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Filtered Products of Copies of Injective Modules

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Abstract: We study the transfer of injectivity to filtered products of copies of an injective module. This leads to the introduction of a generalized Noetherian condition, the so-called $(\aleph, M)$ -Noetherian rings. We prove that M is $\mathcal{F}$ -injective for every filter F with $\text{cpl}(\mathcal{F}) \geq \aleph$ if and only if R is $(\aleph, M)$ -Noetherian. We also examine the behavior of filtered products of τ -injective torsion-free modules, establishing preservation results under suitable conditions.

Keywords: $\mathcal{F}$ -product; $\aleph$ -product; injectivity; $\aleph$ -Noetherian; $\mathcal{F}$ -injectivity; $\aleph$ -A.C.C.

MSC: 18E15; 18G05; 16E05; 16E65

1. Introduction

One of the central problems in module theory is understanding how properties transfer from a family of modules to their sum, product, or more general constructions. A classical result states that the direct sum of injective modules remains injective if and only if the underlying ring is Noetherian, or that the direct product of a family of flat modules is flat if and only if the ring is coherent [1]. This naturally raises a broader question: to what extent can injectivity be preserved under more general constructions beyond direct sums and direct products?

A significant generalization of these constructions is given by the $\aleph$ -product of modules, where $\aleph$ is an arbitrary infinite cardinal. The study of such products has attracted the attention of researchers, who undertook several studies focusing on the properties of these new subdirect products: Dauns [2,3] studied, one way or another, the transfer of the injectivity of a family of modules to their subdirect product; Dauns and Fuchs studied the $\aleph$ -product of slender modules; Loustaunau researched the transferability of projectivity and flatness to $\aleph$ -products [4], even when $\aleph$ -products are direct summands of the direct products [5].

However, it is well known that subdirect products are particular cases of filtered products, so the same type of problems mentioned here have since acquired a special interest for the case of filtered products. This perspective motivates our central question: Under what conditions does injectivity transfer to filtered products of a given module M ? More precisely, given an injective module M , we aim to identify the conditions on the ring that ensure the injectivity of any filtered product of copies of M .

Our main result establishes that this property characterizes a notable class of rings, leading to a structural condition that extends the classical notion of Noetherianity. This insight motivates the introduction of $(\aleph, M)$ -Noetherian modules, a generalization of Noetherianity that depends on the choice of M .

Building on this foundation, we investigate whether fundamental properties of Noetherian modules carry over to $(\aleph, M)$ -Noetherian modules. In particular, we examine how



Academic Editor: Byungdo Park

Received: 3 March 2025

Revised: 14 April 2025

Accepted: 15 April 2025

Published: 16 April 2025

Citation: Bennis, D.; García Rozas, J.R.; Mbarki, M.; Oyonarte, L. Filtered Products of Copies of Injective Modules. *Axioms* **2025**, *14*, 303. <https://doi.org/10.3390/axioms14040303>

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core aspects of Noetherianity adapt to this broader framework, revealing deeper structural insights in module theory.

Additionally, we explore the behavior of filtered products of torsion-free injective (relative to a hereditary torsion theory τ) modules. This investigation enriches our understanding of their injectivity properties.

This paper is structured as follows. In Section 1, we present the preliminaries, definitions, and fundamental propositions needed for the development of our work.

In Section 2, we establish the definition of $(\aleph, M)$ -Noetherian rings and modules, generalizing the classical $\aleph$ -Noetherian properties to fit our context.

In Section 3, we further investigate the relationship between $\mathcal{F}$ -injectivity and $(\aleph, M)$ -Noetherianity. We define and study $\mathcal{F}$ -injective modules, extending classical notions of $\aleph$ -injectivity. Our goal is to understand how the injectivity properties of a module M relative to a filter affect the structural properties of the ring R . This chapter builds foundational results leading to broader generalizations.

Finally, Section 4 explores the behavior of $\mathcal{F}$ -products of injective torsion-free modules, examining how their properties relate to the overarching $(\aleph, M)$ -Noetherian framework. By analyzing these constructions, we further link our generalization of Noetherianity to broader injective and torsion theories.

Through these developments, our work extends classical algebraic structures, offering new insights and opening avenues for future exploration in module and ring theory.

2. Preliminaries

Throughout this paper, R will denote an associative ring with identity, which is not necessarily commutative. Unless specified otherwise, we use $R\text{-Mod}$ to represent the category of left R -modules. Additionally, we reserve $\mathcal{C}$ to denote a locally finitely generated Grothendieck category.

For any $M \in R\text{-Mod}$, we denote by $\mathcal{A}_r(M, R)$ the set of annihilators of all subsets of M , that is,

$$\mathcal{A}_r(M, R) = \{\text{ann}(X); X \subseteq M\}.$$

Now, let $\mathcal{C}$ be a locally finitely generated Grothendieck category, and let M and N be objects of $\mathcal{C}$. For any subset $Z \subseteq \text{Hom}(M, N)$, we define

$$r(Z) = \bigcap_{f \in Z} \ker(f),$$

which is a subobject of M . The collection of all such subobjects of M is denoted by

$$r(M, N) = \{r(Z); Z \subseteq \text{Hom}(M, N)\}.$$

It is worth noting that in the case of module categories, $r(R, M) = \mathcal{A}_r(M, R)$.

For a family of modules $(M_i)_A$ and an arbitrary cardinal number $\aleph$, the $\aleph$ -product is defined as

$$\prod^{\aleph} M_i = \{x \in \prod M_i; \text{supp}(x) < \aleph\},$$

where $\text{supp}(x)$ stands for the support of x , which is defined as follows:

$$\text{supp}(x) = \{i \in I; x_i \neq 0\}.$$

Recall that a filter in set A is a subset $\mathcal{F} \subseteq \mathcal{P}(A)$ that satisfies the following:

1. $\emptyset \notin \mathcal{F}$ and $A \in \mathcal{F}$.
2. If $I, J \in \mathcal{F}$, then $I \cap J \in \mathcal{F}$.

