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On the Strong Atoms of Q-Algebras

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Abstract: The concept of strong atoms in Q-algebras is discussed herein. In this work, some properties of strong atoms are provided. We show that there is no Q-algebra X with $|X| \geq 3$ such that all elements are strong atoms. We also find that any two-element subset of X containing a constant 0 is a subalgebra of X whenever X contains a strong atom. Moreover, any subset of X with the cardinality equal to 3 containing a strong atom and a constant 0 is always a subalgebra. We present some results concerning the concept of an ideal. In a Q-algebra X that contains a strong atom, any ideal of X is a subalgebra of X. An ideal of a Q-algebra X that is induced by any subset containing a strong atom is equal to X. Furthermore, we show that, for any Q-algebra X with a strong atom, there is only one ideal containing a strong atom. In particular, for $|X| \leq 4$, we propose that a finite union of ideals of X is again an ideal of X.

Keywords: Q-algebra; subalgebra; ideals; G-part; atoms; strong atoms

MSC: 03G25; 03G27; 06F35; 20K01

1. Introduction and Preliminaries

In recent years, logical algebras have been widely studied. The starting point of this topic was with the introduction of two kinds of algebras by Japanese mathematicians in the 20th century. In 1966, Iseki, K. and Imai, Y. [1] introduced a notion of BCK-algebra, which is a logical algebra. Shortly thereafter in the same year, a superclass of BCK-algebra was discussed by Iseki, K.; this algebra is called BCI-algebra [2]. For more details of BCI- and BCI-algebras, see also [3,4]. In 1983, Li, X. and Hu, Q.P. [5,6] presented the concept of BCH-algebra, which is a superclass of BCI- and BCK-algebras. In the time between the end of the 20th century and the beginning of the 21st century, many generalizations of and algebras related to BCI- and BCK-algebras were introduced, including the following:

- The class of BCC-algebra by Komori, Y. in 1984 [7];
- The class of BH-algebra by Jun, Y.B. et al. in 1998 [8];
- The class of d-algebra by Neggers, J. and Kim, H.S. in 1999 [9];
- The class of Q-algebra by Neggers, J. et al. in 2001 [10];
- The class of BE-algebra by Kim, H.S. and Kim, Y.H. in 2006 [11];
- The class of CI-algebra by Meng, B.L. in 2010 [12];
- The class of BCL-algebra by Liu, Y. in 2011 [13], etc.

Let us provide more details of some generalizations of BCI- and BCK-algebras. In [10], Neggers, J., Ahn, S., and Kim, H.S. introduced another generalization of BCI- and BCK-algebras, the so-called Q-algebras, in 2001. A Q-algebra consists of a non-empty set, X, and

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a constant $0 \in X$, together with a binary operation, $*$, defined on X such that the identities (Q_1), (Q_2) and (Q_3) are satisfied.

(Q_1) $w * w = 0$;
(Q_2) $w * 0 = w$;
(Q_3) $(w * p) * q = (w * q) * p$ for all $w, p, q \in X$.

We will omit the symbol of a binary operation for convenience. Theorems and properties in BCI-algebras were generalized by Neggers, J., Ahn, S., and Kim, H.S. in [10]. They also discussed the concept of an implicative Q-algebra. There are other discussions of Q-algebras. In [14–16], morphisms and mappings of Q-algebras were discussed. In [14], Ahn, S.S., Lee, H.D., and Kim, H.S. examined some properties of homomorphisms and presented the congruences induced by homomorphisms. They also introduced the notion of mappings on Q-algebra, so-called R-maps and L-maps, and demonstrated that a set of all R-maps form a commutative semigroup. In [15], homomorphisms in Q-algebras and quotient Q-algebras were discussed by So, K. and Ahn, S.S. They also presented the idea of a medial Q-algebra, which is a Q-algebra X that satisfies $(ab)(zu) = (az)(bu)$ for all $a, b, z, u \in X$. They found that a Q-algebra that satisfies an associative property is a medial Q-algebra. In 2011, Kim, K.H. discussed the notion of so-called right-fixed maps on Q-algebras in relation to the concepts of kernels, subalgebras, and ideals [16]. He showed that a set $Ker(\alpha)$, a kernel of a right-fixed map, α, is a subalgebra, and $Ker(\alpha)$ is an ideal whenever α is an endomorphism. The concept of a fuzzy set in Q-algebras was mentioned in [17–19]. The concepts of prime ideals and fuzzy prime ideals were offered in [17,18], while a fuzzy Q-ideal was discussed in [19]. In 2012, Mostafa, S.M., Noby, M.A. and Elgendy, O.R. introduced the concept of fuzzy Q-ideals and Q-ideals in Q-algebras [19]. They showed the connection between the concepts of a fuzzy Q-ideal and a fuzzy BCK-ideal (which is discussed in [20]). A fuzzy Q-ideal of a Q-algebra X is always a fuzzy BCK-ideal. Moreover, they provided a characterization of fuzzy Q-ideals, showing that any fuzzy subset, μ, of a Q-algebra X is a fuzzy Q-ideal if μ is an empty set or it is a Q-ideal of X. Later, in 2020, the concepts of a prime ideal and a fuzzy prime ideal were introduced by Abdullah, H.K. and Tach, M. in [17,18], respectively. They obtained some useful information on ideals and prime ideals in Q-algebras. They showed that there is no prime ideal of order $n - 1$ when $|X| = n$. They also provided a characterization of fuzzy prime ideals in bounded Q-algebras. Another kind of ideal, the so-called pentapartitioned neutrosophic Q-ideal (PN-Q-ideal), was introduced by S. Das et al. in 2021 [21]. The concept of a PN-Q-ideal was motivated by a numerical-valued refined neutrosophic logic. In [21], the authors demonstrated that the intersection of PN-Q-ideals is again PN-Q-ideal. The concepts of the G-part and ideal were also offered in [10]. Let I be a subset of a Q-algebra X and $I \neq \emptyset$. The subset I is an ideal if the conditions (I_1) and (I_2) are satisfied.

