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Positive Solutions for Discrete Robin Problem of Kirchhoff Type Involving p-Laplacian

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Abstract: The aim of this paper is to investigate the existence of positive solutions for a discrete Robin problem of the Kirchhoff type involving the p-Laplacian by the means of critical point theory. Our results demonstrate that the problem admits at least three solutions, or at least two solutions under different conditions on the nonlinear term f. We establish a strong maximum principle for the problem and obtain the existence and multiplicity of positive solutions. Finally, we give three examples to verify our results.

Keywords: p-Laplacian; discrete Robin problem; positive solutions; critical point theory; strong maximum principle

MSC: 39A27

1. Introduction

Let $\mathbb{R}$ be the set of all real numbers. For any positive integer T, $[1,T]$ is defined with the discrete set $\{1,2,\ldots,T\}$. In this paper, we consider the following discrete Robin problem of the Kirchhoff type involving the p-Laplacian:

$$\begin{cases} -g(\|u\|^p)(\Delta\phi_p(\Delta u(k-1))) + q(k)\phi_p(u(k)) = \lambda f(k,u(k)), \quad k \in [1,T], \\ \Delta u(0) = u(T+1) = 0. \end{cases} \tag{1}$$

where $g(s)$ is a continuous function and has a positive lower bound, that is, there exists a positive constant m such that $g(s) \geq m$ for any $s \in \mathbb{R}$; $\phi_p(s) = |s|^{p-2}s$, with $p > 1$ and $s \in \mathbb{R}$; and $f(k,\cdot) \in C(\mathbb{R},\mathbb{R})$ with $q(k) \geq 0$ for each $k \in [1,T]$. Δ is the forward difference operator defined by $\Delta u(k) = u(k+1) - u(k)$; $\Delta^2 u(k) = \Delta(\Delta u(k))$; and λ is a positive parameter.

Over the past two decades, difference equations have received considerable attention from many scientists and have developed rapidly, leading to numerous applications in computer science, neural networks, biology, economics, and other research fields [1–7]. A variety of important tools have been used to investigate the boundary value problem of difference equations. These include invariant sets of descent flow [8], fixed-point methods [9–11], and the method of upper and lower solutions [12–14]. In 2003 , Guo and Yu [15] obtained the existence of periodic and subharmonic solutions for a second-order difference equation by using critical point theory for the first time. Subsequently, critical point theory has been employed to investigate difference equations, leading to the discovery of numerous significant insights pertaining to boundary value problems [16–20], periodic solutions [21,22], homoclinic solutions [23,24], and heteroclinic solutions [25].

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The Kirchhoff equation was presented by Kirchhoff [26] in 1883 and introduced a model given by the following equation:

$$\rho\frac{\partial^2 u}{\partial t^2}-\left(\frac{\rho_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2dx\right)\frac{\partial^2 u}{\partial x^2}=0,$$

where ρ, ρ_0, h, E, and L are constants, which is an extension of the classical D'Alembert wave equation and describes the transversal oscillations of a stretched string. With further research, the relevant results of this equation have been uncovered by many scientists. In particular, the corresponding Dirichlet problem of (1) is a discrete analog of the following boundary value problem of the Kirchhoff type:

$$\begin{cases}-(a+b\int_\Omega|\nabla u|^2dx)\Delta u=\lambda f(k,u), & in \quad \Omega\\ u=0, \quad on \quad \partial\Omega.\end{cases}$$

which is associated with the stationary version of the Kirchhoff equation:

$$u_{tt}-(a+b\int_\Omega|\nabla u|^2dx)\Delta u=f(x,u).$$

In the continuous case, for example, Yang and Zhang [27] applied the local linking theroy to study non-local quasilinear elliptic boundary value problems and successfully acquired non-trivial solutions.Tang and Cheng [28] studied the existence of a ground-state sign-changing solution by applying the non-Nehari manifold method and others [29]. In addition, Heidarkhanil et al. [30] obtained the existence of three solutions for a Kirchhoff-type boundary problem. In our opinion, concerning three solutions for problems depending on a parameter in different settings, it is worth mentioning [31–34].

As for discrete cases, Long and Deng [35] discussed a class of partial discrete Kirchhoff-type problems via minimax methods and invariant sets of descending flow. Owing to the importance of the p-Laplacian operator, Candito and co-author [17,36] disscussed the Dirichlet problem of the ϕ_p-Laplacian by critical point theory. Afrouzi, G.A. and Heidarkhani, S [37] obtained the existence of three solutions for a Dirichlet problem involving the p-Laplacian. Xiong [38] obtained infinitely many solutions for partial discrete Kirchhoff-type problems containing the p-Laplacian. Heidarkhanil et al. [39,40] discussed discrete anisotropic Kirchhoff-type problems by variational methods, and some conlusions on the existence of infinite solutions and non-trivial solutions were obtained. All these boundary value problems are for Dirichlet case. Recently, Ling and Zhou investigated the discrete Robin problem with the ϕ-Laplacian in [16] by critical point theory.

Inspired by the above results, we intend to investigate the multiplicity of the solutions of the problem (1) by applying critical point theory. In Section 2 of this paper, the main lemmas and some theorems of critical point theory are introduced. Our main results and proofs are presented in Section 3, which shows that the problem (1) has at least two or three solutions when the nonlinear term f is under different conditions. Finally, we demonstrate our conclusions through three simple examples in Section 4.

2. Preliminaries

We begin by considering the N-dimensional Banach space $S=\{u:[0,T+1]\to\mathbb{R}\mid \Delta u(0)=u(T+1)=0\}$ equipped with the norm, defined as follows:

$$\|u\|=\left(\sum_{k=1}^{T}|\Delta u(k)|^p\right)^{\frac{1}{p}}.$$

Next, we establish the variational framework corresponding to Problem (1); for each $u \in S$, let

$$\Phi(u) = \frac{1}{p}G(\|u\|^p) + \frac{1}{p}\sum_{k=1}^{T} q(k)|u(k)|^p, \Psi(u) = \sum_{k=1}^{T} F(k, u(k)),$$

where $G(\xi) = \int_0^\xi g(t)dt$ for each $\xi \in \mathbb{R}$; $F(k,\xi) = \int_0^\xi f(k,t)dt$ for each $(k,\xi) \in [1,T] \times \mathbb{R}$; and we define

$$I_\lambda(u) = \Phi(u) - \lambda\Psi(u).$$

Owing to $\Phi, \Psi \in C^1(S,\mathbb{R})$, $I_\lambda(u)$ is also a class of $C^1(S,\mathbb{R})$. Using the summation-by-parts method and the boundary conditions, one has