3. If $I \in \mathcal{F}$ and $I \subseteq J$, then $J \in \mathcal{F}$.

A filtered product relative to $\mathcal{F}$, in short, the $\mathcal{F}$ -product, of a family of modules $(M_i)_{i \in A}$, is

$$\prod^{\mathcal{F}} M_i = \left\{ x \in \prod_A M_i ; A \setminus \text{supp}(x) \in \mathcal{F} \right\}.$$

It is immediately seen that $\prod^{\mathcal{F}} M_i$ is in $R\text{-Mod}$.

Definition 1 ([6], Definition 3.1). A categorical $\mathcal{F}$ -product of a family of left R -modules $(M_i)_{i \in A}$ is defined as a module M , together with a family of morphisms $(\pi_i : M \rightarrow M_i)_{i \in A}$, satisfying the following two properties:

1. For every finitely generated submodule N of M , the set $\{i \in A; \pi_i|_N = 0\}$ is in $\mathcal{F}$.
2. For any R -module O and any family of morphisms, $(f_i : O \rightarrow M_i)_{i \in A}$, satisfying that the set $\{i \in A; f_i|_N = 0\}$ is in $\mathcal{F}$ for every finitely generated submodule N of O ; there is a unique morphism $f : O \rightarrow M$, such that $\pi_i \circ f = f_i$ for all $i \in A$.

It is known that the categorical $\mathcal{F}$ -product always exists (and is unique) in any locally finitely generated Grothendieck category [6]. Moreover, it is easy to see that in $R\text{-Mod}$ both the categorical $\mathcal{F}$ -product and the $\mathcal{F}$ -products are the same construction. Thus, the categorical $\mathcal{F}$ -product is nothing but a new way to understand and work with the $\mathcal{F}$ -product.

It can be observed that $\aleph$ -products are special cases of filtered products. Specifically, if $|A| \geq \aleph$, the filter

$$\mathcal{F}_{\aleph} = \{S \subseteq A; |A \setminus S| < \aleph\}$$

leads to the equivalence

$$\prod^{\mathcal{F}_{\aleph}} M_i = \prod^{\aleph} M_i.$$

However, filtered products are generally more flexible and cannot always be expressed as $\aleph$ -products. For example, consider the filter $\mathcal{F} = \{X \subseteq \mathbb{N} \mid 1 \in X\}$. Then, $x = (0, 1, 1, 1, \dots) \in \prod^{\mathcal{F}} \mathbb{R}$, but there is no cardinal $\aleph$ for which $x \in \prod^{\aleph} \mathbb{R}$, unless, of course, if $\prod^{\aleph} \mathbb{R} = \prod \mathbb{R}$; that is, if $\aleph$ is greater than the cardinality of the index set.

Many properties of subdirect products depend on the assumption that $\aleph$ is regular. A cardinal number $\aleph$ is regular if it cannot be reached by adding cardinal numbers strictly less than $\aleph$, a number of times strictly less than $\aleph$. That is, if $\mathcal{H} = \{\aleph_i \mid \aleph_i < \aleph \text{ is a cardinal number, } i \in A\}$ and $|A| < \aleph$, then

$$\sum_{i \in A} \aleph_i < \aleph.$$

In the context of filters, regularity leads to a similar need for a certain cardinal number called the completion cardinal and denoted as $\text{cpl}(\mathcal{F})$, which we will now define.

A filter $\mathcal{F}$ on a set A is said to be $\aleph$ -complete if for every $S \subseteq \mathcal{F}$ with $|S| \leq \aleph$, $\bigcap_{I \in S} I \in \mathcal{F}$, it is $\aleph$ -incomplete if it is not $\aleph$ -complete, and it is principal if it is generated by a single element, that is, $\exists X \in \mathcal{F}$, such that $\mathcal{F} = \{I \in \mathcal{P}(A); X \subseteq I\}$.

It is clear that a filter on A is principal if and only if it is $\aleph$ -complete for every cardinal number $\aleph$. Thus, for every non-principal filter, there exists a cardinal number for which it is incomplete. Therefore, we can define $\text{cpl}(\mathcal{F})$ to be the smallest cardinal κ for which a non-principal filter $\mathcal{F}$ is not κ -complete.

From now on, $\aleph$ will denote a regular cardinal number and $\aleph^*$ will denote the least ordinal number with cardinality $\aleph$.

After presenting foundational concepts and properties related to the category $R\text{-Mod}$, we recall relevant definitions and properties within the more general framework of locally finitely generated Grothendieck categories.

Definition 2 ([7], Definition 3.2). *An object C in $\mathcal{C}$ satisfies the $\aleph$ ascending chain condition ($\aleph$ -A.C.C.) if every strictly ascending, well-ordered chain of subobjects of C contains less than $\aleph$ terms.*

Proposition 1 ([7], Corollary 3.5). *An object C in $\mathcal{C}$ has the $\aleph$ -A.C.C. if and only if for every subobject C' of C , there exists a cardinal $\alpha < \aleph$ such that C' is α -generated.*

Proposition 2 ([6], Proposition 5.8). *Let $\{G_k; k \in K\}$ be a system of finitely generated generators of $\mathcal{C}$, M an injective object, and $\mathcal{F}$ a filter on a set A with $\text{cpl}(\mathcal{F}) \geq \aleph$. If $r(G_k, M)$ satisfies $\aleph$ -A.C.C. for every $k \in K$, then M is $\mathcal{F}$ -injective.*

3. Definitions and Basic Properties

In this chapter, we generalize the classical concept of $\aleph$ -Noetherian rings through the definition of an $(\aleph, M)$ -Noetherian ring, where the Noetherian condition is adapted relative to the structure of M .

