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Stability Properties of Distributional Solutions for Nonlinear Viscoelastic Wave Equations with Variable Exponents

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Abstract: A system of nonlinear wave equations in viscoelasticity with variable exponents is considered. It is assumed that the kernel included in the integral term of the equations depends on both the time and the spatial variables. Using the Faedo–Galerkin method and the contraction mapping principle, a theorem of unique solvability of the problem is proved. In addition, under appropriate variable assumptions, an estimate of the stability of the solution to the problem of determining the kernel is obtained. The study is based on Komornik's inequality. We expand the class of nonlinear boundary value problems that can be investigated by well-known methods.

Keywords: variable exponents; nonlinear equations; weak solutions; viscoelastic; Komornik's inequality

MSC: 35L35; 35B40; 35B44

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1. Introduction and Problem Statement

There are problems of reconstructing the properties of a medium from data obtained by the properties of viscoelasticity in a main model that arises in many areas of natural science. Recently, there has been an increase in interest in problems of determining the history of a medium in which a particular wave process occurs. This appears to be due to more accurate modeling of some physical fields, processes, or phenomena (electromagnetic, acoustic, seismic, thermal, etc.). Consideration of problems for hyperbolic equations with variable exponents is a relatively new direction in qualitative theory that has arose recently. Under certain conditions, Volterra operators of the convolution type are added to the equations of propagation of elastic and (or) electromagnetic waves, which describe the phenomena of memory.

Let $z = z(x,t)$ be an unknown function. In this paper, we examine the nonlinear viscoelastic hyperbolic problem with variable exponents

$$\begin{cases} z_{tt} + \Delta_x^2 \left[z_{tt} + z + \int_0^t \varpi(t-s) z(s) ds \right] + |z_t|^{\nu(x)-2} z_t = |z|^{r(x)-2} z, & \text{in } \Omega \times (0,\ T) \\ z(x,t) = \frac{\partial z}{\partial \eta}(x,t) = 0, & \text{in } \Gamma \times (0,\ T) \\ z(x,t=0) = z_0(x), z_t(x,t=0) = z_1(x), & \text{in } \Omega. \end{cases} \tag{1}$$

where $\Omega \in \mathbb{R}^n, n \geq 1$, is the bounded domain in $\mathbb{R}^n$, with smooth boundary $\partial\Omega = \Gamma$, η is the unit outer normal to $\partial\Omega$, and ϖ represents the memory kernel; it is a non-increasing

positive function defined in $\mathbb{R}_+$. Conditions (z_0, z_1) are the initial data that belong to a suitable function space. The variable exponents $\nu(.)$ and $r(.)$ are given measurable functions on Ω and satisfy

$$2 \leq \nu^- \leq \nu(x) \leq \nu^+ < \frac{2n}{n-2}, \; n \geq 3, \tag{2}$$

$$2 \leq r^- \leq r(x) \leq r^+ < \frac{2n}{n-2}, \; n \geq 3, \tag{3}$$

here

$$\nu^- := ess \inf_{x \in \bar{\Omega}} \nu(x), \; \nu^+ := ess \sup_{x \in \bar{\Omega}} \nu(x),$$

and r^-, r^+ are defined similarly.

Also $\nu(.), r(.)$ are continuous functions on $\overline{\Omega}$ with the logarithmic module of continuity such that

$$\forall (x,y) \in \Omega^2, \;\; |x-y| < 1, |\nu(x) - \nu(y)| + |r(x) - r(y)| \leq \vartheta(|x-y|), \tag{4}$$

where

$$\lim_{s \longrightarrow 0^+} sup \; \vartheta(s) ln(\frac{1}{s}) = C < \infty. \tag{5}$$

The first equation of problem (1) results from modeling several physical phenomena, including electro-rheological fluids, viscoelasticity, systems controlling the longitudinal motion of a viscoelastic configuration that obeys a nonlinear Boltzmann's model, viscoelastic fluids, temperature-dependent viscosity fluids, filtration processes through porous media, and image processing, resulting in equations with nonstandard growth conditions, that is, equations with variable exponents of nonlinearity. Additional information on these issues can be found in earlier research. Currently, this subject is a subject of scientific interest for many mathematicians (see [1–3]). In [4], a one-dimensional problem of finding a kernel from a single viscoelastic equation is considered in a bounded region with distributed sources of disturbances. Using the Fourier method, the problem is reduced to a system of integral equations of Volterra type for unknown functions depending on the time variable. However, in applications, the most interesting are inverse problems when the data of the direct problem are singular generalized functions. One such problem is considered in that paper.

When $\nu(x)$ and $r(x)$ are constants, ref. [5] discussed the case with the exponential ν and r being constants

$$v_{tt} - \Delta_x v + \int_0^t \varpi(t-\tau) \Delta_x v(\tau) d\tau + |v_t|^{\nu-2} v_t = |v|^{r-2} v.$$

It was shown that for $r > \nu$, any weak solution with negative initial energy blows up in finite time, and for $r \leq \nu$, the existence of a global solution was proved. In [1], the viscoelastic wave equation with power nonlinearity is considered and the local solution is constructed by the Faedo–Galerkin approximation scheme together with the contraction mapping theorem. Finally, the decay rate of the solution was discussed by assuming that the kernel function is convex. Also, Song [6] proved the blow-up of some solutions whose initial data arbitrarily had a high initial energy. Then, Song [7] studied the initial-boundary value problem

$$|v_t|^{\rho} v_{tt} - \Delta_x v + \int_0^t \varpi(t-\tau) \Delta_x v(\tau) d\tau + |v_t|^{\nu-2} v_t = |v|^{r-2} u,$$

and showed that if the initial energy is positive, then there exists no global solution. Cavalcanti, Domingos, and Ferreira [8] tackled the system

$$|v_t|^{\rho}v_{tt} - \Delta_x v - \Delta_x v_{tt} + \int_0^t \varpi(t-\tau)\Delta_x v(\tau)d\tau - \gamma - \Delta_x v_t = 0,$$

