkin introduced the Efimov–Stechkin property and researched its geometric direction [5].

Some concepts of dentability in Banach spaces are known [6–18], and this is primarily because these properties are strongly related to the Radon–Nikodým property, convexity, smoothness, approximative compactness, the continuity of the metric projection operator and the geometric properties of sets in Banach spaces. A variety of dentabilities have been defined and recently studied. In 2018, we defined the notions of weak dentability, weakly locally uniformly dentability and *strong** dentability in Banach space and established the relative connections between weak-denting, weakly locally uniformly denting, strictly convex, very smooth and weakly locally uniformly smooth points [19]. Near-dentability and weak-star near-dentability were studied by S. Q. Shang and Y. A. Cui [20–23]. Fernando García-Castaño, M. A. Melguizo Padial and G. Parzanese focused on denting points of convex sets and the weak property (π) of cones in locally convex spaces [24]. Moreover, weak-star dentability and quasi-weak-star near-dentability were introduced by L.Y. Bao and Suyalatu Wulede in their paper [25]. Three kinds of dentabilities in Banach spaces and their applications have been thoroughly investigated by Z. H. Zhang and J. Zhou [26].

This paper builds on previous research in this direction.

2. Materials and Methods

In this paper, we recall the notions of weak dentability, weakly locally uniformly dentability and *strong** dentability in Banach space that were introduced by us in 2018. Then, we study their further application by establishing the relative connections between weak-denting, weakly locally uniform denting, extreme, strongly convex, weakly locally uniformly smooth and Fréchet smooth points using the Krein–Milman, separation and Bishop–Phelp theorems.

3. Results

3.1. Preliminaries

Let us present three related notions: weak dentability, weakly locally uniformly dentability and *strong** dentability, which were introduced by us in [19].

Definition 1 (see [19]). *Let A be a subset of X. A point $x \in A$ is said to be a weak-denting point of A if $x \notin \overline{co}^w(A \backslash (x + U^w))$ for every weak neighborhood U^w of the origin.*

Definition 2 (see [19]). *A point $x \in S(X)$ is said to be a weakly locally uniformly denting point, if for every $f \in S_x$ there exists $\eta > 0$, such that*

$$\inf_{f \in S_x} \{f(x) - f(y) : y \in \overline{co}^w(U(X) \backslash (x + U^w))\} \geq \eta,$$

where U^w is a weak neighborhood of the origin.

Definition 3 (see [19]). *Let A^* be a subset of a dual Banach space X^*. A point $x^* \in A^*$ is said to be a strong*-denting point if for every $\varepsilon > 0$ there holds $x^* \notin \overline{co}^{w^*}(A^* \backslash B_\varepsilon(x^*))$, where $B_\varepsilon(x^*)$ is*

an ε neighborhood of x^*. If every $x^* \in S(X^*)$ is the strong*-denting point of $U(X^*)$, then X^* is said to be a strong* dentable space.

Let us recall some definitions and some lemmas which will be used in the subsequent sections of this paper.

Definition 4 (see [27]). *We say that Banach space X has the (H) property if for any $x \in S(X)$, $\{x_n\} \subset X$, $\| x_n \| \to \| x \|$ and $x_n \xrightarrow{w} x$; then, $\| x_n - x \| \to 0$.*

Definition 5 (see [28]). *A space X is said to be smooth (or very smooth) if, for any $\{x_n^*\} \subset S(X^*)$, $x \in S(X)$ with $x_n^*(x) \to 1$, ; then, $\{x_n^*\}$ is weakly* convergent (resp. $\{x_n^*\}$ is weakly convergent).*

Definition 6 (see [29]). *A space X is weakly locally uniformly smooth if and only if for every $\varepsilon > 0$, $x \in S(X)$, $f \in S_x$ and $q \in S(X^{**})$, there exists $\delta = \delta(\varepsilon, x, f, q) > 0$, such that $| q(f - g) | < \varepsilon$ where $g \in S(X^*)$ with $\| \frac{f+g}{2} \| > 1 - \delta$.*

