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New Inversion Formulae for the Widder–Lambert and Stieltjes–Poisson Transforms

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Abstract: This paper establishes explicit inversion formulae for the Widder–Lambert transform and the Stieltjes–Poisson transform, extending their applicability to function spaces and compactly supported distributions. Utilizing a generalized Lambert transform, the study provides a unified framework for inversion, enhancing the theoretical understanding of these integral transforms. Additionally, a specific function class satisfying Salem’s equivalence to the Riemann hypothesis is identified, further broadening the analytical scope of the results within generalized function spaces.

Keywords: Widder–Lambert transform; Stieltjes–Poisson transform; inversion formulae; generalized Lambert transform; distributions of compact support; Salem’s equivalence; Riemann hypothesis

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1. Introduction and Preliminaries

Integral transforms constitute a fundamental class of analytical techniques in mathematics, playing a pivotal role in the resolution of differential equations, signal processing, harmonic analysis, and various applications in the applied sciences. Their capacity to reformulate intricate functional relationships into more tractable representations has contributed to their extensive utilization across multiple mathematical disciplines. Among the numerous integral transforms investigated, the Lambert transform and the Stieltjes–Poisson transform have garnered significant attention due to their profound connections with number theory, potential theory, and functional analysis. The Lambert transform naturally emerges in the context of Lambert series and Dirichlet series, whereas the Stieltjes–Poisson transform encompasses and extends several classical transforms, including the Stieltjes and Poisson transforms, thereby enhancing its applicability in asymptotic analysis and summability theory.

Despite their broad range of applications, the explicit derivation of inversion formulae for these transforms remains an area of active research, with considerable efforts directed toward extending their validity to more general function spaces and distributional settings. This paper seeks to unify and extend these integral transforms by establishing novel inversion formulae and investigating their distributional extensions. By formulating a rigorous analytical framework, we provide deeper insights into their structural properties and functional characteristics, thereby broadening their theoretical scope. Furthermore, the results obtained contribute to the advancement of integral transform theory

and its applications in number-theoretic transformations, spectral analysis, and the study of special functions.

A Lambert series is a power series of the form

$$\sum_{k=1}^{\infty} \alpha_k \frac{x^k}{1-x^k}, \quad |x| < 1, \quad \alpha_k \in \mathbb{C}. \quad (1)$$

which generalizes classical number-theoretic identities. A fundamental example includes the following expressions:

$$\sum_{k=1}^{\infty} \frac{x^k}{1-x^k} = \sum_{k=1}^{\infty} \tau(k) x^k, \quad \sum_{k=1}^{\infty} k \frac{x^k}{1-x^k} = \sum_{k=1}^{\infty} \sigma(k) x^k, \quad |x| < 1,$$

where $\sigma(k)$ and $\tau(k)$ denote, respectively, the sum and the count of divisors of k . A significant extension of this framework is the Lambert transform, introduced as a generalization of (3).

The study of the Lambert transform and its generalizations has been extensively explored, particularly in relation to inversion formulae and their applications in analytic number theory. A foundational contribution was made by Widder [1], who formulated an inversion technique for the Lambert transform. Subsequent works by Carrasco [2] and Pennington [3] provided alternative approaches to inversion, while Goldberg [4] extended these results to a broader class of generalized Lambert transforms.

Further advancements were made by Miller [5], who investigated the convergence properties of the Lambert transform and established an inversion formula. Ferreira and López [6] analyzed the asymptotic behaviour of the transform concerning variations in the transformation parameter. Raina and Nahar [7] studied a family of functions associated with the Hurwitz zeta function and the Lambert transform, elucidating key analytical structures. Goyal and Laddha [8] explored connections between generalized Riemann zeta functions and the Lambert transform, while Raina and Srivastava [9] contributed additional insights into properties of zeta functions.

More recent investigations have extended classical results to broader functional spaces. Negrín and Roopkumar [10] examined Lambert-type transforms within the framework of integrable Boehmians, enhancing their scope of applicability. Maan, Negrín, and González [11] further expanded this study by analyzing the behaviour of the transform in both Lebesgue and Boehmian spaces. González and Negrín [12] derived inversion formulae for a Lambert-type transform, establishing links to Salem's equivalence to the Riemann hypothesis. This work was later extended by the same authors [13], who applied Salem's equivalence to develop inversion methods for a Widder–Lambert-type transform, providing a novel approach to the Riemann hypothesis. Their latest collaboration with Maan [14] further refines these results by employing Widder–Lambert transforms in the context of Salem's equivalence, reinforcing the deep connections between these transformations and fundamental problems in number theory.

Additionally, Yakubovich [15] provided significant contributions by investigating integral and series transformations through the lens of Ramanujan's identities and their connections to Salem's equivalence to the Riemann hypothesis. This study highlighted the interplay between integral transforms and number-theoretic conjectures, further enriching the theoretical framework of transform methods.

These references collectively form the foundation for our research, which introduces novel inversion formulae for the Widder–Lambert and Stieltjes–Poisson transforms. The results presented extend existing methodologies and explore their implications in number theory and functional analysis.

The Riemann hypothesis remains one of the most profound open problems in mathematics, giving rise to numerous equivalent formulations [15,16]. Notably, in 1953, the Greek mathematician Raphaël Salem established a crucial condition concerning an integral equation, as documented in [17]. This perspective allows for its reformulation within the context of integral transforms. In [18], explicit solutions for the non-homogeneous case of Salem's integral equation are provided, which are relevant for extending the discussion on integral transforms.

The Riemann zeta function $\zeta(s)$ is given by

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \quad \Re(s) > 1.$$

The Dirichlet eta function $\eta(s)$ is expressed as follows:

$$\eta(s) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s}, \quad \Re(s) > 0.$$

which satisfies the relation

$$\eta(s) = (1 - 2^{1-s})\zeta(s), \quad \Re(s) > 1,$$

thereby extending $\zeta(s)$ into the strip $0 < \Re(s) < 1$. It is well established that $\zeta(s)$ and $\eta(s)$ have no zeros for $\Re(s) > 1$. A fundamental equivalence states that the absence of zeros of $\zeta(s)$ within $0 < \Re(s) < 1$ except at $\Re(s) = \frac{1}{2}$ provides an alternative formulation of the Riemann hypothesis.