(I_1) $0 \in I$;
(I_2) $x \in X, y \in I, xy \in I$ implies $x \in I$.

It is an evidential fact that the sets $\{0\}$ and X are ideals of a Q-algebra X. A subset $G(X)$ of a Q-algebra X defined by

$$G(X) = \{a \in X \mid 0a = a\}$$

is called a G-part of X. By (Q_1), $00 = 0$, then $0 \in G(X)$. Thus, $G(X) \neq \emptyset$. The authors of [10] provided a characterization of the set $G(X)$, which is an ideal of a Q-algebra X with a certain order of X. For $|X| = 3$, $G(X)$ is an ideal of X if and only if $G(X)$ is a singleton set, i.e., $G(X) = \{0\}$. Moreover, they showed that any implicative ideal of X always contains $G(X)$. In the same year, 2001, Sun, D. [22] suggested the concepts of an atom and strong atom in BCK-algebras. He showed that a constant 0 together with a set of all strong atoms is

an ideal. Actually, the concept of atoms was mentioned before in 1995 by Dudek, W.A. and Zhang, X. [23]. In *BCC*-algebras, they examined some properties of atoms and found that any subalgebra is an ideal if all elements in *BCC*-algebra are atoms, and vice versa. Many years later, Kang, S.E. and Ahn, S.S. [24] introduced the concept of an atom in Q-algebras in 2010. An element z of a Q-algebra X is called an atom if z satisfies the following:

$$\text{for } w \in X,\ wz = 0 \ \text{ implies } \ w = z.$$

We denote a set of all atoms of X by $A(X)$. It is easy to see that a constant 0 is an atom of X. The authors of [24] provided some properties of atoms. They also discovered that atoms, subalgebras, and ideals are related. A non-empty subset of a Q-algebra X is a subalgebra if S is closed, i.e., for any $x, y \in S$, $xy \in S$. They proved that, in a Q-algebra X such that all subalgebras are ideals, all elements are atoms.

Example 1. *Let $X = \{0, a, p, q, u\}$ and let $Y = \{0, w, z\}$. We define binary operations $*$ on X and $\bullet$ on Y with the following tables:*

$*$	0	a	p	q	u
0	0	a	u	u	p
a	a	0	p	q	q
p	p	q	0	a	a
q	q	p	a	0	0
u	u	p	a	0	0

$\bullet$	0	w	z
0	0	w	w
w	w	0	0
z	z	0	0

It is not complicated to check that $(X; *, 0)$ and $(Y; \bullet, 0)$ are Q-algebras. From the above tables, we obtain $G(X) = \{0, a\}$, $A(X) = \{0, a, p\}$ and $G(Y) = \{0, w\}$, $A(Y) = \{0\}$. Since $0, p \in A(X)$ but $0p = u \notin A(X)$, $A(X)$ is not a subalgebra of X. Since $qa = p \in A(X)$, $a \in A(X)$, and $q \notin A(X)$, $A(X)$ is not an ideal of X. In addition, $G(X)$ is an ideal, and $A(X)$ is also a subalgebra of X, while $G(Y)$ is not an ideal, but it is a subalgebra of Y. A subset $S = \{0, p\}$ of X is not a subalgebra since $0p = u \notin S$. In addition, a subset S is not an ideal of X since $ap = p \in S$ and $p \in S$ but $a \notin S$.