By defining R as an $(\aleph, M)$ -Noetherian ring, we can explore how the structure of M influences the behavior of R . The importance of this new definition lies in its ability to capture deeper structural properties that were previously inaccessible. We provide a more flexible framework that extends classical results and opens up new perspectives in module theory.

The main result, which will be proved later, is the statement that if M is an injective R -module, then every filter $\mathcal{F}$ with $\text{cpl}(\mathcal{F}) \geq \aleph$ M is $\mathcal{F}$ -injective if and only if R is $(\aleph, M)$ -Noetherian. This property highlights the power of the $(\aleph, M)$ -Noetherian concept: it reveals a direct connection between $\mathcal{F}$ -injectivity and the $(\aleph, M)$ -Noetherian properties in a way that extends traditional theory.

Definition 3. *Let M and K be R -modules. The R -module K is said to be $(\aleph, M)$ -Noetherian if for every ascending chain of submodules $(K_i)_A$ of K there exists a cardinal number $\aleph_0 < \aleph$ such that $\text{Hom}(K_{\alpha+1}/K_\alpha, M) = 0$ for all $\alpha > \aleph_0$.*

Example 1. *Every $\aleph$ -Noetherian R -module is $(\aleph, M)$ -Noetherian for every M . However, not every $(\aleph, M)$ -Noetherian R -module is $\aleph$ -Noetherian, as the next example shows.*

Let K be an $\aleph$ -Noetherian R -module. Then, $K^\aleph$ is not an $\aleph$ -Noetherian $R^\aleph$ -module. However, it is $(\aleph, M)$ -Noetherian with $M = (M_i)_{i \leq \aleph^*}$, where

$$M_i = \begin{cases} R & \text{if } i = 1 \\ 0 & \text{if } i \neq 1. \end{cases}$$

When studying $(\aleph, M)$ -Noetherian R -modules, it is important to understand how this property behaves with submodules and quotient modules. The next result highlights the stability of $(\aleph, M)$ -Noetherianity under these module operations.

Proposition 3. *Let M be an R -module and let $0 \rightarrow N \rightarrow K \rightarrow C \rightarrow 0$ be an exact sequence of R -modules. Then, K is $(\aleph, M)$ -Noetherian if and only if N and C are.*

Proof. The necessity is clear.

Assume now that N and K/N are $(\aleph, M)$ -Noetherian R -modules and let $(K_i)_A$ be an ascending chain of submodules of K .

Since $(K_i \cap N)_A$ is an ascending chain of submodules of N , there exists a cardinal number $\aleph_0 < \aleph$ such that $\text{Hom}(K_{\alpha+1} \cap N / K_\alpha \cap N, M) = 0$ for all $\alpha > \aleph_0$.

Since $((K_i + N)/N)_A$ is an ascending chain of submodules of K/N , there exists a cardinal number $\aleph_1 < \aleph$ such that $\text{Hom}((K_{\alpha+1} + N)/N / (K_\alpha + N)/N, M) = 0$ for all $\alpha > \aleph_1$.

Let $\alpha_2 = \max(\aleph_0, \aleph_1)$. We want to prove that for any $\alpha > \alpha_2$, the following sequence is exact.

$$0 \rightarrow \text{Hom}(L, M) \rightarrow \text{Hom}(B, M) \rightarrow \text{Hom}(C, M)$$

where $L = ((K_{\alpha+1} + N)/N) / (K_\alpha + N)/N$, $B = K_{\alpha+1}/K_\alpha$ and $C = (K_{\alpha+1} \cap N) / (K_\alpha \cap N)$. It is clear that the natural mapping

$$\phi_\alpha : K_\alpha / K_\alpha \cap N \rightarrow K_{\alpha+1} / K_{\alpha+1} \cap N$$

is a monomorphism, so the commutativity of the natural diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & K_\alpha \cap N & \longrightarrow & K_\alpha & \longrightarrow & K_\alpha / K_\alpha \cap N \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \phi_\alpha \\ 0 & \longrightarrow & K_{\alpha+1} \cap N & \longrightarrow & K_{\alpha+1} & \longrightarrow & K_{\alpha+1} / K_{\alpha+1} \cap N \longrightarrow 0 \end{array}$$

gives the exact sequence

$$0 \rightarrow (K_{\alpha+1} \cap N) / K_\alpha \cap N \rightarrow K_{\alpha+1} / K_\alpha \rightarrow (K_{\alpha+1} / (K_{\alpha+1} \cap N)) / (K_\alpha / K_\alpha \cap N) \rightarrow 0$$

But $K_\alpha / (K_\alpha \cap N) \cong (K_\alpha + N) / N$. So, applying the functor $\text{Hom}(-, M)$ to the exact sequence above, by [8] (Proposition 11.10), we obtain the exact sequence

$$0 \rightarrow \text{Hom}(L, M) \rightarrow \text{Hom}(B, M) \rightarrow \text{Hom}(C, M).$$

Then, for every $\alpha > \alpha_2$, we have $\text{Hom}((K_{\alpha+1} + N)/N / (K_\alpha + N)/N, M) = 0$ and $\text{Hom}(K_{\alpha+1} \cap N / K_\alpha \cap N, M) = 0$. So, $\text{Hom}(K_{\alpha+1} / K_\alpha, M) = 0$. $\square$

Corollary 1. A finite direct sum of modules is $(\aleph, M)$ -Noetherian if and only if each of the modules is $(\aleph, M)$ -Noetherian.

Building upon our exploration of $(\aleph, M)$ -Noetherian modules, we now turn our attention to $(\aleph, M)$ -Noetherian rings, a crucial restriction of the concepts we have studied.

Definition 4. Let M be an R -module. The ring R is said to be $(\aleph, M)$ -Noetherian if it is $(\aleph, M)$ -Noetherian as an R -module.