$$\begin{aligned}
& I'(u)(v) \\
= & \lim_{t\to 0}\frac{I_\lambda(u+tv) - I_\lambda(u)}{t} \\
= & \lim_{t\to 0}\left[\frac{\Phi(u+tv)-\Phi(u)}{t} - \lambda\frac{\Psi(u+tv)-\Psi(u)}{t}\right] \\
= & g(\|u\|^p)\sum_{k=1}^{T+1}\phi_p(\Delta u(k-1))\Delta v(k-1) + \sum_{k=1}^{T} q(k)|u(k)|^{p-2}u(k)v(k) - \lambda\sum_{k=1}^{T} f(k,u(k))v(k) \\
= & -g(\|u\|^p)\sum_{k=1}^{T}(\Delta\phi_p(\Delta u(k-1)))v(k) + \sum_{k=1}^{T} q(k)|u(k)|^{p-2}u(k)v(k) - \lambda\sum_{k=1}^{T} f(k,u(k))v(k) \\
= & \sum_{k=1}^{T}[-g(\|u\|^p)(\Delta\phi_p(\Delta u(k-1))) + q(k)\phi_p(u(k))]v(k) - \lambda\sum_{k=1}^{T} f(k,u(k))v(k) \\
= & \sum_{k=1}^{T}[-g(\|u\|^p)(\Delta\phi_p(\Delta u(k-1))) + q(k)\phi_p(u(k)) - \lambda f(k,u(k))]v(k).
\end{aligned}$$

for each $u, v \in S$.

Therefore, the critical point $u \in S$ of the functional I_λ satisfies the following equivalence:

$$u \text{ is a critical point of } I_\lambda \text{ on } S \iff u \text{ solves problem } (1).$$

This equivalence establishes that seeking solutions to (1) is equal to identifying critical points of I_λ in S.

The main lemmas of this paper are as follows.

Lemma 1 (Corollary 3.1 of [41]). *Let X denote a real finite-dimensional Banach and $I_\lambda : X \to \mathbb{R}$ be a functional satisfying the following structure hypothesis:*
(A_1) $I_\lambda(u) = \Phi(u) - \lambda\Psi(u)$ for $u \in X$, where $\lambda > 0, \Phi, \Psi : X \to \mathbb{R}$ are two continuous functions of the class C^1 on X and Φ is coercive, which means that $lim_{\|u\|\to+\infty}\Phi(u) = +\infty$.
(A_2) Φ is convex and $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$.
(A_3) If x_1 and x_2 are local minima for the functional $I_\lambda = \Phi(u) - \lambda\Psi(u)$ such that $\Psi(x_1) \geq 0$ and $\Psi(x_2) \geq 0$, then

$$\inf_{t\in[0,1]} \Psi(tx_1 + (1-t)x_2) \geq 0.$$

Furthermore, let ρ_1 and ρ_2 be positive constants and $\overline{u} \in X$, with $\rho_1 < \Phi(\overline{u}) < \frac{\rho_2}{2}$, such that

(i) $\dfrac{\sup\limits_{u\in\Phi^{-1}(-\infty,\rho_1)}\Psi(u)}{\rho_1} < \frac{1}{2}\dfrac{\Psi(\overline{u})}{\Phi(\overline{u})}$;

(ii) $\dfrac{\sup\limits_{u\in\Phi^{-1}(-\infty,\rho_2)}\Psi(u)}{\rho_2} < \frac{1}{4}\dfrac{\Psi(\overline{u})}{\Phi(\overline{u})}$.

Then, for $\lambda \in \left(\frac{2\Phi(\overline{u})}{\Psi(\overline{u})}, \min\left\{ \frac{\rho_1}{\sup\limits_{u\in\Phi^{-1}(-\infty,\rho_1)} \Psi(u)}, \frac{\rho_2/2}{\sup\limits_{u\in\Phi^{-1}(-\infty,\rho_2)} \Psi(u)} \right\} \right)$, *the functional* I_λ *admits at least three distinct critical points,* u_1, u_2, *and* u_3, *such that* $u_1 \in \Phi^{-1}(-\infty, \rho_1)$, $u_2 \in \Phi^{-1}(\rho_1, \rho_2/2)$ *and* $u_3 \in \Phi^{-1}(-\infty, \rho_2)$.

Lemma 2 (Theorem 4.1 of [42]). *Let* $(X, \|\cdot\|)$ *be a real finite-dimensional Banach space and let* $\Phi, \Psi : X \to \mathbb{R}$ *to two continuously Gâteaux differetiable functions with* Φ *being coercive. Moreover,* Φ *and* Ψ *satisfy*

$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r \in \mathbb{R}$ *and* $\overline{u} \in X$, *with* $0 < r < \Phi(\overline{u})$, *such that we have the following:*

(i) $\frac{\sup\limits_{\Phi(u)\le r} \Psi(u)}{r} < \frac{\Psi(\overline{u})}{\Phi(\overline{u})}$;

(ii) *For each* $\lambda \in \Lambda := \left(\frac{\Phi(\overline{u})}{\Psi(\overline{u})}, \frac{r}{\sup\limits_{\Phi(u)\le r} \Psi(u)} \right)$, *the functional* $I_\lambda = \Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda$, *the functional* I_λ *has at least three different critical points in* X.

To apply the Palais–Smale (PS) condition in the lemma, we first recall its definition. Let X be a real Banach space, and let $I : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional. We say that I satisfies the *Palais–Smale condition* (PS condition for brevity) if every sequence $\{u_n\} \subset X$, for which $\{I(u_n)\}$ is bounded and $I'(u_n) \to 0$ in the dual space X^*, possesses a convergent subsequence in X.

Lemma 3 (Theorem 2.1 of [43]). *Let* X *be a real Banach space and let* $\Phi, \Psi : X \to \mathbb{R}$ *be two continuously Gâteaux differentiable functions such that* $\inf_X \Phi = \Phi(0) = \Psi(0)$. *Assume that there are* $r \in \mathbf{R}$ *and* $\overline{u} \in \mathbb{R}$, *with* $0 < \Phi(\overline{u}) < r$, *such that*

$$\frac{\sup\limits_{\Phi(u)\le r} (\Psi(u))}{r} < \frac{\Psi(\overline{u})}{\Phi(\overline{u})}.$$

Moreover, for each $\lambda \in \Lambda := \left(\frac{\Phi(\overline{u})}{\Psi(\overline{u})}, \frac{r}{\sup\limits_{\Phi(u)\le r} \Psi(u)} \right)$, *the functional* $I_\lambda = \Phi - \lambda\Psi$ *satisfies the PS condition, and it is unbounded from below.*

Then, for each $\lambda \in \Lambda$, *the functional* I_λ *admits at least two non-zero critical points* $u_{\lambda,1}, u_{\lambda,2}$ *such that* $I_\lambda(u_{\lambda,1}) < 0 < I_\lambda(u_{\lambda,2})$.

