

Article

Generalized Bertrand Curve Pairs in Euclidean Four-Dimensional Space

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Abstract: In this study, the existence of Bertrand curves (in the classical sense, i.e., curves with a common principal normal vector field) in four-dimensional Euclidean space is demonstrated using a novel approach. The necessary conditions for a regular curve to be a Bertrand curve pair are obtained. Furthermore, the relationship between Bertrand curves and Combescure-related curves (pairs of curves with parallel Frenet vectors) is established, and several geometric properties are derived. Additionally, examples are constructed for both Bertrand curve pairs and Combescure-related curve pairs, and their orthogonal projections onto three-dimensional subspaces of four-dimensional space are visualized.

Keywords: bertrand curves; combescure-related space curves; parallel frenet frame

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1. Introduction

Bertrand curves are a pair of curves that have been intensively studied by researchers among space curve pairs, and it is a pair of curves that emerged thanks to a problem posed by Venant in 1845. In this problem, it was asked under which conditions the principal normal vectors of one space curve can be the principal normal of another space curve. This problem was answered by Bertrand in 1850, and these types of curves entered the mathematical literature as Bertrand curves or Bertrand curve pairs [1,2].

Bertrand curves have been extensively studied in Euclidean three-dimensional space, and many geometric properties have been revealed [3,4]. The best known example of a Bertrand curve is the helix curve, which has an infinite number of Bertrand conjugates. Bertrand curves other than helix curves and planar curves have only one Bertrand conjugate. Studies on space curves have also explored various generalizations, including the geometric behavior of sweeping surfaces [5–7] and the interactions of osculating curves [8–10] and rectifying curves [11–13] on regular surfaces in Euclidean 3D space. Bertrand curves have been studied in different spaces and in different metric structures. Since our study is in Euclidean 4D space, we will not mention these studies. The first generalization of Bertrand curves from three-dimensional Euclidean space to n -dimensional Euclidean space (and more generally, to Riemannian spaces) was carried out by Pears [14]. In this study, it was demonstrated that Bertrand curves must always remain in a three-dimensional subspace of the n -dimensional space. In other words, all curvatures of the curve, except for the first two, must be zero. Consequently, a special curve with all nonzero curvatures cannot be a Bertrand curve in n -dimensional space according to the classical definition.

Building upon Pears' findings, Matsuda and Yorozu [15] introduced a new type of Bertrand curve in four-dimensional Euclidean space, termed the (1,3)-Bertrand curve, which refers to a space curve pair whose osculating planes, spanned by the first and third normal vectors, remain parallel. They further investigated the geometric properties of these curves. Later, Cheng and Lin proposed the concept of generalized Bertrand curves, arguing that the classical definition of Bertrand curves is not applicable to Frenet curves in higher-dimensional spaces ($n > 3$), where all curvatures are nonzero [16]. This idea, in fact, extends the concept of (1,3)-Bertrand curves introduced by Matsuda and Yorozu. Another generalization was provided by Li et al., who introduced the notion of (1,3)-V Bertrand curves and derived their corresponding geometric properties. This study presents a further extension of (1,3)-Bertrand curves in four-dimensional space.

In these studies, no examples other than helix curves or W-curves (curves with all curvatures different from zero) could be found. If γ is a Bertrand curve, its Bertrand conjugate curve γ^* is parameterized as

$$\gamma^*(\varphi(s)) = \gamma(s) + \lambda N(s)$$

which is referred to as a classical approach. In this formulation, the vector $\overrightarrow{\gamma^* \gamma}$ is observed to be parallel to the normal vector N . However, this alignment is not inherently dictated by the definition of a Bertrand curve but rather results from a specific choice.

Motivated by this perspective, Camcı et al. [17] introduced an alternative parameterization for the Bertrand conjugate curve γ^* , termed the new approach, given by

$$\gamma^*(\varphi(s)) = \gamma(s) + u(s)T(s) + v(s)N_1(s) + w(s)N_2(s)$$

where $u(s)$, $v(s)$ and $w(s)$ are differentiable functions subject to certain conditions. Notably, setting $u(s) = w(s) = 0$ in this formulation reduces it to the classical approach.

This novel perspective has enabled the construction of numerous Bertrand curve pairs in three-dimensional Euclidean space. For instance, while general helices do not qualify as Bertrand curves under the classical approach, they can be considered Bertrand curves under this new framework.

In this paper, the existence of Bertrand curve pairs in four-dimensional Euclidean space is demonstrated by using the new approach to Bertrand curves given by Camcı et al. [17] and by adhering to the classical definition that the principal normal vector fields of the curves are linearly dependent. Moreover, in this space, a subclass of Bertrand curve pairs, namely the Combescure-related space curves, were obtained. After some geometrical properties of the curves are stated, the examples of interest are constructed and the orthogonal projections of the related curves to the three-dimensional subspace are illustrated.

2. Preliminaries

In this section, we introduce some fundamental concepts that are necessary for this paper; for further details, see [18–20]. Let $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$ be a curve in $\mathbb{E}^4$. Recall that the curve γ is said to be of unit speed (or parameterized by arc length s) if $\langle \gamma'(s), \gamma'(s) \rangle = 1$, where $\langle \cdot, \cdot \rangle$ is the standard scalar product of $\mathbb{E}^4$ given by

$$\langle X, Y \rangle = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$$

where $X = (x_1, x_2, x_3, x_4)$, $Y = (y_1, y_2, y_3, y_4) \in \mathbb{E}^4$. In particular, the norm of a vector $X \in \mathbb{E}^4$ is given by $\|X\| = \sqrt{\langle X, X \rangle}$.

Let $\{T, N_1, N_2, N_3\}$ represent an orthonormal Frenet frame along the unit speed curve γ , where T, N_1, N_2 and N_3 correspond to the tangent vector, principal normal, first binormal and second binormal vector fields, respectively. The Frenet equations are then expressed as follows:

$$\begin{bmatrix} T' \\ N_1' \\ N_2' \\ N_3' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & -\kappa_2 & 0 & \kappa_3 \\ 0 & 0 & -\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N_1 \\ N_2 \\ N_3 \end{bmatrix}. \quad (1)$$

The functions κ_1, κ_2 and κ_3 are referred to as the first, second and third curvatures of the curve γ , respectively [21]. This formulation can be utilized in the analysis of the eigenproblem of alignment [22].

Relationship Between Space Curves via a Combescure Transformation

In this subsection, firstly, let us make a brief introduction about Combescure-related curves that we will encounter in the next section, i.e., curves with a parallel Frenet frame. We know that if the tangent vectors of the space curve pair are equal, the other Frenet vectors are also equal. It is well known that such curves are called curves related by a transformation of Combescure by Bianchi [23]. The following theorem can be seen in [24].

Theorem 1. Let $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$ and $\gamma^* : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$ be regular curves in Euclidean 3D space, each with its respective Frenet frame $\{T, N_1, N_2, \kappa, \tau\}$ for α and $\{T^*, N_1^*, N_2^*, \kappa^*, \tau^*\}$ for γ . The tangent vector T of α is equal to the tangent vector T^* of γ if and only if there exists a function $C : I \subset \mathbb{R} \rightarrow \mathbb{R}$ such that

$$\gamma^*(\varphi(s)) = \gamma(s) + \frac{d}{dr} \left(C(s) + \frac{d^2 C(s)}{dt^2} \right) T - \frac{d^2 C(s)}{dt^2} N_1 + \frac{dC(s)}{dt} N_2$$

where $t = \int \tau(s) ds$, $r = \int \kappa(s) ds$ and $\varphi'(s) = ds^*/ds$ where s and s^* are arc-length parameters of γ and γ^* , respectively.