Proposition 4. Let M and K be R -modules. The following statements are equivalent:

1. The R -module K is $(\aleph, M)$ -Noetherian;
2. For any indexed family $\{x_\alpha; \alpha \in A\}$ of elements of K , if O_α denotes the submodule of K generated by $\{x_i; i \leq \alpha\}$ for all $\alpha \in A$, there exists a cardinal number $\aleph_0 < \aleph$, such that $\text{Hom}(O_{\alpha+1} / O_\alpha, M) = 0$ for all $\alpha > \aleph_0$.

Proof. 1. $\Rightarrow$ 2. Immediate from the definition.

2. $\Rightarrow$ 1. Let $(K_\alpha)_{\alpha < \aleph^*}$ be an ascending chain of submodules of K and suppose that for every cardinal number $\beta < \aleph^*$ there exists a cardinal number $\alpha > \beta$ such that

$\text{Hom}(K_{\alpha+1}/K_\alpha, M) \neq 0$. Then, for any such α , there is a non-zero homomorphism $f_\alpha \in \text{Hom}(K_{\alpha+1}/K_\alpha, M)$, and then we can find an element $x_{\alpha+1} \in K_{\alpha+1}$, such that $f_\alpha(x_{\alpha+1} + K_\alpha) \neq 0$.

For any such α , consider the element $x_{\alpha+1}$ and for the rest of the $j < \aleph^*$ choose any $x_{j+1} \in K_{j+1}$.

We have then constructed a family of elements $\{x_\alpha; \alpha < \aleph^*\}$ that, by hypotheses, provides a cardinal number $\alpha_0 < \aleph$, such that $\text{Hom}(O_{\alpha+1}/O_\alpha, M) = 0$ for every $\alpha > \alpha_0$.

For this α_0 , we know by the construction of $\{x_\alpha; \alpha < \aleph^*\}$ that there is some $\alpha > \alpha_0$ with $f_\alpha(x_{\alpha+1} + K_\alpha) \neq 0$. For this α , we have the situation

$$\begin{array}{ccccccc} O_\alpha & \longrightarrow & O_{\alpha+1} & \xrightarrow{g} & O_{\alpha+1}/O_\alpha & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow t & & \\ K_\alpha & \longrightarrow & K_{\alpha+1} & \xrightarrow{h} & K_{\alpha+1}/K_\alpha & \longrightarrow & 0. \end{array}$$

Therefore, $f_\alpha \circ t \circ g(x_{\alpha+1}) = f_\alpha \circ h(x_{\alpha+1}) = f_\alpha(x_{\alpha+1} + K_\alpha) \neq 0$ and so $f_\alpha \circ t \neq 0$. This contradicts the fact that $\text{Hom}(O_{\alpha+1}/O_\alpha, M) = 0$. $\square$

In this context, we will show that the $\aleph$ -Noetherianity is a special case of the $(\aleph, M)$ -Noetherianity.

Proposition 5. *Let M be an R -module, such that R is $(\aleph, M)$ -Noetherian. The following assertions hold:*

1. *If N is a submodule of M , then R is $(\aleph, N)$ -Noetherian.*
2. *If M is a cogenerator of $R\text{-Mod}$ then R is $\aleph$ -Noetherian.*

Proof.

1. Clear.
2. Let $\{I_\alpha\}_{\alpha \in A}$ be a chain of ideals of R . Since R is $(\aleph, M)$ -Noetherian, there exists a cardinal number $\aleph_0 < \aleph$ such that $\text{Hom}(I_{\alpha+1}/I_\alpha, M) = 0$ for all $\alpha > \aleph_0$. Since M is a cogenerator $I_{\alpha+1}/I_\alpha = 0$ for all $\alpha > \aleph_0$. This implies that the chain of ideals eventually stabilizes from $\aleph_0 < \aleph$. Therefore, R is $\aleph$ -Noetherian.

$\square$

One may ask which classes $\mathcal{T}$ of $R\text{-Mod}$ satisfy the condition that for any $C \in \mathcal{T}$, R is $(\aleph, C)$ -Noetherian if and only if R is $\aleph$ -Noetherian. We provide a class of modules that meets this criterion.

Lemma 1. *Let M be an R -module. R is $(\aleph, M)$ -Noetherian if and only if it is $(\aleph, M^A)$ -Noetherian for every index set A .*

Proof. Clear from the natural isomorphism.

$$\text{Hom}(I_{\alpha+1}/I_\alpha, M)^A \cong \text{Hom}(I_{\alpha+1}/I_\alpha, M^A).$$

$\square$

The classical result that a finitely generated module over a Noetherian ring is Noetherian highlights the strong relationship between ring and module properties. This foundational result underscores how the Noetherian property of a ring influences its modules. Motivated by this, we generalize the property to explore what happens to finitely generated R -modules when the ring R is $(\aleph, M)$ -Noetherian.

Proposition 6. *Let M be an R -module. If R is an $(\aleph, M)$ -Noetherian ring and K a finitely generated R -module, then K is $(\aleph, M)$ -Noetherian.*

Proof. Since K is finitely generated, there exists an exact sequence:

$$0 \rightarrow A \rightarrow R^n \rightarrow K \rightarrow 0.$$

Since R is $(\aleph, M)$ -Noetherian, by Corollary 1, R^n is $(\aleph, M)$ -Noetherian, and so, by Proposition 3, it follows that K is $(\aleph, M)$ -Noetherian. $\square$

The next result shows that the $(\aleph, M)$ -Noetherian property is preserved under quotient rings, highlighting the robustness of this generalization of $\aleph$ -Noetherian rings.