where the global existence of solutions was proved. In addition, the uniform decay of solution was obtained by assuming a strong damping $\Delta_x v_t$ acting in the domain and providing the relaxation function. Messaudi [9] established the existence of a unique weak local solution of the system

$$v_{tt} - \Delta_x v + a|v_t|^{\nu(x)-2}v_t = b|v|^{r(x)-2}v.$$

The local existence was established by using the Faedo–Galerkin method with certain conditions on the variable exponents $\nu(x)$ and $r(x)$. In addition, they proved that the local solution is global, and they obtained the stability estimate of the solution. Ding and Zhou [10] considered a Timoshenko-type equation

$$v_{tt} + \Delta_x^2 v_- M(\|\nabla_x v\|_2^2)\Delta_x v + |v_t|^{\nu(x)-2}v_t = |v_t|^{r(x)-2}v. \tag{6}$$

The existence of a weak solution to the problem (6) is proved by making use of the Faedo–Galerkin method together with the contraction mapping principle.

Our aim in the present work is to consider the following: In Section 2, we introduce and state certain notations and preliminary assumptions which will be used later. In Section 3, we prove the existence of a weak solution to the problem (1) using the well-known Faedo–Galerkin method to the extent that it complies with the contraction mapping principle. In Section 4, using Komornik's inequality, we establish the stability of the distributional solution under certain assumptions on the variable exponents $\nu(x)$ and $r(x)$.

2. Preliminary

In this section, necessary materials concerning Lebesgue and Sobolev spaces with variable exponents will be presented (see [11,12]).

Let Ω be a domain of $\mathbb{R}^n$ and let $\nu : \Omega \to [0,\infty]$ be a measurable function. We define the Lebesgue space with a variable exponent $\nu(.)$ as

$$L^{\nu(.)}(\Omega) = \Big\{v : \Omega \to \mathbb{R}, \textit{measurable in } \Omega : Q_{\nu(.)}(\varrho v) < \infty,\ \varrho > 0\Big\}, \tag{7}$$

here

$$Q_{\nu(.)}(v) = \int_\Omega |v(y)|^{\nu(y)}dy, \tag{8}$$

equipped with the next Luxembourg-type norm

$$\|v\|_{\nu(.)} = inf\Big\{\varrho > 0 : \int_\Omega \Big|\frac{v(y)}{\varrho}\Big|^{\nu(y)}dy \le 1\Big\}, \tag{9}$$

$L^{\nu(.)}(\Omega)$ is a Banach space.

Of course, the variable-exponent Sobolev $W^{1,\nu(.)}(\Omega)$ is defined by

$$W^{1,\nu(.)}(\Omega) = \Big\{v \in L^{\nu(.)}(\Omega),\ \nabla_x v \textit{ exists and } |\nabla_x v| \in L^{\nu(.)}(\Omega)\Big\}, \tag{10}$$

with respect to the norm

$$\|v\|_{W^{1,\nu(.)}(\Omega)} = \|v\|_{\nu(.)} + \|\nabla_x v\|_{\nu(.)}.$$

Let $W_0^{1,\nu(.)}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $W^{1,\nu(.)}(\Omega)$.

The dual of $W_0^{1,\nu(.)}(\Omega)$ is defined as $W^{-1,\nu'(.)}(\Omega)$, in the same way as the classical Sobolev spaces, where $\frac{1}{\nu(.)} + \frac{1}{\nu'(.)} = 1$ (see [13]).

When $1 < r^- \leq r^+ < \infty$, where $r^- := ess\inf_{x\in\bar{\Omega}} r(x), r^+ := ess\sup_{x\in\bar{\Omega}} r(x)$. As usual, we denote the conjugate exponent of $r(x)$ by $r'(x) = \frac{r(x)}{r(x)-1}$, and the Sobolev exponent by

$$r^*(x) = \begin{cases} \frac{nr(x)}{n-kr(x)} & \text{if } r(x) < n, \\ \infty & \text{if } r(x) \geq n, \end{cases} \tag{11}$$

where $k \in \mathbb{N}$.

Definition 1. *We say that the function*

$$u \in C\Big([0,T]; W_0^{1,\nu(.)}(\Omega)\Big) \cap C^1\Big([0,T]; L^2(\Omega)\Big),$$

is a distributed solution to (1) *on the interval* $[0,T]$*, if*

$$\begin{aligned} &\int_\Omega z_t(x,t)\phi(x)dx + \int_0^t \left[\int_\Omega \Delta z(x,s)\Delta\phi(x)dx + \int_\Omega |z_t|^{\nu(x)-2} z_t(x,s)\phi(x)dx\right] ds \\ &+ \int_0^t \int_\Omega \int_0^s \varpi(t-\tau)\Delta z(x,\tau)\Delta\phi(x)d\tau dxds + \int_0^t \int_\Omega \Delta z_{tt}(x,s)\Delta\phi(x)dxds \\ &= \int_\Omega z_1\phi(x)dx + \int_0^t \int_\Omega |z|^{r(x)-2} z(x,s)\phi(x)dxds \end{aligned} \tag{12}$$

holds for every test functions $\phi \in W_0^{1,\nu(.)}(\Omega)$, $\forall 0 \leq t \leq T$.