Corollary 1. Let $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$ and $\gamma^* : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$ be regular curves in Euclidean 3D space, each with its respective Frenet frame $\{T, N_1, N_2, \kappa, \tau\}$ for γ and $\{T^*, N_1^*, N_2^*, \kappa^*, \tau^*\}$ for γ^* . Then, the tangent vector T of γ is equal to the tangent vector T^* of γ^* if and only if $N_1 = N_1^*$ and $N_2 = N_2^*$.

Corollary 2. Every curve pair related with a Combescure transformation is also a Bertrand curve pair. However, the opposite is not true.

Theorem 2. Let $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$ and $\gamma^* : I \subset \mathbb{R} \rightarrow \mathbb{E}^3$ be regular curves in Euclidean 3D space, each with its respective Frenet frame $\{T, N_1, N_2, \kappa, \tau\}$ for γ and $\{T^*, N_1^*, N_2^*, \kappa^*, \tau^*\}$ for γ^* . Then, the tangent vector T of γ is equal to the tangent vector T^* of γ^* if and only if

$$\gamma^*(\varphi(s)) = \varphi'(s)\gamma(s) - \int \varphi''(s)\gamma(s)ds \quad (2)$$

for $\varphi'(s) = ds^*/ds$ where s and s^* are arc-length parameters of γ and γ^* , respectively.

Remark 1. As in $\mathbb{E}^3$, pairs of curves defined in $\mathbb{E}^4$ have a parallel Frenet vector if their tangent vectors are the same.

As an application of the above theorems, we can provide the following example:

Example 1. Let us consider the curve in $\mathbb{E}^3$ given by

$$\gamma(s) = \left(\frac{3}{5} \sin s, \frac{3}{5} \cos s, \frac{4}{5} s \right)$$

with the curvatures $\kappa = 3/5$, $\tau = -4/5$ and the Frenet frame as

$$\begin{aligned} T(s) &= \left(\frac{3}{5} \cos s, \frac{-3}{5} \sin s, \frac{4}{5} \right), \\ N_1(s) &= (-\sin s, -\cos s, 0), \\ N_2(s) &= \left(\frac{4}{5} \cos s, -\frac{4}{5} \sin s, \frac{-3}{5} \right). \end{aligned}$$

By taking $\varphi'(s) = s$, from (2), we find

$$\gamma^*(\varphi(s)) = \left(\frac{3}{5} \cos s + \frac{3}{5} s \sin s, \frac{3}{5} s \cos s - \frac{3}{5} \sin s, \frac{2}{5} s^2 \right)$$

with the curvatures $\kappa^* = 3/5s$, $\tau^* = -4/5s$ and the Frenet frame as

$$\begin{aligned} T^*(\varphi(s)) &= \left(\frac{3}{5} \cos s, \frac{-3}{5} \sin s, \frac{4}{5} \right), \\ N_1^*(\varphi(s)) &= (-\sin s, -\cos s, 0), \\ N_2^*(\varphi(s)) &= \left(\frac{4}{5} \cos s, -\frac{4}{5} \sin s, \frac{-3}{5} \right). \end{aligned}$$

It is clear that the curve γ is a circular helix, while γ^* is a general helix. This pair of curves is a pair of Bertrand curves that are also related by a Combescure transformation. It is not possible to obtain this pair of curves with the classical approach. See Figure 1.

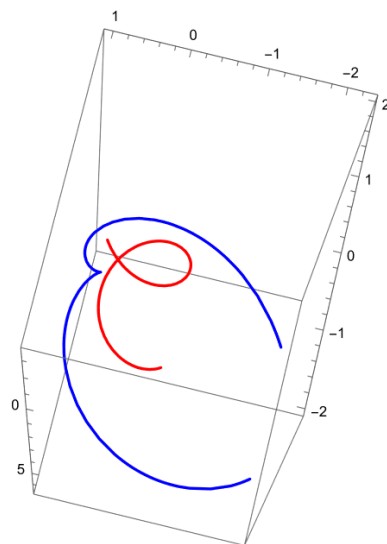


Figure 1. Graphs of curves with parallel Frenet vectors given in Example 1. Main curve γ in red; other curve γ^* in blue).

3. Bertrand Curves in Four-Dimensional Euclidean Space

In this section, the existence of Bertrand curves in four-dimensional Euclidean space is shown. The parameterization of the conjugate curve of a Bertrand curve is examined according to a new approach, while maintaining the classical definition of a Bertrand curve. The necessary conditions for a curve to be a Bertrand curve are derived, and relevant examples are provided.

Definition 1. A curve $\gamma : I \rightarrow \mathbb{E}^4$ with nonzero curvatures is defined as a Bertrand curve if there exists another curve $\gamma^* : I^* \rightarrow \mathbb{E}^4$ along with a bijective function $\varphi : I \rightarrow I^*$ such that the principal normal vectors of $\gamma(s)$ and $\gamma^*(s^*) = \gamma^*(\varphi(s))$ at $s \in I$ and $s^* \in I^*$ coincide, respectively. In this context, $\gamma^*(s)$ is referred to as the Bertrand mate of $\gamma(s)$.

Let $\gamma : I \rightarrow \mathbb{E}^4$ be a Bertrand curve in $\mathbb{E}^4$ equipped with the Frenet frame $\{T, N_1, N_2, N_3\}$ and possessing nonzero curvatures $\kappa_1, \kappa_2, \kappa_3$. Additionally, let $\gamma^* : I^* \rightarrow \mathbb{E}^4$ represent the Bertrand mate of γ , with the corresponding Frenet frame $\{T^*, N_1^*, N_2^*, N_3^*\}$ and nonzero curvatures $\kappa_1^*, \kappa_2^*, \kappa_3^*$. Under these conditions, the curve γ^* can be expressed as

$$\gamma^*(s^*) = \gamma(\varphi(s)) = \gamma(s) + u(s)T(s) + v(s)N_1(s) + w_1(s)N_2(s) + w_2(s)N_3(s)$$

where u, v, w_1 and w_2 are differentiable functions on I .

Theorem 3. Let $\gamma : I \subset \mathbb{R} \rightarrow \mathbb{E}^4$ be a unit speed curve with nonzero curvatures $\kappa_1, \kappa_2, \kappa_3$. Then, the curve γ is a Bertrand curve if and only if it satisfies one of the following conditions.