Proposition 7. *Let M be an R -module and B any ring. If R is $(\aleph, M)$ -Noetherian and $f : R \rightarrow B$ is a surjective ring homomorphism, then B is an $(\aleph, M')$ -Noetherian ring, where $M' = \{x \in M; \ker(f) \cdot x = 0\}$.*

Proof. We see that M' is indeed in $B\text{-Mod}$ with the product $bx = f^{-1}(b)x \in M'$ (where $f^{-1}(b)$ denotes an arbitrary preimage of b). In fact, if $f(a) = f(a')$, then $f(a - a') = 0$ and $(a - a') \in \ker(f)$. Hence, $\forall x \in M', (a - a')x = 0$, so $a'x = ax$.

Now consider $\{I_i\}_{i < \aleph^*}$ an ascending chain of ideals of B . $\{I_i\}_{i < \aleph^*}$ can be seen as a chain of R -modules via f .

Since $B \cong R / \ker(f)$, then $R / \ker(f)$ is $(\aleph, M)$ -Noetherian as an R -module by Proposition 3, so B is $(\aleph, M)$ -Noetherian as an R -module. Then, there exists $\alpha_0 < \aleph$ such that $\text{Hom}_R(I_{\alpha+1}/I_\alpha, M) = 0$ for every $\alpha > \alpha_0$.

Now, for any $\alpha > \alpha_0$, we have $\text{Hom}_B(I_{\alpha+1}/I_\alpha, M') \subset \text{Hom}_R(I_{\alpha+1}/I_\alpha, M)$, which means that $\text{Hom}_B(I_{\alpha+1}/I_\alpha, M') = 0$. $\square$

The final proposition of this chapter illustrates the flexibility and broad applicability of $(\aleph, M)$ -Noetherian concepts in extending properties from subrings to larger rings.

Proposition 8. *Let A be a subring of a ring B and M an A -module. If A is an $(\aleph, M)$ -Noetherian ring and B is a finitely generated A -module, then B is an $(\aleph, \text{Hom}_A(B, M))$ -Noetherian ring.*

Proof. Since A is an $(\aleph, M)$ -Noetherian ring and B is a finitely generated A -module, then by Proposition 6, B is an $(\aleph, M)$ -Noetherian A -module.

Let $\{I_i\}_{i < \aleph^*}$ be an ascending chain of ideals of B . There exists $\alpha_0 < \aleph$ such that $\text{Hom}_A(I_{\alpha+1}/I_\alpha, M) = 0$ for any $\alpha > \alpha_0$.

Suppose there exists $f \in \text{Hom}_B(I_{\beta+1}/I_\beta, \text{Hom}_A(B, M))$ a non-zero morphism, where $\beta > \alpha_0$. Then, there exists $x_0 \in I_{\beta+1}/I_\beta$ such that $f(x_0) = \phi \neq 0$. So, there exists $b \in B$ such that $\phi(b) \neq 0$. Now, we consider the morphism

$$g : I_{\beta+1}/I_\beta \rightarrow M$$

$$x + I_\beta \mapsto f(x)(b)$$

$g \in \text{Hom}_A(I_{\beta+1}/I_\beta, M)$ and $g(x_0) = f(x_0)(b) = \phi(b) \neq 0$. This is a contradiction. Therefore, $\text{Hom}_B(I_{\beta+1}/I_\beta, \text{Hom}_A(B, M)) = 0$. $\square$

4. Filtered Products of Copies of an Injective Module

In this section, we focus on the concept of $\mathcal{F}$ -injective R -modules, relative to a filter $\mathcal{F}$, a generalization of the classical $\aleph$ -injective R -module.

Given an injective R -module M , our primary objective is to explore the relationship between the $(\aleph, M)$ -Noetherian property of the ring R and the $\mathcal{F}$ -injectivity of M .

To build a solid foundation, we first present key definitions and preliminary results that are essential for understanding the more advanced generalizations and theorems introduced later in the chapter. We aim to clarify how the $\mathcal{F}$ -injectivity of an R -module affects the ring R itself.

Definition 5. Let $\mathcal{F}$ be a filter on an index set A . An injective R -module M is said to be $\mathcal{F}$ -injective if any $\mathcal{F}$ -product of copies of M is injective.

The next property generalizes the well-known result where the direct sum commutes with the Hom functor under certain conditions. Here, instead of a direct sum, we use a filtered product with respect to a filter $\mathcal{F}$, allowing us to extend the Hom-isomorphism to a more general setting. This result is important because it provides a way to work with infinite families of modules while keeping the properties of Hom functor, making it useful for studying more advanced structures in module theory.

Proposition 9. Let $\mathcal{F}$ be a filter on an index set A , M an $\mathcal{F}$ -injective R -module, and I an ideal of R . The natural morphism

$$\varphi : \prod^{\mathcal{F}} \text{Hom}(I, M) \rightarrow \text{Hom}(I, \prod^{\mathcal{F}} M)$$

is an isomorphism.

Proof. It is clear that φ is a monomorphism. Let us prove that it is an epimorphism. Let f be a morphism in $\text{Hom}(I, \prod^{\mathcal{F}} M)$.

Since $\prod^{\mathcal{F}} M$ is injective, by Baer's criterion, there exists $y = (y)_{i \in A} \in \prod^{\mathcal{F}} M$ such that $f(x) = yx$ for all $x \in I$. Since $(y)_{i \in A} \in \prod^{\mathcal{F}} M$, $\{i \in A; y_i = 0\} \in \mathcal{F}$.

