

Article

Quasi-Elliptic Cohomology of 4-Spheres

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Abstract: It is a famous hypothesis that orbifold D-brane charges in string theory can be classified in twisted equivariant K-theory. Recently, it is believed that the hypothesis has a non-trivial lift to M-branes classified in twisted real equivariant 4-Cohomotopy. Quasi-elliptic cohomology, which is defined as an equivariant cohomology of a cyclification of orbifolds, potentially interpolates the two statements, by approximating equivariant 4-Cohomotopy classified by 4-sphere orbifolds. In this paper we compute Real and complex quasi-elliptic cohomology theories of 4-spheres under the action by some finite subgroups that are the most interesting isotropy groups where the M5-branes may sit. The computation connects the M-brane charges in the presence of discrete symmetries to Real quasi-elliptic cohomology theories, and those with the symmetry omitted to complex quasi-elliptic cohomology theories.

Keywords: elliptic cohomology; M5-branes; KR-theory

MSC: 55N34; 81T30; 19L50

1. Introduction

String theory and homotopy theory are closely related. It is a famous hypothesis that orbifold D-brane charges in string theory can be classified in twisted equivariant K-theory, or rather, Real twisted equivariant K-theory. A natural question is whether we have a classification for M-branes in term of cocycles in a generalized cohomology theory.

M-branes are membranes in M-theory, which is a theoretical framework in physics that unifies the five superstring theories and 11-dimensional supergravity. M-branes come in two types, M2-branes and M5-branes. These branes are analogous to the D-branes in string theory but exist in the 11-dimensional spacetime and M-theory. An M2-brane acts as a source for the 3-form field, similar to how an electric charge is a source for the electromagnetic field. In lower dimensions, M2-brane charges can manifest as various types of charges in string theory, such as D2-brane charges or fundamental string charges. In addition, M5-brane is a magnetic dual to the M2-brane and couples to the 6-form field. In string theory, M5-brane charge can appear as NS5-brane charge or D4-brane charge after compactification. M-brane charges play a crucial role in the duality web of string theory and M-theory. They play a key role in unifying string theories, understanding black hole entropy, and studying gauge theories.

A key open problem in M-theory is the identification of the degrees of freedom that are expected to be hidden at ADE-singularities in spacetime. Comparison with the classification of D-branes by K-theories suggests that the answer must come from the right choice of generalized cohomology theory for M-branes. Recently it is believed that the hypothesis has a non-trivial lift to M-branes classified in twisted real equivariant 4-Cohomotopy. Then, it is an evident question how the statement for D-branes and this for M-branes may relate.



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As indicated in [1], cyclification of ∞ -stacks is a fundamental and elementary base-change construction over moduli stacks in cohesive higher topos theory. Quasi-elliptic cohomology theory, which is defined as an equivariant cohomology of a cyclification of orbifolds, is a form of twisted equivariant K-theory involving a further S^1 -action, just as one would expect in lifting through the M/II reduction. It is a variant of Tate K-theory. Quasi-elliptic cohomology theory is a generalized cohomology theory that potentially interpolates the two statements, by approximating equivariant 4-Cohomotopy classified by 4-sphere orbifolds. In addition, the universal shifted integral 4-class of equivariant 4-Cohomotopy theory on ADE-orbifolds induces the Platonic 4-twist of ADE-equivariant Tate-elliptic cohomology,

In [2], the author computes a list of examples explicitly, specifically for the domain being orbifolds of 4-spheres by finite subgroups of $SU(2)$ that serve as the transverse geometry to M5-branes probing ADE-singularities. The results in all cases investigated are deformations of familiar twisted equivariant K-theory groups by a further formal variable q , being the character of the extra S^1 -action and playing the role of a kind of M-theoretic quantum deformation of the K-theory groups.

In this paper, we consider the approximation of the Real version of quasi-elliptic cohomology to Real twisted equivariant 4-Cohomotopy. Young and the author construct a Real version of quasi-elliptic cohomology in [3]. Since twisted equivariant KR -theory appears in physical contexts that are unoriented, such as orientifold string theory [4] and unoriented topological field theory [5,6], it is natural to expect a relation between unoriented conformal field theory and a Real generalization of elliptic cohomology. This expectation suggests the promotion of the group of loop rotations $\mathbb{T}$ to the orthogonal group $O_2 = \mathbb{T} \rtimes \mathbb{Z}/2$ so as to include loop reflection, an orientation reversing symmetry of loops. The O_2 -equivariant topology of loop spaces has been studied by a number of authors [7–11]. In [3], Real quasi-elliptic cohomology is constructed as the O_2 -equivariant Freed–Moore K -theory of an inertia groupoid. In this paper, we compute Real and complex quasi-elliptic cohomology of 4-spheres under the specific action of some finite subgroups of $Spin(5)$, which aims to give an approximation to the Real equivariant unstable 4th Cohomotopy, which is especially difficult to compute.

To interpret the relation between the computation and cohomotopy, we start the story by classifying spaces for cohomology theories. For a given cohomology theory $E^*(-)$ with classifying space E , we have, for any good enough space X ,

$$E^0(X) = \pi_0 \operatorname{Map}(X, E).$$

Here, we can regard a map $X \rightarrow E$ as a “cocycle” for the E -cohomology, and a homotopy between such maps as a “boundary” in E -cohomology. Generally, the classifying space of an abelian cohomology theory is its spectrum at level 0. A classical example is complex topological K -theory $K(-)$, whose classifying space can be taken to be $KU = BU \times \mathbb{Z}$.

In addition, instead of using the whole spectrum of E , with only the classifying space, we can define a generalized non-abelian cohomology theory

$$E(X) := \pi_0 \operatorname{Map}(X, E)$$

which makes good sense. One issue is that computing such cohomology theories is generally difficult. One method is approximating the cohomology theory E by another one E' , which is better understood and easier to compute. The method is clearly possible whenever there is a map of classifying spaces $E \rightarrow E'$ because it induces, evidently, a cohomology operation

$$E(-) \rightarrow E'(-),$$

which provides an image of the less-understood E -cohomology in the better-understood E' -cohomology.

The archetypical example here is the Chern-Dold character map [12,13], which approximates any generalized cohomology theory by a rational cohomology theory. For instance, the ordinary Chern character on $K(-)$

$$K(-) \longrightarrow H^{ev}(-; \mathbb{Q})$$

with $H^{ev}(-; \mathbb{Q}) := \prod_{n \in \mathbb{N}} H^{2n}(-; \mathbb{Q})$ is represented by a map of classifying spaces

$$BU \times \mathbb{Z} \longrightarrow \prod_{n \in \mathbb{N}} K(\mathbb{Q}, 2n).$$

This map of classifying spaces is itself a cocycle in the rational cohomology of the classifying space $BU \times \mathbb{Z}$. In other words, the Chern character itself can be viewed as an element in

$$H^{ev}(BU \times \mathbb{Z}; \mathbb{Q}).$$

Generally, a map of classifying spaces $E \longrightarrow E'$, inducing a cohomology operation $E(-) \longrightarrow E'(-)$, is itself a cocycle in the E' -cohomology $E'(E)$ of the classifying space E . Thus, in order to understand E -cohomology, we may try to understand the E' -cohomology of its classifying space E for suitable alternative cohomology theories E' .

Now, we consider the cohomology theory, the n -th Cohomotopy theory

$$nCohomotopy(-),$$

whose classifying space is an n -sphere S^n . Each cocycle in the E' -cohomology $E'(S^n)$ is represented by a map $S^n \longrightarrow E'$. From it, we obtain a cohomology operation

$$nCohomotopy(-) \longrightarrow E'(-),$$

which provides us images of $nCohomotopy$ in E' -cohomology similarly to how the Chern character provides images of K -cohomology in ordinary rational cohomology.

It is suggested by Hypothesis H [14–16] that, specifically, $\text{Spin}(5)$ -twisted equivariant unstable $4Cohomotopy$ classifies the charges carried by M-branes in M-theory in a way that is analogous to the traditional idea that $K(-)$ classifies the charges of D-branes in string theory. Therefore, it is essential to compute the $4Cohomotopy$ of spacetime domains relevant in M-theory. This can be hard, in particular, once we remember that all of these need to be performed in twisted equivariant generality. Thus, we apply the idea to approximate $4Cohomotopy$ of spacetime by using the cocycles

$$S^4 \longrightarrow E'$$

in $E'(S^4)$ for some suitable cohomology theory E' . Instead of $4Cohomotopy$ itself, we will study the image of the corresponding cohomology operation

$$4Cohomotopy(-) \longrightarrow E'(-).$$

Some information of the actual $4Cohomotopy$ may be lost but what is retained can still be valuable and is expected to be better understandable.

Specifically, the classifying spaces for equivariant $4Cohomotopy$ are orbifolds $S^4 // G$ of the 4-sphere acted by a group G , i.e., the orbifolds of representation 4-spheres. Hence, the elements of the G -equivariant E' -cohomology $E'_G(S^4)$ serve, in the above way, as “gen-

eralized equivariant characters" on equivariant 4Cohomotopy , namely, as the equivariant cohomology operation

$$4\text{Cohomotopy}_G(-) \longrightarrow E'_G(-).$$

As conjectured in [1], the choice

$$E'_G(-) := QEll_G(-)$$

should be a particularly suitable approximation to equivariant 4Cohomotopy for the purpose of computing M-brane charge. One motivation for this is that the Witten elliptic genus, which was originally discussed for string [17], actually makes sense for M5-branes [18–22], so that one should expect that it is actually part of the charges carried by M5-branes. But these charges should also be in Cohomotopy , and hence, it is conjectured in [1] that there is a useful approximation of 4Cohomotopy by elliptic cohomology, and specifically by quasi-elliptic cohomology.

This is the motivation for computing the quasi-elliptic cohomology for representation 4-spheres. Moreover, as indicated in [13], the particular choice of equivariance groups G as finite subgroups of $\text{Spin}(5)$ comes from the fact that these are the most interesting isotropy groups for the orbifolds on which these M5-branes may sit. We describe the interesting groups and their action on 4-spheres below.

The space $\mathbb{H}$ of quaternions is isomorphic to $\mathbb{R}^4$ as a real vector space. In addition, the group of the unit quaternions is isomorphic to the special unitary group $SU(2)$, which is isomorphic to $\text{Spin}(3)$. It can be identified with a subgroup of $\text{Spin}(5)$ via the composition

$$\text{Spin}(3) \xrightarrow{p_1} \text{Spin}(3) \times \text{Spin}(3) \cong \text{Spin}(4) \hookrightarrow \text{Spin}(5)$$

where the first homomorphism is the inclusion into the first factor. Under quaternion multiplication, there are two choices of group action by $\mathbb{H}$ on $\mathbb{R}^4$ that we are especially interested in.

$$\begin{aligned} \text{Spin}(4) \simeq \text{Spin}(3) \times \text{Spin}(3) &\longrightarrow \text{SO}(\mathbb{H}) \simeq \text{SO}(4) \\ (e_1, 1) &\mapsto (q \mapsto e_1 \cdot q) \\ (e_1, e_1) &\mapsto (q \mapsto e_1 \cdot q \cdot e_1^*) \end{aligned} \quad (1)$$

The group action can extend to S^4 by keeping the north pole and the south pole fixed. In ([2] Section 6), we compute complex quasi-elliptic cohomology of S^4 under the first group action in (1). In Section 4, we compute the Real quasi-elliptic cohomology for that. Moreover, in Section 5, we compute some examples of complex and Real quasi-elliptic cohomology of S^4 under the second group action in (1). In addition, in Section 6.1, we discuss the relation between the computation and physics a little; in Section 6.2, we give a future problem that we are interested in.

In the computation of Real quasi-elliptic cohomology theories, we pick a Real structure on each equivariance group, with which the corresponding theories are computable. Real quasi-elliptic cohomology is defined as a Freed–Moore K-theory of a Real version of an inertia groupoid. The computation in Section 4 shows how the classification of M-brane charges is described by equivariant complex K-theories when the conjugacy classes are free under the involution, and how the classification is described by equivariant KR-theories when the conjugacy classes are fixed by the involution. In Section 5.2, we compute some Real and complex quasi-elliptic cohomology theories whose equivariance groups are some finite product subgroups of $\text{Spin}(5)$. The conjugacy classes of the equivariance group determine the distinct types of localized charges, while the Real conjugacy classes describe

how these charges behave under the involution. The connection is formalized using equivariant K-theories and equivariant KR-theories, which provide a deep link between geometry, algebraic topology and physics.

In addition, we would like to mention some other methods approximating 4-Cohomotopy, which is a challenging task in equivariant homotopy theory. Generally, a G -equivariant 4-Cohomotopy of a space X is difficult to compute and depends heavily on the equivariance group G and the topological space X . Tools from equivariant homotopy theory, such as the Borel construction, equivariant cohomology, and spectral sequences, are often used to study these groups. We list a few of them below.

- (1) One method is to use obstruction theory to determine whether a G -equivariant map $X \rightarrow S^4$ exists and classify such maps up to homotopy [23]. This method works well for spaces that can be decomposed into G -CW complexes. However, obstruction classes can be difficult to compute explicitly, especially for higher dimensional spaces.
- (2) In addition, we can apply Atiyah–Hirzebruch spectral sequence or the Serre spectral sequence to approximate equivariant 4-Cohomotopy whose E_2 -page involves terms like $H^p(EG \times_G X; \pi_q(S^4))$ [24]. This method breaks down the problem into simpler pieces using cohomology and homotopy groups. It can handle a wide range of spaces X and equivariance groups G . But higher differentials in the spectral sequence are often difficult to compute and the spectral sequences may not always converge cleanly.
- (3) Another method is that we stabilize the problem first by replacing S^4 with its stabilization $\Sigma^\infty S^4$ and use equivariant Adams spectral sequence to approximate 4-Cohomotopy [24]. Reducing the problem to stable homotopy groups certainly simplifies the problem and Adams spectral sequence is a powerful tool. However, stabilization loses some geometric information about X and the 4-sphere, and the unstable problem of interest may not directly be solved.
- (4) In addition, when the space X has a smooth structure and the G -action is geometric, one can study G -equivariant instantons or harmonic maps to S^4 [25,26]. Using these methods, the equivariant maps can be constructed explicitly, which may connect to physical applications. But the method may not be generalized to more abstract or singular spaces.

The method we use in this paper is a novel one. It works well especially when the cohomology theory approximating 4-Cohomotopy is suitably chosen and computable. In this paper, we implemented this idea.

In the Appendices A and B, we give some corollaries of the decomposition formula for complex equivariant K -theories in [27] and the Mackey decomposition formula for Freed–Moore K -theories in [3]. They are used in the computation in Sections 4 and 5, respectively.

In addition, before we present the computation of quasi-elliptic cohomology, we review, in Sections 2 and 3, quasi-elliptic cohomology and twisted Real quasi-elliptic cohomology, respectively.

2. Quasi-Elliptic Cohomology

In this section, we recall the definition of quasi-elliptic cohomology in term of equivariant K -theories and state the conclusions that we need in this paper. For more details on quasi-elliptic cohomology, please refer to [28].

Let G be a compact Lie group and X a G -space. Let $G^{tors} \subseteq G$ denote the set of torsion elements of G . For any $g \in G^{tors}$, the fixed point space X^g is a $C_G(g)$ -space where $C_G(g)$ is the centralizer $\{h \in G \mid hg = gh\}$. This group action can be extended to that by the group

$$\Lambda_G(g) := C_G(g) \times \mathbb{R} / \langle (g, -1) \rangle,$$

which is given explicitly by

$$[h, t] \cdot x := h \cdot x, \quad (2)$$

for any $[h, t] \in \Lambda_G(g)$ and $x \in X^g$.

The inertia orbifold of $X//G$ is defined to be the functor groupoid

$$\text{hom}_{\text{grpd}}(B\mathbb{Z}, X//G).$$

The inertia orbifold plays an import role in the geometry and index theory of orbifolds [29,30]. The rotation action of the circle group $\mathbb{T}$ can be built into it as new morphisms, as we see in Definition 2.

To give a complete description of the loop groupoid $\Lambda(X//G)$, we need the following definitions.

Definition 1. (1) Let g, g' be two elements in G . Define $C_G(g, g')$ to be the set $\{h \in G \mid g'h = hg\}$.
(2) Let $\Lambda_G(g, g')$ denote the quotient of $C_G(g, g') \times \mathbb{R}/l\mathbb{Z}$ under the equivalence

$$(\alpha, t) \sim (g'\alpha, t - 1) = (\alpha g, t - 1),$$

where l is the order of g in G .

Definition 2. Define $\Lambda(X//G)$ to be the groupoid with

- **objects:** the space $\coprod_{g \in G^{\text{tors}}} X^g$
- **morphisms:** the space

$$\coprod_{g, g' \in G^{\text{tors}}} \Lambda_G(g, g') \times X^g.$$

For an object $x \in X^g$, the morphism $([\alpha, t], x) \in \Lambda_G(g, g') \times X^g$ is an arrow from x to $\alpha \cdot x \in X^{g'}$. The composition of the morphisms is defined by

$$([\alpha_1, t_1], \alpha_2 \cdot x) \circ ([\alpha_2, t_2], x) = ([\alpha_1 \alpha_2, t_1 + t_2], x). \quad (3)$$

Let $\mathbb{T}$ denote the circle group $\mathbb{R}/\mathbb{Z}$. We have a homomorphism of orbifolds

$$\pi : \Lambda(X//G) \longrightarrow B\mathbb{T}$$

sending all the objects to the single object in $B\mathbb{T}$, and a morphism $([\alpha, t], x)$ to $e^{2\pi i t}$ in $\mathbb{T}$.

Definition 3. The quasi-elliptic cohomology $QEll_G^*(X)$ is defined to be $K_{\text{orb}}^*(\Lambda(X//G))$.

The groupoid $\Lambda(X//G)$ is equivalent to the disjoint union of action groupoids

$$\coprod_{g \in \pi_0(G^{\text{tors}}//G)} X^g // \Lambda_G(g) \quad (4)$$

where $G^{\text{tors}}//G$ is the conjugation quotient groupoid. Thus, we can unravel Definition 3 and express it via equivariant K-theory.

Definition 4 ([28] Definition 3.11).

$$QEll_G^*(X) := \prod_{g \in \pi_0(G^{\text{tors}}//G)} K_{\Lambda_G(g)}^*(X^g) = \left(\prod_{g \in G^{\text{tors}}} K_{\Lambda_G(g)}^*(X^g) \right)^G. \quad (5)$$

Consider the composition

$$\mathbb{Z}[q^{\pm}] = K_{\mathbb{T}}(\text{pt}) \xrightarrow{\pi^*} K_{\Lambda_G(g)}(\text{pt}) \longrightarrow K_{\Lambda_G(g)}(X)$$

where $\pi : \Lambda_G(g) \longrightarrow \mathbb{T}$ is the projection $[a, t] \mapsto e^{2\pi it}$ and the second map is defined via the collapsing map $X \longrightarrow \text{pt}$. Via it, $QEll_G^*(X)$ is naturally a $\mathbb{Z}[q^{\pm}]$ -algebra.

Proposition 1 ([28] (1.1)). *The relation between quasi-elliptic cohomology and equivariant Tate K-theory $K_{Tate}^*(-//G)$ is*

$$QEll_G^*(X) \otimes_{\mathbb{Z}[q^{\pm}]} \mathbb{Z}((q)) \cong K_{Tate}^*(X//G). \quad (6)$$

This is the main reason why the theory is called quasi-elliptic cohomology.

In addition, we give an example computing quasi-elliptic cohomology, which is ([28] Example 3.3). The conclusions in Example 1 are applied in the computation of Sections 4 and 5.

Example 1 ($G = \mathbb{Z}/N$). *Let $G = \mathbb{Z}/N$ for $N \geq 1$, and let $\sigma \in G$. Given an integer $k \in \mathbb{Z}$ which projects to $\sigma \in \mathbb{Z}/N$, let x_k denote the representation of $\Lambda_G(\sigma)$ defined by*

$$\Lambda_G(\sigma) = (\mathbb{Z} \times \mathbb{R}) / (\mathbb{Z}(N, 0) + \mathbb{Z}(k, 1)) \xrightarrow{[a, t] \mapsto [(kt-a)/N]} \mathbb{R}/\mathbb{Z} = \mathbb{T} \xrightarrow{q} U(1). \quad (7)$$

$R\Lambda_G(\sigma)$ is isomorphic to the ring $\mathbb{Z}[q^{\pm}, x_k] / (x_k^N - q^k)$.

For any finite abelian group $G = \mathbb{Z}/N_1 \times \mathbb{Z}/N_2 \times \cdots \times \mathbb{Z}/N_m$, let $\sigma = (k_1, k_2, \dots, k_m) \in G$. We have

$$\Lambda_G(\sigma) \cong \Lambda_{\mathbb{Z}/N_1}(k_1) \times_{\mathbb{T}} \cdots \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/N_m}(k_m).$$

Then,

$$\begin{aligned} R\Lambda_G(\sigma) &\cong R\Lambda_{\mathbb{Z}/N_1}(k_1) \otimes_{\mathbb{Z}[q^{\pm}]} \cdots \otimes_{\mathbb{Z}[q^{\pm}]} R\Lambda_{\mathbb{Z}/N_m}(k_m) \\ &\cong \mathbb{Z}[q^{\pm}, x_{k_1}, x_{k_2}, \dots, x_{k_m}] / (x_{k_1}^{N_1} - q^{k_1}, x_{k_2}^{N_2} - q^{k_2}, \dots, x_{k_m}^{N_m} - q^{k_m}) \end{aligned}$$

where all the x_{k_j} 's are defined as x_k in (7).

3. Twisted Real Quasi-Elliptic Cohomology

In this section, we review the definition and properties of twisted Real quasi-elliptic cohomology. For more details, please refer to [3].

We will use $\mathbb{Z}/2$ -graded groups to define the Real structure on groups.