- (i) There exist differentiable functions u, v, w_1 , and w_2 on $I \subset \mathbb{R}$ and a constant $\lambda \in (-1, 1) - \{0\}$, satisfying

$$\begin{cases} u\kappa_1 + v' - w_1\kappa_2 = 0, \\ \lambda(v\kappa_2 + w_1' - w_2\kappa_3) = \sqrt{1 - \lambda^2} \cos(\int \kappa_3 ds)(1 + u' - v\kappa_1), \\ 1 + u' - v\kappa_1 \neq 0 \end{cases}$$

or

$$\begin{cases} u\kappa_1 + v' - w_1\kappa_2 = 0, \\ \lambda(w_1\kappa_3 + w_2') = -\sqrt{1 - \lambda^2} \sin(\int \kappa_3 ds)(1 + u' - v\kappa_1), \\ 1 + u' - v\kappa_1 \neq 0 \end{cases}$$

- (ii) There exist differentiable functions u, v, w_1 , and w_2 on $I \subset \mathbb{R}$, satisfying

$$\begin{cases} 1 + u' - v\kappa_1 = 0, \\ u\kappa_1 + v' - w_1\kappa_2 = 0, \\ (w_1\kappa_3 + w_2') \cos(\int \kappa_3 ds) = -\sin(\int \kappa_3 ds)(v\kappa_2 + w_1' - w_2\kappa_3), \\ (v\kappa_2 + w_1' - w_2\kappa_3) \neq 0, \\ (w_1\kappa_3 + w_2') \neq 0 \end{cases}$$

or

$$\begin{cases} 1 + u' - v\kappa_1 = 0, \\ u\kappa_1 + v' - w_1\kappa_2 = 0, \\ (w_1\kappa_3 + w_2') \sin(\int \kappa_3 ds) = \cos(\int \kappa_3 ds)(v\kappa_2 + w_1' - w_2\kappa_3), \\ (v\kappa_2 + w_1' - w_2\kappa_3) \neq 0, \\ (w_1\kappa_3 + w_2') \neq 0 \end{cases}$$

- (iii) There exist differentiable functions u, v, w_1 , and w_2 on $I \subset \mathbb{R}$, satisfying

$$\begin{cases} 1 + u' - v\kappa_1 = \varphi' \neq 0, \\ u\kappa_1 + v' - w_1\kappa_2 = 0, \\ v\kappa_2 - w_1' - w_2\kappa_3 = 0, \\ w_1\kappa_3 + w_2' = 0. \end{cases}$$

where $\varphi' = ds^*/ds$.

Proof. Assume that γ is a Bertrand curve parametrized by arc-length s with the nonzero curvatures $\kappa_1, \kappa_2, \kappa_3$ and γ^* is the Bertrand mate curve of the curve γ parametrized by with arc-length s^* . Then, we can write the curve γ^* as

$$\begin{aligned}\gamma^*(s^*) &= \gamma^*(\varphi(s)) \\ &= \gamma(s) + u(s)T(s) + v(s)N_1(s) \\ &\quad + w_1(s)N_2(s) + w_2(s)N_3(s)\end{aligned}\quad (3)$$

for all $s \in I$, where u, v, w_1 and w_2 are smooth functions on I . By differentiating (3) with respect to s and using Frenet formulae (1), we obtain

$$\begin{aligned}T^*\varphi' &= (1 + u' - v\kappa_1)T + (u\kappa_1 + v' - w_1\kappa_2)N_1 \\ &\quad + (v\kappa_2 + w_1' - w_2\kappa_3)N_2 + (w_1\kappa_3 + w_2')N_3.\end{aligned}\quad (4)$$

Taking the scalar product of Equation (4) with N_1 yields

$$u\kappa_1 + v' - w_1\kappa_2 = 0. \quad (5)$$

By substituting (5) in (4), we find

$$T^*\varphi' = (1 + u' - v\kappa_1)T + (v\kappa_2 + w_1' - w_2\kappa_3)N_2 + (w_1\kappa_3 + w_2')N_3. \quad (6)$$

Taking the scalar product of (6) with itself, we obtain

$$(\varphi')^2 = (1 + u' - v\kappa_1)^2 + (v\kappa_2 + w_1' - w_2\kappa_3)^2 + (w_1\kappa_3 + w_2')^2. \quad (7)$$

If we denote

$$\delta_1 = \frac{(1 + u' - v\kappa_1)}{\varphi'}, \delta_2 = \frac{v\kappa_2 + w_1' - w_2\kappa_3}{\varphi'}, \delta_3 = \frac{w_1\kappa_3 + w_2'}{\varphi'}, \quad (8)$$

then we obtain

$$\delta_1^2 + \delta_2^2 + \delta_3^2 = 1 \quad (9)$$

and

$$T^* = \delta_1 T + \delta_2 N_2 + \delta_3 N_3. \quad (10)$$

By differentiating (10) with respect to s and using Frenet formulae (1), we find

$$\varphi'\kappa_1^*N_1^* = \delta_1'T + (\delta_1\kappa_1 - \delta_2\kappa_2)N_1 + (\delta_2' - \delta_3\kappa_3)N_2 + (\delta_2\kappa_3 + \delta_3')N_3. \quad (11)$$

By taking the scalar product of Equation (11) with T, N_2 and N_3 , respectively, we obtain

$$\delta_1' = 0, \quad (12)$$

$$\delta_2' - \delta_3\kappa_3 = 0, \quad (13)$$

$$\delta_3' + \delta_2\kappa_3 = 0. \quad (14)$$

From (12) and (9), δ_1 is a constant such that $\delta_1 \in [-1, 1]$. For different values that δ_1 can take within this range, the following cases exist:

Case 1. Assume that $\delta_1 = \lambda \in (-1, 1) - \{0\}$. From Equation (9), we obtain

$$\delta_2^2 + \delta_3^2 = 1 - \lambda^2. \quad (15)$$

Moreover, solving Differential Equations (13) and (14) yields

$$\delta_2 = \sqrt{1 - \lambda^2} \cos\left(\int \kappa_3 ds\right), \quad \delta_3 = -\sqrt{1 - \lambda^2} \sin\left(\int \kappa_3 ds\right) \quad (16)$$

or

$$\delta_2 = \sqrt{1 - \lambda^2} \sin\left(\int \kappa_3 ds\right), \quad \delta_3 = \sqrt{1 - \lambda^2} \cos\left(\int \kappa_3 ds\right). \quad (17)$$

From (8) and the above equations, we have

$$\lambda(v\kappa_2 + w'_1 - w_2\kappa_3) = \sqrt{1 - \lambda^2} \cos\left(\int \kappa_3 ds\right) (1 + u' - v\kappa_1) \quad (18)$$

and

$$\lambda(w_1\kappa_3 + w'_2) = -\sqrt{1 - \lambda^2} \sin\left(\int \kappa_3 ds\right) (1 + u' - v\kappa_1). \quad (19)$$

Thus, all the conditions given in (i) have been fully achieved.

Case 2. Assume that $\delta_1 = 0$. In this case, from Equation (9), we obtain

$$\delta_2^2 + \delta_3^2 = 1. \quad (20)$$

Moreover, considering that the curvature κ_3 is nonzero in (13) and (14), it follows that both δ_2 and δ_3 are nonzero. Since δ_2 is nonzero, then, from (8), we have

$$v\kappa_2 + w'_1 - w_2\kappa_3 \neq 0.$$

By considering (20), solving Differential Equations (13) and (14) yields

$$\delta_2 = \cos\left(\int \kappa_3 ds\right), \quad \delta_3 = -\sin\left(\int \kappa_3 ds\right)$$

or

$$\delta_2 = \sin\left(\int \kappa_3 ds\right), \quad \delta_3 = \cos\left(\int \kappa_3 ds\right).$$

From (8) and the above equations, we have

$$(w_1\kappa_3 + w'_2) \cos\left(\int \kappa_3 ds\right) = -\sin\left(\int \kappa_3 ds\right) (v\kappa_2 + w'_1 - w_2\kappa_3)$$

and

$$(w_1\kappa_3 + w'_2) \sin\left(\int \kappa_3 ds\right) = \cos\left(\int \kappa_3 ds\right) (v\kappa_2 + w'_1 - w_2\kappa_3).$$

The same equalities can be obtained for $\delta_3 \neq 0$. Thus, all the conditions given in (ii) have been fully achieved.