For each $u \in S$, we define other norms:

$$\|u\|_\infty = \max_{k\in[1,N]} |u(k)|,$$

$$\|u\|_p = \left(\sum_{k=1}^{T} |u(k)|^p \right)^{\frac{1}{p}}.$$

Lemma 4. *For each* $u \in S$, *the following relation holds:*

$$\|u\|_\infty \le T^{(p-1)/p} \|u\|.$$

Proof. Let $u \in S$ and $y \in [1, T]$ such that

$$|u(y)| = \max_{k\in[1,T]} |u(k)|.$$

Since $\Delta u(0) = u(T+1) = 0$ for any $u \in S$, by the properties of the inequality, we obtain

$$|u(y)| = |\sum_{k=y}^{T} \Delta u(k)| \leq \sum_{k=y}^{T} |\Delta u(k)|,$$

and by the Cauchy–Schwarz inequality, we obtain

$$\|u\|_\infty = |u(y)| \leq \sum_{k=y}^{T} |\Delta u(k)| \leq (T-y+1)^{\frac{1}{q}} \left(\sum_{k=y}^{T} |\Delta u(k)|^p \right)^{\frac{1}{p}} \leq T^{\frac{1}{q}} \|u\|,$$

where q is a conjugative number of p, that is, $\frac{1}{q} + \frac{1}{p} = 1$.

Thus, Lemma 4 is proved. □

In particular, in the context of the Dirichlet problem, reference [44] establishes a series of inequalities involving various norms, which serves as a crucial reference for our research.

In finite-dimensional Banach spaces, finite-dimensional norms are equivalent, Therefore, there exist two positive constants, $c_{2p} > c_{1p} > 0$, such that

$$c_{1p}\|u\|_p \leq \|u\| \leq c_{2p}\|u\|_p,$$

for each $u \in S$.

Now, we establish a strong maximum principle.

Lemma 5. *Fix $u \in S$, such that either*

$$u(k) > 0 \qquad or \qquad -g(\|u\|^p)(\Delta\phi_p(\Delta u(k-1))) + q(k)\phi_p(u(k)) \geq 0,$$

for each $k \in [1, T]$. Then, either $u(k) \equiv 0$ for all $k \in [1, T]$ or $u(k) > 0$ for all $k \in [1, T]$.

Proof. Let $u(y) = \min_{k\in[1,T]} u(k)$. If $u(y) > 0$, the conclusion follows. If $u(y) \leq 0$, then we have

$$-g(\|u\|^p)(\Delta\phi_p(\Delta u(y-1))) + q(y)\phi_p(u(y)) \geq 0.$$

Since $g(s) \geq m > 0, q(y) \geq 0$, $u(y+1) - u(y) \geq 0$ and $u(y) - u(y-1) \leq 0$, we have

$$0 \leq g(\|u\|^p)[|\Delta u(y)|^{p-2}(u(y+1) - u(y)) - |\Delta u(j-1)|^{p-2}(u(y) - u(y-1))] \leq q(y)\phi_p(u(y)) \leq 0.$$

Thus, we obtain $\Delta\phi_p(\Delta u(y)) = 0$, that is, $u(y+1) = u(y) = u(y-1)$. If $y+1 = T+1$, we obtain $u(y) = 0$. Otherwise, $y+1 \in [1, T]$. Replacing y by $y+1$, we obtain $u(y) = u(y+1) = u(y+2) = \cdots = u(T+1) = 0$. Similarly, we obtain $u(y) = u(y-1) = 0$; if $y-1 = 0$ or $y-1 = 1$, we conclude. If $y \in [2, T]$, we replace y by $y-1$, and we have $u(y-2) = u(y-1) = 0$. Accordingly, we prove that $u(k) \equiv 0$ for all $k \in [1, T]$, and the proof is completed. □

Let

$$F^+(k, \xi) = \int_0^\xi f(k, t^+)dt, \quad (k, \xi) \in [1, T] \times \mathbb{R},$$

where $t^+ = \max\{0, t\}$. Now, we define $I_\lambda^+(u) = \Phi - \lambda\Psi^+$, where $\Psi^+(u) = \sum_{k=1}^{T} F^+(k, u(k))$ and Φ is defined as before. Similarly, the critical points of $I_\lambda^+(u)$ are the solutions of the following problem:

$$\begin{cases} -g(\|u\|^p)(\Delta\phi_p(\Delta u(k-1))) + q(k)\phi_p(u(k)) = \lambda f(k, u^+(k)), \quad k \in [1, T], \\ \Delta u(0) = u(T+1) = 0. \end{cases} \tag{2}$$

Lemma 6. *If $f(k, 0) \geq 0$ for each $k \in [1, T]$, then all the non-zero critical points of $I_\lambda^+(u)$ are positive solutions of Problem (1).*

3. Main Results

Let

$$F_u := \sum_{k=1}^{T} F(k, u), \forall u \in \mathbb{R},$$

$$\overline{q}(k) = \max\{q(k) : k \in [1, T]\}, \underline{q} = \min\{q(k) : k \in [1, T]\}, Q = \sum_{k=1}^{T} q(k).$$

Now, we introduce the first theorem.

Theorem 1. *Assume that $G(s)$ is convex, and $\forall k \in [1, T]$ with $f(k, \xi) \geq 0$ and $\xi \in \mathbb{R}$. Moreover, there exist three positive constants, c_1, c_2 and d, with*

$$\frac{m(c_1 c_{2p})^p + \underline{q}c_1^p}{pc_{2p}^p T^{p-1}} < \frac{1}{p}G(d^p) + \frac{d^p}{p}Q < \frac{m(c_2 c_{2p})^p + \underline{q}c_2^p}{2pc_{2p}^p T^{p-1}}, \tag{3}$$

such that

(A) $\max\left\{\frac{pc_{2p}^p T^{p-1} F_{c_1}}{m(c_1 c_{2p})^p + \underline{q}c_1^p}, \frac{2pc_{2p}^p T^{p-1} F_{c_2}}{m(c_2 c_{2p})^p + \underline{q}c_2^p}\right\} < \frac{pF_d}{2G(d^p) + 2Qd^p}$,

Then, for each

$$\lambda \in \left(\frac{2G(d^p) + 2Qd^p}{pF_d}, \min\left\{\frac{m(c_1 c_{2p})^p + \underline{q}c_1^p}{pc_{2p}^p T^{p-1} F_{c_1}}, \frac{m(c_2 c_{2p})^p + \underline{q}c_2^p}{2pc_{2p}^p T^{p-1} F_{c_2}}\right\}\right)$$

the problem has at least three solutions, $u_i (i = 1, 2, 3)$, with $u_1 \in \Phi^{-1}(-\infty, \rho_1)$ and $u_2 \in \Phi^{-1}(\rho_1, \rho_2/2)$, where $\rho_1 = \frac{m(c_1 c_{2p})^p + \underline{q}c_1^p}{pc_{2p}^p T^{p-1}}$, $\rho_2 = \frac{m(c_2 c_{2p})^p + \underline{q}c_2^p}{pc_{2p}^p T^{p-1}}$.