In general, in a Q-algebra, any subalgebra need not be an ideal, as seen in Example 1, where a subset $G(Y) = \{0, w\}$ of a Q-algebra Y is a subalgebra, but it is not an ideal of Y. However, if all elements of a Q-algebra are atoms, then any subalgebra satisfies the ideal properties (I_1) and (I_2), as demonstrated in [24]. Recently, in 2024, the authors of [25,26] studied Q-algebras and approached the concept of ideals and the G-part. In [25], the authors provided some properties of the set G-part. They found that $G(X) \neq X$ if the order of X is odd and greater than 1. All elements of $G(X)$ commute with each other, and $G(X)$ is an abelian group whenever it is an ideal. In addition, an abelian group T with the property that every element except an identity has an order of 2 is a Q-algebra with $G(T) = T$. They also gave a characterization of a G-part that is an ideal. The authors in [26] dealt with atoms and strong atoms of Q-algebras in relation to the concepts of the G-part and ideals. The authors examined some relations between atoms, strong atoms, and G-parts. They showed that, when X has an ideal I, $I \neq \{0\}$, $I \subseteq G(X)$, and X does not contain a strong atom. Moreover, $A(X) \cap G(X)$ is an abelian group when $A(X)$ is an ideal.

In this study, we deal with strong atoms in Q-algebras. We provide that there is no Q-algebra X, $|X| \geq 3$, in which all elements are strong atoms. We also find that any subset S, $0 \in S$, with a cardinality equal to 2 is a subalgebra of X whenever X contains strong atoms. Moreover, any subset of X with a cardinality of 3 containing a strong atom and a constant 0 is a subalgebra of X. We also demonstrate that a finite union of ideals of X containing a strong atom is again an ideal of X.

We will mention some related definitions and properties that we will use later. The concept of strong atoms in Q-algebras was introduced in [25], which was inspired by the concept of strong atoms in BCK-algebra [22].

From here onward, we denote X as a Q-algebra unless otherwise specified.

Definition 1 ([25]). *Let z be an atom of X. An element z is a strong atom if $z \neq 0$ and for all $x \in X \setminus \{z\}$, $zx = z$.*

For a Q-algebra X, we denote

$$SA(X) = \{z \in X \mid z \text{ is a strong atom}\} \cup \{0\}.$$

A Q-algebra, in general, needs not contain a strong atom, as seen in Example 1. Q-algebras X and Y do not contain a strong atom. A Q-algebra X contains atoms $0, a$, and p, but none of them are strong atoms since $ap = p \neq a$, $pu = a \neq p$. In a Q-algebra Y, there is only one atom, namely, a constant 0. Thus, Y does not contain a strong atom.

Example 2. *Let us consider a Q-algebra $(X; *, 0)$ when $X = \{0, a, b, c\}$, as shown in the following table:*

$*$	0	z	α	v
0	0	z	0	0
z	z	0	z	z
α	α	z	0	α
v	v	z	v	0

A set of all atoms of X is $A(X) = \{0, z\}$. Since $zx = z$ for all $x \in \{0, \alpha, v\}$, z is a strong atom. Therefore, $SA(X) = \{0, z\}$. It is not complicated to check that subsets $\{0, z\}$, $\{0, \alpha\}$, $\{0, v\}$, and $X - \{z\} = \{0, \alpha, v\}$ are subalgebras of X. Moreover, we see that subsets $\{0, \alpha\}$, $\{0, v\}$, and $X - \{a\} = \{0, \alpha, v\}$ are ideals of X; however, a subset $I = \{0, z\}$ is not an ideal because of $\alpha z = z \in I$, $z \in I$ but $\alpha \notin I$.

In [24,25], the authors provided some properties of Q-algebra and strong atoms. They showed that the set $SA(X)$ is always a subalgebra of X.

Lemma 1 ([24]). *Let $\alpha, \mu \in X$. Then, $0(\alpha\mu) = (0\alpha)(0\mu)$.*

Proposition 1 ([25]). *$SA(X)$ is a subalgebra of X.*

2. Strong Atoms and Subalgebras

In this section, we present some properties of strong atoms concerning subalgebras. We remind the reader of some facts about atoms concerning the set G-part. For a Q-algebra X, it is clear that $A(X) \cap G(X) \neq \emptyset$ since $0 \in A(X) \cap G(X)$. From Example 1, we can see that the set G-part need not be a subset of a set of all atoms, and vice versa. In [25], the authors gave a condition that yields an inclusion $G(X) \subseteq A(X)$. They showed that $G(X) \subseteq A(X)$ whenever $G(X)$ is an ideal. Next, we would like to show the relations between the G-part and strong atoms. First, we need the following proposition.