Definition 5. *Let G be a finite group. A $\mathbb{Z}/2$ -graded group is a group homomorphism $\pi : \hat{G} \rightarrow \mathbb{Z}/2$. The ungraded group of $\hat{G}$ is $G = \ker \pi$. When π is non-trivial, $\hat{G}$ is called a Real structure on G . The group $\hat{G}$ acts on G by Real conjugation*

$$\varsigma \cdot g = \varsigma g^{\pi(\varsigma)} \varsigma^{-1},$$

$g \in G$, $\varsigma \in \hat{G}$. The Real centralizer of $g \in G$ is

$$C_G^R(g) = \{\varsigma \in \hat{G} \mid \varsigma g^{\pi(\varsigma)} \varsigma^{-1} = g\}.$$

The group $C_G^R(g)$ is $\mathbb{Z}/2$ -graded with ungraded group the centralizer $C_G(g)$.

Example 2. The terminal $\mathbb{Z}/2$ -graded group is $\text{Id} : \mathbb{Z}/2 \rightarrow \mathbb{Z}/2$ and is denoted simply by $\mathbb{Z}/2$. If $\mathbb{Z}/2$ acts on a group $\hat{H}$, then so does any $\mathbb{Z}/2$ -graded group $\hat{G}$ and the resulting semi-direct product $\hat{H} \rtimes_{\pi} \hat{G}$ is naturally $\mathbb{Z}/2$ -graded.

Example 3. The dihedral group D_{2n}

$$\langle r, s \mid r^n = 1, s^2 = 1, (sr)^2 = 1 \rangle.$$

is a Real structure on $\mathbb{Z}/n$. The subgroup $\langle r \rangle \cong \mathbb{Z}/n$ is a normal subgroup of D_{2n} and we have the short exact sequence

$$1 \longrightarrow \mathbb{Z}/n \longrightarrow D_{2n} \longrightarrow \mathbb{Z}/2 \longrightarrow 1$$

with a generator of $\mathbb{Z}/n$ mapped to the rotation r .

Example 4. As computed in ([3] Example 1.8), the Real representation ring $RR(\mathbb{Z}/n)$ with respect to the Real structure D_{2n} is isomorphic to complex representation ring $R(\mathbb{Z}/n) \cong \mathbb{Z}[\zeta]/\langle \zeta^n - 1 \rangle$.

Example 5. For any $g \in G$, the Real centralizer $C_G^R(g)$ is $\mathbb{Z}/2$ -graded with ungraded group the centralizer $C_G(g)$. It is a Real Structure on $C_G(g)$.

In addition, the element $(-1, g) \in \mathbb{R} \rtimes_{\pi} C_G^R(g)$ is Real central and so generates a normal subgroup isomorphic to $\mathbb{Z}$. This leads to the definition of the enhanced Real centralizer of g .

$$\Lambda_G^R(g) := (\mathbb{R} \rtimes_{\pi} C_G^R(g)) / \langle (-1, g) \rangle.$$

It is a Real structure on the group $\Lambda_G(g)$.

The set of connected components $\pi_0(G//G)$ of the conjugation quotient groupoid is the set of conjugacy classes of G . Given a Real structure $\hat{G}$, Real conjugation defines an involution of $\pi_0(G//G)$. This defines a partition

$$\pi_0(G//G) = \pi_0(G//G)_{-1} \sqcup \pi_0(G//G)_{+1} \quad (8)$$

with $\pi_0(G//G)_{-1}$ the fixed point set of the involution. The conjugacy class of $g \in G$ is fixed by the involution if and only if $C_G^R(g) \setminus C_G(g) \neq \emptyset$. The set $\pi_0(G//_R \hat{G})$ of Real conjugacy classes of G inherits from (8) a partition

$$\pi_0(G//_R \hat{G}) = \pi_0(G//G)_{-1} \sqcup \pi_0(G//G)_{+1}/\mathbb{Z}/2. \quad (9)$$

Let X be a $\hat{G}$ -space. Note that for each $g \in G$, the fixed point space X^g is a $C_G^R(g)$ -space. In addition, the $\Lambda_G(g)$ -action on X^g as defined in (2) can extend to an action by $\Lambda_G^R(g)$:

$$[r, \alpha] \cdot x := \alpha \cdot x, \quad (10)$$

for any element $[r, \alpha] \in \Lambda_G^R(g)$, any $x \in X^g$.

The Real loop groupoid $\hat{\Lambda}(X//\hat{G})$, as defined in ([3] Definition 2.7), adds the involution as morphisms into the groupoid $\Lambda(X//G)$, and it is a double cover of the groupoid $\Lambda(X//G)$. In addition, we have the Real version of the decomposition (4), i.e., the decomposition of the groupoid $\hat{\Lambda}(X//\hat{G})$ corresponding to the partition (9).

Proposition 2. There is an equivalence of $B\mathbb{Z}/2$ -graded groupoids

$$\hat{\Lambda}(X//\hat{G}) \cong \coprod_{g \in \pi_0(G//G)_{-1}} X^g // \Lambda_G^R(g) \sqcup \coprod_{g \in \pi_0(G//G)_{+1}/\mathbb{Z}/2} X^g // \Lambda_G(g). \quad (11)$$

The twisted Real quasi-elliptic cohomology is defined in ([3] Definition 3.2, Proposition 3.3) in terms of Freed–Moore K-theories.

Definition 6 ([3] Definition 3.2 Proposition 3.3).

$$\mathrm{QEIR}^{*+\hat{\alpha}}(X//G) := KR^{*+\hat{\tau}_{\pi}^{\mathrm{ref}}(\hat{\alpha})}(\Lambda(X//G)) \cong \prod_{g \in \pi_0(G//_R \hat{G})} \pi K_{\Lambda_G^R(g)}^{*+\hat{\tau}_{\pi}^{\mathrm{ref}}(\hat{\alpha})}(X^g), \quad (12)$$

where $\hat{\alpha}$ is a fixed element in $H^4(B\hat{G}; \mathbb{Z})$ and $\hat{\tau}_{\pi}^{\mathrm{ref}}$ is the Real transgression map.

By the property of the Freed–Moore K-theory [5], if the Real structure $\hat{G}$ splits, each factor in (12) is the equivariant KR-theory defined by Atiyah and Segal [31].

In addition, using the partition (9), the isomorphism (12) can be written as

$$\mathrm{QEIR}^{*+\hat{\alpha}}(X//G) \cong \prod_{g \in \pi_0(G//G)_{-1}} KR_{\Lambda_G(g)}^{*+\hat{\tau}_{\pi}^{\mathrm{ref}}(\hat{\alpha})}(X^g) \times \prod_{g \in \pi_0(G//G)_{+1}/\mathbb{Z}/2} K_{\Lambda_G(g)}^{*+\tau(\alpha)}(X^g). \quad (13)$$

The $B\mathbb{Z}/2$ -graded morphism $\hat{\Lambda}(X//\hat{G}) \rightarrow BO_2$ which tracks loop rotation and reflection makes $\mathrm{QEIR}^*(X//G)$ into a $KR_{\mathbb{T}}^*(\mathrm{pt})$ -algebra and, in particular, a module over $\mathbb{Z}[q^{\pm}] \subset KR_{\mathbb{T}}^*(\mathrm{pt})$.

Theorem 1 ([3] Theorem 3.15). Assume that $\hat{G}$ is non-trivially $\mathbb{Z}/2$ -graded. The relation between twisted Real quasi-elliptic cohomology and twisted Real equivariant Tate K-theory is

$$KR_{\mathrm{Tate}}^{*+\hat{\alpha}}(X//G) \cong \mathrm{QEIR}^{*+\hat{\alpha}}(X//G) \otimes_{KR^*(\mathrm{pt})[q^{\pm}]} KR^*(\mathrm{pt})((q)).$$

In addition, we give an example computing Real quasi-elliptic cohomology, which is ([3] Example 3.7). The conclusions in Example 6 are applied in the computation of Sections 4 and 5.

Example 6. Let $G = \mathbb{Z}/n = \langle r \rangle$ and $\hat{G} = D_{2n}$. The $\mathbb{Z}/2$ -action on $\pi_0(\mathbb{Z}/n//\mathbb{Z}/n) = \mathbb{Z}/n$ is trivial. By the isomorphism (13),

$$\mathrm{QEIR}^*(\mathrm{pt} // \mathbb{Z}/n) \cong \prod_{m=0}^{n-1} KR_{\Lambda_{\mathbb{Z}/n}(r^m)}^*(\mathrm{pt}). \quad (14)$$

As discussed in ([3] Example 3.7),

$$KR_{\Lambda_{\mathbb{Z}/n}(r^m)}^*(\mathrm{pt}) \cong KR^*(\mathrm{pt})[q^{\pm}, x_m] / \langle x_m^n - q^m \rangle. \quad (15)$$

4. Real Quasi-Elliptic Cohomology of S^4 Acted by a Finite Subgroup of $\mathrm{Spin}(3)$

In this section, we compute all the Real quasi-elliptic cohomology theories

$$\mathrm{QEIR}_G^*(S^4)$$

where G goes over all the finite subgroups of $SU(2) \cong \mathrm{Spin}(3)$.

First, we explain how the group G acts on S^4 . We have the standard orthogonal $\mathrm{SO}(5)$ -action on $\mathbb{R}^5$ and also on the subspace $S^4 \subset \mathbb{R}^5$. The covering map

$$\mathrm{Spin}(5) \rightarrow \mathrm{SO}(5)$$

makes S^4 a well-defined $\text{Spin}(5)$ -space. The G -action on S^4 is induced by the composition

$$i_G : G \hookrightarrow \text{Spin}(3) \xrightarrow{p_1} \text{Spin}(3) \times \text{Spin}(3) = \text{Spin}(4) \hookrightarrow \text{Spin}(5) \quad (16)$$

where p_1 is the projection to the first factor of the product group.

We give the explicit formula of the G -action below. The group $S(\mathbb{H})$ of unit quaternions is isomorphic to $SU(2) \cong \text{Spin}(3)$ via the correspondence

$$a + bi + cj + dk \mapsto \begin{bmatrix} a + bi & c + di \\ -c + di & a - bi \end{bmatrix}.$$

In view of this, $\text{Spin}(4)$ can be described as the group

$$\left\{ \begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix} \mid q, r \in \mathbb{H}, |q| = |r| = 1 \right\},$$

and $\text{Spin}(5)$ can be identified with the quaternionic unitary group. Thus, as indicated in ([32] p. 263), the inclusion from $\text{Spin}(4) \hookrightarrow \text{Spin}(5)$ is given by the formula

$$\begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix} \mapsto \begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix}. \quad (17)$$

In addition, as shown in ([32] p. 151), the rotation of $\mathbb{R}^4$ represented by

$$\begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix} \in \text{Spin}(4)$$

is given by the map

$$\begin{bmatrix} y & 0 \\ 0 & \bar{y} \end{bmatrix} \mapsto \begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix} \begin{bmatrix} y & 0 \\ 0 & \bar{y} \end{bmatrix} \left[\widehat{\begin{bmatrix} q & 0 \\ 0 & r \end{bmatrix}} \right]^{-1} = \begin{bmatrix} qy\bar{r} & 0 \\ 0 & r\bar{y}q \end{bmatrix}. \quad (18)$$

where $\mathbb{R}^4$ is identified with the linear space

$$\left\{ \begin{bmatrix} y & 0 \\ 0 & \bar{y} \end{bmatrix} \mid y \in \mathbb{H} \right\}.$$

Then, the group $\text{Spin}(4) \subset \text{Spin}(5)$ acts on $S^4 \subset \mathbb{R}^5$ via the composition

$$\text{Spin}(4) \rightarrow \text{SO}(4) \xrightarrow{A \mapsto \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix}} \text{SO}(5) \quad (19)$$

and the standard orthogonal action.

In the rest part of the paper, we will use the symbol

$$A_\theta$$

to denote the matrix

$$\begin{bmatrix} e^{\theta i} & 0 \\ 0 & e^{-\theta i} \end{bmatrix},$$

and the symbol

$$B_\theta$$

to denote the matrix

$$\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}.$$

First, we need to pick a Real structure $(\widehat{SU(2)}, \pi)$ on the group $SU(2)$ as well as on all its finite subgroups by equipping the group with a reflection s . The choice is definitely not unique. Next, we define the reflection action on S^4 and, thus, together with (18), we define the action on S^4 by $\widehat{SU(2)}$.

Example 7. *Motivated by the Real structure*

$$1 \rightarrow \mathbb{Z}/n \rightarrow D_{2n} \xrightarrow{\pi} \mathbb{Z}/2 \rightarrow 1$$

of the cyclic group $\mathbb{Z}/n < SU(2)$, we want to pick a Real structure $(\widehat{SU(2)}, \pi)$ on $SU(2)$ making the diagrams below commute.

$$\begin{array}{ccccccc} 1 & \longrightarrow & \mathbb{Z}/n & \longrightarrow & D_{2n} & \longrightarrow & \mathbb{Z}/2 \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & SO(2) & \longrightarrow & O(2) & \xrightarrow{\det} & \mathbb{Z}/2 \longrightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \longrightarrow & SU(2) & \longrightarrow & \widehat{SU(2)} & \xrightarrow{\pi} & \mathbb{Z}/2 \longrightarrow 1 \end{array} \quad (20)$$

where the horizontal sequences are all exact. In the left column, the generator r of the rotation group $\mathbb{Z}/n < D_{2n}$ is mapped to the rotation $B_{\frac{2\pi}{n}}$ in $SO(2)$. The lower left vertical map can be chosen to map the rotation $B_{\frac{2\pi}{n}}$ to $A_{\frac{2\pi}{n}} \in SU(2)$. In addition, the reflection in D_{2n} can be mapped to

$$s := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in U(2).$$

It is straightforward to check that $(sA_\theta)^2$ is the identity map for any θ .

In addition, we can take the action of s on $\mathbb{R}^4 \cong \mathbb{H}$ to be

$$(a + bi + cj + dk) \mapsto (a - bi + cj - dk). \quad (21)$$

Note that under the reflection (21), the north and south poles of S^4 are still fixed. On the $\mathbb{R}^4$ -plane, the two pairs of points

$$(0, 1, 0, 0) \text{ and } (0, -1, 0, 0)$$

$$(0, 0, 0, 1) \text{ and } (0, 0, 0, -1)$$

are switched by the reflection, respectively. It is straightforward to check that $(sA_\theta)^2$ acts as the identity map on $\mathbb{R}^4$ for any θ . Thus, it is reasonable to take the Real structure to be the subgroup

$$SU(2)\langle s \rangle$$

of $U(2)$ and take the projection to be the determinant map $\det$.

Instead, we can map the rotation r to the matrix $B_{\frac{2\pi}{n}}$, which is a conjugation of $A_{\frac{2\pi}{n}}$. We have

$$A^{-1}B_\theta A = A_\theta$$

where $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$ and θ is any real number. In addition, the reflection s is fixed under the conjugation. The corresponding Real structure of $SU(2)$ is still $SU(2)\langle s \rangle$ and the diagram (20) still commutes.

Moreover, we would like to mention a different choice of the Real structure $\widehat{SU(2)}$. In the diagram (20), we map the rotation $B_{\frac{2\pi}{n}}$ in $SO(2)$ to the same matrix in $SU(2)$ but map the reflection to

$$s' := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in U(2).$$

Note that $A^{-1}s'A \neq s'$, i.e., s' is not a fixed point under the conjugation taking B_θ to A_θ . We can check, for any θ , $(s'B_\theta)^2 = I$. The action of s' on $\mathbb{R}^4$ can be defined as

$$(a + bi + cj + dk) \mapsto (a + bi - cj - dk). \quad (22)$$

Under the reflection (22), the north and south poles are also fixed. On $\mathbb{R}^4$, the two pairs of points

$$(0, 0, 1, 0) \text{ and } (0, 0, -1, 0)$$

$$(0, 0, 0, 1) \text{ and } (0, 0, 0, -1)$$

are switched by the reflection, respectively. It is straightforward to check that $(s'B_\theta)^2$ acts as identity on $\mathbb{R}^4$ for any θ . Thus, it is reasonable to take the Real structure to be the subgroup

$$SU(2)\langle s' \rangle$$

of $U(2)$ and the projection π to be the determinant $\det$.

Since $SU(2)$ is a normal subgroup of $U(2)$, both Real structures, $SU(2)\langle s \rangle$ and $SU(2)\langle s' \rangle$, split.

Example 8. For any finite subgroup G of $SU(2)$,

$$\hat{G} := (G\langle s \rangle, \det)$$

is the restriction of the Real structure

$$(SU(2)\langle s \rangle, \det)$$

of $SU(2)$ to G . It defines a Real structure on G .

Similarly,

$$\hat{G}' := (G\langle s' \rangle, \det)$$

defines a Real structure on G .

Remark 1. We give in Example 7 some reasonable choices of reflection on the representation sphere S^4 , which all keep the north pole and the south pole fixed. We did not find a canonical choice of reflection that switches the north pole and the south pole.

As indicated in ([33] p. 215), for V , a real vector space equipped with a linear G -action, stereographic projection exhibits a G -equivariant homeomorphism between the representation sphere $S^V := V_{\text{cpt}}$ (the one-point compactification) and the unit sphere $S(V \oplus \mathbb{R}_{\text{triv}})$ (where the $\mathbb{R}$ -summand is equipped with the trivial G -action):

$$S^V \simeq_G S(V \oplus \mathbb{R}_{\text{triv}}).$$

Another reasonable choice of the reflection on $S(V \oplus \mathbb{R}_{\text{triv}})$ is the one sending a point $(v, r) \in S(V \oplus \mathbb{R}_{\text{triv}})$ to $(v, -r)$. The map corresponding to that on S^V , which is

$$\begin{cases} v \mapsto \frac{1}{\|v\|}v, & \text{if } v \neq 0, \infty; \\ \text{the north pole} \mapsto \text{the south pole}, & \text{if } v = \infty; \\ \text{the south pole} \mapsto \text{the north pole}, & \text{if } v = 0, \end{cases}$$

where $\|v\|$ is the length of the vector. The map preserves the angle but not the length of the vector when it is not 1, and, especially, it is not linear. Thus, this is not the right choice of reflection for our computation.

Remark 2. We would like to mention that, other than the choice of the Real structure, the choice of the action of $\hat{G}$ on S^4 is also not unique where G is any finite subgroup of $SU(2)$. Though different choices of the Real structure may lead to different $\text{QEIR}_G^*(S^4)$, different choices of the action of the reflection may lead to little difference.

In the computation of $\text{QEIR}_G^*(S^4)$ with G a finite subgroup of $SU(2)$, for most elements $g \in \pi_0(G//G)$, the fixed point spaces $(S^4)^g$ consist of only the north pole and the south pole, where the reflections, those in Example 7, etc., act trivially. In addition, for the identity element $e \in G$, $(S^4)^e = S^4$ is a representation sphere of the group $\Lambda_G(e)$. Thus, by ([34] Theorem 5.1), the computation of the corresponding factor $KR_{\Lambda_G(e)}^0((S^4)^e)$ can be reduced to that of the Real representation ring of $\Lambda_G(e) \cong G \times \mathbb{T}$.

To compute the Real quasi-elliptic cohomology of 4-spheres

$$\text{QEIR}^*(S^4//G) \cong \prod_{g \in \pi_0(G//G)_{-1}} KR_{\Lambda_G(g)}^*((S^4)^g) \times \prod_{g \in \pi_0(G//G)_{+1}/\mathbb{Z}/2} K_{\Lambda_G(g)}^*((S^4)^g), \quad (23)$$

acted by a finite subgroup

$$G < SU(2) \cong \mathbb{H},$$

we need to find all the fixed points in G under the involution, i.e., the Real conjugation. Below is a conclusion that we will apply in the computation later.

Proposition 3. If we take the Real structure $\hat{G}'$ on a finite subgroup G of $SU(2)$, for any element β in G , we have the conclusions below.

- (1) The element β is a fixed point under the involution s' if and only if $s'\beta^{-1}s'$ is in the conjugacy class of β in G .
- (2) If β is a unit quaternion whose coefficient of i is zero, then $s'\beta^{-1}s' = \beta$ and β is a fixed point under the involution.

Proof. A given element $\beta \in G$ is a fixed point under the involution if and only if the set $C_G^R(\beta) \setminus C_G(\beta)$ is nonempty, i.e., there is an element $x = s'y$ for some $y \in G$ satisfying

$$x\beta x^{-1} = \beta^{-1}.$$

So we obtain the first conclusion.

Since β is an element in $SU(2)$, thus, it has a quaternion representation $\beta = a + bi + cj + dk$. In (ii), we discuss a very special case that $s'\beta s' = \beta^{-1}$. We start the computation below.

$$s'\beta s' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a + bi & c + di \\ -c + di & a - bi \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a + bi & -c - di \\ c - di & a - bi \end{bmatrix}$$

The right hand side should be the inverse of β . So we establish the equation

$$\begin{bmatrix} a+bi & c+di \\ -c+di & a-bi \end{bmatrix} \begin{bmatrix} a+bi & -c-di \\ c-di & a-bi \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solving the equation, we obtain

$$\begin{cases} b = 0 \\ a^2 + c^2 + d^2 = 1 \end{cases}$$

i.e.,

$$s'\beta s' = \beta^{-1}$$

if and only if $\beta = a + bi + cj + dk$ is a unit quaternion with $b = 0$. $\square$

Similarly, we have the conclusion.

Proposition 4. *If we take the Real structure $\hat{G}$ on a finite subgroup G of $SU(2)$, for any element β in G , we have the conclusions below.*

- (1) *The element β is a fixed point under the involution s if and only if $s\beta^{-1}s$ is in the conjugacy class of β in G .*
- (2) *If β is a unit quaternion whose coefficient of k is zero, then we have $s\beta^{-1}s = \beta$ and β is a fixed point under the involution.*

The proof is analogous to that of Proposition 3.

Next, we will compute $\text{QEILR}^*(S^4//G)$ with G a finite subgroup of $SU(2)$ one by one. Before that, we recall the classification of the finite subgroups of $\text{Spin}(3) \cong SU(2)$. There are many references for the classification, including ([35] Chapter XIII), [36,37] etc. The finite subgroups of $SU(2)$ are classified as follows:

- the cyclic group of order n

$$G_n := \left\{ \begin{bmatrix} \cos \frac{2\pi k}{n} & \sin \frac{2\pi k}{n} \\ -\sin \frac{2\pi k}{n} & \cos \frac{2\pi k}{n} \end{bmatrix} \mid k \in \mathbb{Z} \right\};$$

- the dicyclic group of order $4n$

$$2D_{2n} := \langle A_{\frac{2\pi}{2n}}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rangle;$$

- the binary tetrahedral group E_6 ;
- the binary octahedral group E_7 ;
- the binary icosahedral group E_8 ;

where n is any positive integer.