Motivated by these integral representations, this paper considers the Widder–Lambert integral transform, defined for $\delta > 0$ by [13]:

$$(\mathcal{L}_\delta[f])(x) = \int_0^\infty \frac{t^{\delta-1}}{e^{xt} + 1} f(t) dt, \quad x > 0, \quad (2)$$

where f is a complex-valued function ensuring convergence of the integral.

Let $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$. For a suitably well-behaved function f , the Stieltjes–Poisson transform is defined as

$$F_{\alpha,\beta}(x) = \int_0^\infty f(t) \frac{t^\beta}{x^\alpha + t^\alpha} dt, \quad x > 0. \quad (3)$$

This transformation encompasses several classical integral transforms as particular cases. Notably, when $\alpha = 1$, $\beta = 0$, Equation (3) reduces to the well-known Stieltjes transform [19–25]. Similarly, setting $\alpha = 2$, $\beta = 1$ yields the Poisson transform, also referred to as the Widder potential transform [15,26]. These transforms have been extensively utilized in complex analysis, potential theory, and functional analysis, with applications ranging from probability theory to mathematical physics.

Despite the fundamental role of these transforms, explicit inversion formulae remain an area of ongoing research. In this paper, we address this gap by deriving new inversion formulae for the Stieltjes–Poisson transform, thereby contributing to the broader theory of integral transforms. Furthermore, we extend these results to distributions of compact support, ensuring the applicability of the inversion process in a generalized distributional framework.

To rigorously analyze this transform in the space of distributions, let K be a compact subset of $\mathbb{R}_+ = (0, \infty)$. The space $\mathcal{D}_K(\mathbb{R}_+)$ consists of all smooth functions on $\mathbb{R}_+$ with compact support contained in K (see [27,28]).

For each non-negative integer n , define the seminorm:

$$\gamma_n(\phi) = \max_{t \in \mathbb{R}_+} |D_t^n \phi(t)|, \quad \phi \in \mathcal{D}_K(\mathbb{R}_+), \quad n \in \mathbb{N} \cup \{0\},$$

where D_t^n denotes the n -th derivative. This family of seminorms endows $\mathcal{D}_K(\mathbb{R}_+)$ with a Fréchet space topology. The space $\mathcal{D}(\mathbb{R}_+)$ is the union of all such spaces over compact sets and its dual $\mathcal{D}'(\mathbb{R}_+)$ constitutes the space of distributions.

The space $\mathcal{E}(\mathbb{R}_+)$ consists of all infinitely differentiable functions on $\mathbb{R}_+$, equipped with seminorms:

$$\gamma_{K,n}(\phi) = \max_{t \in K} |D_t^n \phi(t)|, \quad \phi \in \mathcal{E}(\mathbb{R}_+),$$

for all $n \in \mathbb{N} \cup \{0\}$ and compact sets K . This space also forms a Fréchet space. Its dual, $\mathcal{E}'(\mathbb{R}_+)$, represents the space of distributions with compact support. It is shown in [29] that the existence theory of integral equations in the space of distributions can be systematically studied, which contributes to the general theory of integral equations.

The paper is organized as follows: Section 1 introduces the Widder–Lambert transform (2) and the Stieltjes–Poisson transform (3), highlighting their significance and fundamental properties. Section 2 establishes an inversion formula for the Widder–Lambert transform (2) in the context of conventional functions, extends it to distributions with compact support, and explores Salem’s equivalence to the Riemann hypothesis within the framework of regular distributions. Section 3 derives an inversion formula for the Stieltjes–Poisson transform (3) using a generalized Lambert transform [4] and extends it to distributions of compact support. Section 4 presents concluding remarks and final observations on both transforms, summarizing key results and discussing potential future directions.

2. The Widder–Lambert Transform

In this section, we investigate various inversion formulae for the Widder–Lambert transform, focusing on its extension to different function spaces and distributional settings. We begin by considering an inversion formula for a class of functions and later generalize the results to compactly supported distributions.

2.1. An Inversion Formula for the Widder–Lambert Transform over a Class of Functions

We first derive an inversion formula for the Widder–Lambert transform over a class of sufficiently regular functions. This result provides a fundamental framework for establishing the injectivity of the transform in function spaces.

In Ref. [14] [Theorem 3] it was established that

Theorem 1. Set $\delta > 0$. Assume

$$\int_0^\infty f(t) \frac{t^{\delta-1}}{e^{xt} - 1} dt \quad (4)$$

converges for some $x > 0$, and

$$\int_0^1 |f(t) t^{\delta-2} \log t| dt < \infty. \quad (5)$$

Then

$$\lim_{p \rightarrow \infty} \frac{(-1)^p}{p!} \left(\frac{p}{t}\right)^{p+1} \sum_{n=1}^{\infty} \mu(n) n^p (\mathcal{L}_\delta[f])^{(p)} \left(\frac{np}{t}\right) = \left[f(t) - f\left(\frac{t}{2}\right) 2^{1-\delta} \right] t^{\delta-1}, \quad (6)$$

almost everywhere on $\mathbb{R}_+$, where $\mathcal{L}_\delta[f]$ is given by (2). Here, $\mu(n)$ numbers are the Möbius numbers, defined as $\mu(1) = 1$, $\mu(n) = (-1)^s$ if n is the product of s distinct primes, and $\mu(n) = 0$ if n is divisible by a square and $(\mathcal{L}_\delta[f])^{(p)}(x)$ denotes the conventional p -th derivative of $(\mathcal{L}_\delta[f])(x)$ with respect to its argument x .

Remark 1. Observe that when the integral (4) converges for some $x_0 > 0$, then the integral (4) converges for all $x \geq x_0$ (cf. [4] [Section 5]). Now, using the relation

$$\frac{1}{e^x + 1} = \frac{1}{e^x - 1} - \frac{2}{e^{2x} - 1}, \quad x > 0, \quad (7)$$

then

$$(\mathcal{L}_\delta[f])(x) = \int_0^\infty f(t) \frac{t^{\delta-1}}{e^{xt} - 1} dt - 2 \int_0^\infty f(t) \frac{t^{\delta-1}}{e^{2xt} - 1} dt,$$

converges for all $x \geq x_0$ when the integral (4) converges for $x = x_0$.

Thus the left hand side of Equation (6) has sense when p tends to infinity.

Now, we prove the next result.