We know that $\{i \in A; y_i = 0\} \subset \{i \in A; \pi_i \circ f = 0\}$. It follows that $\{i \in A; \pi_i \circ f = 0\} \in \mathcal{F}$. Since $(\pi_i \circ f)_{i \in A} \in \prod^{\mathcal{F}} \text{Hom}(I, M_i)$ and $\varphi((\pi_i \circ f)_A) = f$, φ is an epimorphism. $\square$

We observe that, in the particular case of the filter $\mathcal{F}_{\aleph}$ (see Section 2), we have $\aleph \leq \text{cpl}(\mathcal{F}_{\aleph})$. In fact, if $S \subseteq \mathcal{F}_{\aleph}$ is any subset with cardinality less than $\aleph$, then

$$|A \setminus \cap_{X \in S} X| \leq \sum_{X \in S} |A \setminus X| < \aleph,$$

where the last inequality holds because $|S| < \aleph$, $|A \setminus X| < \aleph$ for all $X \in S$, and $\aleph$ is regular. This implies that $\cap_{X \in S} X \in \mathcal{F}_{\aleph}$, and consequently, $\aleph \leq \text{cpl}(\mathcal{F}_{\aleph})$.

With this preliminary observation, we now present the key theorem of this section, which inspired the definition of the $(\aleph, M)$ -Noetherian rings.

Theorem 1. Let M be an injective R -module. The following statements are equivalent:

1. For every filter $\mathcal{F}$ with $\text{cpl}(\mathcal{F}) \geq \aleph$, M is $\mathcal{F}$ -injective.
2. R is $(\aleph, M)$ -Noetherian.
3. Whenever we have a sequence $\{x_i\}_{i < \aleph^*}$ of elements of R , there exists a cardinal number $\aleph_0 < \aleph$, such that

$$\text{Hom}(\langle \{x_i\}_{i \leq \alpha+1} \rangle / \langle \{x_i\}_{i \leq \alpha} \rangle, M) = 0$$

for all $\alpha > \aleph_0$.

4. R satisfies the $\aleph$ -A.C.C. on ideals inside $\mathcal{A}_r(M, R)$.

Proof. $1 \Rightarrow 2$. Assume that M is $\mathcal{F}$ -injective for every filter $\mathcal{F}$ with $\text{cpl}(\mathcal{F}) \geq \aleph$. Let $\{I_i\}_{i < \aleph^*}$ be an ascending chain of ideals of R .

Suppose that for each index $\beta < \aleph^*$, there exists an $\alpha > \beta$ and a non-zero morphism $f_\alpha \in \text{Hom}(I_{\alpha+1}/I_\alpha, M)$, and consider the morphism $f = (f_\alpha)_{\alpha < \aleph^*} \in \prod_{\alpha < \aleph^*} \text{Hom}(I_{\alpha+1}/I_\alpha, M)$. Our goal is to show that there exists some $\alpha_0 < \aleph$ such that $f_\alpha = 0$ for all $\alpha > \alpha_0$, contradicting the original assumption.

Let $I = \bigcup_{i < \aleph^*} I_i$. For every $x \in I$, there exists $\alpha_x < \aleph^*$, such that $x \in I_{\alpha_x}$; then, $f_\alpha(\bar{x}) = 0$ for all $\alpha > \alpha_x$, where $\bar{x}$ denotes the coset of x in $I_{\alpha+1}/I_\alpha$. Consequently, $(f_\alpha(\bar{x}))_{\alpha < \aleph^*} \in \prod_{\alpha < \aleph^*} M$ for any $x \in I$. Therefore, $(f_\alpha)_{\alpha < \aleph^*} \in \text{Hom}(I, \prod_{\alpha < \aleph^*} M)$.

Since M is $\mathcal{F}_\aleph$ -injective, by Proposition 9 $f \in \prod_{\alpha < \aleph^*} \text{Hom}(I_{\alpha+1}/I_\alpha, M)$ up to an isomorphism. This implies that there exists $\alpha_0 < \aleph$ such that $f_\alpha = 0$ for all $\alpha > \alpha_0$, contradicting the original choice of f_α . Therefore, R is $(\aleph, M)$ -Noetherian.

$2 \Leftrightarrow 3$. Proposition 4.

$2 \Rightarrow 4$. Assume that R is $(\aleph, M)$ -Noetherian. Let $\{I_\alpha\}_{\alpha < \aleph^*}$ be an ascending chain of ideals holding inside $\mathcal{A}_r(M, R)$. There exists $\alpha_0 < \aleph$ such that $\text{Hom}(I_{\alpha+1}/I_\alpha, M) = 0$ for every $\alpha > \alpha_0$.

Consider an index $\alpha > \alpha_0$, such that $I_\alpha \subsetneq I_{\alpha+1}$, so there exists $r \in I_{\alpha+1} \setminus I_\alpha$. But $I_\alpha = \text{ann}(S_\alpha)$ for some $S_\alpha \subseteq M$, so there is some $u \in S_\alpha$ such that $r \cdot u \neq 0$.

Define the morphism

$$f : I_{\alpha+1}/I_\alpha \rightarrow M$$

$$x + I_\alpha \mapsto xm$$

By construction, $f \neq 0$, since $f(r + I_\alpha) = rm \neq 0$, so $f \neq 0$. This is a contradiction. Therefore, $\{I_\alpha\}_{\alpha < \aleph^*}$ stabilizes at α_0 .

$4 \Rightarrow 1$. Proposition 2 $\square$

5. Filtered Product Torsion-Free Injective Modules

In this chapter, we investigate the behavior of the $\mathcal{F}$ -product torsion-free injective R -modules. First, we recall some basic notions of torsion theory to provide the necessary foundation for our study.

A hereditary torsion theory in $R\text{-Mod}$ is a pair $\tau = (\mathcal{T}, \mathcal{F})$ where $\mathcal{T}$ is a class of left R -modules that is closed under submodules, homomorphic images, direct sums, and module extensions, and $\mathcal{F}$ comprises all R -modules N , such that $\text{Hom}(M, N) = 0$ for all $M \in \mathcal{T}$.

The modules in $\mathcal{T}$ are called τ -torsion, and those in $\mathcal{F}$ are called τ -torsion free. For each R -module, there is the largest τ -torsion submodule of M , which we denote by M_τ . For further details on this module, see Chapter IX of [9].