Lemma 1 (see [5]). *A space X has the Efimov–Stechkin property if and only if X is reflexive and satisfies the following condition: whenever $\{x_n\} \subset X, x \in X, \| x_n \| \to \| x \|$ and $x_n \xrightarrow{w} x$, then there exists a subsequence $\{x_{n_k}\} \subset \{x_n\}$ such that $\| x_{n_k} - x \| \to 0$.*

Lemma 2. *A space X is Fréchet-smooth if and only if $\{x_n^*\} \subset S(X^*)$ there exists $x \in S(X)$ such that $x_n^*(x) \to 1$; then, there exists $x_0^* \in U(X^*)$ such that $x_n^* \to x_0^*$, i.e., $x_n^*(x)$ is convergent.*

We also conducted research on the applications concerning three dentabilities in Banach space and obtained the following conclusion:

Theorem 1 (see [19]). *Let X be a Banach space. If every point $x \in S(X)$ is a weak-denting point of $U(X)$, then X is strictly convex.*

Theorem 2 (see [19]). *Let X be a reflexive Banach space. Then, X^* is very smooth if and only if every point $x \in S(X)$ is a weak-denting point of $U(X)$.*

Theorem 3 (see [19]). *Let X be a Banach space. If X is weakly locally uniformly convex then for every $x \in S(X)$, the weak neighborhood U^w of the origin and $f \in S_x$, there exists $\delta = \delta(x, U^w) > 0$ such that $F(f, \delta) \subset x + U^w$.*

Theorem 4 (see [19]). *Let X be a reflexive Banach space. If X is weakly locally uniformly smooth, then every $x \in S(X)$ is a weakly locally uniformly denting point of $U(X)$.*

3.2. Results and Discussion

We present the main theorem of this section.

Theorem 5. *If X is a separable Banach space with the Efimov–Stechkin property, and $x_0 \in U(X)$ is an extreme point of $U(X)$, then x_0 is a weak-denting point of $U(X)$.*

Proof of Theorem 5. We prove the contrapositive. Suppose that $x_0 \in U(X)$ is an extreme point of $U(X)$ but not a weak-denting point of $U(X)$. Then, there exists a weak neighborhood U_0^w of the origin such that

$$x_0 \in \overline{co}^w(U(X) \backslash (x_0 + U_0^w)). \tag{1}$$

Since $U(X)$ is a bounded closed set of X,

$$K = \overline{co}^w(U(X)\backslash(x_0 + U_0^w)) \tag{2}$$

is a weak compact convex set. According to the Krein–Milman theorem,

$$K = \overline{co}(ext(K)). \tag{3}$$

Since X is separable and reflexive, X is quantifiable under (X, W), and then there exist $\{u_n\}, \{v_n\} \subset U(X)\backslash(x_0 + U_0^w)$ and $\{\lambda_n\} \subseteq [0, 1]$, such that

$$\lambda_n u_n + (1 - \lambda_n)v_n \xrightarrow{w} x_0. \tag{4}$$

Since X is reflexive, and $U(X)$ is weakly sequentially compact, for $u_0, v_0 \in U(X)$ and $\lambda_0 \in [0, 1]$, there exist $\{u_{n_i}\} \subset \{u_n\}$, $\{v_{n_i}\} \subset \{v_n\}$ and $\{\lambda_{n_i}\} \subseteq \{\lambda_n\}$, such that

$$\lambda_{n_i} u_{n_i} + (1 - \lambda_{n_i})v_{n_i} \xrightarrow{w} x_0 = \lambda_0 u_0 + (1 - \lambda_0)v_0, \tag{5}$$

and since $x_0 \in U(X)$ is an extreme point of $U(X)$,

$$u_0 = v_0 = x_0. \tag{6}$$

Consequently,

$$x_0 \in U(X)\backslash(x_0 + U_0^w). \tag{7}$$

Since

$$extU(X) \cap \overline{co}^w(U(X)\backslash(x_0 + U_0^w)) \subseteq ext(\overline{co}^w(U(X)\backslash(x_0 + U_0^w))) \tag{8}$$

we have

$$ext(\overline{co}^w(U(X)\backslash(x_0 + U_0^w))) \subset U(X)\backslash(x_0 + U_0^w), \tag{9}$$