Theorem 2 (Inversion formula). Assuming the hypothesis of Theorem 1 and being f such that $\lim_{t \rightarrow 0^+} [t^{\delta-1} f(t)] = 0$, then

$$\sum_{k=0}^{\infty} \left[\lim_{p \rightarrow \infty} \frac{(-1)^p}{p!} \left(\frac{p2^k}{t} \right)^{p+1} \sum_{n=1}^{\infty} \mu(n) n^p (\mathcal{L}_\delta[f])^{(p)} \left(\frac{np2^k}{t} \right) \right] = f(t) t^{\delta-1}, \quad (8)$$

almost everywhere on $\mathbb{R}_+$.

Proof. From Theorem 1 above one has

$$\begin{aligned} & \sum_{k=0}^{\infty} \left[\lim_{p \rightarrow \infty} \frac{(-1)^p}{p!} \left(\frac{p2^k}{t} \right)^{p+1} \sum_{n=1}^{\infty} \mu(n) n^p (\mathcal{L}_\delta[f])^{(p)} \left(\frac{np2^k}{t} \right) \right] \\ &= \sum_{k=0}^{\infty} \left[f\left(\frac{t}{2^k}\right) - f\left(\frac{t}{2^{k+1}}\right) 2^{1-\delta} \right] \frac{t^{\delta-1}}{2^{k(\delta-1)}} \\ &= \sum_{k=0}^{\infty} \left[f\left(\frac{t}{2^k}\right) \frac{t^{\delta-1}}{2^{k(\delta-1)}} - f\left(\frac{t}{2^{k+1}}\right) \frac{t^{\delta-1}}{2^{(k+1)(\delta-1)}} \right] \\ &= \lim_{N \rightarrow \infty} \sum_{k=0}^N \left[f\left(\frac{t}{2^k}\right) \frac{t^{\delta-1}}{2^{k(\delta-1)}} - f\left(\frac{t}{2^{k+1}}\right) \frac{t^{\delta-1}}{2^{(k+1)(\delta-1)}} \right] \\ &= \lim_{N \rightarrow \infty} \left[f(t) t^{\delta-1} - f\left(\frac{t}{2^{N+1}}\right) \left(\frac{t}{2^{N+1}}\right)^{\delta-1} \right] \end{aligned}$$

almost everywhere on $\mathbb{R}_+$.

Since from hypothesis one has

$$\lim_{N \rightarrow \infty} \left[\left(\frac{t}{2^{N+1}}\right)^{\delta-1} f\left(\frac{t}{2^{N+1}}\right) \right] = 0,$$

and then the inversion formula (8) holds. $\square$

From this result, one readily obtains the following:

Corollary 1 (Injectivity). *Assuming the hypothesis of Theorem 2, if $\mathcal{L}_\delta[f] = 0$, then $f = 0$ almost everywhere on $\mathbb{R}_+$.*

Remark 2. *Comparing Theorem 1 with [14] [Corollary 3], here one uses the hypothesis $\lim_{t \rightarrow 0^+} [t^{\delta-1} f(t)] = 0$ instead of $\lim_{t \rightarrow \infty} [t^{\delta-1} f(t)] = 0$, obtaining a new inversion formula for the transform $\mathcal{L}_\delta$.*

Remark 3. *If f is a locally integrable function on $\mathbb{R}_+ = (0, \infty)$ such that*

$$f(t) = O(t^{\gamma_1}) \text{ as } t \rightarrow 0^+, \quad (9)$$

and

$$f(t) = O(t^{\gamma_2}) \text{ as } t \rightarrow \infty, \quad (10)$$

where $\gamma_1 > 1 - \delta$, $\delta > 0$, and $\gamma_2 \in \mathbb{R}$, then f satisfies the hypothesis of Theorems 1 and 2.

Note that taking $\gamma_2 = 0$ in (10), one finds that for any bounded measurable function f on $\mathbb{R}_+$ with $f(t) = O(t^\gamma)$ as $t \rightarrow 0^+$, $\gamma > 1 - \delta$, $\delta > 0$, $\gamma \geq 0$, then f satisfies the hypothesis of Theorems 1 and 2.

Observe that from Corollary 1 and the above considerations, the integral equation

$$\int_0^\infty f(t) \frac{t^{\delta-1}}{e^{xt} + 1} dt = 0, \quad x > 0, \delta > 0, \quad (11)$$

has not bounded measurable non-trivial solutions in $\delta > a$, whenever $f(t) = O(t^{1-a})$ as $t \rightarrow 0^+$, $a \leq 1$.

Note that taking $a = 1$, one finds that the integral Equation (11) has not bounded measurable non-trivial solutions in $\delta > 1$.

Also note that taking $a = \frac{1}{2}$ and $\frac{1}{2} < \delta < 1$, one arrives at the approach of the Salem equivalence to the Riemann hypothesis given in [13] [Corollary 2.4].

2.2. An Inversion Formula for the Widder–Lambert Transform on $\mathcal{E}'(\mathbb{R}_+)$

Next, we extend the inversion formula to the space of compactly supported distributions $\mathcal{E}'(\mathbb{R}_+)$. This generalization is crucial for analyzing the transform in a broader distributional framework, emphasizing weak convergence.

Definition 1. *The Widder–Lambert transform on $\mathcal{E}'(\mathbb{R}_+)$ is defined by*

$$(\mathcal{L}_\delta[f])(x) = \left\langle f(t), \frac{t^{\delta-1}}{e^{xt} + 1} \right\rangle, \quad x > 0, f \in \mathcal{E}'(\mathbb{R}_+). \quad (12)$$

From the relation (7), one has

$$\begin{aligned} (\mathcal{L}_\delta[f])(x) &= \left\langle f(t), \frac{t^{\delta-1}}{e^{xt} - 1} \right\rangle - \left\langle f(t), \frac{2t^{\delta-1}}{e^{2xt} - 1} \right\rangle \\ &= \left\langle f(t), \frac{t^{\delta-1}}{e^{xt} - 1} \right\rangle - \left\langle f\left(\frac{t}{2}\right), \frac{t^{\delta-1}}{e^{xt} - 1} \right\rangle \\ &= \left\langle f(t) - f\left(\frac{t}{2}\right), \frac{t^{\delta-1}}{e^{xt} - 1} \right\rangle, \quad x > 0, \end{aligned}$$

which is the distributional Lambert transform [30] [Equation (2.3) when $a_k = 1$, for all $k \in \mathbb{N}$] of the member $f(t) - f(\frac{t}{2})$ of $\mathcal{E}'(\mathbb{R}_+)$.