Proposition 2. *Let a be a strong atom of X; then, each of the following holds:*

(i) $0x = 0$ *for all* $x \in X \setminus \{a\}$.

(ii) $0a = a$.

Proof. (i) Let a be a strong atom of X. Let $x \in X$ and $x \neq a$. If $x = 0$, then $0x = 00 = 0$. Now, let $x \neq 0$. Then, by (Q_1), (Q_3) and since a is a strong atom, we see that $0x = (aa)x = (ax)a = aa = 0$. Therefore, for all $x \in X \setminus \{a\}$, $0x = x$.

(ii) Suppose that $0a \neq a$. Since a is a strong atom, then $a \neq 0$. If $0a = 0$ and since a is an atom, it follows that $0 = a$. This results in a contradiction. Therefore, $0a \neq 0$. Now, let $0a = w$ for some $w \in X \setminus \{0, a\}$. Since $w \neq a$, then by (i) we find that $0w = 0$. It follows that $0 = ww = (0a)w = (0w)a = 0a = w$, which is a contradiction. Therefore, $0a = a$. □

From Proposition 2(ii), we immediately obtain the connection between $SA(X)$ and $G(X)$ in the form of the following corollary:

Corollary 1. *$SA(X) \subseteq G(X)$ for any Q-algebra X.*

We also obtain a sharp inclusion in Corollary 1 when $|X| = 2$.

Proposition 3. *Let $|X| = 2$. Then $SA(X) = G(X)$.*

Proof. Let $X = \{0, a\}$. By Corollary 1, we have that $SA(X) \subseteq G(X)$. Now, let $w \in G(X)$. If $w = 0$, then $w \in SA(X)$. If $w = a$, then $0a = a$. It follows that a is a strong atom. Therefore, $w \in SA(X)$. Altogether, we obtain $G(X) \subseteq SA(X)$. □

We notice that there is no Q-algebra X in which the cardinality is greater than or equal to 3 such that all elements are strong atoms.

Proposition 4. *Let $|X| \geq 3$. Then, we obtain $SA(X) \neq X$.*

Proof. Suppose that $SA(X) = X$. Since $|X| \geq 3$, there exist strong atoms $z, w \in X \setminus \{0\}$ such that $z \neq w$. By Proposition 2(ii), we obtain $0z = z$ and $0w = w$. By (Q_1) and (Q_3), it follows that $w = 0w = (zz)w = (zw)z = zz = 0$, which is a contradiction. □

Atoms and strong atoms have an impact on the concept of subalgebra. In *BCC*-algebras, Dudek, W.A. and Zhang, X. demonstrated that a set of all atoms forms a subalgebra, and any subalgebra is an ideal if all elements are atoms [23]. In *BCK*-algebras, Sun, D. obtained a similar result, demonstrating that a set of all atoms forms a subalgebra [22]. Later, the authors of [26] considered strong atoms in Q-algebras and provided information on how a set of all strong atoms together with a constant 0 is also a subalgebra, as mentioned in Proposition 1. In a Q-algebra that contains a strong atom, we have an intensive result concerning subalgebras. It is not hard to verify that a subset $\{0, \delta\}$ of a Q-algebra X is a subalgebra where δ is a strong atom. In general, any subset S of a Q-algebra X such that $0 \in S$ and $|S| = 2$ need not be a subalgebra, as seen in Example 1. However, as shown in Example 2, when X contains a strong atom, it turns out that all two-element subsets containing 0, namely, $\{0, z\}$, $\{0, \alpha\}$, and $\{0, v\}$, are subalgebras. In general, we can provide the following.

Proposition 5. *If X contains a strong atom, then, for any $\omega \in X$, a subset $\{0, \omega\}$ is a subalgebra of X.*

Proof. Let a be a strong atom. Let $\omega \in X$ and let $I = \{0, \omega\}$. If $\omega = 0$, then it is obvious that I is a subalgebra of X. Now, we assume that $\omega \neq 0$. If $\omega = a$, then $0\omega = 0a = a = \omega \in I$ according to Proposition 2(ii). If $\omega \neq a$, according to Proposition 2(i), we obtain $0\omega = 0 \in I$. Altogether, we can conclude that I is a subalgebra of X. □

In addition, any subset of a Q-algebra X with a cardinality of 3 containing a strong atom and a constant 0 is always a subalgebra of X.