Case 3. Assume that $\delta_1 = \lambda = \mp 1$. From Equations (8) and (9), we obtain

$$1 + u' - v\kappa_1 = \mp \varphi'(s) \neq 0$$

and

$$\begin{aligned} v\kappa_2 + w'_1 - w_2\kappa_3 &= 0 \\ w_1\kappa_3 + w'_2 &= 0. \end{aligned}$$

Finally, we obtain the conditions in (iii).

Conversely, let γ be a curve parameterized by arc-length parameter s , with nonzero curvatures $\kappa_1, \kappa_2, \kappa_3$. Firstly, assume that the first condition in (i) holds for differentiable

functions u, v, w_1, w_2 and a constant $\lambda \in (-1, 1) - \{0\}$. Then, we can define a new curve γ^* as

$$\begin{aligned}\gamma^*(s^*) &= \gamma^*(\varphi(s)) \\ &= \gamma(s) + u(s)T(s) + v(s)N_1(s) + w_1(s)N_2(s) + w_2(s)N_3(s).\end{aligned}\quad (21)$$

By differentiating (21) with respect to s , and using Frenet formulae (1), we find

$$\frac{d\gamma^*}{ds} = (1 + u' - v\kappa_1)T + (v\kappa_2 + w'_1 - w_2\kappa_3)N_2 + (w_1\kappa_3 + w'_2)N_3. \quad (22)$$

From (22) and using the first condition in (i), we have

$$\frac{d\gamma^*}{ds} = (1 + u' - v\kappa_1) \left(T + \frac{\sqrt{1-\lambda^2}}{\lambda} \cos\left(\int \kappa_3 ds\right) N_2 - \frac{\sqrt{1-\lambda^2}}{\lambda} \sin\left(\int \kappa_3 ds\right) N_3 \right). \quad (23)$$

From (23), we have

$$\varphi' = \frac{ds^*}{ds} = \left\| \frac{d\gamma^*}{ds} \right\| = \varepsilon_1 \left(\frac{1 + u' - v\kappa_1}{\lambda} \right) \quad (24)$$

where $\varepsilon_1 = \operatorname{sgn}\left(\frac{1+u'-v\kappa_1}{\lambda}\right)$. By substituting (24) in (23) and using the first condition in (i), we obtain

$$T^* = \varepsilon_1 \lambda T + \varepsilon_1 \frac{\sqrt{1-\lambda^2}}{\lambda} \cos\left(\int \kappa_3 ds\right) N_2 - \varepsilon_1 \frac{\sqrt{1-\lambda^2}}{\lambda} \sin\left(\int \kappa_3 ds\right) N_3. \quad (25)$$

By differentiating the above equation with respect to s and using Frenet equations, we obtain

$$\kappa_1^* N_1^* \varphi' = \left(\varepsilon_1 \lambda \kappa_1 - \varepsilon_1 \kappa_2 \sqrt{1-\lambda^2} \cos\left(\int \kappa_3 ds\right) \right) N_1. \quad (26)$$

Consequently, N_1^* and N_1 are parallel. Thus, γ is a Bertrand curve.

Now, assume that the first condition in (ii) holds. Then, we can define a curve γ^* as

$$\begin{aligned}\gamma^*(s^*) &= \gamma^*(\varphi(s)) \\ &= \gamma(s) + u(s)T(s) + v(s)N_1(s) + w_1(s)N_2(s) + w_2(s)N_3(s).\end{aligned}\quad (27)$$

By differentiating (27) with respect to s , and using Frenet formulae (1), we find

$$\frac{d\gamma^*}{ds} = (v\kappa_2 + w'_1 - w_2\kappa_3)N_2 + (w_1\kappa_3 + w'_2)N_3. \quad (28)$$

From (28) and the first condition in (ii), we have

$$\frac{d\gamma^*}{ds} = (v\kappa_2 + w'_1 - w_2\kappa_3) \left(N_2 - \tan\left(\int \kappa_3 ds\right) N_3 \right). \quad (29)$$

From (29), we obtain

$$\varphi' = \frac{ds^*}{ds} = \left\| \frac{d\gamma^*}{ds} \right\| = \varepsilon_2 \left(\frac{v\kappa_2 + w'_1 - w_2\kappa_3}{\cos\left(\int \kappa_3 ds\right)} \right) > 0 \quad (30)$$

where

$$\varepsilon_2 = \operatorname{sgn}\left(\frac{v\kappa_2 + w'_1 - w_2\kappa_3}{\cos\left(\int \kappa_3 ds\right)}\right).$$

By substituting (30) in (29) and using the first condition in (ii), we obtain

$$T^* = \varepsilon_2 \cos\left(\int \kappa_3 ds\right) N_2 - \varepsilon_2 \sin\left(\int \kappa_3 ds\right) N_3 \quad (31)$$

By differentiating (31) with respect to s and using Frenet equations, we obtain

$$\kappa_1^* N_1^* \varphi' = \varepsilon_2 \cos\left(\int \kappa_3 ds\right) N_1$$

Thus, N_1^* and N_1 are parallel. Thus, γ is a Bertrand curve.

The proof can be similarly conducted for the second conditions in (i) and (ii).

Finally, assume that Condition (iii) holds. Then, we can define a curve γ^* as

$$\begin{aligned} \gamma^*(s^*) &= \gamma^*(\varphi(s)) \\ &= \gamma(s) + u(s)T(s) + v(s)N_1(s) + w_1(s)N_2(s) + w_2(s)N_3(s). \end{aligned} \quad (32)$$

By differentiating (32) with respect to s , and using Frenet formulae (1), we find

$$\begin{aligned} \frac{d\gamma^*}{ds} &= (1 + u' - v\kappa_1)T + (u\kappa_1 + v' - w_1\kappa_2)N_1 \\ &\quad + (v\kappa_2 + w_1' - w_2\kappa_3)N_2 + (w_1\kappa_3 + w_2')N_3. \end{aligned}$$

If the conditions given in (iii) are substituted in the above equation, then we obtain

$$\frac{d\gamma^*}{ds} = (1 + u' - v\kappa_1)T. \quad (33)$$

From (33), we have

$$\varphi' = \frac{ds^*}{ds} = \left\| \frac{d\gamma^*}{ds} \right\| = \varepsilon_3 (1 + u' - v\kappa_1) > 0$$

where $\varepsilon_3 = \text{sgn}(1 + u' - v\kappa_1)$. Thus, we can easily obtain

$$T^* = \varepsilon_3 T, \quad N_1^* = \varepsilon_3 N_1, \quad N_2^* = \varepsilon_3 N_2, \quad N_3^* = \varepsilon_3 N_3.$$