Example 9. *In this example, we compute $\text{QEILR}^*(S^4//G_n)$ where G_n is the finite cyclic subgroup*

$$\left\{ \begin{bmatrix} \cos \frac{2\pi k}{n} & \sin \frac{2\pi k}{n} \\ -\sin \frac{2\pi k}{n} & \cos \frac{2\pi k}{n} \end{bmatrix} \mid k \in \mathbb{Z} \right\} < SU(2).$$

We take the Real structure $\hat{G}'_n$ as defined in Example 8, i.e., the group below together with the determinant map $\det$

$$\langle G_n, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rangle.$$

It is isomorphic to the dihedral group D_{2n} . The involution on $\pi_0(G_n // G_n)$ is trivial.

Thus, by ([3] Example 3.7), we obtain directly that

$$\mathrm{QEIR}^*(S^4 // G_n) \cong \prod_{m=0}^{n-1} \mathrm{KR}^*_{\Lambda_{G_n}(B_{\frac{2\pi m}{n}})}((S^4)^{B_{\frac{2\pi m}{n}}}) \cong \mathrm{KR}^*_{G_n \times \mathbb{T}}(S^4) \oplus \prod_{m=1}^{n-1} \mathrm{KR}^*_{\Lambda_{G_n}(B_{\frac{2\pi m}{n}})}(S^0),$$

where S^0 consists of the fixed points, i.e., the south pole and the north pole of S^4 . Thus,

$$\prod_{m=1}^{n-1} \mathrm{KR}^*_{\Lambda_{G_n}(B_{\frac{2\pi m}{n}})}(S^0) \cong \prod_{m=1}^{n-1} \mathrm{KR}^*_{\Lambda_{G_n}(B_{\frac{2\pi m}{n}})}(\mathrm{pt}) \oplus \mathrm{KR}^*_{\Lambda_{G_n}(B_{\frac{2\pi m}{n}})}(\mathrm{pt}),$$

and by ([3] Example 3.7), the right hand side is isomorphic to

$$\prod_{m=1}^{n-1} \mathrm{KR}^*(\mathrm{pt})[x, q^{\pm}] / \langle x^n - q^m \rangle \oplus \mathrm{KR}^*(\mathrm{pt})[x, q^{\pm}] / \langle x^n - q^m \rangle.$$

In addition, by ([34] Theorem 5.1),

$$\begin{aligned} \mathrm{KR}^*_{G_n \times \mathbb{T}}(S^4) &\cong \mathrm{KR}^*_{G_n \times \mathbb{T}}(S^0) \cong \mathrm{KR}^*_{G_n \times \mathbb{T}}(\mathrm{pt}) \oplus \mathrm{KR}^*_{G_n \times \mathbb{T}}(\mathrm{pt}) \\ &\cong \mathrm{KR}^*(\mathrm{pt})[x, q^{\pm}] / \langle x^n - 1 \rangle \oplus \mathrm{KR}^*(\mathrm{pt})[x, q^{\pm}] / \langle x^n - 1 \rangle. \end{aligned}$$

In conclusion,

$$\mathrm{QEIR}^*(S^4 // G_n) \cong \prod_{m=0}^{n-1} \mathrm{KR}^*(\mathrm{pt})[x, q^{\pm}] / \langle x^n - q^m \rangle \oplus \mathrm{KR}^*(\mathrm{pt})[x, q^{\pm}] / \langle x^n - q^m \rangle.$$

Example 10. In this example, we compute $\mathrm{QEIR}^*(S^4 // 2D_{2n})$ where $2D_{2n}$ is the dicyclic group

$$\langle A_{\frac{2\pi}{2n}}, \tau \rangle,$$

where τ is the reflection

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The Real structure on $2D_{2n}$ is taken to be $2\hat{D}_{2n}$, as that defined in Example 8.

In $2D_{2n}$, there are $n + 3$ conjugacy classes. They are as follows:

- (1) $\{I\}$,
- (2) $\{-I\}$,
- (3) $\{A_{\frac{\pi}{n}}, A_{\frac{\pi}{n}}^{-1}\}, \{A_{\frac{2\pi}{n}}^2, A_{\frac{2\pi}{n}}^{-2}\}, \dots, \{A_{\frac{\pi}{n}}^{n-1}, A_{\frac{\pi}{n}}^{-(n-1)}\}$,
- (4) $\{\tau, \tau A_{\frac{\pi}{n}}^2, \tau A_{\frac{\pi}{n}}^4, \dots, \tau A_{\frac{\pi}{n}}^{2n-2}\}$,
- (5) $\{\tau A_{\frac{\pi}{n}}, \tau A_{\frac{\pi}{n}}^3, \dots, \tau A_{\frac{\pi}{n}}^{2n-1}\}$,

where the first two classes form the centre of the group.

By Proposition 4, all the conjugacy classes are fixed points under the reflection s . Thus, all the factors in $\mathrm{QEIR}^*(S^4 // 2D_{2n})$ are equivariant KR-theories. The factor in $\mathrm{QEIR}^*(S^4 // 2D_{2n})$ corresponding to each conjugacy class is computed below one by one.

- (1) First, we consider the Real conjugacy class represented by I . The centralizer $C_{2D_{2n}}(I) = 2D_{2n}$ and the Real centralizer is

$$C_{2\hat{D}_{2n}}^R(I) = 2\hat{D}_{2n}.$$

Then, the group $\Lambda_{2\hat{D}_{2n}}^R(I) = \mathbb{R} \rtimes_{\pi} 2\hat{D}_{2n} / \langle (-1, I) \rangle$. By ([34] Theorem 5.1),

$$\begin{aligned} KR_{\Lambda_{2D_{2n}}(I)}^*((S^4)^I) &\cong KR_{\mathbb{T} \times 2D_{2n}}^*(S^4) \cong KR_{\mathbb{T} \times 2D_{2n}}^*(S^0) \\ &\cong KR_{\mathbb{T} \times 2D_{2n}}^*(\text{pt}) \oplus KR_{\mathbb{T} \times 2D_{2n}}^*(\text{pt}) \\ &\cong KR_{2D_{2n}}^*(\text{pt})[q^{\pm}] \oplus KR_{2D_{2n}}^*(\text{pt})[q^{\pm}]. \end{aligned}$$

Note that $\Lambda_{2\hat{D}_{2n}}^R(I)$ is a Real structure on $\mathbb{T} \times 2D_{2n}$.

- (2) Then, we consider the Real conjugacy class represented by $-I$. In this case, the centralizer $C_{2D_{2n}}(-I) = 2D_{2n}$ and the Real centralizer

$$C_{2\hat{D}_{2n}}^R(-I) = 2\hat{D}_{2n}.$$

In this case, we have the Real central extension

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \Lambda_{2\hat{D}_{2n}}^R(-I) \longrightarrow \Lambda_{\hat{D}_{2n}}^R(I) \longrightarrow 1.$$

By Corollary A1,

$$\begin{aligned} KR_{\Lambda_{2D_{2n}}(-I)}^*((S^4)^{-I}) &\cong KR_{\Lambda_{2D_{2n}}(-I)}^*(S^0) \\ &\cong \prod_1^2 KR_{\Lambda_{2D_{2n}}(-I)}^*(\text{pt}) \\ &\cong \prod_1^2 KR_{\Lambda_{D_{2n}}(I)}^*(\text{pt}) \oplus KR_{\Lambda_{D_{2n}}(I)}^{*+\hat{\nu}_{\Lambda_{2\hat{D}_{2n}}^R(-I), \text{sign}}}(\text{pt}) \\ &\cong \prod_1^2 KR_{D_{2n}}^*(\text{pt})[q^{\pm}] \oplus KR_{D_{2n}}^{*+\hat{\nu}_{\Lambda_{2\hat{D}_{2n}}^R(-I), \text{sign}}}(\text{pt})[q^{\pm}], \end{aligned}$$

where sign is the sign representation of $\mathbb{Z}/2$.

- (3) Then, we compute the factor in $\text{QEII}^R(S^4//2D_{2n})$ corresponding to $A_{\frac{2\pi m}{2n}}$ which is not $\pm I$. The centralizer $C_{2D_{2n}}(A_{\frac{2\pi m}{2n}})$ is the cyclic group $\langle A_{\frac{2\pi}{2n}} \rangle \cong \mathbb{Z}/(2n)$; and the Real centralizer

$$C_{2\hat{D}_{2n}}^R(A_{\frac{2\pi m}{2n}}) = D_{4n}$$

is the dihedral group of order $4n$. In this case, by ([3] Example 3.7),

$$\begin{aligned} KR_{\Lambda_{2D_{2n}}(A_{\frac{2\pi m}{2n}})}^*(S^4)^{A_{\frac{2\pi m}{2n}}} &\cong KR_{\Lambda_{2D_{2n}}(A_{\frac{2\pi m}{2n}})}^*(S^0) \cong KR_{\Lambda_{2D_{2n}}(A_{\frac{2\pi m}{2n}})}^*(\text{pt}) \oplus KR_{\Lambda_{2D_{2n}}(A_{\frac{2\pi m}{2n}})}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^{\pm}] / \langle x^{2n} - q^{2m} \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^{2n} - q^{2m} \rangle. \end{aligned}$$

- (4) Then, we compute the factor corresponding to the conjugacy class represented by τ . The centralizer $C_{2D_{2n}}(\tau) = \langle \tau \rangle \cong \mathbb{Z}/4$ and the Real centralizer

$$C_{2\hat{D}_{2n}}^R(\tau) = \langle \tau, s \rangle \cong D_4.$$

Thus,

$$\begin{aligned} KR_{\Lambda_{2D_{2n}}}^*(S^4)^\tau &\cong KR_{\Lambda_{\mathbb{Z}/4}(1)}^*(S^0) \\ &\cong RR\Lambda_{\mathbb{Z}/4}(1) \oplus RR\Lambda_{\mathbb{Z}/4}(1) \\ &\cong KR^*(\text{pt})[x, q^\pm] / \langle x^4 - q \rangle \oplus KR^*(\text{pt})[x, q^\pm] / \langle x^4 - q \rangle. \end{aligned}$$

- (5) For the conjugacy class represented by $\tau A_{\frac{\pi}{n}}$, the centralizer $C_{2D_{2n}}(\tau A_{\frac{\pi}{n}}) = \langle \tau A_{\frac{\pi}{n}} \rangle \cong \mathbb{Z}/4$ and the Real centralizer $C_{2\hat{D}_{2n}}^R(\tau A_{\frac{\pi}{n}}) = \langle \tau A_{\frac{\pi}{n}}, s\tau \rangle \cong D_4$. Then, the factor corresponding to $\tau A_{\frac{\pi}{n}}$ is

$$\begin{aligned} KR_{\Lambda_{2D_{2n}}(\tau A_{\frac{\pi}{n}})}^*(S^4)^{\tau A_{\frac{\pi}{n}}} &\cong KR_{\Lambda_{\mathbb{Z}/4}(1)}^*(S^0) \\ &\cong RR\Lambda_{\mathbb{Z}/4}(1) \oplus RR\Lambda_{\mathbb{Z}/4}(1) \\ &\cong KR^*(\text{pt})[x, q^\pm] / \langle x^4 - q \rangle \oplus KR^*(\text{pt})[x, q^\pm] / \langle x^4 - q \rangle. \end{aligned}$$

In conclusion,

$$\begin{aligned} \text{QEIR}^*(S^4//2D_{2n}) &= KR_{\Lambda_{2D_{2n}}(I)}^*((S^4)^I) \times KR_{\Lambda_{2D_{2n}}(-I)}^*((S^4)^{-I}) \\ &\quad \times \prod_{m=1}^{n-1} KR_{\Lambda_{2D_{2n}}(A_{\frac{m}{n}})}^*((S^4)^{A_{\frac{m}{n}}}) \\ &\quad \times KR_{\Lambda_{2D_{2n}}(\tau)}^*((S^4)^\tau) \times KR_{\Lambda_{2D_{2n}}(\tau A_{\frac{\pi}{n}})}^*((S^4)^{\tau A_{\frac{\pi}{n}}}) \\ &\cong KR_{2D_{2n}}^*(\text{pt})[q^\pm] \oplus KR_{2D_{2n}}^*(\text{pt})[q^\pm] \\ &\quad \times \prod_1^2 KR_{D_{2n}}^*(\text{pt})[q^\pm] \oplus KR_{D_{2n}}^{*\hat{\nu}\Lambda_{2D_{2n}}(-I), \text{sign}}^*(\text{pt})[q^\pm] \\ &\quad \times \prod_{m=1}^{n-1} KR^*(\text{pt})[x, q^\pm] / \langle x^{2n} - q^{2m} \rangle \oplus KR^*(\text{pt})[x, q^\pm] / \langle x^{2n} - q^{2m} \rangle \\ &\quad \times KR^*(\text{pt})[x, q^\pm] / \langle x^4 - q \rangle \oplus KR^*(\text{pt})[x, q^\pm] / \langle x^4 - q \rangle \\ &\quad \times KR^*(\text{pt})[x, q^\pm] / \langle x^4 - q \rangle \oplus KR^*(\text{pt})[x, q^\pm] / \langle x^4 - q \rangle, \end{aligned}$$

where sign is the sign representation of $\mathbb{Z}/2$.

Example 11. In this example, we compute $\text{QEIR}^*(S^4//E_6)$ where E_6 is the binary tetrahedral group E_6 . We take the Real structure $\hat{E}_6$ on it, i.e.,

$$\hat{E}_6 = E_6 \langle s' \rangle.$$

The quaternion representation of E_6 is given explicitly in [38,39].

We can compute the conjugacy classes in E_6 explicitly. A multiplication table for the binary tetrahedral group is given in [40]. For the convenience of the readers, we apply the same symbols of the elements as those in [39,40]. A list of representatives are given in Figure 1. This list can be obtained by direct computation. In addition, by Proposition 3, an element in E_6 represents a fixed point in $\pi_0(E_6//_R \hat{E}_6)$ if and only if it is $\pm I$, $\pm i$, $\pm j$ or $\pm k$. Note that, for E_6 , if we take the Real structure $\hat{E}_6$, we will obtain the same set of fixed points under the reflection.

A representative of the conjugacy class	Conjugacy class	Order	Fixed point under the involution?
1	{1}	1	Y
−1	{−1}	2	Y
j	{ $\pm i, \pm j, \pm k$ }	4	Y
a	{ a, b, c, d }	6	N
− a	{ $−a, −b, −c, −d$ }	3	N
a^2	{ a^2, b^2, c^2, d^2 }	3	N
− a^2	{ $−a^2, −b^2, −c^2, −d^2$ }	6	N

Figure 1. Conjugacy classes of E_6 .

Below, we compute the factors of $\mathrm{QEllR}_{E_6}(S^4)$ corresponding to each conjugacy class respectively.

- (1) For the conjugacy class represented by I , the Real centralizer $C_{E'_6}^R(I) = \hat{E}'_6$. By ([34] Theorem 5.1), we have

$$\begin{aligned} KR_{\Lambda_{E_6}(I)}^*((S^4)^I) &\cong KR_{E_6 \times \mathbb{T}}^*(S^4) \cong KR_{E_6 \times \mathbb{T}}^*(S^0) \\ &\cong KR_{E_6 \times \mathbb{T}}^*(\mathrm{pt}) \oplus KR_{E_6 \times \mathbb{T}}^*(\mathrm{pt}) \\ &\cong KR_{E_6}^*(\mathrm{pt})[q^\pm] \oplus KR_{E_6}^*(\mathrm{pt})[q^\pm]. \end{aligned}$$

- (2) For the conjugacy class represented by $−I$, we have $(S^4)^{−I} = S^0$. Let $\hat{T}'_6$ denote the group $T_6\langle s' \rangle$. We have the short exact sequence

$$1 \rightarrow \mathbb{Z}/2 \rightarrow \hat{T}'_6 \rightarrow T_6 \rightarrow 1$$

Especially, we have the commutative diagram below:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & E_6 & \xrightarrow{\pi} & T_6 \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \hat{E}'_6 & \xrightarrow{\pi} & \hat{T}'_6 \longrightarrow 0 \end{array} \quad (24)$$

Note that we have the short exact sequence

$$0 \rightarrow \mathbb{Z}/2 \rightarrow \Lambda_{\hat{E}'_6}^R(−I) \xrightarrow{[(\pi, id), id]} \Lambda_{\hat{T}'_6}^R(I) \rightarrow 0.$$

By Corollary A1,

$$\begin{aligned} KR_{\Lambda_{E_6}(−I)}^*((S^4)^{−I}) &\cong KR_{\Lambda_{E_6}(−I)}^*(S^0) \\ &\cong \prod_1^2 KR_{\Lambda_{E_6}(−I)}^*(\mathrm{pt}) \\ &\cong \prod_1^2 KR_{T_6 \times \mathbb{T}}^*(\mathrm{pt}) \oplus KR_{T_6 \times \mathbb{T}}^{*+\hat{\nu}_{\Lambda_{\hat{E}'_6}(−I)}^R, sign}(\mathrm{pt}) \\ &\cong \prod_1^2 KR_{T_6}^*(\mathrm{pt})[q^\pm] \oplus KR_{T_6}^{*+\hat{\nu}_{\Lambda_{\hat{E}'_6}(−I)}^R, sign}(\mathrm{pt})[q^\pm], \end{aligned}$$

where $sign$ is the sign representation of $\mathbb{Z}/2$.

- (3) For the conjugacy class represented by j , $(S^4)^j = S^0$. The centralizer $C_{E_6}(j) = \langle j \rangle \cong \mathbb{Z}/4$ and the Real centralizer

$$C_{E_6}^R(j) = C_{E_6}(j)\langle s' \rangle \cong D_4.$$

Thus,

$$\begin{aligned} KR_{\Lambda_{E_6}(j)}^*((S^4)^j) &\cong KR_{\Lambda_{\mathbb{Z}/4}(1)}^*(S^0) \cong KR_{\Lambda_{\mathbb{Z}/4}(1)}^*(\text{pt}) \oplus KR_{\Lambda_{\mathbb{Z}/4}(1)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^{\pm}]/\langle x^4 - q \rangle \oplus KR^*(\text{pt})[x, q^{\pm}]/\langle x^4 - q \rangle. \end{aligned}$$

- (4) For the conjugacy class represented by a , we have

$$\begin{aligned} K_{\Lambda_{E_6}(a)}((S^4)^a) &\cong K_{\Lambda_{\mathbb{Z}/6}(1)}(S^0) \cong R(\Lambda_{\mathbb{Z}/6}(1)) \oplus R(\Lambda_{\mathbb{Z}/6}(1)) \\ &\cong \mathbb{Z}[x, q^{\pm}]/\langle x^6 - q \rangle \oplus \mathbb{Z}[x, q^{\pm}]/\langle x^6 - q \rangle. \end{aligned}$$

- (5) For the conjugacy class represented by $-a$, we have

$$\begin{aligned} K_{\Lambda_{E_6}(-a)}((S^4)^{-a}) &\cong K_{\Lambda_{\mathbb{Z}/6}(4)}(S^0) \cong R(\Lambda_{\mathbb{Z}/6}(4)) \oplus R(\Lambda_{\mathbb{Z}/6}(4)) \\ &\cong \mathbb{Z}[x, q^{\pm}]/\langle x^6 - q^4 \rangle \oplus \mathbb{Z}[x, q^{\pm}]/\langle x^6 - q^4 \rangle. \end{aligned}$$

- (6) For the conjugacy class represented by a^2 , we have

$$\begin{aligned} K_{\Lambda_{E_6}(a^2)}((S^4)^{a^2}) &\cong K_{\Lambda_{\mathbb{Z}/6}(2)}(S^0) \cong R(\Lambda_{\mathbb{Z}/6}(2)) \oplus R(\Lambda_{\mathbb{Z}/6}(2)) \\ &\cong \mathbb{Z}[x, q^{\pm}]/\langle x^6 - q^2 \rangle \oplus \mathbb{Z}[x, q^{\pm}]/\langle x^6 - q^2 \rangle. \end{aligned}$$

- (7) For the conjugacy class represented by $-a^2$, we have

$$\begin{aligned} K_{\Lambda_{E_6}(-a^2)}((S^4)^{-a^2}) &\cong K_{\Lambda_{\mathbb{Z}/6}(5)}(S^0) \cong R(\Lambda_{\mathbb{Z}/6}(5)) \oplus R(\Lambda_{\mathbb{Z}/6}(5)) \\ &\cong \mathbb{Z}[x, q^{\pm}]/\langle x^6 - q^5 \rangle \oplus \mathbb{Z}[x, q^{\pm}]/\langle x^6 - q^5 \rangle. \end{aligned}$$

Thus, in conclusion,

$$\begin{aligned} \text{QEIR}^*(S^4 // E_6) &= KR_{\Lambda_{E_6}(1)}^*((S^4)^1) \times KR_{\Lambda_{E_6}(-1)}^*((S^4)^{-1}) \times KR_{\Lambda_{E_6}(j)}^*((S^4)^j) \\ &\quad \times K_{\Lambda_{E_6}(a)}^*((S^4)^a) \times K_{\Lambda_{E_6}(-a)}^*((S^4)^{-a}) \times K_{\Lambda_{E_6}(a^2)}^*((S^4)^{a^2}) \\ &\quad \times K_{\Lambda_{E_6}(-a^2)}^*((S^4)^{-a^2}) \\ &\cong KR_{E_6}^*(\text{pt})[q^{\pm}] \oplus KR_{E_6}^*(\text{pt})[q^{\pm}] \\ &\quad \times \prod_1^2 KR_{T_6}^*(\text{pt})[q^{\pm}] \oplus KR_{T_6}^{*\oplus \Lambda_{E_6}^R(-1)\text{sign}}(\text{pt})[q^{\pm}] \\ &\quad \times KR^*(\text{pt})[x, q^{\pm}]/\langle x^4 - q \rangle \oplus KR^*(\text{pt})[x, q^{\pm}]/\langle x^4 - q \rangle \\ &\quad \times K^*(\text{pt})[x, q^{\pm}]/\langle x^6 - q \rangle \oplus K^*(\text{pt})[x, q^{\pm}]/\langle x^6 - q \rangle \\ &\quad \times K^*(\text{pt})[x, q^{\pm}]/\langle x^6 - q^4 \rangle \oplus K^*(\text{pt})[x, q^{\pm}]/\langle x^6 - q^4 \rangle \\ &\quad \times K^*(\text{pt})[x, q^{\pm}]/\langle x^6 - q^2 \rangle \oplus K^*(\text{pt})[x, q^{\pm}]/\langle x^6 - q^2 \rangle \\ &\quad \times K^*(\text{pt})[x, q^{\pm}]/\langle x^6 - q^5 \rangle \oplus K^*(\text{pt})[x, q^{\pm}]/\langle x^6 - q^5 \rangle, \end{aligned}$$

where sign is the sign representation of $\mathbb{Z}/2$.