The family $\mathcal{L}(\tau) = \{I \subset {}_R R; R/I \in \mathcal{T}\}$ of left ideals of R is called a Gabriel filter on R . And the lattice $\text{Sat}_\tau(R)$ consists of all τ -saturated ideals of R ; that is, $\text{Sat}_\tau(R) = \{I \subset {}_R R; R/I \in \mathcal{F}\}$.

Definition 6 ([9]). An R -module M is said to be τ -injective (resp. τ -closed) if the canonical homomorphisms

$$\phi_I : \text{Hom}(R, M) \rightarrow \text{Hom}(I, M)$$

are epimorphisms (resp. isomorphisms) for all $I \in \mathcal{L}(\tau)$.

Thus, M is τ -closed if and only if M is both τ -torsion free and τ -injective. The category $(R, \tau)\text{-Mod}$ denotes the full subcategory of $R\text{-Mod}$ consisting of τ -closed modules, and it is

called the quotient category of $R\text{-Mod}$ with respect to τ . In practice, we will usually not make any distinction between this category and the category of $R_\tau\text{-Mod}$ of the form M_τ .

In what follows, τ will denote a hereditary torsion theory. We begin by stating the following corollary.

Corollary 2 ([10], Lemmas 2.1 and 2.2). *If R_τ is α -generated, for some $\alpha < \aleph$, then $\mathcal{L}(\tau)$ has a basis consisting of β_i -generated ideals of R , where each $\beta_i < \aleph$.*

The next result examines the behavior of the $\mathcal{F}$ -products of τ -injective R -modules when the ring R_τ is α -generated for some $\alpha < \aleph$. It shows that in this setting, the $\mathcal{F}$ -product of τ -injective R -modules retains the τ -injective property.

Proposition 10. *Let α be a cardinal number, such that $\alpha < \aleph$. If R_τ is α -generated as an (R, τ) -module, then for every filter $\mathcal{F}$ with $\text{cpl}(\mathcal{F}) \geq \aleph$, the $\mathcal{F}$ -product of τ -injective R -modules is τ -injective.*

Proof. Assume that R_τ is α -generated for some cardinal number $\alpha < \aleph$.

Let I be an element of $\mathcal{L}(\tau)$, $(M_i)_A$, a family of τ -injective modules, and f a morphism in $\text{Hom}(I, \prod^{\mathcal{F}} M_i)$.

Since R_τ is α -generated, by Corollary 2, there exists a β -generated ideal $L \in \mathcal{L}(\tau)$ with $\beta < \alpha$, such that $L \subseteq I$. So, we have the situation

$$\begin{array}{ccccc} L & \xrightarrow{\quad} & I & \xrightarrow{\quad} & R \\ & \searrow f & \downarrow f & & \\ & & \prod^{\mathcal{F}} M_i & & \end{array}$$

Suppose $L = \langle l_i; i \in S \rangle$ with $|S| \leq \beta$. Then, $f(L) = \langle f(l_i) : i \in S \rangle$.

For every $i \in S$, there exists $J_i \in \mathcal{F}$, such that $f(l_i) \in \prod_{A \setminus J_i} M_i$.

Now, letting $K = \bigcap_{i \in S} J_i$ makes $f(L) \subset \prod_{A \setminus K} M_i$. Since $\text{cpl}(\mathcal{F}) \geq \aleph > |S|$, $K \in \mathcal{F}$, we obtain $\prod_{A \setminus K} M_i \subset \prod^{\mathcal{F}} M_i$.

Our original diagram can be written as the following form:

$$\begin{array}{ccc} L & \xrightarrow{\quad} & R \\ \downarrow f & \searrow f & \\ \prod^{\mathcal{F}} M_i & & f(L) \\ & \swarrow & \searrow \\ & \prod_{A \setminus K} M_i & \end{array}$$

$k' : \prod_{A \setminus K} M_i \rightarrow \prod^{\mathcal{F}} M_i$

It is clear that the two triangles are commutative.

For every $i \in A \setminus K$, let $p_i : \prod_{A \setminus K} M_i \rightarrow M_i$ be the canonical projection. Then, by the τ -injectivity of M_i , each diagram

$$\begin{array}{ccc} L & \xrightarrow{\quad} & R \\ \downarrow f & & \downarrow h_i \\ \prod_{A \setminus K} M_i & \xrightarrow{p_i} & M_i \end{array}$$

can be commutatively completed by a morphism h_i . But those h_i induce a unique $h : R \rightarrow \prod_{A \setminus K} M_i$ such that $p_i \circ h = h_i$ for every $i \in A$. It is then clear that the upper triangle of our diagram

$$\begin{array}{ccc}
 L & \xrightarrow{\quad} & R \\
 \downarrow f & \searrow f & \downarrow h \\
 & f(L) & \\
 \downarrow & \swarrow & \downarrow \\
 \prod^{\mathcal{F}} M_i & \xleftarrow{k'} & \prod_{A \setminus K} M_i
 \end{array}$$

commutes, and so the outer square is commutative too. Therefore, we have the commutativity of the outer triangle in the diagram

$$\begin{array}{ccccc}
 L & \xrightarrow{\quad} & I & \xrightarrow{\quad} & R \\
 & \searrow f & \downarrow f & \swarrow h & \\
 & & \prod^{\mathcal{F}} M_i & &
 \end{array}$$

A simple application of Zorn's Lemma shows that

$$\begin{array}{ccc}
 I & \xrightarrow{\quad} & R \\
 \downarrow f & \searrow & \downarrow \\
 \prod^{\mathcal{F}} M_i & &
 \end{array}$$

is also commutative and so that $\prod^{\mathcal{F}} M_i$ is τ -injective. $\square$

After establishing that R_{τ} being α -generated ensures that the $\mathcal{F}$ -product preserves τ -injectivity, the next proposition demonstrates that, under the same condition, the $\mathcal{F}$ -products also preserves τ -closure.