From this, it is observed that the γ curve is a Bertrand curve. Notably, the Frenet vectors of the γ and γ^* curves are parallel. In other words, the γ and γ^* curves are associated through a Combescure transformation. $\square$

Corollary 3. Let γ be a Bertrand curve in the four-dimensional Euclidean space $\mathbb{E}^4$. According to the cases presented in Theorem 3, the Bertrand mate of γ is given by one of the following:

1. If the conditions in (i) of Theorem 3 are satisfied, the Bertrand mate γ^* of γ is one of the following:

$$\gamma^*(\varphi(s)) = \int \left(\lambda T + \sqrt{1 - \lambda^2} \cos\left(\int \kappa_3 ds\right) N_2 - \sqrt{1 - \lambda^2} \sin\left(\int \kappa_3 ds\right) N_3 \right) \varphi'(s) ds \quad (34)$$

or

$$\gamma^*(\varphi(s)) = \int \left(\lambda T + \sqrt{1 - \lambda^2} \sin\left(\int \kappa_3 ds\right) N_2 + \sqrt{1 - \lambda^2} \cos\left(\int \kappa_3 ds\right) N_3 \right) \varphi'(s) ds \quad (35)$$

2. If the conditions in (ii) of Theorem 3 are satisfied, the Bertrand mate γ^* of γ is one of the following:

$$\gamma^*(\varphi(s)) = \int \left(\cos \left(\int \kappa_3 ds \right) N_2 - \sin \left(\int \kappa_3 ds \right) N_3 \right) \varphi'(s) ds \quad (36)$$

or

$$\gamma^*(\varphi(s)) = \int \left(\sin \left(\int \kappa_3 ds \right) N_2 + \cos \left(\int \kappa_3 ds \right) N_3 \right) \varphi'(s) ds \quad (37)$$

3. If the conditions in (iii) of Theorem 3 are satisfied, the Bertrand mate γ^* of γ is as follows:

$$\gamma^*(\varphi(s)) = \varphi'(s)\gamma(s) - \int \varphi''(s)\gamma(s) ds \quad (38)$$

The following example is provided to illustrate the first case of Theorem 3.

Example 2. Consider the curve in $\mathbb{E}^4$ given by

$$\gamma(s) = \left(\cos \frac{2s}{\sqrt{5}}, \sin \frac{2s}{\sqrt{5}}, \cos \frac{s}{\sqrt{5}}, \sin \frac{s}{\sqrt{5}} \right)$$

with the curvatures $\kappa_1 = \frac{\sqrt{17}}{5}$, $\kappa_2 = \frac{6}{5\sqrt{17}}$, $\kappa_3 = \frac{2}{\sqrt{17}}$ and Frenet vectors

$$\begin{aligned} T(s) &= \frac{1}{\sqrt{5}} \left(-2 \sin \frac{2s}{\sqrt{5}}, 2 \cos \frac{2s}{\sqrt{5}}, -\sin \frac{s}{\sqrt{5}}, \cos \frac{s}{\sqrt{5}} \right), \\ N_1(s) &= \frac{1}{\sqrt{17}} \left(-4 \cos \frac{2s}{\sqrt{5}}, -4 \sin \frac{2s}{\sqrt{5}}, -\cos \frac{s}{\sqrt{5}}, -\sin \frac{s}{\sqrt{5}} \right), \\ N_2(s) &= \frac{1}{\sqrt{5}} \left(\sin \frac{2s}{\sqrt{5}}, -\cos \frac{2s}{\sqrt{5}}, -2 \sin \frac{s}{\sqrt{5}}, 2 \cos \frac{s}{\sqrt{5}} \right), \\ N_3(s) &= \frac{1}{\sqrt{17}} \left(\cos \frac{2s}{\sqrt{5}}, \sin \frac{2s}{\sqrt{5}}, -4 \cos \frac{s}{\sqrt{5}}, -4 \sin \frac{s}{\sqrt{5}} \right). \end{aligned}$$

By using the first case of Corollary 3 and taking $\varphi(s) = \sqrt{2}s$ and $\lambda = 1/\sqrt{2}$, we obtain

$$\begin{aligned} \gamma^*(s^*) &= \gamma^*(\varphi(s)) \\ &= \begin{pmatrix} \cos \frac{2s}{\sqrt{5}} - \frac{11}{12} \cos \frac{2s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} - \frac{\sqrt{85}}{12} \sin \frac{2s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}}, \\ \sin \frac{2s}{\sqrt{5}} - \frac{11}{12} \cos \frac{2s}{\sqrt{17}} \sin \frac{2s}{\sqrt{5}} + \frac{\sqrt{85}}{12} \cos \frac{2s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}}, \\ \cos \frac{s}{\sqrt{5}} - \frac{74}{3} \cos \frac{s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} - \frac{8\sqrt{85}}{3} \sin \frac{s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}}, \\ \sin \frac{s}{\sqrt{5}} - \frac{74}{3} \cos \frac{2s}{\sqrt{17}} \sin \frac{s}{\sqrt{5}} + \frac{8\sqrt{85}}{3} \cos \frac{s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}} \end{pmatrix}. \end{aligned}$$

with curvatures

$$\kappa_1^*(\varphi(s)) = \frac{1}{10\sqrt{17}} \left(17 - 6 \cos \frac{2s}{\sqrt{17}} \right),$$

$$\kappa_2^*(\varphi(s)) = \frac{1}{10\sqrt{17}} \sqrt{650 - \left(17 - 6 \cos \frac{2s}{\sqrt{17}} \right)^2}$$

and

$$\kappa_3^*(\varphi(s)) = \frac{\sqrt{17} \left(650 - \left(17 - 6 \cos \frac{2s}{\sqrt{17}} \right)^2 \right)}{12 \left(6 + 17 \cos \frac{2s}{\sqrt{17}} \right)}.$$

Moreover, Frenet vectors are obtained as follows:

$$T^*(\varphi(s)) = \begin{pmatrix} -\frac{2}{\sqrt{10}} \sin \frac{2s}{\sqrt{5}} + \frac{1}{\sqrt{10}} \sin \frac{2s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} - \frac{1}{\sqrt{34}} \sin \frac{2s}{\sqrt{17}} \cos \frac{2s}{\sqrt{5}}, \\ -\frac{1}{\sqrt{34}} \sin \frac{2s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}} + \frac{2}{\sqrt{10}} \cos \frac{2s}{\sqrt{5}} - \frac{1}{\sqrt{10}} \cos \frac{2s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}}, \\ -\frac{1}{\sqrt{10}} \sin \frac{s}{\sqrt{5}} - \frac{2}{\sqrt{10}} \sin \frac{s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} + \frac{2\sqrt{2}}{\sqrt{17}} \sin \frac{2s}{\sqrt{17}} \cos \frac{s}{\sqrt{5}}, \\ \frac{2\sqrt{2}}{\sqrt{17}} \sin \frac{s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}} + \frac{1}{\sqrt{10}} \cos \frac{s}{\sqrt{5}} + \frac{2}{\sqrt{10}} \cos \frac{s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} \end{pmatrix}$$