Example 12. In this example, we compute $\text{QEIR}^*(S^4 // E_7)$ where E_7 is the binary octahedral group. We take the Real structure $\hat{E}_7^R$ on it, i.e., $E_7\langle s' \rangle$.

A presentation of E_7 is given as

$$E_7 = \langle \theta, t \mid r^2 = \theta^3 = t^4 = r\theta t = -1 \rangle.$$

We can obtain immediately that $r = \theta t$. Equivalently, there is a quaternion presentation of E_7 given by the embedding

$$E_7 \rightarrow \mathbb{H}$$

sending θ to $\frac{1}{2}(1 + i + j + k)$, t to $\frac{1}{\sqrt{2}}(1 + i)$, and r to $\frac{1}{\sqrt{2}}(i + j)$.

By [41] and direct computation, we obtain Figure 2, which provides a list of the representatives of the conjugacy classes of E_7 , the centralizers of each representative, and the corresponding fixed point spaces.

Representatives β of conjugacy classes	Centralizers $C_{E_7}(\beta)$	Conjugacy class	Fixed points under the involution?	$(S^4)^\beta$
1	E_7	$\{1\}$	Y	S^4
-1	E_7	$\{-1\}$	Y	S^0
$j = \theta t^2 \theta^{-1}$	$\langle \theta t \theta^{-1} \rangle \cong \mathbb{Z}/8$	$\{\pm i, \pm j, \pm k\}$	Y	S^0
θ	$\langle \theta \rangle \cong \mathbb{Z}/6$	$\{\frac{(1 \pm i \pm j \pm k)}{2}\}$	Y	S^0
$-\theta = \theta^4$	$\langle \theta \rangle \cong \mathbb{Z}/6$	$\{\frac{(-1 \pm i \pm j \pm k)}{2}\}$	Y	S^0
r	$\langle r \rangle \cong \mathbb{Z}/4$	$\{\frac{1}{\sqrt{2}}(\pm i \pm j), \frac{1}{\sqrt{2}}(\pm i \pm k), \frac{1}{\sqrt{2}}(\pm j \pm k)\}$	Y	S^0
t	$\langle t \rangle \cong \mathbb{Z}/8$	$\{\frac{1 \pm i}{\sqrt{2}}, \frac{1 \pm j}{\sqrt{2}}, \frac{1 \pm k}{\sqrt{2}}\}$	Y	S^0
$-t = t^5$	$\langle t \rangle \cong \mathbb{Z}/8$	$\{\frac{-1 \pm i}{\sqrt{2}}, \frac{-1 \pm j}{\sqrt{2}}, \frac{-1 \pm k}{\sqrt{2}}\}$	Y	S^0

Figure 2. Conjugacy classes, centralizers, and fixed point spaces.

Below, we give the factor of $\mathrm{QEIR}^*(S^4 // E_7)$ corresponding to each conjugacy class.

- (1) For the conjugacy class represented by I , the Real centralizer $C_{\hat{E}_7}^R(I) = \hat{E}_7'$. The factor corresponding to I

$$\begin{aligned} KR_{\Lambda_{E_7}(I)}^*((S^4)^I) &\cong KR_{E_7 \times \mathbb{T}}^*(S^4) \cong KR_{E_7 \times \mathbb{T}}^*(S^0) \\ &\cong KR_{E_7 \times \mathbb{T}}^*(\mathrm{pt}) \oplus KR_{E_7 \times \mathbb{T}}^*(\mathrm{pt}) \cong KR_{E_7}^*(\mathrm{pt})[q^\pm] \oplus KR_{E_7}^*(\mathrm{pt})[q^\pm]. \end{aligned}$$

- (2) For the conjugacy class represented by $-I$, the Real centralizer $C_{\hat{E}_7}^R(-I) = \hat{E}_7'$. Let T_7 denote the chiral octahedral group and $\hat{T}_7'$ the Real structure $T_7 \langle s' \rangle$, and we have the commutative diagram below.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & E_7 & \xrightarrow{\pi} & T_7 \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \hat{E}_7' & \xrightarrow{(\pi, id)} & \hat{T}_7' \longrightarrow 0 \end{array} \quad (25)$$

Thus, by Corollary A1,

$$\begin{aligned} KR_{\Lambda_{E_7}(-I)}^*(S^4)^{-I} &\cong KR_{\Lambda_{E_7}(-I)}^*(S^0) \\ &\cong \prod_1^2 KR_{\Lambda_{E_7}(-I)}^*(\mathrm{pt}) \\ &\cong \prod_1^2 KR_{T_7 \times \mathbb{T}}^*(\mathrm{pt}) \oplus KR_{T_7 \times \mathbb{T}}^{*+\hat{\vartheta}_{\Lambda_{E_7}^R(-I)} sign}(\mathrm{pt}) \\ &\cong \prod_1^2 KR_{T_7}^*(\mathrm{pt})[q^\pm] \oplus KR_{T_7}^{*+\hat{\vartheta}_{\Lambda_{E_7}^R(-I)} sign}(\mathrm{pt})[q^\pm] \end{aligned}$$

where $sign$ is the sign representation of $\mathbb{Z}/2$.

- (3) For the conjugacy class represented by j is $\{\pm i, \pm j, \pm k\}$, its Real centralizer

$$C_{E_7'}^R(i) \cong D_8.$$

Thus, $KR_{\Lambda_{E_7}(i)}^*((S^4)^i)$ is isomorphic to

$$\begin{aligned} KR_{\Lambda_{\mathbb{Z}/8}(2)}^*(S^0) &\cong KR_{\Lambda_{\mathbb{Z}/8}(2)}^*(\text{pt}) \oplus KR_{\Lambda_{\mathbb{Z}/8}(2)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^{\pm}] / \langle x^8 - q^2 \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^8 - q^2 \rangle. \end{aligned}$$

- (4) For the conjugacy class represented by $\theta = \frac{1}{2}(1 + i + j + k)$, the Real centralizer

$$C_{E_7'}^R(\theta) = \langle \theta, \frac{j+k}{\sqrt{2}}s' \rangle \cong D_6.$$

Note that $(\frac{j+k}{\sqrt{2}}s')^2 = 1$ and $(\frac{j+k}{\sqrt{2}}s'\theta)^2 = 1$. Then $KR_{\Lambda_{E_7}(\theta)}^*((S^4)^\theta)$ is isomorphic to

$$\begin{aligned} KR_{\Lambda_{\mathbb{Z}/6}(1)}^*(S^0) &\cong KR_{\Lambda_{\mathbb{Z}/6}(1)}^*(\text{pt}) \oplus KR_{\Lambda_{\mathbb{Z}/6}(1)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^{\pm}] / \langle x^6 - q \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^6 - q \rangle. \end{aligned}$$

- (5) For the conjugacy class represented $-\theta = -\frac{1}{2}(1 + i + j + k)$, the Real centralizer

$$C_{E_7'}^R(-\theta) = \langle -\theta, \frac{j+k}{\sqrt{2}}s' \rangle \cong D_6.$$

Then, $KR_{\Lambda_{E_7}(-\theta)}^*((S^4)^{-\theta})$ is isomorphic to

$$\begin{aligned} KR_{\Lambda_{\mathbb{Z}/6}(4)}^*(S^0) &\cong KR_{\Lambda_{\mathbb{Z}/6}(4)}^*(\text{pt}) \oplus KR_{\Lambda_{\mathbb{Z}/6}(4)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^{\pm}] / \langle x^6 - q^4 \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^6 - q^4 \rangle. \end{aligned}$$

- (6) For the conjugacy class represented by $r = \frac{1}{\sqrt{2}}(i + j)$, the Real centralizer

$$C_{E_7'}^R(r) \cong D_4.$$

Thus, $KR_{\Lambda_{E_7}(r)}^*((S^4)^r)$ is isomorphic to

$$\begin{aligned} KR_{\Lambda_{\mathbb{Z}/4}(1)}^*(S^0) &\cong KR_{\Lambda_{\mathbb{Z}/4}(1)}^*(\text{pt}) \oplus KR_{\Lambda_{\mathbb{Z}/4}(1)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^{\pm}] / \langle x^4 - q \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^4 - q \rangle. \end{aligned}$$

- (7) For the conjugacy class represented by $t = \frac{1}{\sqrt{2}}(1 + i)$, its Real centralizer

$$C_{E_7'}^R(t) \cong D_8.$$

Thus, $KR_{\Lambda_{E_7}(t)}^*((S^4)^t)$ is isomorphic to

$$\begin{aligned} KR_{\Lambda_{\mathbb{Z}/8}(1)}^*(S^0) &\cong KR_{\Lambda_{\mathbb{Z}/8}(1)}^*(\text{pt}) \oplus KR_{\Lambda_{\mathbb{Z}/8}(1)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^{\pm}] / \langle x^8 - q \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^8 - q \rangle. \end{aligned}$$

(8) For the conjugacy class represented by $-t$, its Real centralizer

$$C_{E_7}^R(-t) \cong D_8.$$

Thus, $KR_{\Lambda_{E_7}(-t)}^*((S^4)^{-t})$ is isomorphic to

$$\begin{aligned} KR_{\Lambda_{\mathbb{Z}/8}(1)}^*(S^0) &\cong KR_{\Lambda_{\mathbb{Z}/8}(1)}^*(\text{pt}) \oplus KR_{\Lambda_{\mathbb{Z}/8}(1)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^{\pm}] / \langle x^8 - q^5 \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^8 - q^5 \rangle. \end{aligned}$$

Thus, in conclusion,

$$\begin{aligned} \text{QEIR}^*(S^4 // E_7) &= KR_{\Lambda_{E_7}(I)}^*((S^4)^I) \times KR_{\Lambda_{E_7}(-I)}^*(S^4)^{-I} \times KR_{\Lambda_{E_7}(i)}^*((S^4)^i) \\ &\quad \times KR_{\Lambda_{E_7}(s)}^*((S^4)^s) \times KR_{\Lambda_{E_7}(-s)}^*((S^4)^{-s}) \times KR_{\Lambda_{E_7}(r)}^*((S^4)^r) \\ &\quad \times KR_{\Lambda_{E_7}(t)}^*((S^4)^t) \times KR_{\Lambda_{E_7}(-t)}^*((S^4)^{-t}) \\ &\cong KR_{E_7}^*(\text{pt})[q^{\pm}] \oplus KR_{E_7}^*(\text{pt})[q^{\pm}] \\ &\quad \times \prod_1^2 KR_{T_7}^*(\text{pt})[q^{\pm}] \oplus KR_{T_7}^{*+\hat{\nu}_{\Lambda_{E_7}^R(-I)} \text{sign}}(\text{pt})[q^{\pm}] \\ &\quad \times KR^*(\text{pt})[x, q^{\pm}] / \langle x^8 - q^2 \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^8 - q^2 \rangle \\ &\quad \times KR^*(\text{pt})[x, q^{\pm}] / \langle x^6 - q \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^6 - q \rangle \\ &\quad \times KR^*(\text{pt})[x, q^{\pm}] / \langle x^6 - q^4 \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^6 - q^4 \rangle \\ &\quad \times KR^*(\text{pt})[x, q^{\pm}] / \langle x^4 - q \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^4 - q \rangle \\ &\quad \times KR^*(\text{pt})[x, q^{\pm}] / \langle x^8 - q \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^8 - q \rangle \\ &\quad \times KR^*(\text{pt})[x, q^{\pm}] / \langle x^8 - q^5 \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^8 - q^5 \rangle, \end{aligned}$$

where sign is the sign representation of $\mathbb{Z}/2$.

Example 13. In this example, we compute $\text{QEIR}^*(S^4 // E_8)$ where E_8 is the binary icosahedral group. A presentation of this group is

$$\langle r, s, t \mid (st)^2 = s^3 = t^5 = -1 \rangle.$$

The cardinality of E_8 is 120. In this example, we use τ to denote $\frac{1+\sqrt{5}}{2}$ and σ to denote the number $\frac{1-\sqrt{5}}{2}$. We take the Real structure $\hat{E}_8'$ on E_8 , i.e., $E_8\langle s' \rangle$.

By ([42] p. 7635, Table 1) and direct computation, we obtain a list of the representatives of the conjugacy classes of E_8 , the centralizers of each representative, whether it is fixed under the involution or not, and the corresponding fixed point spaces in Figure 3.

Representatives ξ of conjugacy classes	Centralizers $C_{E_8}(\xi)$	Fixed points under the involution?	$(S^4)^{\xi}$
1	E_8	Y	S^4
-1	E_8	Y	S^0
$y_3 := \frac{1}{2}(\tau + i + \sigma k)$	$\langle y_3 \rangle \cong \mathbb{Z}/10$	Y	S^0
$y_4 := y_5^2 = \frac{1}{2}(-\tau + \sigma i - j)$	$\langle y_5 \rangle \cong \mathbb{Z}/10$	Y	S^0
$y_5 := \frac{1}{2}(\sigma + i + \tau j)$	$\langle y_5 \rangle \cong \mathbb{Z}/10$	Y	S^0
$y_6 := y_3^2 = \frac{1}{2}(-\sigma + \tau i - k)$	$\langle y_3 \rangle \cong \mathbb{Z}/10$	Y	S^0
$y_7 := \frac{1}{2}(1 + i + j + k)$	$\langle y_7 \rangle \cong \mathbb{Z}/6$	Y	S^0
$y_8 := y_7^2 = \frac{1}{2}(-1 + i + j + k)$	$\langle y_7 \rangle \cong \mathbb{Z}/6$	Y	S^0
$y_9 := i$	$\langle y_9 \rangle \cong \mathbb{Z}/4$	Y	S^0

Figure 3. Conjugacy classes, centralizers, and fixed point spaces.

Next, we compute each factor of $\text{QEIR}^*(S^4 // E_8)$ corresponding to each conjugacy class of E_8 .

- (1) For the conjugacy class $\{I\}$, the Real centralizer $C_{E_8}^R(I) = \hat{E}_8'$. Thus, by ([34] Theorem 5.1),

$$\begin{aligned} KR_{\Lambda_{E_8}(I)}^*((S^4)^I) &\cong KR_{E_8 \times \mathbb{T}}^*(S^4) \cong KR_{E_8 \times \mathbb{T}}^*(S^0) \\ &\cong KR_{E_8 \times \mathbb{T}}^*(\text{pt}) \oplus KR_{E_8 \times \mathbb{T}}^*(\text{pt}) \cong KR_{E_8}^*(\text{pt})[q^\pm] \oplus KR_{E_8}^*(\text{pt})[q^\pm]. \end{aligned}$$

- (2) For the conjugacy class $\{-I\}$, the Real centralizer $C_{E_8}^R(-I) = \hat{E}_8'$. Thus, by Corollary A1,

$$\begin{aligned} KR_{\Lambda_{E_8}(-I)}^*(S^4)^{-I} &\cong KR_{\Lambda_{E_8}(-I)}^*(S^0) \\ &\cong \prod_1^2 KR_{\Lambda_{E_8}(-I)}^*(\text{pt}) \oplus KR_{\Lambda_{E_8}(-I)}^*(\text{pt}) \\ &\cong \prod_1^2 KR_{T_8 \times \mathbb{T}}^*(\text{pt}) \oplus KR_{T_8 \times \mathbb{T}}^{*+\hat{\nu}_{\Lambda_{E_8}^R(-I)} \text{sign}}(\text{pt}) \\ &\cong \prod_1^2 KR_{T_8}^*(\text{pt})[q^\pm] \oplus KR_{T_8}^{*+\hat{\nu}_{\Lambda_{E_8}^R(-I)} \text{sign}}(\text{pt})[q^\pm], \end{aligned}$$

where sign is the sign representation of $\mathbb{Z}/2$.

- (3) For the conjugacy class represented by y_3 , its Real centralizer

$$C_{E_8}^R(y_3) \cong D_{10}.$$

Thus, $KR_{\Lambda_{E_8}(y_3)}^*((S^4)^{y_3})$ is isomorphic to

$$\begin{aligned} KR_{\Lambda_{\mathbb{Z}/10}(1)}^*(S^0) &\cong KR_{\Lambda_{\mathbb{Z}/10}(1)}^*(\text{pt}) \oplus KR_{\Lambda_{\mathbb{Z}/10}(1)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^\pm] / \langle x^{10} - q \rangle \oplus KR^*(\text{pt})[x, q^\pm] / \langle x^{10} - q \rangle. \end{aligned}$$

- (4) For the conjugacy class represented by y_4 , the Real centralizer

$$C_{E_8}^R(y_4) \cong D_{10}.$$

Thus, $KR_{\Lambda_{E_8}(y_4)}^*((S^4)^{y_4})$ is isomorphic to

$$\begin{aligned} KR_{\Lambda_{\mathbb{Z}/10}(2)}^*(S^0) &\cong KR_{\Lambda_{\mathbb{Z}/10}(2)}^*(\text{pt}) \oplus KR_{\Lambda_{\mathbb{Z}/10}(2)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^\pm] / \langle x^{10} - q^2 \rangle \oplus KR^*(\text{pt})[x, q^\pm] / \langle x^{10} - q^2 \rangle. \end{aligned}$$

- (5) For the conjugacy class represented by y_5 , the Real centralizer $C_{E_8}^R(y_5) \cong D_{10}$. Thus, the factor $KR_{\Lambda_{E_8}(y_5)}^*((S^4)^{y_5})$ is isomorphic to

$$\begin{aligned} KR_{\Lambda_{\mathbb{Z}/10}(1)}^*(S^0) &\cong KR_{\Lambda_{\mathbb{Z}/10}(1)}^*(\text{pt}) \oplus KR_{\Lambda_{\mathbb{Z}/10}(1)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^\pm] / \langle x^{10} - q \rangle \oplus KR^*(\text{pt})[x, q^\pm] / \langle x^{10} - q \rangle. \end{aligned}$$

- (6) For the conjugacy class represented by y_6 , the Real centralizer $C_{E_8}^R(y_6) \cong D_{10}$. Thus, the factor $KR_{\Lambda_{E_8}(y_6)}^*((S^4)^{y_6})$ is isomorphic to

$$\begin{aligned} KR_{\Lambda_{\mathbb{Z}/10}(2)}^*(S^0) &\cong KR_{\Lambda_{\mathbb{Z}/10}(2)}^*(\text{pt}) \oplus KR_{\Lambda_{\mathbb{Z}/10}(2)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^\pm] / \langle x^{10} - q^2 \rangle \oplus KR^*(\text{pt})[x, q^\pm] / \langle x^{10} - q^2 \rangle. \end{aligned}$$

- (7) For the conjugacy class represented by y_7 , the Real centralizer $C_{E_8}^R(y_7) \cong D_6$. Thus, the factor $KR_{\Lambda_{E_8}(y_7)}^*((S^4)^{y_7})$ is isomorphic to

$$\begin{aligned} KR_{\Lambda_{\mathbb{Z}/6}(1)}^*(S^0) &\cong KR_{\Lambda_{\mathbb{Z}/6}(1)}^*(\text{pt}) \oplus KR_{\Lambda_{\mathbb{Z}/6}(1)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^{\pm}] / \langle x^6 - q \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^6 - q \rangle. \end{aligned}$$

- (8) For the conjugacy class represented by y_8 , the Real centralizer $C_{E_8}^R(y_8) \cong D_6$. Thus, the factor $KR_{\Lambda_{E_8}(y_8)}^*((S^4)^{y_8})$ is isomorphic to

$$\begin{aligned} KR_{\Lambda_{\mathbb{Z}/6}(2)}^*(S^0) &\cong KR_{\Lambda_{\mathbb{Z}/6}(2)}^*(\text{pt}) \oplus KR_{\Lambda_{\mathbb{Z}/6}(2)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^{\pm}] / \langle x^6 - q^2 \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^6 - q^2 \rangle. \end{aligned}$$

- (9) For the conjugacy class represented by y_9 , the Real centralizer

$$C_{E_8}^R(y_9) \cong D_4.$$

Thus, the corresponding factor $KR_{\Lambda_{E_8}(y_9)}^*((S^4)^{y_9})$ is isomorphic to

$$\begin{aligned} KR_{\Lambda_{\mathbb{Z}/4}(1)}^*(S^0) &\cong KR_{\Lambda_{\mathbb{Z}/4}(1)}^*(\text{pt}) \oplus KR_{\Lambda_{\mathbb{Z}/4}(1)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^{\pm}] / \langle x^4 - q \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^4 - q \rangle. \end{aligned}$$

In conclusion,

$$\begin{aligned} \text{QEIR}(S^4 // E_8) &= KR_{\Lambda_{E_8}(I)}^*((S^4)^I) \times KR_{\Lambda_{E_8}(-I)}^*((S^4)^{-I}) \times KR_{\Lambda_{E_8}(y_3)}^*((S^4)^{y_3}) \\ &\quad \times KR_{\Lambda_{E_8}(y_4)}^*((S^4)^{y_4}) \times KR_{\Lambda_{E_8}(y_5)}^*((S^4)^{y_5}) \times KR_{\Lambda_{E_8}(y_6)}^*((S^4)^{y_6}) \\ &\quad \times KR_{\Lambda_{E_8}(y_7)}^*((S^4)^{y_7}) \times KR_{\Lambda_{E_8}(y_8)}^*((S^4)^{y_8}) \times KR_{\Lambda_{E_8}(y_9)}^*((S^4)^{y_9}) \\ &\cong KR_{E_8}^*(\text{pt})[q^{\pm}] \oplus KR_{E_8}^*(\text{pt})[q^{\pm}] \\ &\quad \times \prod_1^2 KR_{T_8}^*(\text{pt})[q^{\pm}] \oplus KR_{T_8}^{*\hat{\nu}_{\Lambda_{E_8}^R(-I)} \text{sign}}(\text{pt})[q^{\pm}] \\ &\quad \times KR^*(\text{pt})[x, q^{\pm}] / \langle x^{10} - q \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^{10} - q \rangle \\ &\quad \times KR^*(\text{pt})[x, q^{\pm}] / \langle x^{10} - q^2 \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^{10} - q^2 \rangle \\ &\quad \times KR^*(\text{pt})[x, q^{\pm}] / \langle x^{10} - q \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^{10} - q \rangle \\ &\quad \times KR^*(\text{pt})[x, q^{\pm}] / \langle x^{10} - q^2 \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^{10} - q^2 \rangle \\ &\quad \times KR^*(\text{pt})[x, q^{\pm}] / \langle x^6 - q \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^6 - q \rangle \\ &\quad \times KR^*(\text{pt})[x, q^{\pm}] / \langle x^6 - q^2 \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^6 - q^2 \rangle \\ &\quad \times KR^*(\text{pt})[x, q^{\pm}] / \langle x^4 - q \rangle \oplus KR^*(\text{pt})[x, q^{\pm}] / \langle x^4 - q \rangle, \end{aligned}$$

where sign is the sign representation of $\mathbb{Z}/2$.