Lemma 1 ([13])**.** *Let* Ω *be a bounded domain of* $\mathbb{R}^n$ *and* $\nu(.)$ *satisfies* (4)*, then*

$$\|v\|_{\nu(.)} \leq c\|\nabla_x v\|_{\nu(.)}, \ \forall v \in W_0^{1,\nu(.)}(\Omega),$$

where $c > 0$ *depends only on* Ω*. In addition, the space* $W_0^{1,\nu(.)}(\Omega)$ *has an equivalent norm*

$$\|v\|_{W_0^{1,\nu(.)}(\Omega)} = \|\nabla_x u\|_{\nu(.)}.$$

Proposition 1. *Suppose that* $\nu(.), r(.) \in C(\overline{\Omega})$*. Assume that* $\nu(.)$ *and* $r(.)$ *satisfy the log-Hölder continuity condition*

$$|\nu(a) - \nu(b)| \leq \frac{A}{|\log(|a-b|)|}, \forall a, b \in \overline{\Omega}, \text{ such that } 0 < |a-b| < 1, \tag{13}$$

$\forall a, b \in \Omega$, $A > 0$*, then* $\forall (z_0, z_1) \in W_0^{1,\nu(.)}(\Omega) \times L^2(\Omega)$ *the system* (1) *has a unique distributional solution according to Definition* (1)*, such that*

$$\begin{aligned} &z \in L^\infty((0,T), W_0^{1,\nu(.)}(\Omega)), \\ &z_t \in L^\infty((0,T), L^2(\Omega) \cap L^{\nu(.)}(\Omega), \\ &z_{tt} \in L^2((0,T), W_0^{-1,\nu'(.)}(\Omega)). \end{aligned}$$

Suppose that $\nu(.)$ satisfies (13), then

$$\|v\|_{\nu(.)} \leq c\|\nabla_x v\|_{\nu(.)} \qquad \forall v \in W_0^{1,\nu(.)}(\Omega), \tag{14}$$

where $c = c(r^-, \nu^+, |\Omega|) > 0$.

If $p(.) \in C(\overline{\Omega}, [1, \infty))$, and

$$2 \leq r^- \leq r(x) \leq p^+ \leq \frac{2n}{n-2}, \qquad n \geq 3, \tag{15}$$

is satisfied, then the embedding $H_0^1(\Omega) \hookrightarrow L^{r(.)}(\Omega)$ is continuous.

If $p^+ < \infty$ and $p \in C(\overline{\Omega}, [1, \infty))$ are measurable, then $C_0^\infty(\Omega)$ is dense in $L^{r(.)}(\Omega)$. Let $r(x), q(x) \in C^+$, where

$$C^+(\overline{\Omega}) = \{f \in C(\Omega); minf(x) > 1\},$$

assume that $q(x) < r^*(x)$, where the next is a compact and continuous embedding

$$W^{k,r(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega).$$

Lemma 2 ((Hölder's inequality) [14])**.**

1. *Let $p, q, s \geq 1$ be measurable functions defined on Ω and*

$$\frac{1}{s(x)} = \frac{1}{p(x)} + \frac{1}{q(x)}, \qquad \text{for a.e } x \in \Omega, \tag{16}$$

if $f \in L^{r(.)}(\Omega)$ and $g \in L^{q(.)}(\Omega)$, then

$$\|f.g\|_{s(.)} \leq \|f\|_{r(.)} \|g\|_{q(.)}.$$

2. *For each $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$, then*

$$\begin{aligned} \Big| \int_\Omega f g dx \Big| &\leq \Big(\frac{1}{p^-} + \frac{1}{q^-} \Big) \|f\|_p \|g\|_q \\ &\leq 2 \|f\|_p \|g\|_q. \end{aligned} \tag{17}$$

Lemma 3. *(Young's inequality) Let $\varrho_1 > 1, \varrho_2 < \infty, \frac{1}{\varrho_1} + \frac{1}{\varrho_2} = 1$ and $a, b \geq 0$ for all $\eta > 0$, then*

$$ab \leq \eta a^{\varrho_1} + c_\eta b^{\varrho_2}, \qquad \text{such that} \qquad c_\eta = \frac{1}{\varrho_2 (\eta \varrho_1)^{\frac{\varrho_2}{\varrho_1}}}, \tag{18}$$

and if $\varrho_1 = \varrho_2 = 2$, then

$$ab \leq \eta a^2 + \frac{1}{4\eta} b^2. \tag{19}$$

Lemma 4. *If*

$$1 \leq \nu^- := ess \inf_{x \in \Omega} \nu(x) \leq \nu(x) \leq \nu^+ := ess \sup_{x \in \Omega} \nu(x) < \infty,$$

then we have

$$\min \Big\{ \|z\|_{\nu(.)}^{\nu^-}, \|z\|_{\nu(.)}^{\nu^+} \Big\} \leq \rho_{\nu(.)}(z) \leq \max \Big\{ \|z\|_{\nu(.)}^{\nu^-}, \|z\|_{\nu(.)}^{\nu^+} \Big\}, \tag{20}$$

for any $z \in L^\nu(\Omega)$.

Assume that the functional $\varpi \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ satisfies

$$\varpi(0) > 0, \varpi'(s) \leq 0, 1 - \int_0^t \varpi(s) ds = l > 0, \tag{21}$$

and

$$\varpi(t) \geq 0, \varpi'(t) \leq 0, \varpi(t) \leq -\delta\varpi'(t), \qquad \forall t \geq 0, \forall \delta > 0, \tag{22}$$

here

$$(\varpi \circ z) = \int_\Omega \int_0^t \varpi(t-s)|z(s) - z(t)|^2 ds dx.$$

Lemma 5 ([2]). *For $\varpi, \Psi \in C^1([0, +\infty[, R)$, we have*

$$\begin{aligned}\int_\Omega \int_0^t \varpi(t-s)\Psi(s)\Psi_t(t) ds dx &= -\frac{1}{2}\varpi(t)\|\Psi(t)\|_2^2 + \frac{1}{2}(\varpi' \circ \Psi)(t) \\ &= -\frac{1}{2}\frac{d}{dt}\left[(\varpi \circ \Psi)(t) - \left(\int_0^t \varpi(s) ds\right)\|\Psi\|_2^2\right].\end{aligned}$$