Proof. We invoke Lemma 1 to establish the desired result. Let the Banach space S and the functionals Φ and Ψ be defined as in Section 2. These functionals are continuously Gâteaux differentiable, satisfying the following fundamental relation:

$$\inf_{u \in S} \Phi(u) = \Phi(0) = \Psi(0) = 0.$$

Since $\Phi(u) = \frac{1}{p}G(\|u\|^p) + \frac{1}{p}\sum_{k=1}^{T} q(k)|u(k)|^p$ with $G(\xi) = \int_0^\xi g(t)dt$, and $g(t) \geq m > 0, q(k) \geq 0$, then $G(\xi)$ is a monotone increasing function with $G(\xi) \to +\infty$ as $\xi \to +\infty$. Thus, we acquire $\Phi(u) \to +\infty$ when $\|u\| \to +\infty$. Accordingly, we confirm the coercivity of Φ.

Owing to $\|u\|^p$ being convex and $G(s)$ being convex, we obtain that Φ is convex.

Therefore, Φ and Ψ satisfy (A_1) and (A_2) of Lemma 1. At present, let x_1 and x_2 be two local minima for $I_\lambda(u)$ in S. Then, x_1 and x_2 are two critical points of $I_\lambda(u)$. Now, x_1 and x_2 are two solutions of Problem (1). By the strong maximum principle (Lemma 5), we

have $x_1(k) \geq 0$ and $x_2(k) \geq 0$ for $k \in [1, T]$. Thus, we obtain $tx_1(k) + (1-t)x_2(k) \geq 0$ for $t \in [0, 1]$ and $k \in [1, T]$. Next, $\Psi(tx_1 + (1-t)x_2) \geq 0$, and (A_3) is verified.

When $\|u\| \leq c_{2p}\left(\frac{p\rho_1}{mc_{2p}^p + \underline{q}}\right)^{\frac{1}{p}}$ by Lemma 4, we obtain

$$\begin{aligned} \max_{k\in[1,T]} |u(k)| &\leq c_{2p} T^{(p-1)/p} \left(\frac{p\rho_1}{mc_{2p}^p + \underline{q}}\right)^{\frac{1}{p}} \\ &= c_{2p}\left(\frac{p\rho_1 T^{p-1}}{mc_{2p}^p + \underline{q}}\right)^{\frac{1}{p}} \\ &= c_1. \end{aligned}$$

Similarly, when $\|u\| \leq c_{2p}\left(\frac{p\rho_2}{mc_{2p}^p + \underline{q}}\right)^{\frac{1}{p}}$, we obtain

$$\begin{aligned} \max_{k\in[1,T]} |u(k)| &\leq c_{2p} T^{(p-1)/p} \left(\frac{p\rho_2}{mc_{2p}^p + \underline{q}}\right)^{\frac{1}{p}} \\ &= c_{2p}\left(\frac{p\rho_2 T^{p-1}}{mc_{2p}^p + \underline{q}}\right)^{\frac{1}{p}} \\ &= c_2. \end{aligned}$$

for $u \in S$. Then, we have

$$\begin{aligned} \frac{\sup_{u\in\Phi^{-1}(-\infty,\rho_1)} \Psi(u)}{\rho_1} &\leq \frac{\sup_{\|u\|\leq c_{2p}\left(\frac{p\rho_1}{mc_{2p}^p + \underline{q}}\right)^{\frac{1}{p}}} \sum_{k=1}^{T} F(k, u(k))}{\rho_1} \\ &\leq \frac{\sum_{k=1}^{T} F(k, c_1)}{\rho_1} \\ &= \frac{pc_{2p}^p T^{p-1} F_{c_1}}{m(c_1 c_{2p})^p + \underline{q}c_1^p}, \end{aligned}$$

and

$$\begin{aligned} \frac{\sup_{u\in\Phi^{-1}(-\infty,\rho_2)} \Psi(u)}{\rho_2} &\leq \frac{\sup_{\|u\|\leq c_{2p}\left(\frac{p\rho_2}{mc_{2p}^p + \underline{q}}\right)^{\frac{1}{p}}} \sum_{k=1}^{T} F(k, u(k))}{\rho_1} \\ &\leq \frac{\sum_{k=1}^{T} F(k, c_2)}{\rho_2} \\ &= \frac{pc_{2p}^p T^{p-1} F_{c_2}}{m(c_2 c_{2p})^p + \underline{q}c_2^p}, \end{aligned}$$

We verify assumptions (i) and (ii) of Lemma 1.
Now, let $\overline{u} \in S$ be given by

$$\overline{u}(k) = \begin{cases} d, & k \in [0, T], \\ 0, & k = T + 1. \end{cases}$$

We obtain $\Phi(u) = \frac{1}{p}G(d^p) + \frac{1}{p}\sum_{k=1}^{T} q(k)d^p = \frac{1}{p}(G(d^p) + Qd^p)$. Hence, from (3), we have

$$\rho_1 < \Phi(\overline{u}) < \frac{\rho_2}{2}.$$

Moreover, we acquire $\frac{2\Phi(\overline{u})}{\Psi(\overline{u})} = \frac{2G(d^p)+2Qd^p}{pF_d}$.