a contradiction. Thus, the assumption does not hold. □

Example 1. *Let* $X = \ell^2 = \left\{(x_1, x_2, x_3, \ldots) | x_i \in \mathbb{R} \text{ or } \mathbb{C}, \sum_{i=1}^{\infty} |x_i|^2 < +\infty\right\}$. *Since X is a Hilbert space and has a countable basis (e.g., the standard basis* $\{e_n\}$*), X is separable. Since X is a reflexive Banach space, the closed convex hull of a weakly compact set is still weakly compact; thus, X has an Efimov–Stechkin property. In fact, for any extreme point* $x \in U(X)$ *and any weak neighborhood* U^w *of x,* $x \notin \overline{co}^w(U(X)\backslash U^w)$ *holds. We can select* $x_0 = e_1 = (1, 0, 0, \cdots) \in S(X)$*; since X is strictly convex,* x_0 *is an extreme point of* $U(X)$*. We can select functional* $f_1(y) = y_1$ *(projection to the first coordinate) and define* $U^w = \{y \in X : |f_1(y) - 1| < \varepsilon\}$ *with* $\varepsilon > 0$*. Obviously,* $U(X)\backslash U^w$ *contains all unit vectors with* $|y_1| \leq 1 - \varepsilon$*. Any* $z \in co(U(X)\backslash U^w)$ *can be written as*

$$z = \sum_{i=1}^{n} \lambda_i z_i,\ \lambda_i \geq 0,\ \sum \lambda_i = 1,\ z_i = (z_{i,1}, z_{i,2}, z_{i,3}, \cdots) \in U(X)\backslash U^w.$$

Case I. If every $z_{i,1} \leq 1 - \varepsilon$*, then* $z_1 \leq 1 - \varepsilon$*.*

Case II. If there exists $z_{i,1} < 0$*, then* $z_1 \leq \sum \lambda_i(1 - \varepsilon) = 1 - \varepsilon$*.*

It follows that $|z_1| \leq 1 - \varepsilon$*; hence,* $|z| \leq 1 - \varepsilon$*. Since X is a reflexive Banach space, the weak-closed convex hull equals the norm-closed convex hull. Thus, every* $z \in \overline{co}^w(U(X)\backslash U^w)$ *still satisfies* $|z| \leq 1 - \varepsilon$*. However,* $e_1 = (1, 0, 0, \cdots)$ *has the first coordinate, coordinate 1; thus, we can obtain* $e_1 \notin \overline{co}^w(U(X)\backslash U^w)$*, which implies that* e_1 *is a weak-denting point of* $U(X)$*.*

Theorem 6. *Let X be a weakly sequentially complete and separable Banach space and X^* be separable. If $x_0 \in U(X)$ is an extreme point of $U(X)$, then x_0 is a weak-denting point of $U(X)$.*

Proof of Theorem 6. We prove the contrapositive. Suppose that $x_0 \in U(X)$ is an extreme point of $U(X)$ but not a weak-denting point of $U(X)$. Then, there exists a weak neighborhood U_0^w of the origin such that

$$x_0 \in \overline{co}^w(U(X)\backslash(x_0 + U_0^w)). \tag{10}$$

Since $U(X)$ is a bounded closed set of X,

$$K = \overline{co}^w(U(X)\backslash(x_0 + U_0^w)), \tag{11}$$

is a weak compact convex set. According to the Krein–Milman theorem,

$$K = \overline{co}(ext(K)). \tag{12}$$

There exist $\{u_n\}, \{v_n\} \subseteq U(X)\backslash(x_0 + U_0^w)$, and $\{\lambda_n\} \subseteq [0, 1]$ such that

$$\lambda_n u_n + (1 - \lambda_n)v_n \xrightarrow{w} x_0. \tag{13}$$

Since X is separable, $U(X)\backslash(x_0 + U_0^w)$ has a dense subset $\{y_k\}$ that is countable. Since X^* is separable, there exists a dense subset $\{f_k\}$ that is countable. Based on the diagonal rule, we can obtain a Cauchy sequence $\{f_k(y_{k_i})\}$, and since X is weakly sequentially complete, there exists $y \in U(X)\backslash(x_0 + U_0^w)$, such that