Lemma 6. *Let ϖ satisfy (21) and (22), then, for $\Psi \in H_0^1(\Omega)$, we have*

$$\int_\Omega \left(\int_0^t \varpi(t-s)(\Psi(t) - \Psi(s)) ds\right)^2 dx \leq c(\varpi \circ \nabla\Psi)(t),$$

and

$$\int_\Omega \left(\int_0^t \varpi'(t-s)(\Psi(t) - \Psi(s)) ds\right)^2 dx \leq -c(\varpi' \circ \nabla\Psi)(t).$$

We introduce and show our main result, including the contribution of the memory term using integral inequalities. We define

$$\begin{aligned}E(t) = &\frac{1}{2}\|z_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t \varpi(s) ds\right)\|\Delta_x z\|_2^2 + \frac{1}{2}\|\Delta_x z_t\|_2^2 + \frac{1}{2}(\varpi \circ \Delta_x z) \\ &- \int_\Omega \frac{1}{r(x)}|z|^{r(x)} dx,\end{aligned} \tag{23}$$

Lemma 7. *Let z be a solution of (1) and assume that (21) and (22) are satisfied. Then, the energy functional $E(t)$ given by (23) satisfies*

$$E'(t) = -\int_\Omega |z_t|^{\nu(x)} dx + \frac{1}{2}(\varpi' \circ \Delta_x z) - \frac{1}{2}\varpi(t)\|\Delta_x z\|_2^2 \leq 0, \quad \forall t \geq 0, \tag{24}$$

Proof. Multiplication of $(1)_1$ by z_t, and applying integration over Ω leads to

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\|z_t\|_2^2 + \frac{1}{2}\|\Delta_x z_t\|_2^2 + \frac{1}{2}\|\Delta_x z\|_2^2\right] - \int_\Omega z_t|v|^{r(x)-2}v dx \\ = &\int_\Omega \int_0^t \varpi(t-s)\Delta_x z(s)\Delta_x z_t(t) ds dx - \int_\Omega |z_t|^{\nu(x)} dx.\end{aligned} \tag{25}$$

Owing to Lemma 5, the term on the RHS of (25) can be rewritten as

$$\begin{aligned}&\int_\Omega \int_0^t \varpi(t-s)\Delta_x z(s)\Delta_x z_t(s) ds dx \\ = &\frac{1}{2}\frac{d}{dt}\left[\int_0^t \varpi(s)\|\Delta_x z(s)\|_2^2 - (\varpi \circ \Delta_x z)(t)\right] \\ - &\frac{1}{2}\varpi(t)\|\Delta_x z\|_2^2 + \frac{1}{2}(\varpi' \circ \Delta_x z)(t).\end{aligned} \tag{26}$$

Combining (25), (26) and using (21) and (22) give (7), which concludes the proof. □

Lemma 8 ([15]). *Let $E : \mathbb{R}^+ \to \mathbb{R}^+$ be a nonincreasing function assuming that there are constants ω, ζ such that*

$$\int_{\sigma}^{+\infty} E^{\omega+1}(t)dt \leq \frac{1}{|\Omega|}E^{\omega}(0)E(\sigma) = cE(\sigma), \qquad \forall \sigma > 0, \tag{27}$$

then

$$\begin{cases} E(t) \leq \dfrac{cE(0)}{(1+t)^{\frac{1}{\omega}}}, & t > 0, \text{ if } \omega > 0, \\ E(t) \leq cE(0)e^{-\zeta t}, & t > 0, \text{ if } \omega = 0. \end{cases} \tag{28}$$

3. Local Existence of Weak Solution

We use the Faedo–Galerkin method together with the contraction mapping principle to prove the existence of a weak solution for problem (1) in this section. Fix $T > 0$, and consider

$$V = C(0, T, H_0^2(\Omega)) \cap C^1(0, T, L^2(\Omega)),$$

equipped by the norm

$$\|z\|_V^2 = \max(\|\Delta_x z_t\|_2^2 + e\|\Delta_x z\|_2^2).$$

We study the local existence of the problem (1), for $z \in V$. To this end, we first fixed $z = v \in C(0, T, H_0^2(\Omega))$ in the source to obtain the related problem (29) and we will prove the local existence of this problem by using the Faedo–Galerkin method.

$$\begin{cases} z_{tt} + \Delta_x^2\left[z_{tt} + z - \int_0^t \varpi(t-s)z(s)ds\right] + |z_t|^{\nu(x)-2}z_t = |v|^{r(x)-2}v, \\ z(x,t) = \frac{\partial z}{\partial \eta}(x,t) = 0, \\ z(x,t=0) = z_0(x), z_t(x,t=0) = z_1(x). \end{cases} \tag{29}$$

Then, using the well-known contraction mapping theorem, we can show the local existence of (1). The following technical lemma will play an important role in the sequel.

Lemma 9 ([2]). *For any $z \in C^1(0, T, H_0^2(\Omega))$, we have*

$$\begin{aligned} & \int_\Omega \int_0^t \varpi(t-s)\Delta_x z(s)z'(t)dsdx \\ = \; & \frac{1}{2}\frac{d}{dt}(\nabla_x \varpi(t-s)z(s))(t) - \frac{1}{2}\frac{d}{dt}[\int_0^t \varpi(s)\int_\Omega |\nabla_x z(t)|^2 dxds] \\ - \; & \frac{1}{2}(\varpi \circ \nabla_x z)(t) + \frac{1}{2}\varpi(t)\int_\Omega |\nabla_x z(t)|^2 dxds, \end{aligned} \tag{30}$$

where

$$(\varpi \circ z)(t) = \int_0^t \varpi(t-s)\int_\Omega |z(s) - z(t)|^2 dxds.$$

Local Existence Result

Theorem 1. *Let $(z_0, z_1) \in H_0^2(\Omega) \times L^2(\Omega)$ be given, suppose that (2) and (3) hold. Then, under conditions (21) and (22), the problem (1) has a unique local solution $z(t, x) \in V$ for T small enough.*

The proof of Theorem 1 will be established through several lemmas.