By Lemma 1, for $\lambda \in \left(\frac{2\Phi(\overline{u})}{\Psi(\overline{u})}, \min\left\{\frac{m(c_1c_{2p})^p+\underline{q}c_1^p}{pc_{2p}^pT^{p-1}F_{c_1}}, \frac{m(c_2c_{2p})^p+\underline{q}c_2^p}{2pc_{2p}^pT^{p-1}F_{c_2}}\right\}\right)$ Problem (1) has at least three solutions, $u_i (i = 1, 2, 3)$, and $u_1' \in \Phi^{-1}\left(-\infty, \frac{m(c_1c_{2p})^p+\underline{q}c_1^p}{pc_{2p}^pT^{p-1}}\right)$, $u_2' \in \Phi^{-1}\left(\frac{m(c_1c_{2p})^p+\underline{q}c_1^p}{pc_{2p}^pT^{p-1}}, \frac{m(c_2c_{2p})^p+\underline{q}c_2^p}{2pc_{2p}^pT^{p-1}}\right)$, $u_3' \in \Phi^{-1}\left(-\infty, \frac{m(c_2c_{2p})^p+\underline{q}c_2^p}{pc_{2p}^pT^{p-1}}\right)$. □

Theorem 2. *Assume that there exist two positive constants, c and d, with*

$$G(d^p) + Qd^p > \frac{m(cc_{2p})^p + \underline{q}c^p}{c_{2p}^pT^{p-1}} \tag{4}$$

such that we have the following:

(N_1) $f(k, \xi) \geq 0$ *for each* $(k, \xi) \in [1, T] \times [-c, c]$;

(N_2) $\frac{pF_d}{G(d^p)+Qd^p} > \frac{pc_{2p}^pT^{p-1}F_c}{m(cc_{2p})^p+\underline{q}c^p}$;

(N_3) *There exists a non-negative constant* β *with*

$$\max_{k\in[1,T]} \limsup_{|\xi|\to+\infty} \frac{F(k, \xi)}{|\xi|} = \beta,$$

and it satisfies

$$\frac{mc_{2p}^p + \underline{q}}{pc_{2p}^pT^{(2p-1)/p}} > \frac{\beta(G(d^p) + Qd^p)}{pF_d},$$

If $\beta = 0$, *the above inequation is clearly valid.*

Moreover, let

$$\Lambda_1 = \frac{G(d^p) + Qd^p}{pF_d},$$

$$\Lambda_2 = \min\left\{\frac{mc_{2p}^p + \underline{q}}{p\beta c_{2p}^pT^{(2p-1)/p}}, \frac{m(cc_{2p})^p + \underline{q}c^p}{pc_{2p}^pT^{p-1}F_c}\right\},$$

Then, for each $\lambda \in \Lambda := (\Lambda_1, \Lambda_2)$, *Problem (1) has at least three solutions.*

Proof. We use Lemma 2 to prove our conclusion. S, Φ, Ψ, and I_λ are defined as in Section 2. Thus, we know that Φ and Ψ are two Gâteaux differentible functions, and one has $\inf_S \Phi = \Phi(0) = \Psi(0)$.

From Theorem 1 , it follows that Φ is coercive.

Let

$$r = \frac{m(cc_{2p})^p + \underline{q}c^p}{pc_{2p}^pT^{p-1}}$$

If $\Phi(u) \leq r$, we obtain

$$\frac{m}{p}\|u\|^p + \frac{\underline{q}}{pc_{2p}^p}\|u\|^p \leq \frac{1}{p}G(\|u\|^p) + \frac{1}{p}\sum_{k=1}^{T} q(k)|u(k)|^p \leq r,$$

and thus, we obtain $\|u\| \leq c_{2p}\left(\frac{pr}{mc_{2p}^{p}+\underline{q}}\right)^{\frac{1}{p}}$; by Lemma 4, one has

$$\|u\|_{\infty} \leq T^{(p-1)/p}\|u\| \leq c,$$

From condition (N_1) of Theorem 2, we obtain

$$\begin{aligned}\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} &\leq \frac{\sup_{\|u\|_{\infty}\leq c}\sum_{k=1}^{T}F(k,u(k))}{r}\\ &\leq \frac{pc_{2p}^{p}T^{p-1}\sum_{k=1}^{T}\max_{|\xi|\leq c}F(k,\xi)}{m(cc_{2p})^{p}+\underline{q}c^{p}}\\ &= \frac{pc_{2p}^{p}T^{p-1}F_c}{m(cc_{2p})^{p}+\underline{q}c^{p}}.\end{aligned}$$

According to $\lambda \in \Lambda$, it is easy to gain

$$\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r} < \frac{1}{\lambda}.$$

Let $\overline{u} \in S$ be given by

$$\overline{u}(k) = \begin{cases} d, & k \in [0,T],\\ 0, & k = T+1.\end{cases}$$

According to the definition of $\overline{u}$ and (4), we could know that

$$\Phi(\overline{u}) = \frac{1}{p}G(d^{p}) + \frac{1}{p}Qd^{p} > \frac{m(cc_{2p})^{p}+\underline{q}c^{p}}{pc_{2p}^{p}T^{p-1}} > 0.$$

Moreover, we have

$$\frac{\Psi(\overline{u})}{\Phi(\overline{u})} = \frac{p\sum_{k=1}^{T}F(k,\overline{u}(k))}{G(d^{p})+Qd^{p}} = \frac{pF_d}{G(d^{p})+Qd^{p}},$$

Therefore, the following holds:

$$\frac{\Psi(\overline{u})}{\Phi(\overline{u})} > \frac{1}{\lambda}.$$

Condition (i) of Lemma 2 follows.

Next, we prove the coercivity of the functionals $I_\lambda(u)$. Owing to $\lambda < \frac{mc_{2p}^{p}+\underline{q}}{p\beta c_{2p}^{p}T^{(2p-1)/p}}$, when we fix $\lambda \in \Lambda$, we can obtain that there is a positive constant ϵ such that

$$\lambda < \frac{mc_{2p}^{p}+\underline{q}}{p(\beta+\epsilon)c_{2p}^{p}T^{(2p-1)/p}}.$$

According to (N_3) of Theorem 2, we obtain

$$\limsup_{|\xi|\to+\infty}\frac{F(k,\xi)}{|\xi|} \leq \beta < \beta+\epsilon.$$

Therefore, there is a positive constant h such that

$$F(k,\xi) \leq (\beta+\epsilon)|\xi| + h$$

for each $(k, \xi) \in [1, T] \times \mathbb{R}$. By applying Lemma 4, we obtain

$$\begin{aligned}
\lambda \sum_{k=1}^{T} F(k, u(k)) &\leq \lambda \sum_{k=1}^{T} [(\beta + \epsilon)|u(k)| + h] \\
&\leq \lambda \sum_{k=1}^{T} [(\beta + \epsilon)\|u\|_\infty + h] \\
&\leq \lambda T(\beta + \epsilon) T^{(p-1)/p} \|u\| + \lambda\, Th \\
&= \lambda(\beta + \epsilon) T^{(2p-1)/p} \|u\| + \lambda Th,
\end{aligned}$$