$$f_k(y_{k_i}) \to f_k(y). \tag{14}$$

From (13) it follows that

$$\lambda_n f_k(u_n) + (1 - \lambda_n)f_k(v_n) \to f_k(x_0). \tag{15}$$

Since $\{y_k\}$ is dense in $U(X)\backslash(x_0 + U_0^w)$, for any $\varepsilon > 0$, there exists $\{y_{k_i}\} \subseteq \{y_k\}$ such that

$$\|u_n - y_{k_i}\| < \varepsilon, \tag{16}$$

for $\{u_n\}$. For the same reason, for any $\varepsilon > 0$, there exists $\left\{y_{k_j}\right\} \subseteq \{y_k\}$ such that

$$\left\|v_n - y_{k_j}\right\| < \varepsilon, \tag{17}$$

for $\{v_n\}$. From (13), we can obtain

$$\lambda_n f_k(y_{k_i}) + (1 - \lambda_n)f_k(y_{k_j}) \to f_k(x_0). \tag{18}$$

Furthermore,

$$f_k(y_{k_i}) \to f_k(y), f_k(y_{k_j}) \to f_k(y). \tag{19}$$

Thus, we can obtain

$$\lambda_n f_k(y_{k_i}) + (1 - \lambda_n)f_k(y_{k_j}) \to f_k(y). \tag{20}$$

Consequently,

$$f_k(y) = f_k(x_0). \tag{21}$$

Since $\{f_k\}$ is dense in X^*, we can obtain $y = x_0$, and then

$$x_0 \in U(X)\backslash(x_0 + U_0^w). \tag{22}$$

Furthermore,

$$x \in ext(K) \cap K \subset ext(K). \tag{23}$$

Hence, we have

$$ext(K) = ext(\overline{co}^w(U(X)\backslash(x_0 + U_0^w))) \subset U(X)\backslash(x_0 + U_0^w), \tag{24}$$

a contradiction. Thus, the assumption does not hold. □

Theorem 7. *Let X be a reflexive Banach space with the (H) property. If every point $x \in S(X)$ is a weakly locally uniformly denting point of $U(X)$, then X is strongly convex.*

Proof of Theorem 7. We prove the contrapositive. Suppose that X is not strongly convex. Then, for any $f \in S_{x_0}$, there exist $x_0 \in S(X)$ and $\{x_n\} \subset S(X)$, which are weakly locally uniformly denting points of $U(X)$, indicating that $\{x_n\}$ does not converge to x_0, although

$$f(x_n) \to 1. \tag{25}$$

Since $x_0 \in S(X)$ and $\{x_n\} \subset S(X)$ are weakly locally uniformly denting points of $U(X)$, there exists $\eta > 0$, such that

$$f(x_0) - f(\overline{co}^w(U(X)\backslash(x_0 + U^w)) \geq \eta > 0. \tag{26}$$

From (25), it follows that for any $\varepsilon > 0$, we select $\varepsilon < \eta$, and there exists $n_0 \in N$ such that

$$f(x_0) - f(x_n) < \varepsilon < \eta. \tag{27}$$

This implies that

$$\{x_n\}_{n=n_0+1}^{\infty} \not\subset \overline{co}^w(U(X)\backslash(x_0 + U)^w). \tag{28}$$

Since $f(x_n) \to 1$ is reflexive, there exists $\{x_{n_i}\} \subset \{x_n\}, x_1 \in U(X)$, such that

$$\{x_{n_i}\} \xrightarrow{w} x_1, \| x_{n_i} \| \to \| x_1 \| . \tag{29}$$

Hence,

$$f(x_{n_i}) \to f(x_1). \tag{30}$$

Moreover, according to (25),

$$f(x_0) = 1 = f(x_1). \tag{31}$$

Since $f(x_n) \to 1$ has the (H) property,

$$x_{n_i} \to x_1, \tag{32}$$

Thus

$$x_0 \neq x_1 \text{ i.e. } x_0 - x_1 \neq 0. \tag{33}$$

Note that

$$f(x_0) - f(x_1) = 0 = f(x_0 - x_1), \tag{34}$$

which contradicts $f \in S(X^*)$. Therefore, the assumption is not true. □

Theorem 8. *Let X be a reflexive Banach space that has the (H) property. If every point $x \in S(X)$ is a weakly locally uniformly denting point of $U(X)$, then X^* is weakly locally uniformly smooth.*