Lemma 10. *Let $(z_0, z_1) \in H_0^2(\Omega) \times L^2(\Omega)$, assume that (2) and (3) hold. Then, under conditions (21) and (22), there exists a unique weak solution $z \in C(0, T, H_0^2(\Omega))$ of the problem (29) for any $v \in C(0, T, H_0^2(\Omega))$ given.*

The proof of Lemma 10 follows the techniques of Lions [16] to deal with nonlinear damping. Our main tool is the Faedo–Galerkin method, which consists of constructing approximations of the solutions. Then, we obtain a prior estimate necessary to guarantee the convergence of the approximations.

Lemma 11. *Suppose that (2), (3), (21) and (22) hold. Let* $(z_0, z_1) \in H_0^2(\Omega) \times L^2(\Omega)$ *for all* $T > 0$, $v \in C(0, T, H_0^2(\Omega))$. *Thus, there exists* $z \in V \cap C^2(0, T, H_0^2(\Omega))$, *with* $z_{tt} \in L^2(0, T, L^2(\Omega))$ *satisfying the system* (29).

Proof. The existence of a distributional solution is ensured by the standard of the Faedo–Galerkin method. Introducing a sequence function $\{w_j\}_{j=1}^{\infty}$ as an orthogonal basis of $H_0^2(\Omega)$ with w_j yields:

$$-\Delta_x w_j = \varrho_j w_j, \qquad x \in \Omega, \qquad w_j = 0, \qquad x \in \partial\Omega.$$

Let $V_k = span\{w_1, \ldots, w_k\}$ be the subspace generated by the first k vectors of the basis $\{w_j\}_j^{\infty}$. We can choose $z_{n0},\ z_{n1} \in [w_1, ..., w_n]$ such that

$$z_{n0} \equiv \sum_{j=1}^{n} \alpha_{jn} w_j \longrightarrow z_0, \tag{31}$$

$$z_{n1} \equiv \sum_{j=1}^{n} \beta_{jn} w_j \longrightarrow z_1, \tag{32}$$

where

$$\alpha_{jn} = \int_\Omega z_0 w_j dx,$$

$$\beta_{jn} = \int_\Omega z_1 w_j dx.$$

By normalization, we have $\|w_j\|_2 = 1$, and $\forall k \geq 1$; we have k functions

$$c_1^k(t), c_2^k(t), ..., c_k^k(t) \in C^2([0, T]),$$

such that

$$z_k = \sum_{j=1}^{k} c_j^k w_j,$$

satisfies the following equation

$$\begin{aligned} & \left(z_{tt}^k, w_j\right) + \left(\Delta_x z_{tt}^k, \Delta_x w_j\right) + \left(\Delta_x z^k, \Delta_x w_j\right) \\ - & \int_0^t \varpi(t-s)\left(\Delta_x z^k(s), \Delta_x w_j\right) ds + \left(z_t^k |z_t^k|^{\nu(x)-2}, w_j\right) \\ = & \left(v^k |v^k|^{r(x)-2}, w_j\right). \end{aligned} \tag{33}$$

The system (33) becomes a nonlinear system of ordinary differential equations and it will be associated with IC

$$z^k(x, t=0) = z_0^k = \sum_{j=1}^{k} \omega_j^k(t) w_j \to z_0 \ \text{ when } \ k \to \infty \text{ in } H_0^2(\Omega), \qquad \text{for any } w_j \in V_k, \tag{34}$$

and

$$z_t^k(x,t=0)=z_1^k=\sum_{j=1}^{k}\chi_j^k(t)w_j\to z_1 \text{ when } k\to\infty \text{ in } H^2(\Omega), \qquad \text{for any } w_j\in V_k. \quad (35)$$

Thus, (33) generates the initial value problem for the system of second-order differential equations with respect to $c_j^k(t)$

$$(1+\lambda_j^2)c_{jtt}^k(t)+\lambda_j^2c_j^k(t)=G_j(c_{1t}^k(t)),...,c_{kt}^k(t))+\varpi_j(c_i^k(t)), \quad j=1,2,...,k,$$

$$c_i^k(0)=\int_\Omega z_0w_jdx, c_{it}^k(0)=\int_\Omega z_1w_jdx, \quad j=1,2,...,k.$$

where

$$G_j(c_{1t}^k(t),..,c_{kt}^k(t))=-\int_\Omega\Big|\sum_{i=1}^{k}c_{it}^k(t)w_i(x)\Big|^{\nu(x)-2}\sum_{i=1}^{k}c_{it}^k(t)w_i(x)w_j(x)dx,$$

and

$$\varpi_j(c_j^k(t))=\lambda_j^2\int_0^t\varpi(t-s)c_j^k(s)ds+|v|^{r(x)-2}vw_jdx.$$

Then, for any given $w^k\in span\{w_1,w_2,w_3,...,w_k\}$, the Equation (33) holds.

By the theory of the ordinary equation differential system, we show that problem (29) admits a local unique solution c_j^k in $[0,t_k]$, where $t_k>0$. Then, we can obtain an approximate solution $z^k(t)$ for (1), in V_k, over $[0,t_k)$.