for each $u \in S$. Thus, we can determine that

$$\begin{aligned}
I_\lambda(u) &= \Phi(u) - \lambda \Psi(u) \\
&= \frac{1}{p} G(\|u\|^p) + \frac{1}{p} \sum_{k=1}^{T} q(k)|u(k)|^p - \lambda \sum_{k=1}^{T} F(k, u(k)) \\
&\geq \frac{m}{p} \|u\|^p + \frac{\underline{q}}{p} \sum_{k=1}^{T} |u(k)|^p - \lambda(\beta + \epsilon) T^{(2p-1)/p} \|u\| - \lambda\, Th \\
&\geq \frac{m}{p} \|u\|^p + \frac{\underline{q}}{p c_{2p}^p} \|u\|^p - \lambda(\beta + \epsilon) T^{(2p-1)/p} \|u\| - \lambda\, Th \\
&\geq (\frac{m}{p} + \frac{\underline{q}}{p c_{2p}^p}) \|u\| - \lambda(\beta + \epsilon) T^{(2p-1)/p} \|u\| - \lambda\, Th \\
&= \left[\frac{m}{p} + \frac{\underline{q}}{p c_{2p}^p} - \lambda(\beta + \epsilon) T^{(2p-1)/p} \right] \|u\| - \lambda Th,
\end{aligned}$$

for all $\|u\| \geq 1$. It is easy to see that $\left[\frac{m}{p} + \frac{\underline{q}}{p c_{2p}^p} - \lambda(\beta + \epsilon) T^{(2p-1)/p} \right] > 0$, and when $\|u\| \to +\infty$, we have $I_\lambda(u) \to +\infty$. All assumptions of Lemma 2 are proven. In conclusion, Problem (1) has at least three solutions. □

Remark 1. *If condition (N_1) of Theorem 2 transforms into*

$$f(k, \xi) > 0, \quad (k, \xi) \in [1, T] \times \mathbb{R},$$

and the other conditions of Theorem 2 remain, then $u = 0$ is not a solution of Problem (1). Then, for each $\lambda \in \Lambda := (\Lambda_1, \Lambda_2)$, Problem (1) has at least three positive solutions.

Theorem 3. *Assume that there exists $M \in \mathbb{R}^+$ such that $g(s) \leq M^2 s^{M-1}$ for $s > 1$ and there exist two positive constants, c and d, with*

$$G(d^p) + Q d^p < \frac{m(c c_{2p})^p + \underline{q} c^p}{c_{2p}^p T^{p-1}} \tag{5}$$

and the following conditions hold:
(H_1) $f(k, 0) \geq 0, k \in [1, T]$;
(H_2) $\frac{p F_d}{G(d^p) + Q d^p} > \frac{p c_{2p}^p T^{p-1} \sum_{k=1}^{T} \max_{|\xi| \leq c} F(k, \xi)}{m(c c_{2p})^p + \underline{q} c^p}$;
(H_3) *There is a positive constant β such that*

$$\min_{k \in [1,T]} \liminf_{|\xi| \to +\infty} \frac{F(k, \xi)}{|\xi|^{Mp}} = \beta,$$

and it satisfies

$$\frac{Mc_{1p}^{p}c_{2Mp}^{Mp}+\overline{q}c_{2Mp}^{Mp}}{p\beta c_{1p}^{p}}<\frac{m(cc_{2p})^{p}+\underline{q}c^{p}}{pc_{2p}^{p}T^{p-1}\sum_{k=1}^{T}\max_{|\xi|\leq c}F(k,\xi)}.$$

Furthermore, let

$$Y_1=\max\left\{\frac{G(d^p)+Qd^p}{pF_d},\frac{Mc_{1p}^{p}c_{2Mp}^{Mp}+\overline{q}c_{2Mp}^{Mp}}{p\beta c_{1p}^{p}}\right\},$$

$$Y_2=\frac{m(cc_{2p})^{p}+\underline{q}c^{p}}{pc_{2p}^{p}T^{p-1}\sum_{k=1}^{T}\max_{|\xi|\leq c}F(k,\xi)},$$

Then, for each $\lambda\in Y:=(Y_1,Y_2)$, *Problem (1) admits at least two positive solutions.*

Proof. Our goal is to use Lemma 3 to demonstrate our conclusion as above. Let S,Φ,Ψ, and I_λ be defined as in Section 2, so we obtain that Φ and Ψ are two continuously G$\widehat{a}$teaux differentiable funtions, and $\inf_S\Phi=\Phi(0)=\Psi(0)$.

Now, let r and $\overline{u}$ be defined as in Theorem 2. According to (H_2) of Theorem and $\lambda\in Y$, we have

$$\begin{aligned}\frac{\sup_{\Phi(u)\leq r}\Psi(u)}{r}&\leq\frac{\sup_{\Phi(u)\leq r}\sum_{k=1}^{T}F(k,u(k))}{r}\\&\leq\frac{pc_{2p}^{p}T^{p-1}\sum_{k=1}^{T}\max_{|\xi|\leq c}F(k,\xi)}{m(cc_{2p})^{p}+\underline{q}c^{p}}\\&<\frac{1}{\lambda},\end{aligned}$$

and

$$\frac{\Psi(\overline{u})}{\Phi(\overline{u})}=\frac{pF_d}{G(d^p)+Qd^p}>\frac{1}{\lambda},$$

Meanwhile, from (5), we acquire

$$0<\Phi(\overline{u})=\frac{1}{p}[G(d^p)+Qd^p]<\frac{m(cc_{2p})^{p}+\underline{q}c^{p}}{pc_{2p}^{p}T^{p-1}}=r.$$

Therefore, the inequation $0<\Phi(\overline{u})<r$ of Lemma 3 is true. Next, we prove that I_λ is unbounded from below. According to $\lambda>\frac{Mc_{1p}^{p}c_{2Mp}^{Mp}+\overline{q}c_{2Mp}^{Mp}}{p\beta c_{1p}^{p}},(\lambda\in Y)$. when we fix $\lambda\in Y$, it is easy to see that there is a positive solution constant $\epsilon(\epsilon<\beta)$ such that

$$\lambda>\frac{Mc_{1p}^{p}c_{2Mp}^{Mp}+\overline{q}c_{2Mp}^{Mp}}{p(\beta-\epsilon)c_{1p}^{p}},(\lambda\in Y).$$

Owing to (H_3) of Theorem 3, we have

$$\liminf_{|\xi|\to+\infty}\frac{F(k,\xi)}{|\xi|^{Mp}}\geq\beta>\beta-\epsilon.$$

Undoubtedly, there exists a positive constant l such that

$$F(k,\xi)\geq(\beta-\epsilon)|\xi|^{Mp}-l,$$

for each $(k, \xi) \in [1, T] \times \mathbb{R}$. Thus, we acquire

$$\begin{aligned}
\lambda \sum_{k=1}^{T} F(k, u(k)) &\geq \lambda \sum_{k=1}^{T} [(\beta - \epsilon)|u(k)|^{Mp} - l] \\
&= \lambda(\beta - \epsilon) \sum_{k=1}^{T} |u(k)|^{Mp} - \lambda\, Tl \\
&= \lambda(\beta - \epsilon)\|u\|_{Mp}^{Mp} - \lambda\, Tl \\
&\geq \lambda(\beta - \epsilon)\frac{1}{c_{2Mp}^{Mp}}\|u\|^{Mp} - \lambda Tl,
\end{aligned}$$

for each $u \in S$.