Proof of Theorem 8. We prove the contrapositive. Suppose for any $0 < \delta \leq 1$, there exists

$$\varepsilon_0 > 0, x_0^* \in S(X^*), x_0 \in A_{x_0^*}, y_0^* \in S(X^*), \tag{35}$$

such that

$$|y_0^*(x_0 - y)| \geq \varepsilon_0, \tag{36}$$

and

$$\| \frac{x_0 + y}{2} \| \leq 1 - \delta. \tag{37}$$

According to Formula (36), we have

$$y \notin x_0 + U_0^w, \tag{38}$$

where U_0^w is a weak neighborhood of the origin. Therefore, we obtain

$$y \in U(X) \backslash (x_0 + U_0^w) \subseteq \overline{\text{co}^w}(U(X) \backslash (x_0 + U_0^w)). \tag{39}$$

Since x_0 is a weakly locally uniformly denting point of $U(X)$, there exists $\eta > 0$ such that

$$f_0(x_0) - f_0(\overline{co}^w(U(X) \backslash (x_0 + U^w)) \geq \eta, \tag{40}$$

where U^w is any weak neighborhood of the origin. Then,

$$f_0(x_0) - f_0(y) \geq \eta. \tag{41}$$

Therefore, according to Formula (37), we find that

$$\left| f_0(\frac{x_0 + y}{2}) \right| \leq \| f_0 \| \cdot \frac{x_0 + y}{2} \leq 1 - \delta. \tag{42}$$

Then,

$$f_0(x_0) + f_0(y) \leq |f_0(x_0) + f_0(y)| \leq 2(1 - \delta). \tag{43}$$

Furthermore,

$$\eta + 2f_0(y) \leq f_0(x_0) + f_0(y) \leq 2(1 - \delta). \tag{44}$$

The proof requires a consideration of the two cases separately.

Case I. When $f_0(y) \geq 0$, we can set $\delta = 1 - \frac{1}{n}$. Hence, according to Formula (44), we find that

$$\eta + 2f_0(y) \leq \frac{2}{n}. \tag{45}$$

Then,

$$\eta + 2f_0(y) \leq 0 \; as \; n \to \infty, \tag{46}$$

which contradicts $\eta > 0$, $f_0(y) \geq 0$.

Case II. When $-1 \leq f_0(y) < 0$, since

$$f_0(x_0) - f_0(y) \geq \eta, \tag{47}$$

it holds that

$$|f_0(y) + \eta| < 2. \tag{48}$$

Select

$$1 - \frac{|f_0(y) + \eta|}{2} < \delta \leq 1. \tag{49}$$

Moreover, according to Formula (44), we find that

$$|\eta + 2f_0(y)| \leq 2(1 - \delta), \tag{50}$$

which shows that

$$\delta \leq 1 - \frac{|\eta + 2f_0(y)|}{2}, \tag{51}$$

which contradicts Formula (49). Therefore, the assumption is not true. □

Following from Theorems 4 and 7, we will determine a necessary and sufficient condition regarding weakly locally uniformly denting points and conclude the following corollary.

Corollary 1. *Let X be a Banach space. Every $x \in S(X)$ is a weakly locally uniformly denting point of $U(X)$ if and only if X^* is weakly locally uniformly smooth.*

Theorem 9. *If X is a reflexive Fréchet-smooth Banach space, then every point $x^* \in S(X^*)$ is a strong*-denting point of $U(X^*)$.*

Proof of Theorem 9. We prove the contrapositive. Suppose that $x_0^* \in S(X^*)$ is not a *strong**-denting point of $U(X^*)$. Then, there exists a weak neighborhood $U_0^{w^*}$ of the origin such that

$$x_0^* \in \overline{co}^{w^*}(U(X^*) \backslash (x_0^* + U_0^{w^*})). \tag{52}$$