Remark 3. As we can see in the examples of this section, most computations lead to the equivariant KR-theory of a single point. The whole data of the equivariant KR-theory, by the computation in ([31] Section 8) and ([43] Proposition 3.1), are given as

$$\begin{aligned} KR_G^*(\text{pt}) &:= \sum_{n=0}^7 KR_G^{-n}(\text{pt}) \\ &= RR(G) \oplus RR(G)/\rho(R(G)) \oplus R(G)/j(RH(G)) \oplus 0 \\ &\quad \oplus RH(G) \oplus RH(G)/\eta(R(G)) \oplus R(G)/i(RR(G)) \oplus 0 \end{aligned}$$

where $i : RR(G) \rightarrow R(G)$ and $j : RH(G) \rightarrow R(G)$ are the forgetful functors, the map ρ is given explicitly in ([43] Proposition 2.17), and the map η is given explicitly in ([43] Proposition 2.24).

In addition, there is a graded ring isomorphism (see [31, Section 8])

$$KR^*(pt) \cong \mathbb{Z}[\eta, \mu] / \langle 2\eta, \eta^3, \eta\mu, \mu^2 - 4 \rangle, \quad \deg \eta = -1, \quad \deg \mu = -4.$$

5. Quasi-Elliptic Cohomology of S^4 Acted by a Finite Subgroup of $\text{Spin}(4)$

In this section, we compute $\text{QEII}^*_G(S^4)$ with G a finite subgroup of $\text{Spin}(4)$. The $\text{Spin}(4)$ -action on S^4 that we are interested in is that given by the Formulas (18) and (19).

Denote by $\mathbb{H} \simeq_{\mathbb{R}} \mathbb{R}^4$ the space of quaternions, to be regarded mainly as a real module under quaternion multiplication from the left and right, in particular by unit quaternions

$$q \in \mathbb{H} \quad \vdash \quad q \cdot q^* = 1 \quad \Leftrightarrow \quad q \in S(\mathbb{H}).$$

We have group isomorphism

$$\text{Spin}(3) \simeq S(\mathbb{H})$$

and

$$\text{Spin}(4) \simeq \text{Spin}(3) \times \text{Spin}(3)$$

under which the spin double cover of $\text{SO}(4)$ is given by

$$\begin{aligned} \text{Spin}(4) \simeq \text{Spin}(3) \times \text{Spin}(3) &\longrightarrow \text{SO}(\mathbb{H}) \simeq \text{SO}(4) \\ (e_1, e_2) &\mapsto (q \mapsto e_1 \cdot q \cdot e_2^*) \end{aligned} \quad (26)$$

5.1. Warm-Up Examples

We start with a simple example.

Example 14. In ([2] Section 6) and Section 4, we compute complex and Real quasi-elliptic cohomology of S^4 under the action of the finite subgroups of $\text{Spin}(3) \times 1 \subset \text{Spin}(4) \subset \text{Spin}(5)$. In this example We consider the “dual” of them, i.e., the finite subgroup of $1 \times \text{Spin}(3) \subset \text{Spin}(4) \subset \text{Spin}(5)$, which are the groups

$$1 \times G_n, 1 \times 2D_{2n}, 1 \times E_6, 1 \times E_7, 1 \times E_8.$$

For a point $(1, r) \in 1 \times \text{Spin}(3)$, it acts on a point $y \in \mathbb{H}$ by

$$(1, r) \cdot y = y\bar{r} = \overline{r\bar{y}}.$$

For any finite subgroup G of $1 \times \text{Spin}(3)$, for any torsion point $(1, r) \in G$, $(S^4)^{(1, r)} = \overline{(S^4)^{(r, 1)}}$; and the centralizer $C_{1 \times G}(1, r) = 1 \times C_G(r) \cong C_G(r) \times 1 = C_{G \times 1}(r, 1)$. Thus, $\Lambda_{1 \times G}(1, r) = \Lambda_{G \times 1}(r, 1)$. For the Real case, the Real centralizer $C_{1 \times G}^R(1, r) = 1 \times C_G^R(r) \cong C_G^R(r) \times 1 = C_{G \times 1}^R(r, 1)$ It is straightforward to check case by case that

$$\text{QEII}^*_{1 \times G}(S^4) \cong \text{QEII}^*_{G \times 1}(S^4)$$

and the Real quasi-elliptic cohomology

$$\text{QEII}^*_{1 \times G}(S^4) \cong \text{QEII}^*_{G \times 1}(S^4).$$

Example 15. In this example, we study the $\mathbb{Z}/2$ -action on S^4 induced by the involution x on $\mathbb{H}$

$$x : a + bi + cj + dk \mapsto (-a) + bi + cj + dk.$$

The north pole and south pole are both fixed points under the involution.

There are two conjugacy classes in $\mathbb{Z}/2 = \{1, \tau\}$ corresponding to its two elements.

Below, we compute the factors of $\text{QEll}_{\mathbb{Z}/2}(S^4)$.

- For the conjugacy class 1, $(S^4)^1$ is S^4 itself. $\Lambda_{\mathbb{Z}/2}(1) \cong \mathbb{Z}/2 \times \mathbb{T}$.

$$\begin{aligned} K_{\Lambda_{\mathbb{Z}/2}(1)}(S^4)^1 &\cong K_{\mathbb{Z}/2 \times \mathbb{T}}(S^4) \cong K_{\mathbb{Z}/2 \times \mathbb{T}}^*(S^0) \\ &\cong \mathbb{Z}[x, q^{\pm}] / \langle x^2 - 1 \rangle \oplus \mathbb{Z}[y, q^{\pm}] / \langle y^2 - 1 \rangle. \end{aligned}$$

- For the conjugacy class τ , $(S^4)^{\tau} = \{bi + cj + dk \in \mathbb{H} \mid b, c, d \in \mathbb{R}\} \cup \{\infty\} \cong S^3$.

$$\begin{aligned} K_{\Lambda_{\mathbb{Z}/2}(\tau)}(S^4)^{\tau} &\cong K_{\Lambda_{\mathbb{Z}/2}(\tau)}(S^3) \cong K_{\Lambda_{\mathbb{Z}/2}(\tau)}(S^0) \\ &\cong \mathbb{Z}[x, q^{\pm}] / \langle x^2 - q \rangle \oplus \mathbb{Z}[y, q^{\pm}] / \langle y^2 - q \rangle. \end{aligned}$$

Next we compute $\text{QEll}_{\mathbb{Z}/2}^*(S^4)$. If we take the Real structure on $\mathbb{Z}/2$ to be the Dihedral Real structure, we can take the reflection to be

$$y : \mathbb{H} \longrightarrow \mathbb{H}, \quad (a + bi + cj + dk) \mapsto (-a - bi - cj - dk).$$

The composition $x \circ y$ sends a point $a + bi + cj + dk$ to $a - bi - cj - dk$, i.e., the quaternion conjugation. The group generated by x and y is the dihedral group D_4 .

Additionally, the Real centralizers $C_{D_4}^R(\alpha) = D_4$ for $\alpha = 1, \tau$ in $\mathbb{Z}/2$. The factors of $\text{QEll}_{\mathbb{Z}/2}^*(S^4)$ are computed below.

- For the conjugacy class 1, $\Lambda_{\mathbb{Z}/2}(1) \cong \mathbb{Z}/2 \times \mathbb{T}$.

$$\begin{aligned} KR_{\Lambda_{\mathbb{Z}/2}(1)}^*(S^4)^1 &\cong KR_{\Lambda_{\mathbb{Z}/2}(1)}^*(S^4) \cong KR_{\Lambda_{\mathbb{Z}/2}(1)}^*(S^0) \\ &\cong KR_{\Lambda_{\mathbb{Z}/2}(1)}^*(\text{pt}) \oplus KR_{\Lambda_{\mathbb{Z}/2}(1)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^{\pm}] / \langle x^2 - 1 \rangle \oplus KR^*(\text{pt})[y, q^{\pm}] / \langle y^2 - 1 \rangle. \end{aligned}$$

- For the conjugacy class τ ,

$$\begin{aligned} KR_{\Lambda_{\mathbb{Z}/2}(\tau)}^*(S^4)^{\tau} &\cong KR_{\Lambda_{\mathbb{Z}/2}(\tau)}^*(S^3) \cong KR_{\Lambda_{\mathbb{Z}/2}(\tau)}^*(S^0) \\ &\cong KR_{\Lambda_{\mathbb{Z}/2}(\tau)}^*(\text{pt}) \oplus KR_{\Lambda_{\mathbb{Z}/2}(\tau)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x, q^{\pm}] / \langle x^2 - q \rangle \oplus KR^*(\text{pt})[y, q^{\pm}] / \langle y^2 - q \rangle. \end{aligned}$$

Next, we compute $\text{QEll}_G(S^4)$ and $\text{QEll}_G^*(S^4)$ with G a cyclic subgroup of $\text{Spin}(4)$.

Example 16. Let

$$G = \left\langle \begin{bmatrix} e^{\frac{2\pi i m_1}{n_1}} & 0 \\ 0 & e^{\frac{2\pi i m_2}{n_2}} \end{bmatrix} \in U(2, \mathbb{H}) \mid m_1, m_2 \in \mathbb{Z} \right\rangle.$$

Let

$$\alpha := \begin{bmatrix} e^{\frac{2\pi i p_1}{n_1}} & 0 \\ 0 & e^{\frac{2\pi i p_2}{n_2}} \end{bmatrix}$$

denote a generator of the cyclic group. We can assume that p_1 and n_1 are coprime, and p_2 and n_2 are coprime. The order of G is the least common multiple N of n_1 and n_2 .

Then, for any $\alpha^m \in G$, the centralizer

$$C_G(\alpha^m) = G.$$

And

$$(S^4)^{\alpha^m} = \begin{cases} S^4, & \text{if } \alpha^m = I; \\ S^0, & \text{otherwise.} \end{cases}$$

The group $G = \langle \alpha \rangle$ is isomorphic to $\mathbb{Z}/N$. Then, we can apply the results in [2] and Example 9 directly.

The complex quasi-elliptic cohomology is

$$\begin{aligned} \text{QEll}_G(S^4) &= \prod_{m=0}^N K_{\Lambda_G(\alpha^m)}((S^4)^{\alpha^m}) \\ &\cong \prod_{m=0}^N \mathbb{Z}[q^\pm, x] / \langle x^N - q^m \rangle \oplus \mathbb{Z}[q^\pm, x] / \langle x^N - q^m \rangle. \end{aligned}$$

The Real quasi-elliptic cohomology is

$$\text{QEIR}^*(S^4 // G) \cong \prod_{m=0}^{N-1} KR^*(\text{pt})[x, q^\pm] / \langle x^N - q^m \rangle \oplus KR^*(\text{pt})[x, q^\pm] / \langle x^N - q^m \rangle.$$

5.2. Product of Finite Subgroups

I did not find all the finite subgroups of $\text{Spin}(5)$ that have a well-defined action on $\mathbb{H}$. I will discuss some finite subgroups of the form $H \times K$ where both H and K are finite subgroups of $\text{Spin}(3)$.

Example 17. For any $(h, k) \in H \times K$, and $y \in \mathbb{H}$, as given in (26),

$$(h, k) \cdot y := hy\bar{k}.$$

The set of conjugacy classes $\pi_0(H \times K // H \times K)$ is a one-to-one correspondent to $\pi_0(H // H) \times \pi_0(K // K)$. In addition,

$$\Lambda_{H \times K}(h, k) \cong \Lambda_H(h) \times_{\mathbb{T}} \Lambda_K(k). \quad (27)$$

If $(\hat{H}, \pi_H)$ is a Real structure on H and $(\hat{K}, \pi_K)$ is a Real structure on K , then we have the product Real structure

$$(\hat{H} \times_{\mathbb{Z}/2} \hat{K}, \pi)$$

where the projection

$$\pi = \pi_H \times_{\mathbb{Z}/2} \pi_K : \hat{H} \times_{\mathbb{Z}/2} \hat{K} \longrightarrow \mathbb{Z}/2$$

sends (h, k) to $\pi_H(h) = \pi_K(k)$. For the Real centralizers,

$$C_{\hat{H} \times_{\mathbb{Z}/2} \hat{K}}^R(h, k) \cong C_{\hat{H}}^R(h) \times_{\mathbb{Z}/2} C_{\hat{K}}^R(k).$$

Thus,

$$\Lambda_{\hat{H} \times_{\mathbb{Z}/2} \hat{K}}^R(h, k) \cong \Lambda_{\hat{H}}^R(h) \times_{O(2)} \Lambda_{\hat{K}}^R(k), \quad (28)$$

where $O(2)$ is the 2-dimensional orthogonal group.

In addition, if the reflection in $\hat{H}$ and $\hat{K}$ on $\mathbb{R}^4$ are represented by the same matrix $\alpha \in U(2)$ with $\alpha^2 = I$, it defines a $\mathbb{C}$ -linear map

$$\begin{aligned} SU(2) &\longrightarrow SU(2) \\ A &\mapsto \alpha A \bar{\alpha}. \end{aligned}$$

Then, by direct computation, if we take α to be the reflection s defined in Example 7, the resulting reflection on $\mathbb{H} \cong \mathbb{R}^4$ is defined by

$$(a + bi + cj + dk) \mapsto (a - bi - cj + dk).$$

And if we take α to be the reflection s' defined in Example 7, the resulting reflection is

$$(a + bi + cj + dk) \mapsto (a + bi - cj - dk).$$

In addition, if we take the reflection in $\hat{H}$ to be s and that on $\hat{K}$ to be s' , the resulting reflection on $\mathbb{H}$ is

$$(a + bi + cj + dk) \mapsto (-c + di - aj + bk).$$

And if we take the reflection in $\hat{H}$ to be s' and that on $\hat{K}$ to be s , the resulting reflection on $\mathbb{H}$ is

$$(a + bi + cj + dk) \mapsto (c + di + aj + bk).$$

In fact, we have a conclusion generalizing Example 14.

Proposition 5. Let H and K denote two finite subgroups of $\text{Spin}(3)$. The product $H \times K$ acts on S^4 in the way as in (26). Then,

$$QEll_{H \times K}^*(S^4) \cong QEll_{K \times H}^*(S^4).$$

Moreover, if $(\hat{H}, \pi_H)$ is Real structure on H and $(\hat{K}, \pi_K)$ is Real structure on K , then,

$$QEIR_{H \times K}^*(S^4) \cong QEIR_{K \times H}^*(S^4).$$

Proof. We first notice the factors of both $QEll_{H \times K}^*(S^4)$ and $QEll_{K \times H}^*(S^4)$ go through the set $\pi_0(H//H) \times \pi_0(K//K)$.

Then, by (27), for any $\sigma \in H$, and $\tau \in K$,

$$\Lambda_{H \times K}(\sigma, \tau) \cong \Lambda_H(\sigma) \times_{\mathbb{T}} \Lambda_K(\tau) \cong \Lambda_K(\tau) \times_{\mathbb{T}} \Lambda_H(\sigma) \cong \Lambda_{K \times H}(\tau, \sigma).$$

In addition, for any fixed point $a + bi + cj + dk \in \mathbb{H}$ of (σ, τ) , we have the equality

$$\sigma(a + bi + cj + dk)\bar{\tau} = a + bi + cj + dk.$$

Taking the complex conjugate of both sides, we obtain

$$\tau(a - bi - cj - dk)\bar{\sigma} = a - bi - cj - dk.$$

Thus, the complex conjugate of the quaternion induces a one-to-one correspondence

$$(S^4)^{(\sigma, \tau)} \xrightarrow{\overline{(-)}} (S^4)^{(\tau, \sigma)}.$$

Moreover, it is direct to show that for any element $(u, v) \in C_{H \times K}(\sigma, \tau)$, any $x = a + bi + cj + dk \in (S^4)^{(\sigma, \tau)}$, we have the equality

$$\overline{(u, v)} \cdot x = (v, u) \cdot \bar{x}.$$

Note that $(v, u) \in C_{K \times H}(\tau, \sigma)$. This leads to the isomorphism

$$K_{\Lambda_{H \times K}(\sigma, \tau)}^*(S^4)^{(\sigma, \tau)} \cong K_{\Lambda_{K \times H}(\tau, \sigma)}^*(S^4)^{(\tau, \sigma)}.$$

Thus,

$$QEll_{H \times K}^*(S^4) \cong QEll_{K \times H}^*(S^4).$$

For the Real case, the factors of both $QEll_{H \times K}^*(S^4)$ and $QEll_{K \times H}^*(S^4)$ go through the same set $\pi_0((H \times K) //_{\mathbb{R}}(\hat{H} \times_{\mathbb{Z}/2} \hat{K}))$. In addition, by (28),

$$\Lambda_{\hat{H} \times_{\mathbb{Z}/2} \hat{K}}^R(\sigma, \tau) \cong \Lambda_{\hat{K} \times_{\mathbb{Z}/2} \hat{H}}^R(\tau, \sigma).$$

And the complex conjugate

$$(S^4)^{(\sigma, \tau)} \longrightarrow (S^4)^{(\tau, \sigma)}$$

commutes with the reflections, as shown below.

$$\begin{array}{ccc} (S^4)^{(\sigma, \tau)} & \xrightarrow{\overline{(-)}} & (S^4)^{(\tau, \sigma)} \\ (s_H, s_K) \downarrow & & \downarrow (s_K, s_H) \\ (S^4)^{(s_H \sigma s_H^{-1}, s_K \tau s_K^{-1})} & \xrightarrow{\overline{(-)}} & (S^4)^{(s_K \tau s_K^{-1}, s_H \sigma s_H^{-1})} \end{array}$$

where s_H is the reflection in $\hat{H}$ and s_K is the reflection in $\hat{K}$.

Thus, we obtain

$$QEll_{H \times K}^*(S^4) \cong QEll_{K \times H}^*(S^4).$$

□

Proposition 6. Let H and K denote two finite subgroups of $\text{Spin}(3)$. The product $H \times K$ acts on S^4 by the action given in (18). Let $(\hat{H}, \pi_H)$ denote a Real structure on H and $(\hat{K}, \pi_K)$ a Real structure on K . Then, we have the conclusions below.

- (1) The factor in $QEll_{H \times K}(S^4)$ corresponding to the conjugacy class (h, k) , i.e., $K_{\Lambda_{H \times K}(h, k)}(S^4)^{(h, k)}$, is isomorphic to

$$\prod_1^2 R(\Lambda_H(h)) \otimes_{\mathbb{Z}[q^{\pm}]} R(\Lambda_K(k)).$$

Then, we have the isomorphism

$$\begin{aligned} QEll_{H \times K}(S^4) &= \prod_{(h, k) \in \pi_0((H \times K) // (H \times K))} K_{\Lambda_{H \times K}(h, k)}(S^4)^{(h, k)} \\ &\cong \prod_{h \in \pi_0(H // H), k \in \pi_0(K // K)} \prod_1^2 R(\Lambda_H(h)) \otimes_{\mathbb{Z}[q^{\pm}]} R(\Lambda_K(k)). \end{aligned}$$

- (2) The factor in $QEll_{H \times K}^*(S^4)$ corresponding to the Real conjugacy class (h, k) , i.e., ${}^{\pi}K_{\Lambda_{H \times K}(h, k)}(S^4)^{(h, k)}$, is isomorphic to

•

$$\prod_1^2 KR_{\Lambda_H(h)}^*(\text{pt}) \otimes_{KR_{\mathbb{T}}^*(\text{pt})} KR_{\Lambda_K(k)}^*(\text{pt}),$$

if (h, k) is a fixed point under the involution;

•

$$\prod_1^2 K_{\Lambda_H(h)}^*(\text{pt}) \otimes_{\mathbb{Z}[q^{\pm}]} K_{\Lambda_K(k)}^*(\text{pt}),$$

if (h, k) is a free point under the involution.

Proof. We prove the conclusions one by one.

- (1) Note that (18) defines a 4-dimensional representation of $H \times K$. Thus, $(S^4)^{(h,k)}$ is a representation sphere of $\Lambda_{H \times K}(h, k)$ and contains S^0 as a subspace. Whatever $(S^4)^{(h,k)}$ is, by ([34] Theorem 4.3), we have

$$K_{\Lambda_{H \times K}(h,k)}(S^4)^{(h,k)} \cong K_{\Lambda_{H \times K}(h,k)}(S^0).$$

And the right hand side is isomorphic to

$$\begin{aligned} K_{\Lambda_{H \times K}(h,k)}(\text{pt}) \oplus K_{\Lambda_{H \times K}(h,k)}(\text{pt}) &\cong R(\Lambda_{H \times K}(h, k)) \oplus R(\Lambda_{H \times K}(h, k)) \\ &\cong R(\Lambda_H(h)) \otimes_{\mathbb{Z}[q^{\pm}]} R(\Lambda_K(k)) \oplus R(\Lambda_H(h)) \otimes_{\mathbb{Z}[q^{\pm}]} R(\Lambda_K(k)). \end{aligned}$$

So the first conclusion is proved.