First estimate. Multiplying (33) by $(c_j^k(t))_t$ and adding with respect to j, will give

$$\begin{aligned}&\frac{d}{dt}\left[\frac{1}{2}\left\|z_t^k\right\|_2^2+\frac{1}{2}\left\|\Delta_xz_t^k\right\|_2^2+\frac{1}{2}\left\|\Delta_xz^k\right\|_2^2\right]\\ -&\int_\Omega\int_0^t\varpi(t-s)\Delta_xz^k(s)\Delta_xz_t^k(s)dsdx+\int_\Omega\left|z_t^k\right|^{\nu(x)}dx\\ =&\int_\Omega z_t^k|v|^{r(x)-2}vdx. \end{aligned}\quad (36)$$

By simple calculation, we have

$$\begin{aligned}&-\int_0^t\varpi(t-s)\int_\Omega\Delta_xz^k(s)\Delta_xz_t^k(s)dxds \quad (37)\\ =&\frac{1}{2}\frac{d}{dt}(\varpi\circ\Delta_xz^k)-\frac{1}{2}(\varpi'\circ\Delta_xz^k)-\frac{1}{2}\frac{d}{dt}\int_0^t\varpi(s)ds\|\Delta_xz^k\|_2^2+\frac{1}{2}\varpi(s)\|\Delta_xz^k\|_2^2,\end{aligned}$$

where

$$(\varpi\circ\Delta_xz)(t)=\int_0^t\varpi(t-s)\|z(s)-z(t)\|_2^2ds,$$

inserting (2) into (1), using Hölder's inequality and Young's inequality, we obtain

$$\begin{aligned}&\frac{d}{dt}[\frac{1}{2}\|z_t^k\|_2^2+\frac{1}{2}\|\Delta_xz_t^k\|_2^2+\frac{1}{2}(\varpi\circ\Delta_xz^k)+\frac{1}{2}(1-\int_0^t\varpi(s)ds\|\Delta_xz^k\|_2^2)]\\ =&\frac{1}{2}(\varpi'\circ\Delta_xz^k)-\frac{1}{2}\varpi(t)\|\Delta z^k\|_2^2+\int_\Omega\|v\|^{r(x)-2}vz_t^kdx-\int_\Omega\|z_t^k\|^\nu dx\\ \le&\int_\Omega\|v\|^{r(x)-2}vz_t^kdx\le\||v|^{r(x)-2}v\|_2\|z_t^k\|_2\\ \le&\frac{n}{2}\int_\Omega\|v\|^{2(r(x)-1)}dx+\frac{1}{2n}\|z_t^k\|_2^2, \end{aligned}\quad (38)$$

using the embedding $H_0^2(\Omega) \hookrightarrow L^{2(r(x)-1)}(\Omega)$, we obtain

$$\begin{aligned} \int_\Omega |v|^{2(r(x)-1)}dx &\leq max\{\|v\|^{2(r^--1)}_{2(r(x)-1)}, \|v\|^{2(r^+-1)}_{2(r(x)-1)}\} \\ &\leq Cmax\{\|\Delta v\|_2^{2(r^--1)}, \|v\|_2^{2(r^+-1)}\} \\ &\leq C, \end{aligned} \tag{39}$$

where C is a positive constant. We denote by C various positive constants that may be different at different occurrences.

Combining (38) and (39), we obtain

$$\begin{aligned} &\frac{d}{dt}[\frac{1}{2}\|z_t^k\|_2^2 + \frac{1}{2}\|\Delta_x z_t^k\|_2^2 + \frac{1}{2}(\varpi \circ \Delta z^k) + \frac{1}{2}\Big(1 - \int_0^t \varpi(s)ds\|\Delta_x z^k\|_2^2\Big)] \\ \leq\ & \frac{n}{2}C + \frac{1}{2n}\|z_t\|_2^2, \end{aligned} \tag{40}$$

By Gronwall's inequality, there exists a positive constant C_T such that

$$\|z_t^k\|_2^2 + \|\Delta_x z_t^k\|_2^2 + (\varpi \circ \Delta_x z^k) + e\|\Delta z_t^k\|_2^2 \leq C_T. \tag{41}$$

Therefore, there exists a subsequence of $\{z^k\}_{k=1}^\infty$ such that

$$z^k \hookrightarrow z \text{ is weakly in } L^\infty\Big(0, T; H_0^2(\Omega)\Big), \tag{42}$$

$$\begin{aligned} &z_t^k \hookrightarrow z_t \text{ is weakly in } L^\infty\Big(0, T; L^2(\Omega)\Big), \\ &z^k \hookrightarrow z \text{ is weakly in } L^2\Big(0, T; H_0^2(\Omega)\Big), \end{aligned}$$

$$z_t^k \hookrightarrow z_t \text{ is weakly in } L^2\Big(0, T; H_0^2(\Omega)\Big). \tag{43}$$

Second estimate. In this part, multiplying (33) by $(c_i^k(t))_{tt}$, then summing over j, will result in the following:

$$\begin{aligned} &\|z_{tt}^k\|_2^2 + \|\Delta_x z_{tt}^k\|_2^2 - \int_0^t \varpi(t-s)\int_\Omega \Delta_x z^k \Delta_x z_{tt}^k dsdx + \frac{d}{dt}\Big(\frac{1}{\nu(x)}|z_t^k|^{\nu(x)}\Big) \\ &= -\int_\Omega \Delta_x z^k \Delta_x z_{tt}^k dx + \int_\Omega v^k |v^k|^{r(x)-2} z_{tt}^k dx. \end{aligned} \tag{44}$$

Note that, Young's inequality gives

$$\Big|\int_\Omega \Delta_x z^k \Delta_x z_{tt}^k dx\Big| \leq \eta\|\Delta_x z_{tt}^k\|_2^2 + \frac{1}{4\eta}\|\Delta_x z^k\|_2^2, \quad \eta > 0,$$

$$\int_\Omega |v|^{r(x)-2} v z_{tt}^k dx \leq \||v|^{r(x)-2}v\|_2 \|z_{tt}^k\|_2 \leq \eta\|z_{tt}^k\|_2^2 + \frac{1}{4\eta}\int_\Omega |v|^{2(r(x)-1)}dx,$$

and

$$\begin{aligned} &\Big|\int_0^t \varpi(t-s)\int_\Omega \Delta_x z^k(s)\Delta_x z_{tt}^k(t)dxds\Big| \leq \eta\|z_{tt}^k\|_2^2 + \frac{1}{4\eta}\int_\Omega\Big(\int_0^t \varpi(t-s)\Delta_x z^k(s)ds\Big)^2 dx \\ &\leq \eta\|\Delta_x z_{tt}^k\|_2^2 + \frac{1}{4\eta}\int_0^t \varpi(\tau)d\tau \int_0^t \varpi(t-s)\int_\Omega \Big|\Delta_x z^k(s)\Big|^2 dxds \\ &\leq \eta\|\Delta_x z_{tt}^k\|_2^2 + \frac{(1-l)\varpi(0)}{4\eta}\int_0^t \|\Delta_x z^k(s)\|_2^2 ds, \end{aligned}$$