Since X is reflexive, there exist $x_0 \in S(X)$ and $\{x_n^*\} \subset S(X^*)$ such that

$$x_0^*(x_0) = 1, x_n^*(x_0) \to 1. \tag{53}$$

Since X is Fréchet-smooth, according to Lemma 2, there exists $x_1^* \in U(X^*)$, such that $x_n^* \to x_1^*$, and from (53), it follows that

$$x_1^*(x_0) = x_0^*(x_0). \tag{54}$$

Thus, $x_1^* = x_0^*$, i.e., $x_n^* \to x_0^*$. Then, there exists $n_0 \in N$ such that

$$\{x_n^*\}_{n=n_0+1}^{\infty} \subset x_0^* + U(X^*). \tag{55}$$

Consequently,

$$\{x_n^*\}_{n=n_0+1}^{\infty} \not\subset U(X^*) \backslash (x_0^* + U^*) \subseteq \overline{co}^{w^*}(U(X^*) \backslash (x_0^* + U^*)), \tag{56}$$

when $n > n_0$ for any neighborhood U^* of the origin. According to the separation theorem, there exists $y \in S(X)$ such that

$$x_n^*(y) > \sup(\overline{co}^{w^*}(U(X^*) \backslash (x_0^* + U_0^{w^*})))(y). \tag{57}$$

However,

$$x_0^* \in \overline{co}^{w^*}(U(X^*) \backslash (x_0^* + U_0^{w^*})). \tag{58}$$

Then, there exists $r > 0$ such that

$$x_n^*(y) - x_0^*(y) > r, n = n_0 + 1, n_0 + 2, \tag{59}$$

which contradicts $x_n^* \to x_0^*$. Therefore, the assumption is not true. □

Theorem 10. *If X is a smooth Banach space, then the strong*-denting points are norm-dense in $S(X^*)$.*

Proof of Theorem 10. The proof requires us to consider two cases separately.

Case I. If $y^* \in S(X^*)$ is norm-attaining on $U(X)$, there exists $y_0 \in S(X)$ such that $y^*(y_0) = 1$. Select $\{y_n^*\} \subset S(X^*)$, satisfying $y_n^*(y) \to 1$ for every $y \in S(X)$. Since X is smooth, then according to Definition 5, there exists $y_0^* \in S_y$ such that $y_n^* \xrightarrow{w^*} y_0^*$. Then, we obtain

$$y_n^*(y_0) \to y_0^*(y_0). \tag{60}$$

Thus,

$$y_0^*(y_0) = 1 = y^*(y_0). \tag{61}$$

Consequently,

$$y^* = y_0^*, \tag{62}$$

Then,

$$y_n^* \xrightarrow{w^*} y^*. \tag{63}$$

There exists $\{z_n^*\} \subset U(X^*)\backslash(y_0^* + U^{w^*})$ such that

$$z_n^*(y_0) \to y_0^*(y_0), \tag{64}$$

where U^{w^*} is a w^* neighborhood of the origin. Following from the previous proof, we obtain

$$z_n^* \xrightarrow{w^*} y_0^*, \tag{65}$$

which contradicts $\{z_n^*\} \subset U(X^*)\backslash(y_0^* + U^{w^*})$. Therefore, there exists $s > 0$ such that $z_n^*(y_0) + s < y_0^*(y_0)$ for every $\{z_n^*\} \subset U(X^*)\backslash(y_0^* + U^{w^*})$. Let

$$K = U(X^*)\backslash(y_0^* + U^{w^*}), \tag{66}$$

Then,

$$\begin{aligned} y_0^*(y_0) - s \geq & \text{minsup}\{z_n^*(y_0) : z_n^* \in K\} \\ \geq & \sup\{z_n^*(y_0) : z_n^* \in co(K)\} \\ \geq & \sup\left\{z_n^*(y_0) : z_n^* \in \overline{co}^{w^*}(K)\right\}, \end{aligned}$$

which implies

$$y_0^*(y_0) \notin \overline{co}^{w^*}(K), \tag{67}$$

i.e., y_0^* is a *strong**-denting point of $U(X^*)$, since $y^* \in S(X^*)$ is norm-attaining on $U(X)$. According to the Bishop–Phelps theorem, functionals reaching the norm on $U(X)$ are dense in $S(X^*)$.