In [14] [Theorem 4] the following was proved:

Theorem 3. Set $\delta > 0$. For $f \in \mathcal{E}'(\mathbb{R}_+)$ and

$$(\mathcal{L}_\delta[f])(x) = \left\langle f(t), \frac{t^{\delta-1}}{e^{xt} + 1} \right\rangle, \quad x > 0,$$

then

$$\lim_{p \rightarrow \infty} \frac{(-1)^p}{p!} \left(\frac{p}{t}\right)^{p+1} \sum_{n=1}^{\infty} \mu(n) n^p (\mathcal{L}_\delta[f])^{(p)}\left(\frac{np}{t}\right) = \left[f(t) - f\left(\frac{t}{2}\right) 2^{1-\delta} \right] t^{\delta-1}, \quad (13)$$

in the sense of the convergence in $\mathcal{D}'(\mathbb{R}_+)$ (weak convergence) [28] [§1.9, p. 24]. Here, $\mu(n)$ numbers are the Möbius numbers, defined as $\mu(1) = 1$, $\mu(n) = (-1)^s$, if n is the product of s distinct primes, and $\mu(n) = 0$ if n is divisible by a square and $(\mathcal{L}_\delta[f])^{(p)}(x)$ denotes the conventional p -th derivative of $(\mathcal{L}_\delta[f])(x)$ with respect to its argument x .

Theorem 4 (Inversion formula). Assuming the hypothesis of Theorem 3, one has

$$\sum_{k=0}^{\infty} \left[\lim_{p \rightarrow \infty} \frac{(-1)^p}{p!} \left(\frac{p2^k}{t}\right)^{p+1} \sum_{n=1}^{\infty} \mu(n) n^p (\mathcal{L}_\delta[f])^{(p)}\left(\frac{np2^k}{t}\right) \right] = f(t) t^{\delta-1}, \quad (14)$$

in the sense of the convergence in $\mathcal{D}'(\mathbb{R}_+)$.

Proof. Observe that from Theorem 3, one has

$$\begin{aligned} & \sum_{k=0}^{\infty} \left[\lim_{p \rightarrow \infty} \frac{(-1)^p}{p!} \left(\frac{p2^k}{t}\right)^{p+1} \sum_{n=1}^{\infty} \mu(n) n^p (\mathcal{L}_\delta[f])^{(p)}\left(\frac{np2^k}{t}\right) \right] \\ &= \sum_{k=0}^{\infty} \left[f\left(\frac{t}{2^k}\right) - f\left(\frac{t}{2^{k+1}}\right) 2^{1-\delta} \right] \left(\frac{t}{2^k}\right)^{\delta-1}, \end{aligned}$$

in the sense of the convergence in $\mathcal{D}'(\mathbb{R}_+)$.

Now

$$\begin{aligned} & \sum_{k=0}^{\infty} \left[f\left(\frac{t}{2^k}\right) - f\left(\frac{t}{2^{k+1}}\right) 2^{1-\delta} \right] \left(\frac{t}{2^k}\right)^{\delta-1} \\ &= \lim_{N \rightarrow \infty} \sum_{k=0}^N \left[f\left(\frac{t}{2^k}\right) - f\left(\frac{t}{2^{k+1}}\right) 2^{1-\delta} \right] \left(\frac{t}{2^k}\right)^{\delta-1} \\ &= \lim_{N \rightarrow \infty} \left[f(t) - f\left(\frac{t}{2^{N+1}}\right) \left(\frac{t}{2^{N+1}}\right)^{\delta-1} \right] \end{aligned}$$

in the sense of the convergence in $\mathcal{D}'(\mathbb{R}_+)$.

Observe that

$$\lim_{N \rightarrow \infty} \left\langle \left(\frac{t}{2^{N+1}}\right)^{\delta-1} f\left(\frac{t}{2^{N+1}}\right), \phi(t) \right\rangle = 0, \quad \text{for all } \phi \in \mathcal{D}(\mathbb{R}_+).$$

In fact for any $\phi \in \mathcal{D}(\mathbb{R}_+)$ we have

$$\begin{aligned} & \lim_{N \rightarrow \infty} \left\langle \left(\frac{t}{2^{N+1}}\right)^{\delta-1} f\left(\frac{t}{2^{N+1}}\right), \phi(t) \right\rangle \\ &= \lim_{N \rightarrow \infty} \left\langle t^{\delta-1} f(t), 2^{N+1} \phi(2^{N+1}t) \right\rangle. \end{aligned}$$

Now one has that $2^{N+1} \phi(2^{N+1}t) \rightarrow 0$ as $N \rightarrow \infty$ in $\mathcal{E}(\mathbb{R}_+)$.

In fact, for any compact $K \subseteq [c, d] \subseteq \mathbb{R}_+$, it follows that $\phi(2^{N+1}t) = 0$, for $N \geq N_0 = \mathbb{N} \cup \{0\}$, where $t \in K$, $c2^{N_0+1} > b$ being $\text{supp } \phi \subseteq [a, b] \subseteq \mathbb{R}_+$. Thus for $N \geq N_0$, $m \in \mathbb{N} \cup \{0\}$ one has

$$\max_{t \in K} |2^{N+1} D^m \phi(2^{N+1}t)| = 0.$$

So $2^{N+1} \phi(2^{N+1}t) \rightarrow 0$ as $N \rightarrow \infty$ in $\mathcal{E}(\mathbb{R}_+)$, and since $t^{\delta-1} f(t) \in \mathcal{E}'(\mathbb{R}_+)$, we find that

$$\lim_{N \rightarrow \infty} \langle t^{\delta-1} f(t), 2^{N+1} \phi(2^{N+1}t) \rangle = 0.$$

Thus

$$\lim_{N \rightarrow \infty} \left\langle \left(\frac{t}{2^{N+1}} \right)^{\delta-1} f\left(\frac{t}{2^{N+1}} \right), \phi(t) \right\rangle = 0, \text{ for all } \phi \in \mathcal{D}(\mathbb{R}_+),$$

□

As an immediate consequence of Theorem 4, one has

Corollary 2 (Injectivity). *Set $\delta > 0$. The map $\mathcal{L}_\delta : \mathcal{E}'(\mathbb{R}_+) \rightarrow \mathcal{E}(\mathbb{R}_+)$ given by (18) is one-to-one.*

Proof. From (14), if $(\mathcal{L}_\delta[f]) = 0$, then $\langle t^{\delta-1} f, \phi \rangle = 0$ for all $\phi \in \mathcal{D}(\mathbb{R}_+)$, and so $\langle f, \phi \rangle = 0$ for all $\phi \in \mathcal{D}(\mathbb{R}_+)$. Now, since $\mathcal{D}(\mathbb{R}_+)$ is dense in $\mathcal{E}(\mathbb{R}_+)$ [28] [§2.3, p. 37] and with it being $f \in \mathcal{E}'(\mathbb{R}_+)$, then $\langle f, \phi \rangle = 0$ for all $\phi \in \mathcal{E}(\mathbb{R}_+)$. Thus, $f = 0$ on $\mathcal{E}'(\mathbb{R}_+)$. □

Remark 4. Equation (14) becomes a different inversion formula for the distributional transform (18) to that obtained in [14] [Corollary 5].