- (2) The proof is similar to the complex case. Since $(S^4)^{(h,k)}$ is both a Real representation sphere of $\Lambda_{\hat{H} \times_{\mathbb{Z}/2} \hat{K}}^R(h, k)$ and a complex representation sphere of $\Lambda_{H \times K}(h, k)$, thus, by ([34] Theorem 4.3, Theorem 5.1), the Freed–Moore K-theory ${}^{\pi}K_{\Lambda_{H \times K}(h,k)}^*(S^4)^{(h,k)}$ is isomorphic to

$${}^{\pi}K_{\Lambda_{H \times K}(h,k)}^*(S^0) \cong {}^{\pi}K_{\Lambda_{H \times K}(h,k)}^*(\text{pt}) \oplus {}^{\pi}K_{\Lambda_{H \times K}(h,k)}^*(\text{pt}).$$

In addition,

$${}^{\pi}K_{\Lambda_{H \times K}(h,k)}^*(\text{pt}) = \begin{cases} KR_{\Lambda_{H \times K}(h,k)}^*(\text{pt}), & \text{if } (h, k) \text{ is a fixed point under the involution;} \\ K_{\Lambda_{H \times K}(h,k)}^*(\text{pt}), & \text{if } (h, k) \text{ is a free point under the involution.} \end{cases}$$

And $KR_{\Lambda_{H \times K}(h,k)}^*(\text{pt}) \cong KR_{\Lambda_H(h)}^*(\text{pt}) \otimes_{KR_{\mathbb{T}}^*(\text{pt})} KR_{\Lambda_K(k)}^*(\text{pt})$.

Then, we obtain the conclusion immediately. $\square$

Remark 4. One probably subtle point is that, as indicated in [3], the Real structure we takes in the $\mathbb{R}$ in the general definition of the enhanced Real stabilizer

$$\Lambda_G^R(g) \simeq (\mathbb{R} \rtimes_{\pi} \hat{G}) / \langle (-1, g) \rangle.$$

It is the reflection of $r \mapsto -r$. This coincides with the dihedral Real structure on $\mathbb{T}$. More explicitly, the involution defined from the dihedral Real structure on $\mathbb{T}$ is given by $t \mapsto -t$, which is the quotient of the reflection on $\mathbb{R}$.

The Real representation ring $RR(\mathbb{T})$ for $\mathbb{T}$ with the dihedral Real structure, i.e., $O(2)$, is exactly $RR(\mathbb{T}; \mathbb{R})$, which is isomorphic to $\mathbb{Z}[q^{\pm}]$. Thus, the isomorphism

$$\Lambda_{\hat{G} \times_{\mathbb{Z}/2} \hat{H}}^R(g, h) \cong \Lambda_{\hat{G}}^R(g) \times_{O(2)} \Lambda_{\hat{H}}^R(h)$$

gives us the isomorphism of Real representation rings, i.e.,

$$RR(\Lambda_{G \times H}(g, h)) \cong RR(\Lambda_G(g)) \otimes_{RR\mathbb{T}} RR(\Lambda_H(h)).$$

Example 18. In this example we compute $QEll_{G_n \times G_m}(S^4)$ with

$$G_n = \{e^{\frac{2\pi i k}{n}} \in \mathbb{H} \mid k \in \mathbb{Z}\}; \quad G_m = \{e^{\frac{2\pi i j}{m}} \in \mathbb{H} \mid j \in \mathbb{Z}\}.$$

By ([2] Example 6.3) and Proposition 6(1),

$$\begin{aligned} QEll_{G_n \times G_m}(S^4) &\cong \prod_{k=0}^{n-1} \prod_{l=0}^{m-1} K_{\Lambda_{G_n \times G_m}(k, l)}(S^4)^{(k, l)} \\ &\cong \prod_{k=0}^{n-1} \prod_{l=0}^{m-1} \mathbb{Z}[x_1, x_2, q^{\pm}] / \langle x_1^n - q^k, x_2^m - q^l \rangle \oplus \mathbb{Z}[y_1, y_2, q^{\pm}] / \langle y_1^n - q^k, y_2^m - q^l \rangle. \end{aligned}$$

We take the Real structure $\hat{G}'_n$ as defined in Example 8, i.e., the group below together with the determinant map $\det$

$$\langle G_n, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \rangle.$$

It is isomorphic to the dihedral group D_{2n} . As discussed in Example 9, all the elements in $\pi_0(G_n //_{\mathbb{R}} \hat{G}'_n)$ and $\pi_0(G_m //_{\mathbb{R}} \hat{G}'_m)$ are fixed points under the involution; thus, so are those in $\pi_0(G_n \times G_m //_{\mathbb{R}} G_n \times_{\mathbb{Z}/2} G_m)$.

By Example 9 and Proposition 6(2),

$$\begin{aligned} QEIR^*_{G_n \times G_m}(S^4) &\cong \prod_{k=0}^{n-1} \prod_{l=0}^{m-1} KR^*_{\Lambda_{G_n \times G_m}(k, l)}(S^4)^{(k, l)} \\ &\cong \prod_{k=0}^{n-1} \prod_{l=0}^{m-1} KR^*(\text{pt})[x_1, x_2, q^{\pm}] / \langle x_1^n - q^k, x_2^m - q^l \rangle \oplus KR^*(\text{pt})[y_1, y_2, q^{\pm}] / \langle y_1^n - q^k, y_2^m - q^l \rangle. \end{aligned}$$

Example 19. Let n and m be positive integers. Let $G_n < \text{Spin}(3)$ denote the cyclic group

$$\{e^{\frac{2\pi i k}{n}} \in \mathbb{H} \mid k \in \mathbb{Z}\}$$

and $2D_{2m}$ denote the binary Dihedral group

$$\langle G_{2m}, \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \rangle < \text{Spin}(3).$$

In this example, we compute $QEll^*_{G_n \times 2D_{2m}}(S^4)$ and $QEIR^*_{G_n \times 2D_{2m}}(S^4)$.

Let τ denote $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ in $2D_{2m}$, which is $-j$ in terms of quaternions.

The factors of $QEll_{G_n \times 2D_{2m}}(S^4)$ corresponding to each conjugacy class are computed one by one below. We first compute the factors corresponding to the conjugacy classes represented by

$$\alpha := \begin{bmatrix} e^{\frac{2\pi i k}{n}} & 0 \\ 0 & e^{\frac{2\pi i j}{2m}} \end{bmatrix} \in U(2, \mathbb{H}). \quad (29)$$

(1) If $\alpha = I$,

$$\begin{aligned} K_{\Lambda_{G_n \times 2D_{2m}}(I)}(S^4)^I &= K_{\Lambda_{G_n \times 2D_{2m}}(I)}(S^4) \stackrel{(*)}{\cong} K_{\Lambda_{G_n \times 2D_{2m}}(I)}(S^0) \cong K_{G_n \times 2D_{2m}}(S^0) \otimes \mathbb{Z}[q^\pm] \\ &\cong (R(G_n \times 2D_{2m}) \oplus R(G_n \times 2D_{2m})) \otimes \mathbb{Z}[q^\pm] \\ &\cong R(2D_{2m})[x_1, x_2, q^\pm] / \langle x_1^n - 1, x_2^n - 1 \rangle \end{aligned}$$

where the isomorphism $(*)$ is by ([34] Theorem 4.3).

(2) If the $e^{\frac{2\pi ip}{2m}}$ in (29) is not $\pm I$, the centralizer $C_{G_n \times 2D_{2m}}(\alpha) = G_n \times G_{2m}$.

$$\begin{aligned} K_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}(S^4)^\alpha &\cong K_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}(S^0) \cong R(\Lambda_{G_n \times 2D_{2m}}(\alpha)) \oplus R(\Lambda_{G_n \times 2D_{2m}}(\alpha)) \\ &\cong \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^n - q^k, x_2^{2m} - q^p \rangle \oplus \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^n - q^k, x_2^{2m} - q^p \rangle. \end{aligned}$$

(3) If the $e^{\frac{2\pi ip}{2m}}$ in (29) is $\pm I$, the centralizer $C_{G_n \times 2D_{2m}}(\alpha) = G_n \times 2D_{2m}$.

• If $e^{\frac{2\pi ip}{2m}} = I$,

$$\begin{aligned} K_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}(S^4)^\alpha &\cong K_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}(S^0) \cong R(\Lambda_{G_n \times 2D_{2m}}(\alpha)) \oplus R(\Lambda_{G_n \times 2D_{2m}}(\alpha)) \\ &\cong R(2D_{2m})[x, q^\pm] / \langle x^n - q^k \rangle \oplus R(2D_{2m})[x', q^\pm] / \langle x'^n - q^k \rangle. \end{aligned}$$

• If $e^{\frac{2\pi ip}{2m}} = -I$, applying Proposition A1, we obtain

$$\begin{aligned} K_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}(S^4)^\alpha &\cong K_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}(S^0) \cong R(\Lambda_{G_n \times 2D_{2m}}(\alpha)) \oplus R(\Lambda_{G_n \times 2D_{2m}}(\alpha)) \\ &\cong \prod_{i=1}^2 R(\Lambda_{G_n}(e^{\frac{2\pi ik}{n}})) \otimes_{\mathbb{Z}[q^\pm]} R(\Lambda_{2D_{2m}}(-I)) \\ &\cong \prod_{i=1}^2 R(\Lambda_{G_n}(e^{\frac{2\pi ik}{n}})) \otimes_{\mathbb{Z}[q^\pm]} (R(D_{2n})[q^\pm] \oplus R(\widetilde{D_{2n\rho}})(D_{2n})[q^\pm]) \\ &\cong \prod_{i=1}^2 (R(D_{2n}) \oplus R(\widetilde{D_{2n\rho}})(D_{2n}))[x, q^\pm] / \langle x^n - q^k \rangle \end{aligned}$$

where ρ is the sign representation of $\mathbb{Z}/2$.

(4) For the conjugacy class of $(e^{\frac{2\pi ik}{n}}, \tau) \in G_n \times 2D_{2m}$, The centralizer

$$C_{G_n \times 2D_{2m}}(e^{\frac{2\pi ik}{n}}, \tau) = G_n \times \langle \tau \rangle \cong G_n \times \mathbb{Z}/4.$$

$$\begin{aligned} K_{\Lambda_{G_n \times 2D_{2m}}(e^{\frac{2\pi ik}{n}}, \tau)}(S^4)^{(e^{\frac{2\pi ik}{n}}, \tau)} &\cong K_{\Lambda_{G_n \times 2D_{2m}}(e^{\frac{2\pi ik}{n}}, \tau)}(S^0) \\ &\cong R(\Lambda_{G_n \times 2D_{2m}}(e^{\frac{2\pi ik}{n}}, \tau)) \oplus R(\Lambda_{G_n \times 2D_{2m}}(e^{\frac{2\pi ik}{n}}, \tau)) \\ &\cong \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^n - q^k, x_2^4 - q \rangle \oplus \mathbb{Z}[y_1, y_2, q^\pm] / \langle y_1^n - q^k, y_2^4 - q \rangle. \end{aligned}$$

(5) Then, we study the case corresponding to the conjugacy class of

$$(e^{\frac{2\pi ik}{n}}, \tau A_{\frac{2\pi i}{2m}}) \in G_n \times 2D_{2m}.$$

The centralizer $C_{G_n \times 2D_{2m}}(e^{\frac{2\pi i k}{n}}, \tau A_{\frac{2\pi i}{2m}}) = G_n \times \langle \tau A_{\frac{2\pi i}{2m}} \rangle \cong G_n \times \mathbb{Z}/4$. Thus,

$$\begin{aligned} & K_{\Lambda_{G_n \times 2D_{2m}}(e^{\frac{2\pi i k}{n}}, \tau A_{\frac{2\pi i}{2m}})}(S^4)^{(e^{\frac{2\pi i k}{n}}, \tau A_{\frac{2\pi i}{2m}})} \cong K_{\Lambda_{G_n}(e^{\frac{2\pi i k}{n}}) \times_{\mathbb{T}} \Lambda_{2D_{2m}}(\tau A_{\frac{2\pi i}{2m}})}(S^0) \\ & \cong R(\Lambda_{G_n}(e^{\frac{2\pi i k}{n}}) \times_{\mathbb{T}} \Lambda_{2D_{2m}}(\tau A_{\frac{2\pi i}{2m}})) \oplus R(\Lambda_{G_n}(e^{\frac{2\pi i k}{n}}) \times_{\mathbb{T}} \Lambda_{2D_{2m}}(\tau A_{\frac{2\pi i}{2m}})) \\ & \cong \mathbb{Z}[x_1, x_2, q^{\pm}] / \langle x_1^n - q^k, x_2^4 - q \rangle \oplus \mathbb{Z}[y_1, y_2, q^{\pm}] / \langle y_1^n - q^k, y_2^4 - q \rangle. \end{aligned}$$

where the isomorphism $(*)$ is by ([34] Theorem 4.3).

Example 20. We compute $\text{QEIR}_{G_n \times 2D_{2m}}^*(S^4)$ in this example. We take the Real structure $\hat{G}'_n$ and $2\hat{D}_{2m}$ as discussed in Example 8. From them, we formulate a Real structure

$$G_n \times \widehat{2D_{2m}} := \hat{G}'_n \times_{\mathbb{Z}/2} 2\hat{D}_{2m}$$

on the product $G'_n \times 2D_{2m}$. By Example 9, all the elements in $\pi_0(G_n //_{\mathbb{R}} \hat{G}'_n)$ are fixed points under the involution; and by Example 10, all the elements in $\pi_0(2D_{2m} //_{\mathbb{R}} 2\hat{D}_{2m})$ are fixed points under the involution. Thus, all the points in $\pi_0(G_n \times 2D_{2m} //_{\mathbb{R}} G_n \times \widehat{2D_{2m}})$ are fixed points.

We compute the factors of $\text{QEIR}_{G_n \times 2D_{2m}}^*(S^4)$ below one by one. We start with those corresponding to the conjugacy classes represented by

$$\alpha = (e^{\frac{2\pi i k}{n}}, e^{\frac{2\pi i p}{2m}}) \in G_n \times 2D_{2m}$$

with $k, p \in \mathbb{Z}$.

(1) If $\alpha = I$, by ([34] Theorem 5.1),

$$\begin{aligned} KR_{\Lambda_{G_n \times 2D_{2m}}(I)}^*(S^4)^I &= KR_{\Lambda_{G_n \times 2D_{2m}}(I)}^*(S^4) \cong KR_{\Lambda_{G_n \times 2D_{2m}}(I)}^*(S^0) \\ &\cong \prod_1^2 KR_{\Lambda_{G_n \times 2D_{2m}}(I)}^*(\text{pt}) \cong \prod_1^2 KR_{\Lambda_{2D_{2m}}(I)}^*(\text{pt}) \otimes_{KR_{\mathbb{T}}^*(\text{pt})} KR_{\Lambda_{G_n}(I)}^*(\text{pt}) \\ &\cong \prod_1^2 KR_{2D_{2m}}^*(\text{pt})[x, q^{\pm}] / \langle x^n - 1 \rangle. \end{aligned}$$

(2) If the $e^{\frac{2\pi i p}{2m}}$ in α is not $\pm I$,

$$\begin{aligned} KR_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}^*(S^4)^{\alpha} &\cong KR_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}^*(S^0) \cong KR_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}^*(\text{pt}) \oplus KR_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}^*(\text{pt}) \\ &\cong KR^*(\text{pt})[x_1, x_2, q^{\pm}] / \langle x_1^n - q^k, x_2^{2m} - q^p \rangle \oplus KR^*(\text{pt})[x_1, x_2, q^{\pm}] / \langle x_1^n - q^k, x_2^{2m} - q^p \rangle. \end{aligned}$$

(3) If the $e^{\frac{2\pi i p}{2m}}$ in α is I ,

$$\begin{aligned} KR_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}^*(S^4)^{\alpha} &\cong KR_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}^*(S^0) \cong KR_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}^*(\text{pt}) \oplus KR_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}^*(\text{pt}) \\ &\cong KR_{2D_{2m}}^*(\text{pt})[x, q^{\pm}] / \langle x^n - q^k \rangle \oplus KR_{2D_{2m}}^*(\text{pt})[x', q^{\pm}] / \langle x'^n - q^k \rangle. \end{aligned}$$

(4) If the $e^{\frac{2\pi i p}{2m}}$ in α is $-I$, applying Corollary A1, we get

$$\begin{aligned} & KR_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}^*(S^4)^{\alpha} \cong KR_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}^*(S^0) \\ & \cong KR_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}^*(\text{pt}) \oplus KR_{\Lambda_{G_n \times 2D_{2m}}(\alpha)}^*(\text{pt}) \\ & \cong \prod_1^2 KR_{\Lambda_{G_n}(e^{\frac{2\pi i k}{n}})}^*(\text{pt}) \otimes_{KR_{\mathbb{T}}^*(\text{pt})} KR_{\Lambda_{2D_{2m}}(-1)}^*(\text{pt}) \\ & \cong \prod_1^2 KR_{\Lambda_{G_n}(e^{\frac{2\pi i k}{n}})}^*(\text{pt}) \otimes_{KR_{\mathbb{T}}^*(\text{pt})} \left(KR_{2D_{2m}}^*(\text{pt})[q^{\pm}] \oplus KR_{2D_{2m}}^{*+\hat{\nu} \Lambda_{2\hat{D}_{2m}}^{\mathbb{R}}(-1), \text{sign}}(\text{pt})[q^{\pm}] \right) \\ & \cong \prod_1^2 \left(KR_{2D_{2m}}^*(\text{pt}) \oplus KR_{2D_{2m}}^{*+\hat{\nu} \Lambda_{2\hat{D}_{2m}}^{\mathbb{R}}(-1), \text{sign}}(\text{pt}) \right) [x, q^{\pm}] / \langle x^n - q^k \rangle \end{aligned}$$

where sign is the sign representation of $\mathbb{Z}/2$.

(5) For the conjugacy class of $(e^{\frac{2\pi i k}{n}}, \tau) \in G_n \times 2D_{2m}$,

$$\begin{aligned} KR^*_{\Lambda_{G_n \times 2D_{2m}}(e^{\frac{2\pi i k}{n}}, \tau)}(S^4)^{(e^{\frac{2\pi i k}{n}}, \tau)} &\cong KR^*_{\Lambda_{G_n \times 2D_{2m}}(e^{\frac{2\pi i k}{n}}, \tau)}(S^0) \\ &\cong \prod_1^2 KR^*_{\Lambda_{G_n \times 2D_{2m}}(e^{\frac{2\pi i k}{n}}, \tau)}(\text{pt}) \\ &\cong \prod_1^2 KR^*(\text{pt})[x_1, x_2, q^{\pm}] / \langle x_1^n - q^k, x_2^4 - q \rangle \end{aligned}$$

(6) For the conjugacy class of $(e^{\frac{2\pi i k}{n}}, \tau r) \in G_n \times 2D_{2m}$,

$$\begin{aligned} KR^*_{\Lambda_{G_n \times 2D_{2m}}(e^{\frac{2\pi i k}{n}}, \tau r)}(S^4)^{(e^{\frac{2\pi i k}{n}}, \tau r)} &\cong KR^*_{\Lambda_{G_n}(e^{\frac{2\pi i k}{n}}) \times_{\mathbb{T}} \Lambda_{2D_{2m}}(\tau r)}(S^0) \\ &\cong \prod_1^2 KR^*_{\Lambda_{G_n}(e^{\frac{2\pi i k}{n}}) \times_{\mathbb{T}} \Lambda_{2D_{2m}}(\tau r)}(\text{pt}) \\ &\cong \prod_1^2 KR^*(\text{pt})[x_1, x_2, q^{\pm}] / \langle x_1^n - q^k, x_2^4 - q \rangle. \end{aligned}$$

Example 21. In this example, we deal with the finite subgroup $E_6 \times E_7$ of $\text{Spin}(5)$ where E_6 is the binary tetrahedral group and E_7 is the binary octahedral group, and compute the complex quasi-elliptic cohomology $\text{QEll}_{E_6 \times E_7}(S^4)$.

First, for the conjugacy classes $(\alpha, 1)$ where α is a conjugacy class in E_6 and 1 represents the conjugacy class consisting of itself in E_7 , we have

$$\begin{aligned} K_{\Lambda_{E_6 \times E_7}(\alpha, 1)}(S^4)^{(\alpha, 1)} &\cong K_{\Lambda_{E_6}(\alpha) \times_{\mathbb{T}} \Lambda_{E_7}(1)}(S^4)^{\alpha} \\ &\cong K_{\Lambda_{E_6}(\alpha)}(S^0) \otimes RE_7 \end{aligned}$$

Note that $\Lambda_{E_6}(\alpha) \times_{\mathbb{T}} \Lambda_{E_7}(1) \cong \Lambda_{E_6}(\alpha) \times_{\mathbb{T}} (C_{E_7}(1) \times \mathbb{T}) \cong \Lambda_{E_6}(\alpha) \times E_7$. The first factor $K_{\Lambda_{E_6}(\alpha)}(S^0)$ above is the factor in $\text{QEll}_{E_6}(S^4)$ corresponding to the conjugacy class α , which is computed explicitly in ([2] Example 6.5).

For the factors corresponding to the conjugacy classes $(1, \beta)$ where β is a conjugacy class in E_7 , as we discuss in Example 14, $(S^4)^{(1, \beta)} \cong (S^4)^{\beta}$, and

$$\begin{aligned} K_{\Lambda_{E_6 \times E_7}(1, \beta)}(S^4)^{(1, \beta)} &\cong K_{\Lambda_{E_6}(1) \times_{\mathbb{T}} \Lambda_{E_7}(\beta)}(S^4)^{\beta} \\ &\cong K_{\Lambda_{E_7}(\beta)}(S^4)^{\beta} \otimes RE_6 \end{aligned}$$

where $K_{\Lambda_{E_7}(\beta)}(S^4)^{\beta}$ is the factor of $\text{QEll}_{E_7}(S^4)$ corresponding to the conjugacy class represented by β , which is computed explicitly in ([2] Example 6.6).

Then, we think about the case corresponding to the conjugacy classes of the form $(\alpha, -1)$. By direct computation,

$$(S^4)^{(\alpha, -1)} = (S^4)^{-\alpha}.$$

We provide the conjugacy class of each $-\alpha$ and each fixed point space $(S^4)^{-\alpha}$ in Figure 4, where $a = \frac{1}{2}(1 - i - j - k)$.