Proposition 6. *Let a be a strong atom of X. Then, a subset $\{0, a, w\}$ is a subalgebra of X for any $w \in X$.*

Proof. Let a be a strong atom of X. Let $w \in X$ and let $S = \{0, a, w\}$. To show that S is a subalgebra, it is enough to show that $aw, wa, 0w, 0a \in S$. If $w = 0$ or $w = a$, then $S = \{0, a\}$. It follows by Proposition 5 that $S = \{0, a\}$ is a subalgebra. Now, let $w \neq 0$ and $w \neq a$. Since a is a strong atom and $w \notin \{0, a\}$, then by Proposition 2 we find that $aw = a$, $0w = 0$, and $0a = a$. Next, we calculate wa. Suppose that $wa \notin \{0, w, a\}$. Let $wa = \gamma$ for some $\gamma \in X \setminus \{0, a, w\}$. Since $\gamma \neq a$ and a is a strong atom, we obtain $a\gamma = a$. It follows that $a = 0a = (ww)a = (wa)w = \gamma w$. Then, we obtain $0 = 0w = (\gamma\gamma)w = (\gamma w)\gamma = a\gamma = \gamma$, which is a contradiction. Thus, $wa \in \{0, a, w\}$. Altogether, $S = \{0, a, w\}$ is closed. □

According to Propositions 5 and 6, we can conclude that, for a Q-algebra X containing a strong atom, any subset S with $0 \in S$ is a subalgebra whenever $|X| \leq 2$. In the case of $|X| = 3$, a subset S, $0 \in S$, is a subalgebra whenever S contains a strong atom. When the size of S is greater than 3 and $S \neq X$, a subset S needs not be a subalgebra. The counterexample is illustrated as follows.

Example 3. *Let us consider a Q-algebra $(X; *, 0)$ that is defined with the following table:*

$*$	0	ϵ	γ	ρ	κ
0	0	ϵ	0	0	0
a	ϵ	0	ϵ	ϵ	ϵ
γ	γ	ϵ	0	κ	0
ρ	ρ	ϵ	0	0	0
κ	κ	ϵ	0	0	0

Since ϵ is an atom and $\epsilon z = \epsilon$ for all $z \in X \setminus \{\epsilon\}$, then ϵ is a strong atom of X. Let us consider a subset $S = \{0, \epsilon, \gamma, \rho\}$ of X. Since $\gamma, \rho \in S$ but $\gamma\rho = \kappa \notin S$, S is not a subalgebra. In general, a subset S of X, $|S| \geq 4$, containing a constant 0 and a strong atom needs not be a subalgebra. In this example, there is a small clue that guides us to obtain a sufficient condition for a subset of a Q-algebra to be a subalgebra. Let us exclude a strong atom from X and consider a subset $X - \{\epsilon\} = \{0, \gamma, \rho, \kappa\}$. Using the above table, we can easily check that a subset $X - \{\epsilon\}$ is a subalgebra. In the general case, we can provide the following.

Proposition 7. *Let a be a strong atom of X. Then, $X - \{a\}$ is a subalgebra of X.*

Proof. Let $\omega, \gamma \in X - \{a\}$. By Lemma 1 and Proposition 2(i), it follows that $0(\omega\gamma) = (0\omega)(0\gamma) = 00 = 0$. It follows that $\omega\gamma \neq a$; otherwise, by Proposition 2(ii), $0(\omega\gamma) = 0a = a$, which is a contradiction. Therefore, we obtain $\omega\gamma \in X - \{a\}$. Thus, $X - \{a\}$ is a subalgebra of X. □

In general, a finite union of subalgebras of a Q-algebra X need not be a subalgebra. However, in the case of $|X| = 4$, if X contains a strong atom, we have a positive result.

Proposition 8. *Let $|X| = 4$ and let $\{S_i \mid i \in I\}$ be a family of subalgebras of X. If X contains a strong atom, then we find that $\bigcup_{i \in I} S_i$ is a subalgebra of X.*