Meanwhile, according to $g(s) \leq M^2 s^{M-1}$, we have $G(s) = \int_0^s g(t)dt \leq Ms^M + \int_0^1 g(t)dt - M$ when $s > 1$. So, we can determine that

$$\begin{aligned}
I_\lambda &= \Phi(u) - \lambda\Psi(u) \\
&= \frac{1}{p}G(\|u\|^p) + \frac{1}{p}\sum_{k=1}^{T} q(k)|u(k)|^p - \lambda \sum_{k=1}^{T} F(k, u(k)) \\
&\leq \frac{M}{p}\|u\|^{Mp} + \frac{\overline{q}}{p}\sum_{k=1}^{T} |u(k)|^p - \lambda(\beta - \epsilon)\frac{1}{c_{2Mp}^{Mp}}\|u\|^{Mp} + \lambda Tl + L - M \\
&\leq \frac{M}{p}\|u\|^{Mp} + \frac{\overline{q}}{pc_{1p}^{p}}\|u\|^p - \lambda(\beta - \epsilon)\frac{1}{c_{2Mp}^{Mp}}\|u\|^{Mp} + \lambda Tl + L - M \\
&\leq \frac{M}{p}\|u\|^{Mp} + \frac{\overline{q}}{pc_{1p}^{p}}\|u\|^{Mp} - \lambda(\beta - \epsilon)\frac{1}{c_{2Mp}^{Mp}}\|u\|^{Mp} + \lambda Tl + L - M \\
&= \left[\frac{M}{p} + \frac{\overline{q}}{pc_{1p}^{p}} - \lambda(\beta - \epsilon)\frac{1}{c_{2Mp}^{Mp}}\right]\|u\|^{Mp} + \lambda Tl + L - M \\
&= \left[\frac{Mc_{1p}^{p} + \overline{q}}{pc_{1p}^{p}} - \frac{\lambda(\beta - \epsilon)}{c_{2Mp}^{Mp}}\right]\|u\|^{Mp} + \lambda Tl + L - M.
\end{aligned}$$

for each $\|u\| \geq 1$, where $L = \int_0^1 g(s)ds$. Owing to $\left[\frac{Mc_{1p}^{p} + \overline{q}}{pc_{1p}^{p}} - \frac{\lambda(\beta - \epsilon)}{c_{2Mp}^{Mp}}\right] < 0$, when $\|u\| \to \infty$, we know that $I_\lambda(u) \to -\infty$, meaning that $I_\lambda(u)$ is bounded from below. Meanwhile, we obtain that $-I_\lambda(u)$ is coercive, which explains that the function $I_\lambda(u)$ satisfies the PS condition. All assumptions of Lemma 3 are proven; thus, Problem (1) admits at least two non-zero solutions.

Next, our goal is the acquisition of positive solutions to Problem (1). Based on $I_\lambda^+(u)$ and condition (H_1) of Theorem 3, it follows that each non-zero point on S of the functional $I_\lambda^+(u)$ is a positive solution to Problem (1) by the strong maximum principle (Lemma 6). Therefore, Problem (1) admits at least two positive solutions. □

Remark 2. *If $f(k, \xi)$ is non-negative for each $(k, \xi) \in [1, T] \times [0, +\infty)$, we obtain that condition (H_1) follows, and condition (H_2) becomes*

$$\frac{pF_d}{G(d^p) + Qd^p} > \frac{pc_{2p}^{p} T^{p-1} F_c}{m(cc_{2p})^p + \underline{q}c^p}.$$

4. Examples

In this section, we present three simple examples to illustrate our conclusions.

Example 1. *Let* $T = 5; p = 2; g(s) = 1 + 2s$; *and* $g(\|u\|) = 1 + 2\|u\|^2 \geq m = 1$. *We have* $G(s) = s + s^2$.

Let

$$f(k,\xi) = f(\xi) = \begin{cases} 0, & \xi \in (-\infty, 0), \\ \xi^9, & \xi \in [0,2], \\ (4-\xi)^9, & \xi \in [2,3], \\ 1, & \xi \in (3,+\infty), \end{cases} \tag{6}$$

for each $k \in [1,5]$. *Thus, we have*

$$F(k,\xi) = F(\xi) = \begin{cases} 0, & \xi \in (-\infty, 0), \\ \frac{1}{10}\xi^{10}, & \xi \in [0,2], \\ -\frac{1}{10}(4-\xi)^{10} + \frac{2048}{10}, & \xi \in [2,3], \\ \xi - 3 + \frac{2047}{10}, & \xi \in (3,+\infty). \end{cases} \tag{7}$$

Let $c_1 = 1; d = 2; c_2 = 40$; *and* $q(k) = |\frac{2}{3}\sin(\frac{k\pi}{2})|$. *Thus, we have* $\underline{q} = 0, Q = 2$. *Then, condition (3) of the theorem holds, since*

$$\frac{1}{10} < \frac{1}{2}[G(4) + 4Q] = 14 < \frac{40^2}{20} = 80.$$

Meanwhile, we have

$$\frac{pc_{2p}^p T^{p-1} F_{c_1}}{m(c_1 c_{2p})^p + \underline{q}c_1^p} = 1, \frac{2pc_{2p}^p T^{p-1} F_{c_2}}{m(c_2 c_{2p})^p + \underline{q}c_2^p} = \frac{4834}{1600} = 3.02125,$$

and

$$\frac{pF_d}{2G(d^p) + 2Qd^p} \approx 3.6571.$$

Therefore, the conditions of Theorem 1 follows. Then, for each $\lambda \in (0.2734375, 0.33099)$, *the problem*

$$\begin{cases} -(1+2\|u\|^2)(\Delta^2 u(k-1)) + |\sin(\frac{k\pi}{2})|u(k) = \lambda f(k, u(k)), \quad k \in [1,5], \\ \Delta u(0) = u(6) = 0. \end{cases} \tag{8}$$

has at least three solutions, $u_i (i = 1,2,3)$, *with* $u_1 \in \Phi^{-1}(-\infty, 0.1); u_2 \in \Phi^{-1}(0.1, 80)$; *and* $u_3 \in \Phi^{-1}(-\infty, 160)$.