In addition, we have the short exact sequence

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \Lambda_{E_6 \times E_7}(\alpha, -1) \longrightarrow \Lambda_{E_6 \times E_7}(\alpha, 1) \longrightarrow 1 \quad (30)$$

Note that the image of $\mathbb{Z}/2 \cong \{(1, \pm 1)\}$ is contained in the center of $\Lambda_{E_6 \times E_7}(\alpha, -1)$; thus, we can apply Proposition A1.

Representatives α of conjugacy classes	Centralizers $C_{E_6}(\alpha)$	$(S^4)^\alpha$	Conjugacy classes of $-\alpha$	$(S^4)^{-\alpha}$
1	E_6	S^4	−1	S^0
−1	E_6	S^0	1	S^4
i	$\mathbb{Z}/4$	S^0	i	S^0
a	$\mathbb{Z}/6$	S^0	− a	S^0
− a	$\mathbb{Z}/6$	S^0	a	S^0
a^2	$\mathbb{Z}/6$	S^0	− a^2	S^0
− a^2	$\mathbb{Z}/6$	S^0	a^2	S^0

Figure 4. Centralizers and fixed point spaces of $(\alpha, -1) \in E_6 \times E_7$.

For $\alpha \neq -1$, the action of $\mathbb{Z}/2$ on $(S^4)^{-\alpha}$ is trivial. So, we have

$$\begin{aligned} K_{\Lambda_{E_6 \times E_7}(\alpha, -1)}(S^4)^{-\alpha} &\cong K_{\Lambda_{E_6 \times E_7}(\alpha, -1)}(\text{pt}) \oplus K_{\Lambda_{E_6 \times E_7}(\alpha, -1)}(\text{pt}) \\ &\cong \prod_1^2 R(\Lambda_{E_6}(\alpha)) \otimes (R(T_7) \oplus R_{[(T_7)_\rho]}(T_7)) \end{aligned}$$

where ρ is the sign representation of $\mathbb{Z}/2$. Applying the computation in ([2] Example 6.5, Example 6.6), we list the result of the computation of $K_{\Lambda_{E_6 \times E_7}(\alpha, -1)}(S^4)^{-\alpha}$ ($\alpha \neq -1$) below.

Representatives α of conjugacy classes	The factor $K_{\Lambda_{E_6 \times E_7}(\alpha, -1)}(S^4)^{-\alpha}$
1	$\prod_1^2 R(E_6) \otimes (R(T_7) \oplus R_{[(T_7)_\rho]}(T_7))[q^\pm]$
i	$\prod_1^2 (R(T_7) \oplus R_{[(T_7)_\rho]}(T_7))[x, q^\pm] / \langle x^4 - q \rangle$
a	$\prod_1^2 (R(T_7) \oplus R_{[(T_7)_\rho]}(T_7))[x, q^\pm] / \langle x^6 - q \rangle$
− a	$\prod_1^2 (R(T_7) \oplus R_{[(T_7)_\rho]}(T_7))[x, q^\pm] / \langle x^6 - q^4 \rangle$
a^2	$\prod_1^2 (R(T_7) \oplus R_{[(T_7)_\rho]}(T_7))[x, q^\pm] / \langle x^6 - q^2 \rangle$
− a^2	$\prod_1^2 (R(T_7) \oplus R_{[(T_7)_\rho]}(T_7))[x, q^\pm] / \langle x^6 - q^5 \rangle$

Then, we discuss the case that $\alpha = -1$.

$$\begin{aligned} K_{\Lambda_{E_6 \times E_7}(-1, -1)}(S^4) &\cong K_{\Lambda_{E_6 \times E_7}(-1, -1)}(S^0) \cong K_{\Lambda_{E_6 \times E_7}(-1, -1)}(\text{pt}) \oplus K_{\Lambda_{E_6 \times E_7}(-1, -1)}(\text{pt}) \\ &\cong \prod_1^2 K_{\Lambda_{E_6}(-1)}(\text{pt}) \otimes_{\mathbb{Z}[q^\pm]} K_{\Lambda_{E_7}(-1)}(\text{pt}) \\ &\cong \prod_1^2 \left(R(T_6) \oplus R_{[(T_6)_\rho]}(T_6) \right) \otimes \left(R(T_7) \oplus R_{[(T_7)_\rho]}(T_7) \right) [q^\pm] \end{aligned}$$

where ρ is the sign representation of $\mathbb{Z}/2$.

Next, we deal with the factor corresponding to the conjugacy classes

$$(\alpha, i)$$

By direct computation, we can obtain

Representatives α of conjugacy classes	The fixed point space $(S^4)^{(\alpha,i)}$	$\Lambda_{E_6 \times E_7}(\alpha, i)$ is isomorphic to
1	S^0	$E_6 \times \Lambda_{\mathbb{Z}/8}(2)$
-1	S^0	$\Lambda_{E_6}(-1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(2)$
i	$\{(a, b, 0, 0) \in \mathbb{R}^4\} \cup \{\infty\} \cong S^2$	$\Lambda_{\mathbb{Z}/4}(1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(2)$
a	S^0	$\Lambda_{\mathbb{Z}/6}(1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(2)$
$-a$	S^0	$\Lambda_{\mathbb{Z}/6}(4) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(2)$
a^2	S^0	$\Lambda_{\mathbb{Z}/6}(2) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(2)$
$-a^2$	S^0	$\Lambda_{\mathbb{Z}/6}(5) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(2)$

For $\alpha = -1$,

$$\begin{aligned}
 K_{\Lambda_{E_6 \times E_7}(\alpha, i)}(S^4)^{(\alpha, i)} &\cong K_{\Lambda_{E_6}(-1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(2)}(S^0) \\
 &\cong \prod_{i=1}^2 R(\Lambda_{E_6}(-1)) \otimes_{\mathbb{Z}[q^{\pm}]} R(\Lambda_{\mathbb{Z}/8}(2)) \\
 &\cong \prod_{i=1}^2 \left(R(T_6) \oplus R_{[(\widehat{T_6})_{\rho}]}(T_6) \right) [x, q^{\pm}] / \langle x^8 - q^2 \rangle
 \end{aligned}$$

where ρ is the sign representation of $\mathbb{Z}/2$.

We list the computation of the other cases $K_{\Lambda_{E_6 \times E_7}(\alpha, i)}(S^4)^{(\alpha, i)}$ ($\alpha \neq -1$) below.

Representatives α of conjugacy classes	The factor $K_{\Lambda_{E_6 \times E_7}(\alpha, i)}(S^4)^{(\alpha, i)}$
1	$\prod_{i=1}^2 R(E_6)[x, q^{\pm}] / \langle x^8 - q^2 \rangle$
i	$\prod_{i=1}^2 \mathbb{Z}[x_1, x_2, q^{\pm}] / \langle x_1^4 - q, x_2^8 - q^2 \rangle$
a	$\prod_{i=1}^2 \mathbb{Z}[x_1, x_2, q^{\pm}] / \langle x_1^6 - q, x_2^8 - q^2 \rangle$
$-a$	$\prod_{i=1}^2 \mathbb{Z}[x_1, x_2, q^{\pm}] / \langle x_1^6 - q^4, x_2^8 - q^2 \rangle$
a^2	$\prod_{i=1}^2 \mathbb{Z}[x_1, x_2, q^{\pm}] / \langle x_1^6 - q^2, x_2^8 - q^2 \rangle$
$-a^2$	$\prod_{i=1}^2 \mathbb{Z}[x_1, x_2, q^{\pm}] / \langle x_1^6 - q^5, x_2^8 - q^2 \rangle$

Next, we deal with the conjugacy classes

$$(\alpha, s = \frac{1}{2}(1 + i + j + k)),$$

and compute the factors $K_{\Lambda_{E_6 \times E_7}(\alpha, i)}(S^4)^{(\alpha, s)}$.

By direct computation, we can obtain

α	$(S^4)^{(\alpha, s)}$	$\Lambda_{E_6 \times E_7}(\alpha, s)$
1	S^0	$E_6 \times \Lambda_{\mathbb{Z}/6}(1)$
-1	S^0	$\Lambda_{E_6}(-1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/6}(1)$
i	S^0	$\Lambda_{\mathbb{Z}/4}(1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/6}(1)$
a	$\{(0, -c - d, c, d) \in \mathbb{R}^4\} \cap \{\infty\} \cong S^2$	$\Lambda_{\mathbb{Z}/6}(1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/6}(1)$
$-a$	S^0	$\Lambda_{\mathbb{Z}/6}(4) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/6}(1)$
a^2	S^0	$\Lambda_{\mathbb{Z}/6}(2) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/6}(1)$
$-a^2$	S^0	$\Lambda_{\mathbb{Z}/6}(5) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/6}(1)$

We list the computation of $K_{\Lambda_{E_6 \times E_7}(\alpha, s)}(S^4)^{(\alpha, s)}$ below.

Representatives α of conjugacy classes	The factor $K_{\Lambda_{E_6 \times E_7}(\alpha, s)}(S^4)^{(\alpha, s)}$
1	$\prod_1^2 R(E_6)[x, q^\pm] / \langle x^6 - q \rangle$
-1	$\prod_1^2 \left(R(T_6) \oplus R_{[(T_6)_\rho]}(T_6) \right) [x, q^\pm] / \langle x^6 - q \rangle$
i	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^4 - q, x_2^6 - q \rangle$
a	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^6 - q, x_2^6 - q \rangle$
$-a$	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^6 - q^4, x_2^6 - q \rangle$
a^2	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^6 - q^2, x_2^6 - q \rangle$
$-a^2$	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^6 - q^5, x_2^6 - q \rangle$

Next, we deal with the conjugacy classes

$$(\alpha, -s = -\frac{1}{2}(1 + i + j + k)).$$

By direct computation, we can obtain

α	$(S^4)^{(\alpha, -s)}$	$\Lambda_{E_6 \times E_7}(\alpha, -s)$
1	S^0	$E_6 \times \Lambda_{\mathbb{Z}/6}(4)$
-1	S^0	$\Lambda_{E_6}(-1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/6}(4)$
i	S^0	$\Lambda_{\mathbb{Z}/4}(1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/6}(4)$
a	S^0	$\Lambda_{\mathbb{Z}/6}(1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/6}(4)$
$-a$	S^0	$\Lambda_{\mathbb{Z}/6}(4) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/6}(4)$
a^2	S^0	$\Lambda_{\mathbb{Z}/6}(2) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/6}(4)$
$-a^2$	S^0	$\Lambda_{\mathbb{Z}/6}(5) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/6}(4)$

We list the computation of $K_{\Lambda_{E_6 \times E_7}(\alpha, -s)}(S^4)^{(\alpha, -s)}$ below.

Representatives α of conjugacy classes	The factor $K_{\Lambda_{E_6 \times E_7}(\alpha, -s)}(S^4)^{(\alpha, -s)}$
1	$\prod_1^2 R(E_6)[x, q^\pm] / \langle x^6 - q^4 \rangle$
-1	$\prod_1^2 \left(R(T_6) \oplus R_{[(T_6)_\rho]}(T_6) \right) [x, q^\pm] / \langle x^6 - q^4 \rangle$
i	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^4 - q, x_2^6 - q^4 \rangle$
a	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^6 - q, x_2^6 - q^4 \rangle$
$-a$	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^6 - q^4, x_2^6 - q^4 \rangle$
a^2	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^6 - q^2, x_2^6 - q^4 \rangle$
$-a^2$	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^6 - q^5, x_2^6 - q^4 \rangle$

Next, we deal with the conjugacy classes

$$(\alpha, r = \frac{1}{\sqrt{2}}(i + j)),$$

and compute the factors $K_{\Lambda_{E_6 \times E_7}(\alpha, r)}(S^4)^{(\alpha, r)}$.

By direct computation, we can obtain

α	$(S^4)^{(\alpha, r)}$	$\Lambda_{E_6 \times E_7}(\alpha, r)$
1	S^0	$E_6 \times \Lambda_{\mathbb{Z}/4}(1)$
−1	S^0	$\Lambda_{E_6}(-1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/4}(1)$
i	S^0	$\Lambda_{\mathbb{Z}/4}(1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/4}(1)$
a	S^0	$\Lambda_{\mathbb{Z}/6}(1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/4}(1)$
− a	S^0	$\Lambda_{\mathbb{Z}/6}(4) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/4}(1)$
a^2	S^0	$\Lambda_{\mathbb{Z}/6}(2) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/4}(1)$
− a^2	S^0	$\Lambda_{\mathbb{Z}/6}(5) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/4}(1)$

We list the computation of $K_{\Lambda_{E_6 \times E_7}(\alpha, r)}(S^4)^{(\alpha, r)}$ below.

Representatives α of conjugacy classes	The factor $K_{\Lambda_{E_6 \times E_7}(\alpha, r)}(S^4)^{(\alpha, r)}$
1	$\prod_1^2 R(E_6)[x, q^{\pm}] / \langle x^4 - q \rangle$
−1	$\prod_1^2 \left(R(T_6) \oplus R_{[(T_6)_{\rho}]}(T_6) \right) [x, q^{\pm}] / \langle x^4 - q \rangle$
i	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^{\pm}] / \langle x_1^4 - q, x_2^4 - q \rangle$
a	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^{\pm}] / \langle x_1^6 - q, x_2^4 - q \rangle$
− a	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^{\pm}] / \langle x_1^6 - q^4, x_2^4 - q \rangle$
a^2	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^{\pm}] / \langle x_1^6 - q^2, x_2^4 - q \rangle$
− a^2	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^{\pm}] / \langle x_1^6 - q^5, x_2^4 - q \rangle$

Next, we deal with the conjugacy classes

$$(\alpha, t = \frac{1}{\sqrt{2}}(1 + i)),$$

and compute the factors $K_{\Lambda_{E_6 \times E_7}(\alpha, t)}(S^4)^{(\alpha, t)}$.

By direct computation, we can obtain

α	$(S^4)^{(\alpha, t)}$	$\Lambda_{E_6 \times E_7}(\alpha, t)$
1	S^0	$E_6 \times \Lambda_{\mathbb{Z}/8}(1)$
−1	S^0	$\Lambda_{E_6}(-1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(1)$
i	S^0	$\Lambda_{\mathbb{Z}/4}(1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(1)$
a	S^0	$\Lambda_{\mathbb{Z}/6}(1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(1)$
− a	S^0	$\Lambda_{\mathbb{Z}/6}(4) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(1)$
a^2	S^0	$\Lambda_{\mathbb{Z}/6}(2) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(1)$
− a^2	S^0	$\Lambda_{\mathbb{Z}/6}(5) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(1)$

We list the computation of $K_{\Lambda_{E_6 \times E_7}(\alpha, t)}(S^4)^{(\alpha, t)}$ below.

Representatives α of conjugacy classes	The factor $K_{\Lambda_{E_6 \times E_7}(\alpha, t)}(S^4)^{(\alpha, t)}$
1	$\prod_1^2 R(E_6)[x, q^\pm] / \langle x^8 - q \rangle$
-1	$\prod_1^2 \left(R(T_6) \oplus R_{[(T_6)_\rho]}(T_6) \right) [x, q^\pm] / \langle x^8 - q \rangle$
i	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^4 - q, x_2^8 - q \rangle$
a	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^6 - q, x_2^8 - q \rangle$
$-a$	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^6 - q^4, x_2^8 - q \rangle$
a^2	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^6 - q^2, x_2^8 - q \rangle$
$-a^2$	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^6 - q^5, x_2^8 - q \rangle$

Next, we deal with the conjugacy classes

$$(\alpha, -t = -\frac{1}{\sqrt{2}}(1+i)),$$

and compute the factors $K_{\Lambda_{E_6 \times E_7}(\alpha, -t)}(S^4)^{(\alpha, -t)}$.

By direct computation, we can obtain

α	$(S^4)^{(\alpha, -t)}$	$\Lambda_{E_6 \times E_7}(\alpha, -t)$
1	S^0	$E_6 \times \Lambda_{\mathbb{Z}/8}(5)$
-1	S^0	$\Lambda_{E_6}(-1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(5)$
i	S^0	$\Lambda_{\mathbb{Z}/4}(1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(5)$
a	S^0	$\Lambda_{\mathbb{Z}/6}(1) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(5)$
$-a$	S^0	$\Lambda_{\mathbb{Z}/6}(4) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(5)$
a^2	S^0	$\Lambda_{\mathbb{Z}/6}(2) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(5)$
$-a^2$	S^0	$\Lambda_{\mathbb{Z}/6}(5) \times_{\mathbb{T}} \Lambda_{\mathbb{Z}/8}(5)$

We list the computation of $K_{\Lambda_{E_6 \times E_7}(\alpha, -t)}(S^4)^{(\alpha, -t)}$ below.

Representatives α of conjugacy classes	The factor $K_{\Lambda_{E_6 \times E_7}(\alpha, -t)}(S^4)^{(\alpha, -t)}$
1	$\prod_1^2 R(E_6)[x, q^\pm] / \langle x^8 - q^5 \rangle$
-1	$\prod_1^2 \left(R(T_6) \oplus R_{[(T_6)_\rho]}(T_6) \right) [x, q^\pm] / \langle x^8 - q^5 \rangle$
i	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^4 - q, x_2^8 - q^5 \rangle$
a	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^6 - q, x_2^8 - q^5 \rangle$
$-a$	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^6 - q^4, x_2^8 - q^5 \rangle$
a^2	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^6 - q^2, x_2^8 - q^5 \rangle$
$-a^2$	$\prod_1^2 \mathbb{Z}[x_1, x_2, q^\pm] / \langle x_1^6 - q^5, x_2^8 - q^5 \rangle$

Example 22. In this example, we compute the Real quasi-elliptic cohomology $\mathrm{QEIR}_{E_6 \times E_7}^*(S^4)$. We take the Real structure of E_6 given in Example 11 and the Real structure of E_7 given in Example 12.

Note that an element $(h, k) \in \widehat{E_6 \times E_7}$ is a fixed point under the reflection if and only if h is a fixed point in E_6 and k is a fixed point in E_7 ; in addition, an element $(h, k) \in \widehat{E_6 \times E_7}$ is a free point under the reflection if and only if h is a free point in E_6 and k is a free point in E_7 .

As shown in Example 12, all the representatives of the conjugacy classes in E_7 , as given in Figure 2, are fixed points under the reflection. Thus, all the representatives of the conjugacy classes in $E_6 \times E_7$ are fixed points and they are represented by the elements $(h, k) \in E_6 \times E_7$ with h a fixed point. Then, by Figure 1, h can only be 1, -1 and j .

We first deal with the conjugacy classes

$$(1, \beta),$$

where β goes over all the representatives of the conjugacy classes in E_7 and compute the factors $KR_{\Lambda_{E_6 \times E_7}(1, \beta)}^*(S^4)^{(1, \beta)}$.

Applying Proposition 6, we list the computation of $KR_{\Lambda_{E_6 \times E_7}(1, \beta)}^*(S^4)^{(1, \beta)}$ below.

Representatives β of conjugacy classes	The factor $KR_{\Lambda_{E_6 \times E_7}(1, \beta)}^*(S^4)^{(1, \beta)}$
1	$\prod_1^2 KR_{E_6}^*(\text{pt}) \otimes_{KR^*(\text{pt})} KR_{E_7}^*(\text{pt})[q^\pm]$
-1	$\prod_1^2 KR_{E_6}^*(\text{pt}) \otimes_{KR^*(\text{pt})} \left(KR_{T_7}^*(\text{pt}) \oplus KR_{T_7}^{*+\vartheta_{\Lambda_{E_7}^R(-I)} \text{sign}}(\text{pt}) \right) [q^\pm]$
j	$\prod_1^2 KR_{E_6}^*(\text{pt})[x, q^\pm] / \langle x^8 - q^2 \rangle$
θ	$\prod_1^2 KR_{E_6}^*(\text{pt})[x, q^\pm] / \langle x^6 - q \rangle$
$-\theta$	$\prod_1^2 KR_{E_6}^*(\text{pt})[x, q^\pm] / \langle x^6 - q^4 \rangle$
r	$\prod_1^2 KR_{E_6}^*(\text{pt})[x, q^\pm] / \langle x^4 - q \rangle$
t	$\prod_1^2 KR_{E_6}^*(\text{pt})[x, q^\pm] / \langle x^8 - q \rangle$
$-t$	$\prod_1^2 KR_{E_6}^*(\text{pt})[x, q^\pm] / \langle x^8 - q^5 \rangle$

Next, applying Proposition 6, we list the computation of $KR_{\Lambda_{E_6 \times E_7}(-1, \beta)}^*(S^4)^{(-1, \beta)}$ below.

Representatives β of conjugacy classes	The factor $KR_{\Lambda_{E_6 \times E_7}(-1, \beta)}^*(S^4)^{(-1, \beta)}$
1	$\prod_1^2 \left(KR_{T_6}^*(\text{pt}) \oplus KR_{T_6}^{*+\vartheta_{\Lambda_{E_6}^R(-I)} \text{sign}}(\text{pt}) \right) \otimes_{KR^*(\text{pt})} KR_{E_7}^*(\text{pt})[q^\pm]$
-1	$\prod_1^2 \left(KR_{T_6}^*(\text{pt}) \oplus KR_{T_6}^{*+\vartheta_{\Lambda_{E_6}^R(-I)} \text{sign}}(\text{pt}) \right) \otimes_{KR^*(\text{pt})} \left(KR_{T_7}^*(\text{pt}) \oplus KR_{T_7}^{*+\vartheta_{\Lambda_{E_7}^R(-I)} \text{sign}}(\text{pt}) \right) [q^\pm]$
j	$\prod_1^2 \left(KR_{T_6}^*(\text{pt}) \oplus KR_{T_6}^{*+\vartheta_{\Lambda_{E_6}^R(-I)} \text{sign}}(\text{pt}) \right) [x, q^\pm] / \langle x^8 - q^2 \rangle$
θ	$\prod_1^2 \left(KR_{T_6}^*(\text{pt}) \oplus KR_{T_6}^{*+\vartheta_{\Lambda_{E_6}^R(-I)} \text{sign}}(\text{pt}) \right) [x, q^\pm] / \langle x^6 - q \rangle$
$-\theta$	$\prod_1^2 \left(KR_{T_6}^*(\text{pt}) \oplus KR_{T_6}^{*+\vartheta_{\Lambda_{E_6}^R(-I)} \text{sign}}(\text{pt}) \right) [x, q^\pm] / \langle x^6 - q^4 \rangle$
r	$\prod_1^2 \left(KR_{T_6}^*(\text{pt}) \oplus KR_{T_6}^{*+\vartheta_{\Lambda_{E_6}^R(-I)} \text{sign}}(\text{pt}) \right) [x, q^\pm] / \langle x^4 - q \rangle$
t	$\prod_1^2 \left(KR_{T_6}^*(\text{pt}) \oplus KR_{T_6}^{*+\vartheta_{\Lambda_{E_6}^R(-I)} \text{sign}}(\text{pt}) \right) [x, q^\pm] / \langle x^8 - q \rangle$
$-t$	$\prod_1^2 \left(KR_{T_6}^*(\text{pt}) \oplus KR_{T_6}^{*+\vartheta_{\Lambda_{E_6}^R(-I)} \text{sign}}(\text{pt}) \right) [x, q^\pm] / \langle x^8 - q^5 \rangle$

In addition, we list computation of $KR^*_{\Lambda_{E_6 \times E_7}(j, \beta)}(S^4)^{(j, \beta)}$ below.