Proof. Let a be a strong atom. Let S_i be a subalgebra of X for all $i \in I$. Now, let $x, y \in \bigcup_{i\in I} S_i$. Then, there are $j, k \in I$ such that $x \in S_j$ and $y \in S_k$. If $x = 0$, then, by Proposition 2, we have $xy = 0$ and $xy = a$ when $y \neq a$ and $y = a$, respectively. Therefore, $xy \in \bigcup_{i\in I} S_i$. Now, we assume that $x \neq 0$. Suppose that $xy \notin \{0, x, y\}$. It follows that $x \neq y$ and $y \neq 0$; otherwise, $xy \in \{0, x\} \subseteq \{0, x, y\}$, which is a contradiction. We may assume that $xy = w$ for some $w \notin \{0, x, y\}$. Since $|X| = 4$, then $\{x, y, w\} \cap \{a\} = \{a\}$. Therefore, $(xy)w = a$. Therefore, $0 = ww = (xy)w = a$, which is a contradiction. Hence, $xy \in \{0, x, y\}$, and it follows that $xy \in S_j \cup S_k \subseteq \bigcup_{i\in I} S_i$. Thus, $\bigcup_{i\in I} S_i$ is a subalgebra of X. □

The converse of Proposition 8 is not true, i.e., for a Q-algebra X, $|X| = 4$, such that any finite union of its subalgebras is a subalgebra, X need not contain a strong atom. The following example reveals this fact.

Example 4. *Let us consider a Q-algebra $(X; *, 0)$ that is defined with the following table:*

$*$	0	ω	v	δ
0	0	ω	δ	v
ω	ω	0	v	δ
v	v	δ	0	ω
δ	δ	v	ω	0

All subalgebras of X are $\{0\}$, $\{0, \omega\}$, and X. It is not difficult to verify that any finite union of subalgebras of X is again a subalgebra. Moreover, all elements of X are atoms, i.e., $A(X) = \{0, \omega, v, \delta\}$. Since $\omega v = v \neq \omega$, $v\delta = \omega \neq v$, and $\delta\omega = v \neq \delta$, then ω, v, and δ are not strong atoms. Therefore, X does not contain a strong atom. This assures that the converse of Proposition 8 is invalid.

3. Strong Atoms and Ideals

In this section, we discuss the concept of strong atoms involved with ideals. There are some links between strong atoms, ideals, and subalgebras. An ideal of a Q-algebra need not be a subalgebra. In this section, we show that, for a Q-algebra X containing a strong atom, every ideal of X is always a subalgebra of X. In BCK-algebra, Sun, D. determined that a set of all strong atoms together with a constant 0 is an ideal [22]. However, in Q-algebras, we obtain the opposite result, as shown in the next example.

Example 5. *Let us consider a Q-algebra X that is defined with the following table:*

$*$	0	ω	b	z
0	0	ω	0	0
ω	ω	0	ω	ω
b	ω	ω	0	0
z	z	ω	b	0

It is easy to check that an element ω is only a strong atom of X. Then, $SA(X) = \{0, \omega\}$. Since $b\omega = \omega \in SA(X)$ and $\omega \in SA(X)$ but $b \notin SA(X)$, $SA(X)$ is not an ideal.

According to Example 5, in Q-algebras, a constant 0 together with a set of all strong atoms need not be an ideal. Next, we will determine all ideals of X that contain a strong atom. To achieve this aim, we need the following.

Lemma 2. *If a is a strong atom of X, then, for all $x \in X \setminus \{a\}$, $xa = a$.*

Proof. Assume that an element a is a strong atom. Suppose that there is $x \in X \setminus \{a\}$ such that $xa \neq a$. We may assume that $xa = b$ for some $b \in X \setminus \{a\}$. Then, $0(xa) = 0b = 0$ according to Proposition 2(i). From this and together with Lemma 1 and Proposition 2, we obtain $0 = 0(xa) = (0x)(0a) = 0(0a) = 0a = a$, which is a contradiction. Thus, $xa = a$ for all $x \in X \setminus \{a\}$. □

Proposition 2 and Lemma 2 imply that a strong atom commutes with every element of X.

Corollary 2. *Let ω be a strong atom of X. Then, for all $x \in X$, $\omega x = x\omega$.*

Let T and W be non-empty subsets of a Q-algebra X. We define TW in the usual way using $TW = \{tw \mid t \in T, w \in W\}$. If $W = \{x\}$, we denote $T\{x\}$ by Tx. By Corollary 2, we obtain the following proposition.

Proposition 9. *Let ω be a strong atom of X. Then,*

(i) $\omega X = X\omega$.
(ii) $(\omega x)X = \omega(xX) = x(\omega X) = (x\omega)X$ *for any* $x \in X$.
(iii) $(\omega X)(\omega X) = \omega X$.
(iv) ωX *is a subalgebra of* X.