from $H_0^2(\Omega) \hookrightarrow L^2(\Omega)$, we have

$$\int_\Omega |v|^{r(x)-2} v z_{tt}^k dx \leq \eta C \|z_{tt}^k\|_2^2 + \frac{C}{4\eta}, \tag{45}$$

taking into (44) and (45), we obtain

$$\begin{aligned} &\|z_{tt}^k\|_2^2 + (1 + 2\eta - C\eta)\|\Delta z_{tt}^k\|_2^2 + \frac{d}{dt}\Big(\int_\Omega \frac{1}{\omega(x)} |z_t^k|^{\omega(x)} dx\Big) \\ &\leq \frac{1}{4\eta}\Delta_x z_{tt}^k\|_2^2 + \frac{(1-l)\omega(0)}{4\eta}\int_0^t \|\Delta_x z^k(s)\|_2^2 ds + \frac{C}{4\eta}, \end{aligned} \tag{46}$$

integrating (46) over $(0; t)$, we obtain

$$\begin{aligned} &\int_0^t \|z_{tt}^k\|_2^2 + (1 + 2\eta - C\eta)\int_0^t \|\Delta z_{tt}^k\|_2^2 + \int_\Omega \frac{1}{\omega(x)} |z_t^k|^{\omega(x)} dx \\ &\leq \frac{1}{4\eta}\int_0^t (|\Delta_x z_{tt}^k\|_2^2 + \int_0^s \|\Delta_x z^k(s)\|_2^2 ds) dt + C_T, \end{aligned} \tag{47}$$

taking η small enough in (47), for some positive constant C_T, we obtain

$$\int_0^t \|z_{tt}^k\|_2^2 + \int_0^t \|\Delta z_{tt}^k\|_2^2 \leq C_T, \tag{48}$$

we observe that estimate (48) implies that there exists a subsequence of $\{z^k\}_{k=1}^\infty$ such that

$$z_{tt}^k \rightharpoonup z_{tt} \quad \text{weakly star in} \quad L^2(0, T, H_0^2(\Omega)), \tag{49}$$

In addition, from (42), we have

$$(z_{tt}^k, w_i) = \frac{d}{dt}(z_t^k, w_i) \hookrightarrow \frac{d}{dt}(z_t, w_i) = (z_{tt}, w_i) \quad \text{weakly star in} \quad L^\infty(0, T, H^{-2}(\Omega)) \tag{50}$$

Next, we will deal with the nonlinear term. Combining (42), (49), and the Aubin–Lions theorem [16], we deduce that there exists a subsequence of $\{z^k\}_{k=1}^\infty$ such that

$$z_t^k \rightharpoonup z_t \quad \text{strongly in} \quad L^2(0, T, L^2(\Omega)),$$

then,

$$|z_t^k|^{\omega(x)-2} z_t^k \rightharpoonup |z_t|^{\omega(x)-2} z_t, \qquad \forall (x, t) \in \Omega \times (0, T), \tag{51}$$

using the embedding $H_0^2(\Omega) \hookrightarrow L^{2(\omega(x)-1)}(\Omega)$, we have

$$\||z_t^k|^{\omega(x)-2} z_t^k\|_2^2 = \int_\Omega |z_t^k|^{\omega(x)-2} dx \leq \max\{\|\Delta z_t^k\|^{2(\omega^+-1)}, \|\Delta z_t^k\|^{2(\omega^--1)}\} \leq C, \tag{52}$$

using (51) and (52), we obtain

$$|z_t^k|^{\omega(x)-2} z_t^k \rightharpoonup |z_t|^{\omega(x)-2} z_t \quad \text{weakly star in} \quad L^\infty(0, T, L^2(\Omega)), \tag{53}$$

Setting up $k \to \infty$ in (33) combining with (42), (49), (50) and (53), we obtain

$$\begin{aligned} & (z_{tt}, w) + (\Delta_x z_{tt}, \Delta_x w) + (\Delta_x z, \Delta_x w) \\ - & \int_0^t \omega(t-s)(\Delta_x z(s), \Delta_x w) ds + \Big(z_t |z_t|^{\nu(x)-2}, w\Big) \\ = & \Big(v|v|^{r(x)-2}, w\Big). \end{aligned}$$

To handle the initial conditions. From (42) and the Aubin–Lions theorem, we can easily obtain $z^k \to z$ in $C(0,T,L^2(\Omega))$ and we also have that $z^k(0) = z_0$ in $H_0^2(\Omega)$, so $z(0) = z_0$ in $H_0^2(\Omega)$. Similarly, we obtain $z_t(0) = z_1$. Otherwise

$$\begin{aligned}\||z^k|^{r(x)-2}z^k z_{tt}^k\| &\le b\eta\|z_{tt}^k\|_2^2 + \frac{1}{4\eta}\||z^k|^{r(x)-2}z^k\|_2^2\\ &\le \eta\|z_{tt}^k\|_2^2 + \frac{1}{4\eta}\int_\Omega\left(|z^k|^{r(x)-2}z^k\right)^2 dx.\end{aligned}$$