$$N_1^*(\varphi(s)) = \frac{1}{\sqrt{17}} \left(-4 \cos \left(\frac{2s}{\sqrt{5}} \right), -4 \sin \left(\frac{2s}{\sqrt{5}} \right), -\cos \left(\frac{s}{\sqrt{5}} \right), -\sin \left(\frac{s}{\sqrt{5}} \right) \right)$$

$$N_2^*(\varphi(s)) = \frac{10\sqrt{17}}{\sqrt{650 - \left(17 - 6 \cos \left(\frac{2s}{\sqrt{17}} \right) \right)^2}} \times \begin{pmatrix} \sqrt{\frac{32}{85}} \sin \frac{2s}{\sqrt{5}} + \frac{(17 - 6 \cos \frac{2s}{\sqrt{17}}) \left(-2 \sin \frac{2s}{\sqrt{5}} + \sin \frac{2s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} - \frac{\sqrt{10}}{\sqrt{34}} \sin \frac{2s}{\sqrt{17}} \cos \frac{2s}{\sqrt{5}} \right)}{10\sqrt{170}}, \\ -\sqrt{\frac{32}{85}} \cos \frac{2s}{\sqrt{5}} + \frac{(17 - 6 \cos \frac{2s}{\sqrt{17}}) \left(-\frac{\sqrt{10}}{\sqrt{34}} \sin \frac{2s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}} + 2 \cos \frac{2s}{\sqrt{5}} - \cos \frac{2s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} \right)}{10\sqrt{170}}, \\ \frac{1}{\sqrt{170}} \sin \frac{s}{\sqrt{5}} + \frac{(17 - 6 \cos \frac{2s}{\sqrt{17}}) \left(-\sin \frac{s}{\sqrt{5}} - 2 \sin \frac{s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} + \frac{2\sqrt{20}}{\sqrt{17}} \sin \frac{2s}{\sqrt{17}} \cos \frac{s}{\sqrt{5}} \right)}{10\sqrt{170}}, \\ -\frac{1}{\sqrt{170}} \cos \frac{s}{\sqrt{5}} + \frac{(17 - 6 \cos \frac{2s}{\sqrt{17}}) \left(\frac{2\sqrt{20}}{\sqrt{17}} \sin \frac{s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}} + \cos \frac{s}{\sqrt{5}} + 2 \cos \frac{s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} \right)}{10\sqrt{170}} \end{pmatrix}$$

and

$$N_3^*(\varphi(s)) = \frac{1}{2\sqrt{17} \sqrt{650 - \left(17 - 6 \cos \left(\frac{2s}{\sqrt{17}} \right) \right)^2}} \times \begin{pmatrix} 12 \cos \frac{2s}{\sqrt{5}} + 34 \cos \frac{2s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} + 2\sqrt{85} \sin \frac{2s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}}, \\ 12 \sin \frac{2s}{\sqrt{5}} + 34 \cos \frac{2s}{\sqrt{17}} \sin \frac{2s}{\sqrt{5}} - 2\sqrt{85} \cos \frac{2s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}}, \\ -48 \cos \frac{s}{\sqrt{5}} - 136 \cos \frac{s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} - 16\sqrt{85} \sin \frac{s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}}, \\ 16\sqrt{85} \cos \frac{s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}} - 136 \cos \frac{2s}{\sqrt{17}} \sin \frac{s}{\sqrt{5}} - 48 \sin \frac{s}{\sqrt{5}} \end{pmatrix}.$$

Thus, we obtain $N_1(s) = N_1^*(\varphi(s))$. Accordingly, γ is a Bertrand curve and γ^* is a Bertrand mate of γ . See Figure 2.

Corollary 4. In four-dimensional Euclidean space, a Bertrand curve has an infinite number of Bertrand mates.

Corollary 5. If a pair of Bertrand curves in four-dimensional Euclidean space satisfies Conditions (iii) given in Theorem 3, i.e., if the Bertrand curve pair is a Combescure-related curve pair, then the curvature functions of the curves satisfy the following relation:

$$\kappa_i^* = \frac{\kappa_i}{\varphi'}, \quad i = 1, 2, 3,$$

where $\varphi' = \varepsilon_1(1 + u' - v\kappa_1) > 0$ and $\varepsilon_1 = \text{sgn}(1 + u' - w_1\kappa_2)$.

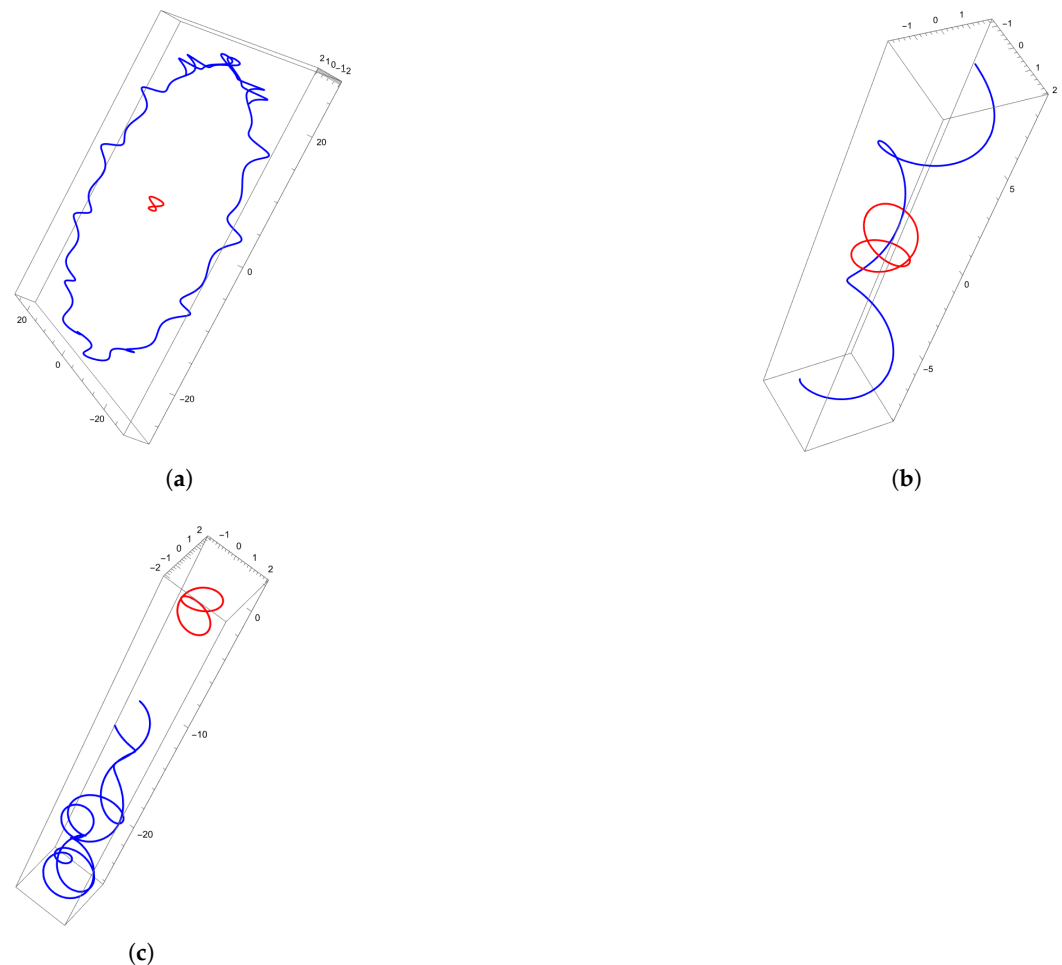


Figure 2. Orthogonal projections of the curves to the following spaces: $x_1 = 0$ (a), $x_3 = 0$ (b), and $x_4 = 0$ (c).