Let Y be a non-empty subset of a Q-algebra X. We denote the smallest ideal of X containing Y by $[Y]_I$. Then,

$$[Y]_I = \bigcap\{W \subseteq X \mid W \text{ is an ideal and } Y \subseteq W\}.$$

Remark 1. *Let Y_1, Y_2 be non-empty subsets of X. If $Y_1 \subseteq Y_2$, then $[Y_1]_I \subseteq [Y_2]_I$.*

Proposition 10. *Let Y_1, Y_2 be non-empty subsets of X and let ω be a strong atom. If $\omega \in Y_1 \cap Y_2$, then $[Y_1]_I = [Y_2]_I$.*

Proof. Assume that $\omega \in Y_1 \cap Y_2$. Let $z \in [Y_1]_I$. Then, by Lemma 2, we find that $z\omega = \omega$. Since $z\omega = \omega \in Y_2 \subseteq [Y_2]_I$, $\omega \in [Y_2]_I$, and $[Y_2]_I$ is an ideal, then $z \in [Y_2]_I$. Therefore, $[Y_1]_I \subseteq [Y_2]_I$. By the same argument, we obtain $[Y_2]_I \subseteq [Y_1]_I$. Thus, $[Y_1]_I = [Y_2]_I$. □

Moreover, for a Q-algebra X containing strong atoms, there is exactly one ideal that contains strong atoms, that is, X itself.

Proposition 11. *Let $I \subseteq X$. If I is an ideal containing a strong atom of X, then $I = X$.*

Proof. Let ω be a strong atom of X. Assume that I is an ideal and $\omega \in I$. Let $x \in X$. If $x = \omega$, then $x \in I$. If $x \neq \omega$, then, by Lemma 2, we find that $x\omega = \omega$. Since $x\omega \in I$ and $\omega \in I$, then $x \in I$ because I is an ideal. Altogether, we conclude that $X \subseteq I$. Hence, $I = X$. □

The following theorem is immediately obtained.

Theorem 1. *Let X be a Q-algebra containing a strong atom. Then, we find that I is an ideal containing a strong atom of X if and only if $I = X$.*

Propositions 10 and 11 give the following corollary.

Corollary 3. *Let ω be a strong atom of X and let $\emptyset \neq Y \subseteq X$. If $\omega \in Y$, then $[Y]_I = X$.*

Proposition 12. *Let X be a Q-algebra containing a strong atom. If I is an ideal of X, then I is a subalgebra.*

Proof. Let ω be a strong atom. Assume that I is an ideal. If $\omega \in I$, by Proposition 11, we obtain $I = X$. Therefore, I is a subalgebra. We assume now that I does not contain a strong atom. Let $x, y \in I$. If $x = 0$, then, by Proposition 2, we obtain $xy = 0y = 0 \in I$. If $y = 0$, then, by (Q_2), we obtain $xy = x0 = x \in I$. Now, let $x \neq 0$ and $y \neq 0$. Then, by Lemma 1 and Proposition 2, we obtain $0(xy) = (0x)(0y) = 00 = 0$. It follows that $xy \neq \omega$; otherwise, by Proposition 2(ii), $0(xy) = 0\omega = \omega$, which is a contradiction. Assume that $xy = c$ for some $c \in X$ and that c is not a strong atom. Then, we can calculate that $cx = (xy)x = (xx)y = 0y = 0$. Since $cx = 0 \in I$, $x \in I$, and I is an ideal, then $c \in I$. Therefore, $xy \in I$. Altogether, we can conclude that I is a subalgebra. □

The converse direction of Proposition 12 is invalid, i.e., any subalgebra of X need not be an ideal of X. The following example shows this fact.

Example 6. *Let us consider a Q-algebra $(X; *, 0)$ that is defined with the following table:*

$*$	0	ω	b	c	ν
0	0	ω	0	0	0
ω	ω	0	ω	ω	ω
b	b	a	0	ν	0
c	c	ω	ν	0	0
ν	ν	ω	0	0	0

Let us consider a subset $S = \{0, c\}$. It is easy to see that S is closed, i.e., S is a subalgebra. Since $\nu c = 0 \in S$ and $c \in S$ but $\nu \notin S$, S is not an ideal. This shows that the converse direction of Proposition 12 is not true.

In Example 2, we have a Q-algebra X such that a subset $X - \{z\}$ is a subalgebra where z is a strong atom, and then we obtain this result in general, as demonstrated in Proposition 7. This subset also fulfills the properties of an ideal.

Proposition 13. *Let ω be a strong atom of X. Then, $X - \{\omega\}$ is an ideal of X.*

Proof. Let $I = X - \{\omega\}$. It is enough to show only the property (I_2). Let $x \in X$ and $y \in I$. Assume that $xy \in I$. Suppose that $x = \omega$. Since ω is a strong atom and $y \neq \omega$, then $xy = \omega y = \omega$. Therefore, $\omega = xy \in I$, which is a contradiction. Thus, $x \neq \omega$; it follows that $x \in X - \{\omega\} = I$. Hence, I is an ideal. □

Normally, the union of ideals of X need not be an ideal. The next example illustrates this fact.