Proposition 11. *Let α be a cardinal number such that $\alpha < \aleph$. If R_{τ} is α -generated as an (R, τ) -module, then for every filter $\mathcal{F}$ with $\text{cpl}(\mathcal{F}) \geq \aleph$, the $\mathcal{F}$ -product of τ -closed R -modules is τ -closed.*

Proof. We assume that $(M_i)_A$ is a family of τ -closed R -modules. By Proposition 10, $\prod^{\mathcal{F}} M_i$ is τ -injective.

Since $\prod^{\mathcal{F}} M_i$ is a submodule of $\prod_{i \in A} M_i$, and the latter is a τ -torsion-free R -module, it follows that $\prod^{\mathcal{F}} M_i$ is τ -torsion free. $\square$

We recall the following definition.

Definition 7 ([10], Definition 2). *A ring R is said to satisfy $(\aleph, \tau)$ -A.C.C. if R_{τ} satisfies $\aleph$ -A.C.C. as an (R, τ) -module.*

As a direct consequence of Proposition 11, we obtain the following corollary.

Corollary 3. *If R satisfies $(\aleph, \tau)$ -A.C.C., then for every filter $\mathcal{F}$ with $\text{cpl}(\mathcal{F}) \geq \aleph$, the $\mathcal{F}$ -product of τ -closed R -module is τ -closed.*

Proof. Since R_{τ} satisfies $(\aleph, \tau)$ -A.C.C., it is σ -generated for some $\sigma \leq \aleph$ (by [7], Proposition 3.4). Then, Proposition 11 gives the result. $\square$

Proposition 12 ([9], Chapter IX, Proposition 4.6). *Let E be an injective R -module cogenerating the torsion theory τ . Then, $\text{Sat}_{\tau}(R) = \{\text{ann}(S); S \subseteq E\}$.*

The following is the main theorem of this section. It establishes several important equivalences, which help to clarify the key connections between different concepts in the theory.

Theorem 2. *Let E be an injective cogenerator of τ . The following statements are equivalent:*

1. R satisfies $(\aleph, \tau)$ -A.C.C.
2. $\text{Sat}_\tau(R)$ satisfies $\aleph$ -A.C.C.
3. R satisfies $\aleph$ -A.C.C. on ideals inside $\mathcal{A}_r(E, R)$.
4. For every filter $\mathcal{F}$ with $\text{cpl}(\mathcal{F}) \geq \aleph$, E is $\mathcal{F}$ -injective.
5. For every filter $\mathcal{F}$ with $\text{cpl}(\mathcal{F}) \geq \aleph$, the $\mathcal{F}$ -product of the τ -torsion-free injective R -modules is injective.

Proof. $1 \Leftrightarrow 2$. R satisfies $(\aleph, \tau)$ -A.C.C., which means that R_τ verifies $\aleph$ -A.C.C., and the lattice of submodules of R_τ is isomorphic to $\text{Sat}_\tau(R)$.

$2 \Leftrightarrow 3$. By Proposition 12, $\text{Sat}_\tau(R) = \mathcal{A}_r(E, R)$.

$3 \Leftrightarrow 4$. Theorem 1.

$1 \Leftrightarrow 5$. Let $(M_i)_A$ be a family of τ -torsion-free injective R -modules. By Proposition 11, $\prod^\mathcal{F} M_i$ is τ -closed since R_τ verifies that $\aleph$ -A.C.C. $\prod^\mathcal{F} M_i$ is injective in the category $(R, \tau)\text{-Mod}$. Moreover, since $\prod^\mathcal{F} M_i$ is τ -closed, it is injective in $R\text{-Mod}$.

$5 \Rightarrow 4$. E is τ -torsion-free injective. $\square$

We conclude the paper by establishing an equivalence between two different notions derived from the results developed throughout this work.

Corollary 4. *Let E be an injective cogenerator of τ . The following statements are equivalent:*

1. R satisfies $(\aleph, \tau)$ -A.C.C.
2. R is $(\aleph, E)$ -Noetherian.

Proof. $1 \Leftrightarrow 2$. By Proposition 1, R is $(\aleph, E)$ -Noetherian if and only if R satisfies $\aleph$ -A.C.C. on ideals inside $\mathcal{A}_r(E, R)$. Through Theorem 2, this is equivalent to R satisfying $(\aleph, \tau)$ -A.C.C. $\square$

6. Conclusions

This work builds a clear framework to understand how the property of injectivity is maintained when developing filtered products of R -modules. We introduce a new concept called $(\aleph, M)$ -Noetherian rings. These rings extend the classic idea of Noetherian rings by including a regular cardinal $\aleph$ and a specific R -module M in their definition. Our main finding is that an injective module M is $\mathcal{F}$ -injective for every filter $\mathcal{F}$ ($\text{cpl}(\mathcal{F}) \geq \aleph$) if and only if the ring R is $(\aleph, M)$ -Noetherian. This result not only confirms the known results about direct sums and subdirect products in Noetherian rings but also applies to more general cases involving filtered products.

In addition, we show that under certain conditions on R_τ , the filtered products of τ -injective torsion-free modules continue to have the properties of τ -injectivity and τ -closure. This links torsion theory with our generalized Noetherian concept and offers new methods for studying how modules behave.

Overall, our work combines and extends classical results into one unified framework. This has important implications for future research in module and ring theory and lays the foundation for new applications involving filtered constructions and torsion theories.

These results broaden the classical understanding of injectivity and Noetherianity and open avenues for further research in module theory and torsion theories, particularly in the study of categorical and filtered constructions.

Author Contributions: All authors have contributed equally to the development of the paper. All authors have read and agreed to the published version of the manuscript

Funding: This research received no external funding.

Data Availability Statement: The data supporting the findings of this study are available within the article.

Conflicts of Interest: The authors declare no conflicts of interest.

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