The following example is provided to illustrate the second and third cases of Theorem 3.

Example 3. Let us consider the curve γ given in Example 2 as the main curve. By using the second case of Corollary 3 and taking $\varphi(s) = 2s$, we obtain

$$\begin{aligned} \gamma^*(\varphi(s)) &= \int \left(\cos \frac{2s}{\sqrt{17}} N_2 - \sin \frac{2s}{\sqrt{17}} N_3 \right) 2ds \\ &= \begin{pmatrix} -\frac{11}{6} \cos \frac{2s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} - \frac{\sqrt{85}}{6} \sin \frac{2s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}}, \\ \frac{\sqrt{85}}{6} \cos \frac{2s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}} - \frac{11}{6} \cos \frac{2s}{\sqrt{17}} \sin \frac{2s}{\sqrt{5}}, \\ -\frac{16\sqrt{85}}{3} \sin \frac{s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}} - \frac{148}{3} \cos \frac{s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}}, \\ -\frac{148}{3} \sin \frac{s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} + \frac{16\sqrt{85}}{3} \sin \frac{2s}{\sqrt{17}} \cos \frac{s}{\sqrt{5}} \end{pmatrix} \end{aligned}$$

with the curvatures

$$\kappa_1^*(\varphi(s)) = \frac{3}{5\sqrt{17}} \cos \frac{2s}{\sqrt{17}},$$

$$\kappa_2^*(\varphi(s)) = \frac{1}{10\sqrt{17}} \sqrt{307 - 18 \cos \frac{4s}{\sqrt{17}}}$$

and

$$\kappa_3^*(\varphi(s)) = \frac{6\sqrt{17} \cos \frac{2s}{\sqrt{17}}}{307 - 18 \cos \frac{4s}{\sqrt{17}}}$$

Via direct calculations, we obtain the Frenet vectors as follows:

$$T^*(\varphi(s)) = \begin{pmatrix} \frac{1}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} \sin \frac{2s}{\sqrt{5}} - \frac{1}{\sqrt{17}} \cos \frac{2s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}}, \\ -\frac{1}{\sqrt{5}} \cos \frac{2s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} - \frac{1}{\sqrt{17}} \sin \frac{2s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}}, \\ \frac{4}{\sqrt{17}} \cos \frac{s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}} - \frac{2}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} \sin \frac{s}{\sqrt{5}}, \\ \frac{2}{\sqrt{5}} \cos \frac{s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} + \frac{4}{\sqrt{17}} \sin \frac{s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}} \end{pmatrix}$$

$$N_1^*(\varphi(s)) = \left(\frac{4}{\sqrt{17}} \cos \frac{2s}{\sqrt{5}}, \frac{4}{\sqrt{17}} \sin \frac{2s}{\sqrt{5}}, \frac{1}{\sqrt{17}} \cos \frac{s}{\sqrt{5}}, \frac{1}{\sqrt{17}} \sin \frac{s}{\sqrt{5}} \right)$$

$$N_2^*(\varphi(s)) = \frac{10\sqrt{17}}{\sqrt{307 - 18 \cos \frac{4s}{\sqrt{17}}}} \times \begin{pmatrix} \frac{3}{10\sqrt{85}} \cos \frac{4s}{\sqrt{17}} \sin \frac{2s}{\sqrt{5}} - \frac{3}{170} \cos \frac{2s}{\sqrt{5}} \sin \frac{4s}{\sqrt{17}} - \frac{37}{10\sqrt{85}} \sin \frac{2s}{\sqrt{5}}, \\ \frac{37}{10\sqrt{85}} \cos \frac{2s}{\sqrt{5}} - \frac{3}{170} \sin \frac{2s}{\sqrt{5}} \sin \frac{4s}{\sqrt{17}} - \frac{3}{10\sqrt{85}} \cos \frac{2s}{\sqrt{5}} \cos \frac{4s}{\sqrt{17}}, \\ \frac{6}{85} \cos \frac{s}{\sqrt{5}} \sin \frac{4s}{\sqrt{17}} - \frac{11}{10\sqrt{85}} \sin \frac{s}{\sqrt{5}} - \frac{3}{5\sqrt{85}} \cos \frac{4s}{\sqrt{17}} \sin \frac{s}{\sqrt{5}}, \\ \frac{11}{10\sqrt{85}} \cos \frac{s}{\sqrt{5}} + \frac{6}{85} \sin \frac{s}{\sqrt{5}} \sin \frac{4s}{\sqrt{17}} + \frac{3}{425} \sqrt{85} \cos \frac{s}{\sqrt{5}} \cos \frac{4s}{\sqrt{17}} \end{pmatrix}$$

$$N_3^*(\varphi(s)) = \frac{1}{\sqrt{5219 - 306 \cos \left(\frac{4s}{\sqrt{17}} \right)}} \times \begin{pmatrix} -\sqrt{85} \sin \frac{2s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}} - 17 \cos \frac{2s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}}, \\ \sqrt{85} \sin \frac{2s}{\sqrt{17}} \cos \frac{2s}{\sqrt{5}} - 17 \sin \frac{2s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}}, \\ 8\sqrt{85} \sin \frac{s}{\sqrt{5}} \sin \frac{2s}{\sqrt{17}} + 68 \cos \frac{s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}}, \\ 68 \sin \frac{s}{\sqrt{5}} \cos \frac{2s}{\sqrt{17}} - 8\sqrt{85} \sin \frac{2s}{\sqrt{17}} \cos \frac{s}{\sqrt{5}} \end{pmatrix}$$

Thus, we obtain $N_1(s) = -N_1^*(\varphi(s))$. Accordingly, γ is a Bertrand curve and γ^* is a Bertrand mate of γ .

Example 4. Let us consider the curve γ given in Example 2 as the main curve. If we take

$$u = \frac{4}{3}\sqrt{17}, \quad v = -\frac{10}{3}s, \quad w_1 = \sqrt{17}, \quad w_2 = -2s,$$

then Condition (iii) in Theorem 3 holds and we have

$$\begin{aligned} \gamma_1^*(s^*) = \gamma_1^*(\varphi(s)) &= \gamma(s) + \frac{4}{3}\sqrt{17}T(s) - \frac{10s}{3}N_1(s) + \sqrt{17}N_2(s) - 2sN_3(s) \\ &= \begin{pmatrix} \left(1 + \frac{2\sqrt{17}}{3}s\right) \cos \frac{2s}{\sqrt{5}} - \frac{\sqrt{85}}{3} \sin \frac{2s}{\sqrt{5}}, \\ \left(1 + \frac{2}{3}\sqrt{17}s\right) \sin \frac{2s}{\sqrt{5}} + \frac{\sqrt{85}}{3} \cos \frac{2s}{\sqrt{5}}, \\ \left(1 + \frac{2\sqrt{17}}{3}s\right) \cos \frac{1s}{\sqrt{5}} - \frac{2\sqrt{85}}{3} \sin \frac{1s}{\sqrt{5}}, \\ \left(1 + \frac{2\sqrt{17}}{3}s\right) \sin \frac{1s}{\sqrt{5}} + \frac{2\sqrt{85}}{3} \cos \frac{1s}{\sqrt{5}} \end{pmatrix} \end{aligned}$$