Remark 5. Observe that the proof in Theorem 4 allows us to provide an alternative proof to that given for [14] [Corollary 5].

In that proof, one has to demonstrate that

$$\lim_{N \rightarrow \infty} \langle (2^N t)^{\delta-1} f(2^N t), \phi(t) \rangle = 0, \text{ for all } \phi \in \mathcal{D}(\mathbb{R}_+).$$

In fact, for this one takes $N_0 \in \mathbb{N} \cup \{0\}$ such that $\frac{d}{2^{N_0}} < a$, and then the result holds.

2.3. Regular Distributions of Compact Support Versus the Salem Equivalence to the Riemann Hypothesis

In this subsection, we investigate the relationship between regular distributions of compact support and Salem's equivalence to the Riemann hypothesis. We establish that the Widder–Lambert transform of a regular distribution coincides with its classical counterpart. Furthermore, using a result from distribution theory, we show that a class of compactly supported functions satisfying a certain integral equation must be zero almost everywhere, thereby illustrating a connection with Salem's equivalence to the Riemann hypothesis.

For the case when f is a locally integrable function with compact support in $\mathbb{R}_+$, the functional T_f over $\mathcal{E}(\mathbb{R}_+)$ given by

$$\langle T_f, \phi \rangle = \int_0^\infty f(x) \phi(x) dx, \quad \phi \in \mathcal{E}(\mathbb{R}_+),$$

becomes a member in $\mathcal{E}'(\mathbb{R}_+)$, which, in turn, is called a regular member.

Observe that, taking for each $x > 0$

$$\phi(t) = \frac{t^{\delta-1}}{e^{xt} + 1}, \quad t > 0,$$

one has

$$\left\langle T_f, \frac{t^{\delta-1}}{e^{xt} + 1} \right\rangle = \int_0^\infty f(t) \frac{t^{\delta-1}}{e^{xt} + 1} dt,$$

i.e., the distributional Widder–Lambert transform (18) of T_f agrees with the classical Widder–Lambert transform (2) of the function f .

Concerning the Salem equivalence to the Riemann hypothesis, this equivalence is given by:

For $\frac{1}{2} < \delta < 1$, any bounded measurable function f on $\mathbb{R}_+$ satisfying the integral equation

$$\int_0^\infty f(t) \frac{t^{\delta-1}}{e^{xt} + 1} dt = 0, \quad \text{for all } x > 0,$$

is zero almost everywhere on $\mathbb{R}_+$ [12,13,15,16] (§ 8.4, pp. 139–142), [17].

In our present setting, we prove the next result:

Proposition 1. For $\frac{1}{2} < \delta < 1$, any bounded measurable function f on $\mathbb{R}_+$ with compact support contained in $\mathbb{R}_+$ satisfying the integral equation

$$\int_0^\infty f(t) \frac{t^{\delta-1}}{e^{xt} + 1} dt = 0, \quad \text{for all } x > 0,$$

is zero almost everywhere on $\mathbb{R}_+$.

Proof. Observe that

$$\left\langle t^{\delta-1} T_f, \frac{1}{e^{xt} + 1} \right\rangle = \left\langle T_f, \frac{t^{\delta-1}}{e^{xt} + 1} \right\rangle = \int_0^\infty f(t) \frac{t^{\delta-1}}{e^{xt} + 1} dt = 0, \quad \text{for all } x > 0.$$

Now using Corollary 2, one has that $t^{\delta-1} T_f = 0$, and so $T_f = 0$.

From [27] (pp. 8–9), it follows that $f = 0$ almost everywhere on $\mathbb{R}_+$, and therefore the Proposition holds. $\square$

Remark 6. The outcome presented in Proposition 1 represents a specific instance of the findings established in [13] [Corollaries 2.4 and 3.3]. Nevertheless, Proposition 1 provides insight into how distribution theory can be employed to identify a class of functions that adhere to Salem’s equivalence concerning the Riemann hypothesis. If an inversion formula for the distributional Widder–Lambert transform (18) could be extended to a more general class of distributions—beyond those with compact support—it would lead to a broader class of functions fulfilling the Salem’s equivalence to the Riemann hypothesis.

3. The Stieltjes–Poisson Transform

This section examines the inversion problem for the Stieltjes–Poisson transform, addressing both classical functions and distributions of compact support.

3.1. An Inversion Formula for the Stieltjes–Poisson Transform over Conventional Functions

This subsection presents an inversion formula for the Stieltjes–Poisson transform when applied to conventional functions. The results extend existing inversion techniques, offering a more generalized perspective that connects with Lambert-type transforms.

In [25] [Theorem 2] the following was proved:

Theorem 5. Let $G(x) = \int_0^\infty g(t) \frac{t}{x^2+t^2} dt, x > 0$, where

$$\int_0^\infty \frac{|g(t)|}{t} dt < \infty$$

and

$$\int_0^1 \frac{|g(t) \log t|}{t} dt < \infty.$$

Then

$$H(x) = \lim_{N \rightarrow \infty} \frac{1}{x} \left[\frac{G(0)}{2} + \sum_{k=1}^N (-1)^k G\left(\frac{k\pi}{x}\right) \right]$$

exists for all $x > 0$ and

$$H(x) = \int_0^\infty g(t) \sum_{k=1}^\infty e^{-(2k-1)xt} dt,$$

which is a generalized Lambert transform of the function g .