Example 2. *Let* $g(s) = 2 + \cos(s)$ *and*

$$f(k,\xi) = f(\xi) = \begin{cases} \xi^{10}, & \xi \in (-\infty, 2), \\ x + 2^{10} - 2, & \xi \in [2,4], \\ \frac{2^{10}+2}{x-3}, & \xi \in (4,+\infty), \end{cases}$$

for each $k \in [1, T]$. *Then, we obtain* $g(s) \geq 1 = m, G(s) = 2s + \sin(s)$ *and*

$$F(k,\xi) = F(\xi) = \begin{cases} \frac{1}{11}\xi^{11}, & \xi \in (-\infty, 2), \\ \frac{1}{2}x^2 + 1022x - 2046 + \frac{2^{11}}{11}, & \xi \in [2, 4], \\ 1026\ln(x-3) + 2050 + \frac{2^{11}}{11}, & \xi \in (4, +\infty). \end{cases}$$

Let $T = 4; p = 3; c = 1; d = 2;$ *and* $q(k) = \frac{1}{2} + \frac{1}{2}\cos(k\pi)$. *We obtain* $Q = \sum_{k=1}^{4} q(k) = 2, \underline{q} = 0$. *Then, condition (4) of Theorem 2 holds, since*

$$32 + \sin(8) = G(d^p) + Qd^p > \frac{m(cc_{2p})^p + \underline{q}c^p}{c_{2p}^p T^{p-1}} = \frac{1}{16}.$$

Meanwhile, we have

$$\frac{pF_d}{G(d^p) + Qd^p} = \frac{3F_2}{G(8) + 16} = \frac{3 \times 2^{11}}{11 \times (32 + \sin(8))} \approx 16.93108,$$

and

$$\frac{pc_{2p}^p T^{p-1} F_c}{m(cc_{2p})^p + \underline{q}c^p} = \frac{48}{11} \approx 4.36364,$$

Therefore, condition (N_2) *of Theorem 2 follows. Moreover, we find that*

$$\beta = \max_{k\in[1,T]} \limsup_{|\xi|\to+\infty} \frac{F(k,\xi)}{|\xi|} = \limsup_{|\xi|\to+\infty} \frac{1026\ln(x-3) + 2050 + \frac{2^{11}}{11}}{|\xi|} = 0.$$

Consequently, applying the result of Theorem 2, for each $\lambda \in (0.05906, 0.22917)$, *the problem*

$$\begin{cases} -(2 + \cos(\|u\|^4))(\Delta\phi_4(\Delta u(k-1))) +)(1 + \cos(k\pi))\phi_4(u(k)) = \lambda f(k, u(k)), & k \in [1, 4], \\ \Delta u(0) = u(5) = 0. \end{cases}$$

admits at least three positive solutions.

Example 3. *Let* $g(s) = 2^{-x} + 1$ *and*

$$f(k,\xi) = f(\xi) = \begin{cases} 1, & \xi \in (-\infty, 15], \\ 6^{\xi-15}, & \xi \in (15, +\infty), \end{cases}$$

for each $k \in [1, T]$. *Then, we have* $g(s) \geq 1 = m; G(s) = \frac{1-2^{-s}}{\ln 2} + s;$ *and*

$$F(k,\xi) = F(\xi) = \begin{cases} \xi, & \xi \in (-\infty, 15], \\ \frac{6^{\xi-15}-1}{\ln 15} + \xi, & \xi \in (15, +\infty). \end{cases}$$

Let $T = 6; p = 4; c = 15; d = 1;$ *and* $q(k) = \frac{2k-2}{3}$. *We have* $Q = \sum_{1}^{5} q(k) = 10$ *and* $\underline{q} = 0$. *Then, we obtain*

$$11 + \frac{1}{2\ln 2} = G(d^p) + Qd^p < \frac{m(cc_{2p})^p + \underline{q}c^p}{c_{2p}^p T^{p-1}} = 234.375.$$

Meanwhile, we obtain

$$\frac{pF_d}{G(d^p)+Qd^p}=\frac{4}{11+\frac{1}{2\ln(2)}}\approx 0.34126,$$

$$\frac{pc_{2p}^p T^{p-1}\sum_{k=1}^{T}\max_{|\xi|\le c}F(k,\xi)}{m(cc_{2p})^p+\underline{q}c^p}=\frac{4\times 6^3}{15^3}=0.256.$$

Owing to $g(s)\le 16s^3$ *when* $s\ge 1$, *one has*

$$\beta=\min_{k\in[1,T]}\liminf_{|\xi|\to+\infty}\frac{F(k,\xi)}{|\xi|^{16}}=+\infty,$$

Thus, we have

$$0=\frac{Mc_{1p}^p c_{2Mp}^{Mp}+\bar{q}c_{2Mp}^{Mp}}{p\beta c_{1p}}<\frac{m(cc_{2p})^p+\underline{q}c^p}{pc_{2p}^p T^{p-1}\sum_{k=1}^{T}\max_{|\xi|\le c}F(k,\xi)}=3.90625.$$

Consequently, applying the result of Theorem 3, for each $\lambda\in(2.9303,3.90625)$, *the problem*

$$\begin{cases}-(2^{\|u\|^4}+1)(\Delta\phi_4(\Delta u(k-1)))+(k-1)\phi_4(u(k))=\lambda f(k,u(k)), & k\in[1,6],\\ \Delta u(0)=u(7)=0.\end{cases}$$

admits at least two positive solutions.

5. Discussion

The discrete Robin problem of the Kirchhoff type with the p-Laplacian is widely used in areas like nonlinear material mechanics, non-Newtonian fluid flow, and physical coupling. In contrast, Neumann problems are especially important in optimal design, so studying Neumann problems of the Kirchhoff type involving the p-Laplacian is necessary. Notably, homoclinic solutions and heteroclinic solutions are also key topics in difference equations. Whether critical point theory can prove their existence for these problems needs further research. All these questions deserve continued exploration.

6. Conclusions

The Kirchhoff model and p-Laplacian operator exhibit extensive applications in both continuous and discrete mechanical systems. This work investigates a generalized discrete Robin problem that integrates Kirchhoff-type problems with the p-Laplacian difference operator. Using critical point theory, we establish the existence and multiplicity of positive solutions across diverse parameter regimes. Notably, when the system parameters are specialized as $g(s)=a+bs$, $p=2$, and $q(k)\equiv 0$, the proposed formulation transforms into the classical Kirchhoff-type difference equation, for which we derive enhanced existence criteria compared to prior studies.

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