with the curvatures

$$\kappa_1^*(\varphi(s)) = \frac{\sqrt{17}}{5} \frac{3}{2\sqrt{17}s + 3},$$

$$\kappa_2^*(\varphi(s)) = \frac{6}{5\sqrt{17}} \frac{3}{2\sqrt{17}s + 3}$$

and

$$\kappa_3^*(\varphi(s)) = \frac{2}{\sqrt{17}} \frac{3}{2\sqrt{17}s + 3}.$$

Moreover, Frenet vectors are obtained as follows:

$$T^*(s^*) = T^*(\varphi(s)) = \left(-\frac{2}{\sqrt{5}} \sin \frac{2s}{\sqrt{5}}, \frac{2}{\sqrt{5}} \cos \frac{2s}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \sin \frac{s}{\sqrt{5}}, \frac{1}{\sqrt{5}} \cos \frac{s}{\sqrt{5}} \right),$$

$$N_1^*(s^*) = N_1^*(\varphi(s)) = \left(-\frac{4}{\sqrt{17}} \cos \frac{2s}{\sqrt{5}}, -\frac{4}{\sqrt{17}} \sin \frac{2s}{\sqrt{5}}, -\frac{1}{\sqrt{17}} \cos \frac{s}{\sqrt{5}}, -\frac{1}{\sqrt{17}} \sin \frac{s}{\sqrt{5}} \right),$$

$$N_2^*(s^*) = N_2^*(\varphi(s)) = \left(\frac{1}{\sqrt{5}} \sin \frac{2s}{\sqrt{5}}, -\frac{1}{\sqrt{5}} \cos \frac{2s}{\sqrt{5}}, -\frac{2}{\sqrt{5}} \sin \frac{s}{\sqrt{5}}, \frac{2}{\sqrt{5}} \cos \frac{s}{\sqrt{5}} \right),$$

$$N_3^*(s^*) = N_3^*(\varphi(s)) = \left(\frac{1}{\sqrt{17}} \cos \frac{2s}{\sqrt{5}}, \frac{1}{\sqrt{17}} \sin \frac{2s}{\sqrt{5}}, -\frac{4}{\sqrt{17}} \cos \frac{s}{\sqrt{5}}, -\frac{4}{\sqrt{17}} \sin \frac{s}{\sqrt{5}} \right).$$

It is clear that the γ curve and the γ_1^* curve are curves with parallel Frenet vectors. In other words, they are Combesure-related curves. It is clear that (γ, γ_1^*) is a pair of Bertrand curves.

The following graphs show the orthogonal projections of the curves: the main curve γ (red), Bertrand mate curve in Example 3 γ^* (blue), and Bertrand mate curve in Example 4 γ_1^* (green) on the $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ and $x_4 = 0$ spaces, respectively, which are located together. See Figure 3.

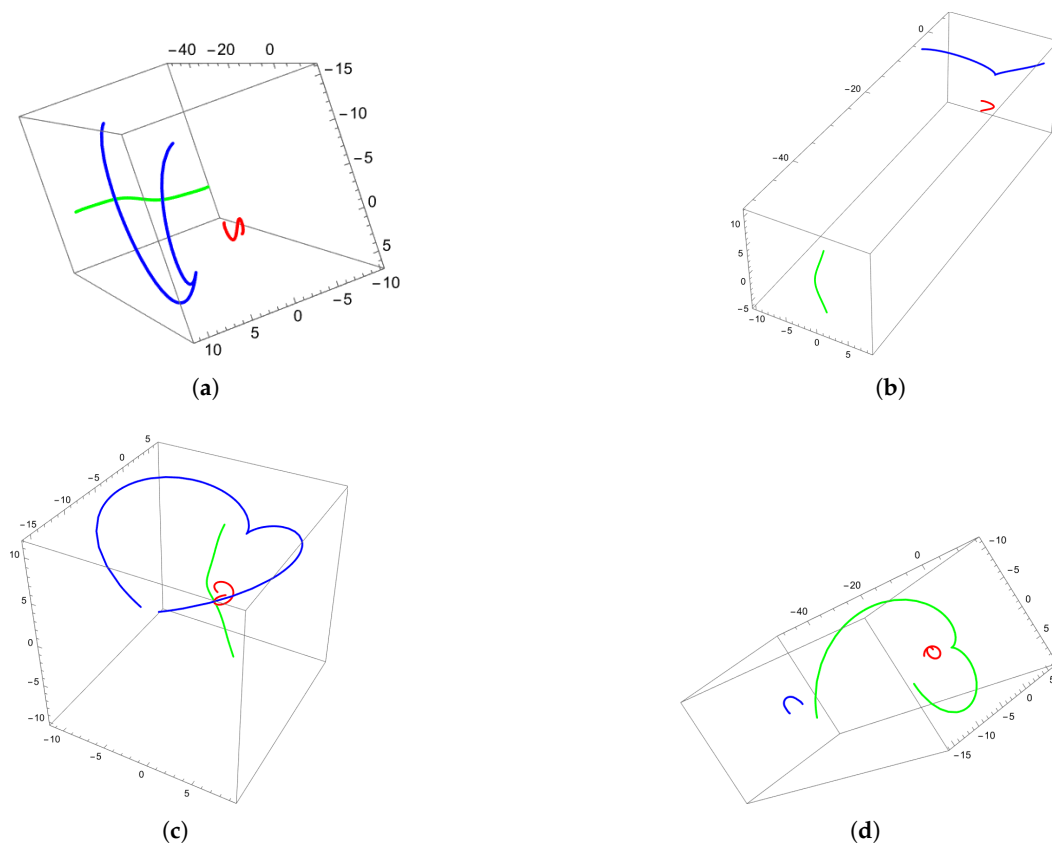


Figure 3. Orthogonal projections of the curves given in Examples 3 and 4 to the following spaces: $x_1 = 0$ (a), $x_2 = 0$ (b), $x_3 = 0$ (c), and $x_4 = 0$ (d).

4. Conclusions

In this study, the conditions for a curve to be a Bertrand curve in a four-dimensional Euclidean space were obtained by adhering to the classical definition of Bertrand curves (i.e., the curve pair sharing a common principal normal vector) and employing the new approach defined by Camcı et al. [17]. Furthermore, a relationship between Bertrand curves and Combescure-related curves was established. Derived Bertrand curve conditions were shown to transform into Combescure-related curve conditions under certain circumstances. Consequently, it was demonstrated that every pair of curves associated with a Combescure transformation constitutes a Bertrand curve pair. Moreover, several illustrative examples were provided to clarify the theoretical framework.

The findings of this study are expected to form a foundation for future research in various metric spaces, particularly in Minkowski spacetime, four-dimensional semi-Euclidean spaces with an index of two, and other spaces where the curve theory is extensively studied [25–27].

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