Representatives β of conjugacy classes	The factor $KR^*_{\Lambda_{E_6 \times E_7}(j, \beta)}(S^4)^{(j, \beta)}$
1	$\prod_1^2 KR^*_{E_7}(\text{pt})[y, q^{\pm}] / \langle y^4 - q \rangle$
-1	$\prod_1^2 \left(KR^*_{T_7}(\text{pt}) \oplus KR^*_{T_7}{}^{*+\hat{\nu}_{E_7}^R(-1)^{\text{sign}}}(\text{pt}) \right) [y, q^{\pm}] / \langle y^4 - q \rangle$
j	$\prod_1^2 KR^*(\text{pt})[x, y, q^{\pm}] / \langle y^4 - q, x^8 - q^2 \rangle$
θ	$\prod_1^2 KR^*(\text{pt})[x, y, q^{\pm}] / \langle y^4 - q, x^6 - q \rangle$
$-\theta$	$\prod_1^2 KR^*(\text{pt})[x, y, q^{\pm}] / \langle y^4 - q, x^6 - q^4 \rangle$
r	$\prod_1^2 KR^*(\text{pt})[x, y, q^{\pm}] / \langle y^4 - q, x^4 - q \rangle$
t	$\prod_1^2 KR^*(\text{pt})[x, y, q^{\pm}] / \langle y^4 - q, x^8 - q \rangle$
$-t$	$\prod_1^2 KR^*(\text{pt})[x, y, q^{\pm}] / \langle y^4 - q, x^8 - q^5 \rangle$

6. Conclusions and Future Problems

6.1. Relation to Physics

In this section, we sketch the relation between the computation in Sections 4 and 5 and physics. We would like to leave a more detailed interpretation of the relation to physicists.

The results in Section 4 are the Real analogues of those in [2]. In most cases investigated, the results are deformations of twisted equivariant KR-theory groups or twisted equivariant K-theory groups by a further formal variable q , being the character of the extra S^1 -action and playing the role of a kind of M-theoretic quantum deformation of the K-theory groups.

The interesting points lie in the decomposition (9). In the computation of Real quasi-elliptic cohomology, for the free points in the equivariance group under the involution, the factors corresponding to them are complex equivariant K-theories, while for the fixed points under the involution, the corresponding factors are equivariant KR-theories.

Each conjugacy class corresponds to a distinct type of fixed point, and M-branes near them have charges quantized according to the conjugacy class. For example, in the binary tetrahedral group, there are seven conjugacy classes, which correspond to seven distinct types of fixed points. They lead to seven distinct charge sectors for M-branes.

When we take an involution on the finite subgroups of $SU(2)$ into account, the Real conjugacy classes refine the classification of fixed points. M-brane charges in the presence of the involution are classified by the Real conjugacy classes, which describe how the charges transform under the involution. Take the binary dihedral group, for example. The Real conjugacy classes partition the charge spectrum into sectors that are invariant under the involution.

In brief, the conjugacy classes and Real conjugacy classes of the finite subgroups of $SU(2)$ classify the fixed points in orbifold compactification of M-theory. They determine the distinct types of M-brane charges and how they transform under involutions. The geometric meaning lies in the classification of fixed points, while the physical meaning lies in the quantization of M-brane charges in the presence of discrete symmetries. This connection is formalized using equivariant K-theories and equivariant KR-theories.

The computation in Section 4 shows how the classification of M-brane charges is described by equivariant complex K-theories when the points are free under the involution,

and how the classification is described by equivariant KR-theories when the points are fixed by the involution.

In Section 5.2, we compute some Real and complex quasi-elliptic cohomology theories whose equivariance groups are some finite product subgroups of $\text{Spin}(5)$. The finite subgroups of $\text{Spin}(5)$ describe discrete symmetries of 5-dimensional spaces, S^4 and $\mathbb{R}^5$, etc. The conjugacy classes of these groups classify the distinct ways the group acts on the space, leading to different types of fixed points or singularities. For example, the binary icosahedral group acting on S^4 creates fixed points that correspond to the vertices of a 4-dimensional polytope.

Similar to our analysis for the finite subgroups of $SU(2)$, the conjugacy classes of the equivariance group determine the distinct types of localized charges, while the Real conjugacy classes describe how these charges behave under the involution. The connection is formalized using equivariant K-theories and equivariant KR-theories, which provide a deep link between geometry, algebraic topology, and physics.

6.2. Further Exploration

The triality of the $\text{Spin}(8)$ group is a remarkable and unique symmetry that plays a significant role in both mathematics and theoretical physics [44,45]. Note that $\text{Spin}(8)$ is the double cover of $SO(8)$. Triality is an outer automorphism of order 3, meaning it cyclically permutes certain representations of $\text{Spin}(8)$ in a symmetric way. Additionally, it acts in the center of $\text{Spin}(8)$ by permuting the nontrivial elements.

We first recall the definition of triality in $\text{Spin}(8)$. There are three 8-dimensional representations of $\text{Spin}(8)$. They are as follows:

- (1) The vector representation 8_v , corresponding to the fundamental representation of $SO(8)$;
- (2) The positive-chirality spinor representation 8_+ ;
- (3) The negative-chirality spinor representation 8_- .

Triality is an outer automorphism of $\text{Spin}(8)$ that cyclically permutes these three representations:

$$8_v \rightarrow 8_+ \rightarrow 8_- \rightarrow 8_v.$$

This symmetry is unique to $\text{Spin}(8)$ and does not generalize to other $\text{Spin}(n)$ groups with $n \neq 8$.

Triality has profound geometric implications, particularly in the study of Clifford algebras, spinors, and exceptional geometries. Moreover, triality plays a central role in the triality symmetries of string theory—we use the term “triality symmetries” here because they are symmetries of order 3, which is the uniqueness of triality—and M-theory and it relates different types of branes and fluxes in a symmetric way. This rich symmetry is a cornerstone of modern theoretical physics and mathematics ([14] Section 3.3).

The term “trialitarian” refers to structures or objects that are related to or exhibit properties of triality, such as trialitarian groups, trialitarian algebras, etc. Trialitarianism is deeply connected to the theory of exceptional Lie groups and Galois cohomology [46,47]. The triality of $\text{Spin}(8)$ is a key example how exceptional phenomena in Lie theory can lead to new algebraic structures and symmetries.

In addition, triality, which has been deeply studied, has important geometric applications, such as those in terms of triality actions in the moduli space of $\text{Spin}(8)$ –Higgs bundles [48] and the Hitchin integrable system [49,50]. The structure group of $\text{Spin}(8)$ –Higgs bundles is $\text{Spin}(8)$ and the Higgs field takes values in the Lie algebra $\mathfrak{spin}(8)$. The triality symmetry of $\text{Spin}(8)$ leads to interesting symmetries in the moduli space of $\text{Spin}(8)$ –Higgs bundles. Moreover, the Hitchin system is an integrable system constructed from

the moduli space of Higgs bundles. It involves a map to a base space, i.e., the Hitchin fibration, whose fibers are typically abelian varieties. For $\text{Spin}(8)$ –Higgs bundles, the triality symmetry induces actions on the Hitchin base and fibers, leading to symmetries in the integrable system.

Thus, it is meaningful to compute complex and Real quasi-elliptic cohomology theories of the representation spheres of some finite subgroups of $\text{Spin}(8)$. These theories should contain information of M-brane charges that is not contained in the computation in this paper.

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Appendix A. Corollaries of Ángel-Gómez-Uribe Decomposition Formula

In this section, we prove some corollaries of ([27] Theorem 3.6, Corollary 3.7). They all apply to compact Lie groups.

Lemma A1. *Let Q and G be compact Lie groups. We have a short exact sequence*

$$1 \longrightarrow \mathbb{Z}/2 \xrightarrow{l} G \xrightarrow{\pi} Q \longrightarrow 1$$

and $l(A)$ is contained in the center of G . Let X be a G -space with $l(\mathbb{Z}/2)$ acting on it trivially. Then, we have the isomorphism

$$K_G^*(X) \cong K_Q^*(X) \oplus K_Q^{[\tilde{Q}_{\text{sign}}]^+*}(X)$$

Proof. As given in ([27] Section 2.1), there is a well-defined G -action on the irreducible $\mathbb{Z}/2$ -representations by

$$(g \cdot \rho)(a) = \rho(g^{-1}ag) = \rho(a),$$

for any $g \in G$, $a \in \mathbb{Z}/2$ and any irreducible $\mathbb{Z}/2$ -representation ρ .

Since the irreducible representations (ρ, V_ρ) of $\mathbb{Z}/2$ are all 1-dimensional and fixed by G , the group $PU(1)$ of inner automorphism of $U(1)$ consists of exactly one element, i.e., the identity map. As in ([27] (1), page 6), we use the symbol $\tilde{G}_\rho$ to denote the pullback

$$\begin{array}{ccc} \tilde{G}_\rho & \xrightarrow{\tilde{f}} & U(1) \\ \downarrow \tau_\rho & & \downarrow \\ G & \longrightarrow & PU(1) \end{array}$$

We obtain $\tilde{G}_\rho = G \times U(1)$ immediately. Moreover, the map τ_ρ is the projection map to G and $\tilde{f}$ is the projection map to $U(1)$.

Next we consider the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}/2 & \xrightarrow{\tilde{l}} & \tilde{G}_\rho \\ \downarrow = & & \downarrow \\ \mathbb{Z}/2 & \xrightarrow{l} & G \end{array}$$

where $\tilde{l}$ is defined to be the unique map so that $\rho = \tilde{f} \circ \tilde{l}$. Thus, $\tilde{l}$ is the product of l and the representation ρ .

Moreover, in the commutative diagram

$$\begin{array}{ccccc} & \mathbb{Z}/2 & & \mathbb{Z}/2 & \\ & \downarrow \tilde{l} & & \downarrow l & \\ \mathbb{T} & \longrightarrow & \tilde{G}_\rho & \longrightarrow & G \\ & & \downarrow \tilde{\pi} & & \downarrow \pi \\ \mathbb{T} & \xrightarrow{i_Q} & \tilde{Q}_\rho & \xrightarrow{p_Q} & Q, \end{array} \quad (A1)$$

the vertical sequences are both exact, the horizontal sequences are $\mathbb{T}$ -central extensions, and the square is a pullback square. If ρ is the trivial representation of $\mathbb{Z}/2$, we have the equality

$$\tilde{Q}_\rho \cong Q \times \mathbb{T}.$$

In addition, by ([27] Proposition 2.2), ρ extends to an irreducible representation of G . However, if ρ is the sign representation of $\mathbb{Z}/2$, it may not extend to the whole group G , and the central extension

$$1 \longrightarrow \mathbb{T} \xrightarrow{i_Q} \tilde{Q}_\rho \xrightarrow{p_Q} Q \longrightarrow 1$$

may correspond to a nontrivial element $[\tilde{Q}_\rho]$ in $H^3(BQ; \mathbb{Z})$.

By ([27] Corollary 3.7),

$$K_G^*(X) \cong \bigoplus_{\rho \in G/\text{Irr}(\mathbb{Z}/2)} K_{Q_\rho}^{[\tilde{Q}_\rho]^+}(X), \quad (A2)$$

where ρ runs over representatives of the orbits of the G -action on the set of isomorphism classes of irreducible $\mathbb{Z}/2$ -representations, i.e., $\{1, \text{sign}\}$, the action of

$$Q_\rho = G_\rho / (\mathbb{Z}/2)$$

on X is induced from the G -action on X , and G_ρ is the isotropy group of ρ under the G -action. Note that the two irreducible $\mathbb{Z}/2$ -representations are fixed by the G -action and $G_\rho = G$ for each ρ . Thus, the isomorphism (A2) is exactly

$$K_G^*(X) \cong K_Q^*(X) \oplus K_Q^{[\tilde{Q}_{\text{sign}}]^+*}(X)$$

In each component, the Q -action on X is induced from the quotient map $\pi : G \rightarrow Q$. $\square$

Let

$$1 \rightarrow \mathbb{Z}/2 \xrightarrow{l} G \xrightarrow{\pi} Q \rightarrow 1$$

be a short exact sequence of compact groups and $l(A)$ is contained in the center of G . For any torsion element α in G , we have the short exact sequence

$$0 \rightarrow \mathbb{Z}/2 \xrightarrow{i} \Lambda_G(\alpha) \xrightarrow{[\pi, id]} \Lambda_Q(\pi(\alpha)) \rightarrow 0$$

with

$$i(\mathbb{Z}/2) = \{[\beta, 0] \in \Lambda_G(\alpha) \mid \beta \in l(\mathbb{Z}/2)\}$$

contained in the center of $\Lambda_G(\pi(\alpha))$. In addition, X^α is a $\Lambda_G(\alpha)$ -space with the action by $i(\mathbb{Z}/2)$ trivial.

Especially, if α is the nontrivial element in $l(\mathbb{Z}/2)$, then $\pi(\alpha) = 1$ and we have

$$\Lambda_Q(\pi(\alpha)) \cong Q \times \mathbb{T}; \quad \widetilde{\Lambda_Q(\pi(\alpha))}_\rho \cong \tilde{Q}_\rho \times \mathbb{T}.$$

In this case, the central extension

$$1 \rightarrow \mathbb{T} \rightarrow \widetilde{\Lambda_Q(\pi(\alpha))}_\rho \rightarrow \Lambda_Q(\pi(\alpha)) \rightarrow 1$$

is completely determined by

$$1 \rightarrow \mathbb{T} \xrightarrow{i_Q} \tilde{Q}_\rho \xrightarrow{p_Q} Q \rightarrow 1;$$

thus, by the 3-cocycle $[\tilde{Q}_\rho]$.

Then, we can obtain a corollary of Lemma A1.

Proposition A1. *Let*

$$1 \rightarrow \mathbb{Z}/2 \xrightarrow{l} G \xrightarrow{\pi} Q \rightarrow 1$$

be a short exact sequence of compact groups and $l(A)$ is contained in the center of G . Let X be a G -space with $l(\mathbb{Z}/2)$ acting on it trivially. For any torsion element α in G , we have the isomorphism

$$K_{\Lambda_G(\alpha)}^*(X^\alpha) \cong K_{\Lambda_Q(\pi(\alpha))}^*(X^\alpha) \oplus K_{\Lambda_Q(\pi(\alpha))}^{[\widetilde{\Lambda_Q(\pi(\alpha))}_{\text{sign}}]^+*}(X^\alpha).$$

Especially, if α is the nontrivial element in $l(\mathbb{Z}/2)$,

$$K_{\Lambda_G(\alpha)}^*(X^\alpha) \cong K_Q^*(X^\alpha) \otimes \mathbb{Z}[q^\pm] \oplus K_Q^{[\tilde{Q}_{\text{sign}}]^+*}(X^\alpha) \otimes \mathbb{Z}[q^\pm].$$

Appendix B. An Application of Real Mackey-Type Decomposition

In this section, we give a corollary of ([3] Theorem 1.10), which is a Real generalization of the Mackey-type decomposition of complex K -theory ([51] §5) and, when it is specialized

to the complex case, we obtain ([27] Theorem 3.6, Corollary 3.7). Then, we apply it in the computation of Real quasi-elliptic cohomology of 4-spheres.

First, we recall the setting of the theorem. Let

$$1 \longrightarrow H \xrightarrow{l} \hat{G} \xrightarrow{p} \hat{Q} \longrightarrow 1 \quad (\text{A3})$$

be an exact sequence of $\mathbb{Z}/2$ -graded compact Lie groups where $\hat{Q}$ is nontrivially graded. The ungraded groups of $\hat{G}$ and $\hat{Q}$ are denoted by G and Q , respectively. Given $\epsilon \in \mathbb{Z}/2$ and a complex vector space V , write

$$\epsilon V = \begin{cases} V & \text{if } \epsilon = 1, \\ \bar{V} & \text{if } \epsilon = -1, \end{cases} \quad (\text{A4})$$

where $\bar{V}$ is the complex conjugate vector space of V .

The group $\hat{G}$ acts on the set $\text{Irr}(H)$ of isomorphism classes of irreducible unitary representations of H : For an irreducible H -representation ρ_V and $\omega \in \hat{G}$, $\omega \cdot \rho_V$ is defined by

$$(\omega \cdot \rho_V)(h) = \rho_{\pi(\omega)V}(\omega^{-1}h\omega), \quad \text{for any } h \in H.$$

For any $x \in H$, the map $\rho_V \rightarrow x \cdot \rho_V$ is an H -equivariant isometry. In particular, H acts trivially on $\text{Irr}(H)$ and there is an induced action of $\hat{Q}$ on $\text{Irr}(H)$.

Fix a representative V of each $[V] \in \text{Irr}(H)$. By Schur's Lemma, for any representative W of $\omega \cdot [V]$,

$$L_{[V],\omega} := \text{hom}_H(W, \omega \cdot V)$$

is a hermitian line. Following ([5] Section 9.4), the composition maps

$$L_{\omega_1 \cdot [V], \omega_2} \otimes \pi(\omega_2) L_{[V], \omega_1} \longrightarrow L_{[V], \omega_2 \omega_1}, \quad f_2 \otimes f_1 \mapsto (\omega_2 \cdot f_1) \circ f_2 \quad (\text{A5})$$

define a π -twisted extension of $\text{Irr}(H) // \hat{G}$. For $q \in \hat{Q}$, let

$$\mathbb{L}_{[V],q}$$

be the set of all sections s of

$$\bigcup_{\omega \in p^{-1}(q)} L_{[V],\omega} \longrightarrow p^{-1}(q) \subset \hat{G}$$

such that the image of $\rho_W(h) \otimes s(\omega)$ under (A5) is $s(h\omega)$ for all $h \in H$, where W is the representative of $q \cdot V$. Exactness of the sequence (A3) implies that $\mathbb{L}_{[V],q}$ is one dimensional. The maps (A5) induce on $\{\mathbb{L}_{[V],q}\}_{[V],q}$ the structure of a π -twisted extension of $\text{Irr}(H) // \hat{Q}$, which we denote by

$$\hat{\nu}_{\hat{G}}.$$

Then, we have the decomposition formula.

Theorem A1. *Let $1 \rightarrow H \rightarrow \hat{G} \rightarrow \hat{Q} \rightarrow 1$ be an exact sequence of $\mathbb{Z}/2$ -graded compact Lie groups with $\hat{Q}$ non-trivially graded. Let $\hat{G}$ act on a compact Hausdorff space X with contractible local slices (existence of contractible local slices means that each $x \in X$ admits a closed $\hat{G}$ -stable neighbourhood of the form $\hat{G} \times_{\text{Stab}_{\hat{G}}(x)} S_x$ for a slice S_x which is $\text{Stab}_{\hat{G}}(x)$ -equivariantly contractible) such that H acts trivially. Then, there is an isomorphism*

$$KR_G^*(X) \cong KR_{Q,\text{cpt}}^{*+\hat{\nu}_{\hat{G}}}(X \times \text{Irr}(H)),$$

where $\hat{Q}$ acts diagonally on $X \times \text{Irr}(H)$; the pullback of $\hat{v}_{\hat{G}}$ along $(X \times \text{Irr}(H))//\hat{Q} \rightarrow \text{Irr}(H)//\hat{Q}$ is again denoted by $\hat{v}_{\hat{G}}$ and $KR_{\text{cpt}}(-)$ is KR-theory with compact supports.

We refer the readers ([3] Section 1.5) for the proof of the theorem and more details.

We are especially in the case when H is $\mathbb{Z}/2$. The irreducible unitary representations of $\mathbb{Z}/2$ are 1 and the sign representation sign . They are both of the real type. Thus, $\hat{G}$ acts trivially on $\text{Irr}(\mathbb{Z}/2)$. So $\hat{G}$ acts trivially on the product $S^0 \times \text{Irr}(\mathbb{Z}/2)$. Thus, $\text{Irr}(H)//\hat{Q} = \{1\}//\hat{Q} \sqcup \{\text{sign}\}//\hat{Q}$. Additionally, we use

$$\hat{v}_{\hat{G},1}, \quad \hat{v}_{\hat{G},\text{sign}}$$

to denote the restriction of π -twisted extension of $\hat{v}_{\hat{G}}$ to the components $\{1\}//\hat{Q}$ and $\{\text{sign}\}//\hat{Q}$ respectively. In addition, $\hat{v}_{\hat{G},1}$ gives the trivial twist. Thus, by Theorem A1,

$$KR_G^*(S^0) \cong KR_Q^{*+\hat{v}_{\hat{G}}}(S^0 \times \text{Irr}(\mathbb{Z}/2)) \cong \prod_1^2 KR_Q^*(\text{pt}) \oplus KR_Q^{*+\hat{v}_{\hat{G},\text{sign}}}(\text{pt}).$$

So, we obtain the corollary below.

Corollary A1. Let $1 \rightarrow \mathbb{Z}/2 \rightarrow \hat{G} \rightarrow \hat{Q} \rightarrow 1$ be an exact sequence of $\mathbb{Z}/2$ -graded compact Lie groups with $\hat{Q}$ non-trivially graded. Let $\hat{G}$ act on S^0 trivially. Then, we have the isomorphism

$$KR_G^*(S^0) \cong \prod_1^2 KR_Q^*(\text{pt}) \oplus KR_Q^{*+\hat{v}_{\hat{G},\text{sign}}}(\text{pt}).$$

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