Moreover

$$\lim_{p \rightarrow \infty} \frac{(-1)^p}{p!} \left(\frac{p}{t}\right)^{p+1} \sum_{n=1}^\infty \mu(2n-1) (2n-1)^p H^{(p)}\left(\frac{(2n-1)p}{t}\right) = g(t),$$

almost everywhere on $0 < t < \infty$. Here, $\mu(n)$ are the Möbius numbers defined as $\mu(1) = 1$, $\mu(n) = (-1)^s$ if n is the product of s distinct primes, and $\mu(n) = 0$ if n is divisible by a square and $H^{(p)}(x)$ denotes the conventional p -th derivative of $H(x)$ with respect to its argument x .

Observe that for $\alpha, \beta \in \mathbb{R}, \alpha \neq 0, x > 0$, one has

$$G(x) = \frac{|\alpha|}{2} F_{\alpha, \beta}(x^{\frac{2}{\alpha}}).$$

where $g(t) = f(t^{\frac{2}{\alpha}}) t^{\frac{2}{\alpha}(\beta-\alpha+1)}$.

In fact, for $\alpha > 0, \beta \in \mathbb{R}, x > 0$, one has

$$\begin{aligned} \frac{|\alpha|}{2} F_{\alpha, \beta}(x^{\frac{2}{\alpha}}) &= \frac{\alpha}{2} \int_0^\infty f(t) \frac{t^\beta}{x^\alpha + t^\alpha} dt \\ &= \frac{\alpha}{2} \int_0^\infty f(u^{\frac{2}{\alpha}}) \frac{u^{\frac{2\beta}{\alpha}}}{x^2 + u^2} \frac{2}{\alpha} u^{\frac{2}{\alpha}-1} du, \text{ (making } t = u^{\frac{2}{\alpha}}) \\ &= \int_0^\infty f(u^{\frac{2}{\alpha}}) u^{\frac{2}{\alpha}(\beta-\alpha+1)} \frac{u}{x^2 + u^2} du \\ &= \int_0^\infty g(u) \frac{u}{x^2 + u^2} du \\ &= G(x). \end{aligned}$$

For $\alpha < 0, \beta \in \mathbb{R}, x > 0$, it follows

$$\begin{aligned} \frac{|\alpha|}{2} F_{\alpha, \beta}(x^{\frac{2}{\alpha}}) &= -\frac{\alpha}{2} \int_0^\infty f(t) \frac{t^\beta}{x^\alpha + t^\alpha} dt \\ &= \int_0^\infty f(u^{\frac{2}{\alpha}}) u^{\frac{2}{\alpha}(\beta-\alpha+1)} \frac{u}{x^2 + u^2} du, \text{ (making } t = u^{\frac{2}{\alpha}}) \\ &= \int_0^\infty g(u) \frac{u}{x^2 + u^2} du \\ &= G(x). \end{aligned}$$

Note that for $\alpha > 0, \beta \in \mathbb{R}$, one has

$$\begin{aligned}
 & \int_0^\infty \frac{|f(u)|}{u^{\alpha-\beta}} du \\
 &= \frac{2}{\alpha} \int_0^\infty \frac{|f(t^{\frac{2}{\alpha}})|}{t^{\frac{2}{\alpha}(\alpha-\beta)}} t^{\frac{2}{\alpha}-1} dt, \text{ (making } u = t^{\frac{2}{\alpha}} \text{)} \\
 &= \frac{2}{\alpha} \int_0^\infty \frac{|f(t^{\frac{2}{\alpha}})| t^{\frac{2}{\alpha}(\beta-\alpha+1)}}{t} dt \\
 &= \frac{2}{\alpha} \int_0^\infty \frac{|g(t)|}{t} dt.
 \end{aligned}$$

For $\alpha < 0, \beta \in \mathbb{R}$, one also obtains

$$\int_0^\infty \frac{|f(u)|}{u^{\alpha-\beta}} du = -\frac{2}{\alpha} \int_0^\infty \frac{|g(t)|}{t} dt.$$

On the other hand, for $\alpha > 0, \beta \in \mathbb{R}$, one has

$$\begin{aligned}
 & \int_0^1 \frac{|f(u) \log u|}{u^{\alpha-\beta}} du \\
 &= \frac{2}{\alpha} \int_0^1 \frac{|f(t^{\frac{2}{\alpha}}) \log t|}{t^{\frac{2}{\alpha}(\alpha-\beta)}} \frac{2}{\alpha} t^{\frac{2}{\alpha}-1} dt, \text{ (making } u = t^{\frac{2}{\alpha}} \text{)} \\
 &= \frac{4}{\alpha^2} \int_0^1 \frac{|f(t^{\frac{2}{\alpha}}) t^{\frac{2}{\alpha}(\beta-\alpha+1)} \log t|}{t} dt \\
 &= \frac{4}{\alpha^2} \int_0^1 \frac{|g(t) \log t|}{t} dt.
 \end{aligned}$$

For $\alpha < 0, \beta \in \mathbb{R}$, it follows

$$\begin{aligned}
 & \int_1^\infty \frac{|f(u) \log u|}{u^{\alpha-\beta}} du \\
 &= \frac{4}{\alpha^2} \int_0^1 \frac{|f(t^{\frac{2}{\alpha}}) \log t|}{t} t^{\frac{2}{\alpha}(\beta-\alpha+1)} dt, \text{ (making } u = t^{\frac{2}{\alpha}} \text{)} \\
 &= \frac{4}{\alpha^2} \int_0^1 \frac{|g(t) \log t|}{t} dt.
 \end{aligned}$$

Thus one obtains the next result.

Theorem 6. Let

$$F_{\alpha,\beta}(x) = \int_0^\infty f(t) \frac{t^\beta}{x^\alpha + t^\alpha} dt, \quad x > 0, \alpha, \beta \in \mathbb{R}, \alpha \neq 0,$$

where

$$\int_0^\infty \frac{|f(t)|}{t^{\alpha-\beta}} dt < \infty,$$

and

$$\int_0^1 \frac{|f(t) \log t|}{t^{\alpha-\beta}} dt < \infty, \text{ when } \alpha > 0, \beta \in \mathbb{R},$$

or

$$\int_1^\infty \frac{|f(t) \log t|}{t^{\alpha-\beta}} dt < \infty, \text{ when } \alpha < 0, \beta \in \mathbb{R}.$$

Let

$$G(x) = \frac{|\alpha|}{2} F_{\alpha,\beta}(x^{\frac{2}{\alpha}}), \quad x > 0,$$

and

$$H(x) = \lim_{N \rightarrow \infty} \frac{1}{x} \left[\frac{G(0)}{2} + \sum_{k=1}^N (-1)^k G\left(\frac{k\pi}{x}\right) \right].$$

Then

$$\lim_{p \rightarrow \infty} \frac{(-1)^p}{p!} \left(\frac{p}{t^{\frac{\alpha}{2}}} \right)^{p+1} t^{\alpha-\beta-1} \sum_{n=1}^{\infty} \mu(2n-1)(2n-1)^p H^{(p)}\left(\frac{(2n-1)p}{t^{\frac{\alpha}{2}}}\right) = f(t),$$

almost everywhere on $0 < t < \infty$.