Example 7. *Let us consider the Q-algebra $(X; *, 0)$, which is defined with the following table:*

$*$	0	ω	ν	c	d	f
0	0	ω	0	0	ω	ω
ω	ω	0	ω	ω	0	0
ν	ν	f	0	0	ω	ω
c	c	d	0	0	ω	0
d	d	ν	ω	ω	0	0
f	f	ν	ω	ω	0	0

We find that subsets $X_1 = \{0, \omega\}$ and $X_2 = \{0, \nu, c\}$ are ideals of X. However, $X_1 \cup X_2 = \{0, \omega, \nu, c\}$ is not an ideal since $d\omega = \nu \in X_1 \cup X_2$ and $\omega \in X_1 \cup X_2$ but $d \notin X_1 \cup X_2$.

In the case that X contains a strong atom and $2 \leq |X| \leq 4$, when we form a union of ideals of X, it turns out that the union is an ideal.

Proposition 14. *Let X be a Q-algebra containing a strong atom and $2 \leq |X| \leq 4$. If X_1 and X_2 are ideals of X, then $X_1 \cup X_2$ is an ideal.*

Proof. In the case of $|X| = 2$, this is not difficult to verify. Let $|X| = 3$, $X = \{0, \omega, b\}$, and let ω be a strong atom. Assume that X_1 and X_2 are ideals of X. If $\omega \in X_1$ or $\omega \in X_2$, then, by Proposition 11, we find $X_1 = X$ or $X_2 = X$. It follows that $X_1 \cup X_2 = X$. Hence, $X_1 \cup X_2$ is an ideal. If $\omega \notin X_1 \cup X_2$, then $X_1 \subseteq \{0, b\}$ and $X_2 \subseteq \{0, b\}$. Note that $\{0, b\}$ is an ideal of X. It follows that $X_1 \cup X_2 = \{0\}$ 0r $X_1 \cup X_2 = \{0, b\}$. Hence, $x_1 \cup X_2$ is an ideal.

Now, let $|X| = 4$, $X = \{0, \omega, b, c\}$, and ω be a strong atom. Assume that X_1 and X_2 are ideals of X. Let $xy \in X_1 \cup X_2$ and $y \in X_1 \cup X_2$. We would like to show that $x \in X_1 \cup X_2$. If $\omega \in X_1 \cup X_2$, then, by Proposition 11, we obtain $X_1 \cup X_2 = X$. Thus, $x \in X_1 \cup X_2$. Now, let $\omega \notin X_1 \cup X_2$, i.e., $X_1 \cup X_2 \subseteq \{0, b, c\}$. If $X_1 \subseteq X_2$, then we obtain $xy \in X_1 \cup X_2 = X_2$ and $y \in X_1 \cup X_2 = X_2$. Since X_2 is an ideal, then $x \in X_2 = X_1 \cup X_2$. If $X_2 \subseteq X_1$, we can apply the same argument to find that $x \in X_1 \cup X_2$. Now, let $X_1 \nsubseteq X_2$ and $X_1 \nsubseteq X_2$. Since $X_1 \cup X_2 \subseteq \{0, b, c\}$, it follows that $|X_1| = |X_2| = 2$ and then $X_1 \cup X_2 = \{0, b, c\}$. Suppose that $x = \omega$; it follows that $\omega = \omega y = xy \in X_1 \cup X_2$, which is a contradiction. Then, $x \in \{0, b, c\} = X_1 \cup X_2$. Altogether, we find that $X_1 \cup X_2$ is an ideal of X. □

4. Conclusions

The concepts of subalgebras, ideals, and atoms play an important role in the study of Q-algebra structures. Many studies have discussed these concepts. In this work, we obtain precise information on strong atoms concerning subalgebras and ideals. In a Q-algebra X containing a strong atom, an ideal of X is always a subalgebra of X. In addition, we find that, for any ideal containing a strong atom, such an ideal is exactly X itself. This means that in a Q-algebra X with a strong atom there is only one ideal that contains a strong atom, i.e., X. Moreover, in a particular case when $2 \leq |X| \leq 4$, we find that a union of ideals is again an ideal. In future study, morphisms among Q-algebras containing a strong atom would be a good further step. In addition, the concepts of congruence and quotient Q-algebras on a set of all Q-algebras containing a strong atom should be considered.

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