Case II. If $y_0^* \in S(X^*)$ is not norm-attaining on $U(X)$, then there exists $\eta \in (0, \frac{1}{3})$ such that $\frac{4}{1-3\eta} < \varepsilon$ for every $\varepsilon > 0$, defining a neighborhood of ηy_0^*,

$$\{y^* \in U(X^*) : \|y^* - \eta y_0^*\| < \frac{1-\eta}{2}\}, \tag{68}$$

i.e.,

$$\frac{1-3\eta}{2} < \|y^*\| < \frac{1+\eta}{2}. \tag{69}$$

since

$$\begin{aligned}\left\| \frac{y^*}{\|y^*\|} - y_0^* \right\| &\leq \left\| \frac{y^*}{\|y^*\|} - \frac{\eta y_0^*}{\|y^*\|} \right\| + \left\| \frac{\eta y_0^*}{\|y^*\|} - \frac{y_0^*}{\|y^*\|} \right\| + \left\| \frac{y_0^*}{\|y^*\|} - y_0^* \right\| \\ &< \frac{2}{1-3\eta} \cdot \frac{1-\eta}{2} + \frac{2}{1-3\eta} \cdot (1-\eta) + (\frac{2}{1-3\eta} - 1) \\ &= \frac{4}{1-3\eta} \\ &< \varepsilon.\end{aligned}$$

According to the Bishop–Phelps theorem, there exists $y_1^* \in U(X^*)$, satisfying $\|y_1^* - \eta y_0^*\| < \frac{1-\eta}{2}$ such that y_1^* is norm-attaining on $U(X)$, and then $\frac{y_1^*}{\|y_1^*\|}$ is norm-attaining on $U(X)$, which implies that

$$\left\| \frac{y_1^*}{\|y_1^*\|} - y_0^* \right\| < \varepsilon. \tag{70}$$

Therefore, the *strong**-denting points are norm-dense in $S(X^*)$. □

Example 2. *Let X be the Hilbert space; X is strictly convex, smooth, and reflexive. In fact, all points in $S(X^*)$ are strong*-denting points. Since X is smooth, for every $x \in S(X)$, there exists a unique unit functional $x^* \in S(X^*)$ such that $x^*(x) = 1$. According to the Riesz theorem, we have $x^* = x$. For any $\varepsilon > 0$ and $x^* \in S(X^*)$, consider the neighborhood $B_\varepsilon(x^*)$. Suppose $x^* \in \overline{co}^{w^*}(S(X^*) \backslash B_\epsilon(x^*))$. Then, there exists a convex combination $\sum \lambda_i x_i^*$ with $x_i^* \notin B_\epsilon(x^*)$ converging weakly to x^*. However, weak convergence is equivalent to norm convergence in X. Thus, $\| \sum \lambda_i x_i^* - x^* \| \to 0$, which contradicts $\| x_i^* - x^* \| \geq \varepsilon$. Therefore, $x^* \notin \overline{co}^{w^*}(S(X^*) \backslash B_\epsilon(x^*))$ and all points in $S(X^*)$ are strong*-denting points, and the strong*-denting points are norm-dense in $S(X^*)$.*

4. Discussion

The discussions in our paper concerning these dentabilities indicate that they play a very significant role in some recent trends related to the geometric theory of Banach spaces, and they can further be used to solve the problem of accurately reflecting the shape and geometric structure of unit spheres in Banach space.

5. Conclusions

In this paper, we study three kinds of dentabilities and present important applications of these dentabilities in the geometric theory of Banach spaces. First, the relationship between weak-denting points and extreme points is investigated (1) if Banach space X is separable and then has an Efimov–Stechkin property; (2) if X is weakly sequentially complete and separable, and if dual-space X^* is separable. Second, we characterize strong convex spaces and weakly locally uniformly smooth spaces for (1) if X is a reflexive and has the (H) property and only (2) if X is reflexive. Furthermore, we study the *strong** dentability in Fréchet-smooth Banach spaces. Finally, we prove that the *strong**-denting point is dense in the closed unit sphere $S(X^*)$ of the dual Banach space X^* if Banach space X is smooth.

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