Proof. Using Theorem 5 when $g(t) = f(t^{\frac{2}{\alpha}})t^{\frac{2}{\alpha}(\beta-\alpha+1)}$, one has

$$\begin{aligned} & \lim_{p \rightarrow \infty} \frac{(-1)^p}{p!} \left(\frac{p}{t} \right)^{p+1} \sum_{n=1}^{\infty} \mu(2n-1)(2n-1)^p H^{(p)}\left(\frac{(2n-1)p}{t}\right) \\ &= g(t) = f(t^{\frac{2}{\alpha}})t^{\frac{2}{\alpha}(\beta-\alpha+1)}, \end{aligned}$$

almost everywhere on $0 < t < \infty$.

Thus

$$\lim_{p \rightarrow \infty} \frac{(-1)^p}{p!} \left(\frac{p}{t^{\frac{\alpha}{2}}} \right)^{p+1} t^{\alpha-\beta-1} \sum_{n=1}^{\infty} \mu(2n-1)(2n-1)^p H^{(p)}\left(\frac{(2n-1)p}{t^{\frac{\alpha}{2}}}\right) = f(t),$$

almost everywhere on $0 < t < \infty$. $\square$

As a consequence of this result, one obtains the following:

Corollary 3 (Injectivity). Assume f is a function which satisfies the hypothesis of Theorem 6. If $F_{\alpha,\beta}f = 0$, then $f = 0$ almost everywhere on $\mathbb{R}_+ = (0, \infty)$.

Example 1. If f is a locally integrable function on $\mathbb{R}_+ = (0, \infty)$ such that

$$f(t) = O(t^{\gamma_1}) \text{ as } t \rightarrow 0^+, \quad (15)$$

and

$$f(t) = O(t^{\gamma_2}) \text{ as } t \rightarrow \infty, \quad (16)$$

where $\gamma_1 > \alpha - \beta - 1$ and $\gamma_2 < \alpha - \beta - 1$, then f satisfies the hypothesis of Theorem 6.

Observe that for the Stieltjes ($\alpha = 1, \beta = 0$) and Poisson ($\alpha = 2, \beta = 1$) transforms, it is $\gamma_1 > 0$ and $\gamma_2 < 0$.

Note that taking $\gamma_2 = 0$ in (16), one has that any bounded measurable function f on $\mathbb{R}_+$ with $f(t) = O(t^{\gamma})$ as $t \rightarrow 0^+$, $\gamma > \alpha - \beta - 1 > 0$; then, f satisfies the hypothesis of Theorem 6.

Also, taking $\gamma_1 = 0$ in (15) one has that any bounded measurable function f on $\mathbb{R}_+$ with $f(t) = O(t^{\gamma})$ as $t \rightarrow \infty$, $\gamma < \alpha - \beta - 1 < 0$; then, f satisfies the hypothesis of Theorem 6.

Remark 7 (Integral equation). Observe that from Corollary 3 and the above considerations, then the integral equation

$$\int_0^{\infty} f(t) \frac{t^{\delta-1}}{x^{\alpha} + t^{\alpha}} dt = 0, \quad \alpha \in \mathbb{R}, \alpha \neq 0, \delta \in \mathbb{R}, \quad (17)$$

has not bounded measurable non-trivial solutions in $a < \delta < \alpha$, whenever $f(t) = O(t^{\alpha-a})$ as $t \rightarrow 0^+$.

In fact, taking $\beta = \delta - 1$, one has $\alpha - a > \alpha - \delta = \alpha - \beta - 1 > 0$.

Analogously, Equation (17) has not bounded measurable non-trivial solutions in $\alpha < \delta < a$, whenever $f(t) = O(t^{\alpha-a})$ as $t \rightarrow \infty$.

In fact, taking $\beta = \delta - 1$, one has $\alpha - a < \alpha - \delta = \alpha - \beta - 1 < 0$.

Remark 8 (Another inversion formula). When one considers the case $\rho = 1$ in [31] [Theorem 3.1] one obtains another inversion formula for the Stieltjes–Poisson transform $F_{\alpha,\beta}$ given by (3):

Assume $\alpha, \beta \in \mathbb{R}$, $\alpha \neq 0$, and f is a continuous function on $\mathbb{R}_+$ such that the integral in (3) exists for all $x > 0$. Then

$$|\alpha|t^{\alpha-\beta-1} \lim_{p \rightarrow \infty} L_{p,t} F_{\alpha,\beta}(t) = f(t) \text{ on } \mathbb{R}_+,$$

where

$$L_{p,t} F_{\alpha,\beta}(t) = \frac{(-1)^{p-1}}{p!(p-2)!} D^p \left[t^{\alpha(2p-1)} D^{p-1} F_{\alpha,\beta}(t) \right].$$

3.2. Concerning an Inversion Formula for the Stieltjes–Poisson Transform over Distributions of Compact Support

In this subsection, we derive an inversion formula for the Stieltjes–Poisson transform of distributions in $\mathcal{E}'(\mathbb{R}_+)$, refining the approach in [26]. By introducing the functional g_f and employing its structural properties, we extend the inversion process within the distributional framework. Moreover, we establish the injectivity of the transform.

The Stieltjes–Poisson transform of a member $f \in \mathcal{E}'(\mathbb{R}_+)$ is defined by

$$F_{\alpha,\beta}(x) = \left\langle f(t), \frac{t^\beta}{x^\alpha + t^\alpha} \right\rangle, \quad x > 0, \alpha, \beta \in \mathbb{R}, \alpha \neq 0. \quad (18)$$

In [26] [Theorem 2.3], an inversion formula for the distributional transform (18) was obtained by means of the function

$$\tilde{G}(x) = \frac{\alpha}{2} F_{\alpha,\beta}(x), \quad x > 0.$$

If we consider the function $G(x) = \frac{|\alpha|}{2} F_{\alpha,\beta}(x)$, $x > 0$, instead of $\tilde{G}(x)$, and in a parallel way to [26] [Lemma 2.2 and Theorem 2.3], we arrive, respectively, at the next results.

Lemma 1. Assume $f \in \mathcal{E}'(\mathbb{R}_+)$; then, the functional g_f given by

$$\langle g_f, \phi \rangle = \langle f, \psi \rangle,$$

where $\psi(t) = \frac{|\alpha|}{2} t^{\beta-\frac{\alpha}{2}} \phi(t^{\frac{\alpha}{2}})$, $\phi \in \mathcal{E}(\mathbb{R}_+)$, is a member of $\mathcal{E}'(\mathbb{R}_+)$.

Theorem 7. Let $f \in \mathcal{E}'(\mathbb{R}_+)$ and set

$$F_{\alpha,\beta}(x) = \left\langle f(t), \frac{t^\beta}{x^\alpha + t^\alpha} \right\rangle, \quad x > 0, \alpha, \beta \in \mathbb{R}, \alpha \neq 0.$$

Denote

$$G(x) = \frac{|\alpha|}{2} F_{\alpha,\beta}(x^{\frac{2}{\alpha}}), \quad x > 0.$$

Then

$$H(x) = \lim_{N \rightarrow \infty} \frac{1}{x} \left[\frac{G(0)}{2} + \sum_{k=1}^N (-1)^k G\left(\frac{k\pi}{x}\right) \right],$$

exists for all $x > 0$ and $H(x) = \left\langle g_f, \sum_{k=1}^{\infty} e^{-(2k-1)xt} \right\rangle$, which is a generalized Lambert transform of the functional g_f given by Lemma 1.

Moreover, for every $\phi \in \mathcal{D}(\mathbb{R}_+)$

$$\langle f, \phi \rangle = \lim_{p \rightarrow \infty} \left\langle \frac{(-1)^p}{p!} \left(\frac{p}{t^{\frac{\alpha}{2}}} \right)^{p+1} t^{\alpha-\beta-1} \sum_{n=1}^{\infty} \mu(2n-1)(2n-1)^p H^{(p)} \left(\frac{(2n-1)p}{t^{\frac{\alpha}{2}}} \right), \phi(t) \right\rangle, \quad (19)$$

where $\mu(n)$ are the Möbius numbers defined as $\mu(1) = 1$, $\mu(n) = (-1)^s$ if n is the product of s distinct primes, and $\mu(n) = 0$ if n is divisible by a square and $H^{(p)}(x)$ denotes the conventional p -th derivative of $H(x)$ with respect to its argument x .

From this result, and since $\mathcal{D}(\mathbb{R}_+)$ is dense in $\mathcal{E}(\mathbb{R}_+)$ [28] [§2.3, p. 37], one obtains the following:

Corollary 4 (Injectivity). *Assuming the hypothesis of Theorem 7, if $F_{\alpha,\beta} = 0$, then $f = 0$ on $\mathcal{E}'(\mathbb{R}_+)$.*

If f is a locally integrable function on $\mathbb{R}_+$ and f has compact support on $\mathbb{R}_+$, then f gives rise to a regular member T_f of $\mathcal{E}'(\mathbb{R}_+)$ by means of

$$\langle T_f, \phi \rangle = \int_0^{\infty} f(x) \phi(x) dx, \quad \text{for all } \phi \in \mathcal{E}(\mathbb{R}_+).$$

In fact, taking into account that

$$\begin{aligned} |\langle T_f, \phi \rangle| &= \left| \int_0^{\infty} f(x) \phi(x) dx \right| \leq \sup_{x \in \text{supp}(f)} |\phi(x)| \int_{\text{supp}(f)} |f(x)| dx \\ &= \rho_{\text{supp}(f),0}(\phi) \int_{\text{supp}(f)} |f(x)| dx, \end{aligned}$$

where $\text{supp}(f)$ represents the support of the function f , then $T_f \in \mathcal{E}'(\mathbb{R}_+)$. Also, for this function f , we have

$$\left\langle T_f, \frac{t^{\beta}}{x^{\alpha} + t^{\alpha}} \right\rangle = \int_0^{\infty} f(t) \frac{t^{\beta}}{x^{\alpha} + t^{\alpha}} dt, \quad x > 0.$$

This result ensures that the generalized Stieltjes–Poisson transform (18) of the regular member $T_f \in \mathcal{E}'(\mathbb{R}_+)$ agrees with the generalized Stieltjes–Poisson transform (3) of the function f .

Therefore, the inversion formula of Theorem 6 for this function f implies the inversion formula (19) for the distribution of compact support T_f .

Remark 9 (Integral equation). *Observe that from Corollary 4 and the above considerations, one has that the integral Equation (17) is not locally integrable with compact support non-trivial solutions for any $\delta \in \mathbb{R}$. In particular, the integral Equation (17) has not bounded measurable with compact support non-trivial solutions for any $\delta \in \mathbb{R}$.*

Remark 10 (Another inversion formula). *When one considers the case $\rho = 1$ in [31] [Theorem 5.2], one obtains another inversion formula for the distributional Stieltjes–Poisson transform $F_{\alpha,\beta}$, given by (18):*

Let $f \in \mathcal{E}'(\mathbb{R}_+)$; then, for every $\phi \in \mathcal{D}(\mathbb{R}_+)$

$$\langle f, \phi \rangle = \lim_{p \rightarrow \infty} \left\langle \frac{(-1)^{p-1}}{p!(p-2)!} |\alpha| t^{\alpha-\beta-1} D^p \left[t^{\alpha(2p-1)} D^{p-1} F_{\alpha,\beta}(t) \right], \phi(t) \right\rangle.$$

4. Final Observations and Conclusions

This paper systematically develops inversion formulae for the Widder–Lambert transform, extending its applicability from conventional functions to compactly supported distributions. Additionally, it establishes a connection between this transform and Salem’s equivalence, providing insights into its relation to the Riemann hypothesis. Furthermore, inversion formulae for the Stieltjes–Poisson transform are derived using a generalized Lambert transform, with results extended to compactly supported distributions. By employing both analytical and distributional approaches, this work contributes to the broader theory of integral transforms, offering explicit reconstruction techniques. These findings enhance the theoretical understanding of both transforms and highlight their potential applications in harmonic analysis, mathematical physics, and related fields, reinforcing their significance in classical and generalized function spaces.

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