We have

$$\begin{aligned}\|z_{tt}^k\|_2^2 &\le C^2\|\Delta_x z_{tt}^k\|_2^2 \le C^4\|\Delta_x z_{tt}^k\|_2^2,\\ \int_\Omega\left(|z^k|^{r(x)-2}z^k\right)^2 dx &= \int_\Omega |z^k|^{2(r(x)-1)}dx\\ &\le \max\left\{\int_\Omega |z^k|^{2(r^--1)}dx, \int_\Omega |z^k|^{2(r^+-1)}dx\right\}\\ &\le \max\left\{C^{*\frac{1}{2(r^--1)}}\|\nabla_x z^k\|^{\frac{1}{2(r^--1)}}, C^{*\frac{1}{2(r^--1)}}\|\nabla_x z^k\|^{\frac{1}{2(r^--1)}}\right\},\end{aligned} \tag{54}$$

where C and C^* are embedding constants. Thus, from (44)–(54), it can be observed that

$$\begin{aligned}&\left\|z_{tt}^k\right\|^2 + (1-2\eta c_r^2 - \eta b C_r^2)\left\|\Delta_x z_{tt}^k\right\|_2^2 + \frac{d}{dt}\left(\frac{1}{\nu(x)}\left|u_t^k\right|^{\nu(x)}\right)\\ &\le \frac{1}{4\eta}\|\Delta_x z_k\|_2^2 + \frac{(1-l)\varpi(0)}{4\eta}\int_0^t\|\Delta_x z_k(s)\|_2^2 ds\\ &+ \max\left\{C^*\frac{1}{2(r^--1)}\|\Delta_x z_k\|^{\frac{1}{r^--1}}, C^{*\frac{1}{2(r^+-1)}}\|\Delta_x z_k\|^{\frac{1}{r^+-1}}\right\}.\end{aligned} \tag{55}$$

Integrating (55) over $(0,t)$ yields

$$\begin{aligned}&\int_0^t\left\|z_{tt}^k\right\|^2 ds + (1-2\eta-\alpha\eta C)\int_0^t\left\|\Delta_x z_{tt}^k\right\|_2^2 ds + \int_\Omega\frac{1}{\nu(x)}\left|z_t^k\right|^{\nu(x)}dx\\ &\le \frac{1}{4\eta}(C_2 + (1-l)\varpi(0)T) + C_3,\end{aligned} \tag{56}$$

where $C_3 > 0$ is a constant that depends only on $\|z_1\|_{H_0^1}$.

Setting b, η to be small enough in (56), gives the estimate

$$\int_0^t\|z_{tt}^k\|_2^2 ds + \int_\Omega\frac{1}{\nu(x)}|z_t^k|^{\nu(x)}dx \le C_4.$$

Hence, by Lemma 1, we have

$$\int_0^t\|z_{tt}^k\|_2^2 ds + \min\left\{\frac{1}{\nu^+}\|z_t^k\|_{\nu(x)}^{\nu^-}, \frac{1}{\nu^+}\|z_t^k\|_{\nu(x)}^{\nu^+}\right\} \le C_4, \tag{57}$$

where $C_4 > 0$ is a constant that depends only on $\|z_0\|_{H_0^2}, \|z_1\|_{H_0^2}, l, \varpi(0)$, and T. From the estimate (57), we obtain

$$z_{tt}^k \text{ is uniformly bounded in } L^2(0,T;L^2(\Omega)).$$

From (57), it can be deduced that there exist a subsequence $\{z^\mu\}$ of $\{z^k\}$ and a function u such that

$$\begin{aligned} z^k &\rightharpoonup u && \text{weakly star in } L^\infty\Big(0,T;H_0^2(\Omega)\Big),\\ z_t^k &\rightharpoonup z_t && \text{weakly star in } L^\infty\Big(0,T;L^2(\Omega)\Big),\\ z_{tt}^k &\rightharpoonup z_{tt} && \text{weakly star in } L^2(0,T;L^2(\Omega)). \end{aligned}$$

Next, by making use of Lion's Lemma we have

$$\Big|v^k\Big|^{r(x)-1}v^k \to |v|^{r(x)-1}v \text{ weakly in } L^2\Big(0,T;L^2(\Omega)\Big),$$

then,

$$\Big|v^k\Big|^{r(x)-1}v^k \to |v|^{r(x)-1}v \text{ almost in } \Omega\times(0,T).$$

Passing to the limit and using the first and second a priori estimates to find $z\in V$, with fixed v in the source.

Uniqueness. Assume that the first equation of problem (29) has two solutions z', z'', then $u = z' - z''$ satisfies the system and $u = 0$. This completes the proof of Lemma 11. □

Proof. (Of Theorem 1) Let $(z_0, z_1) \in H_0^2(\Omega)\times H_0^2(\Omega)$, and

$$R^2 = \Big(\|\Delta z_0\|_2^2 + \|z_1\|_2^2\Big).$$

For any $T > 0$, consider

$$M_T = \Big\{z\in V; z(0) = z_0, z_t(0) = z_1 \text{ and } \|z\|_V \le R\Big\}.$$

Let

$$\Phi: \quad M_T \to M_T$$

$$v \mapsto z = \Phi(v).$$

We will prove this point.

- $\Phi(M_T)\subseteq M_T$.
- Φ is a contraction in M_T.

For any $v\in M_T$, we define $z = \Phi(v)$, the unique solution of problem (29). We claim that, for a suitable $T > 0$, Φ is a contractive map satisfying

$$\Phi(M_T)\subseteq M_T.$$

Multiplying the first equation of the Problem (29) by z_t and integrating it over (Ω), we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}[\|z_t(t)\|_2^2 + \|\Delta_x z_t\|_2^2 + (1-\int_0^t \varpi(s)ds)\|\Delta_x z\|_2^2 + (\varpi\circ\Delta_x z)] \\ &+\frac{1}{2}\varpi(t)\|\Delta_x z\|_2^2 - \frac{1}{2}(\varpi'\circ\Delta_x z) + \int_\Omega |z_t|^{\nu(x)}dx = \frac{1}{2}\int_\Omega |v|^{r(x)-2}vz_t dx, \end{aligned} \tag{58}$$

let $v \in M_T$, the corresponding solution $z = \Phi(v)$, integrating (58) over $[0, T]$, we obtain

$$\begin{aligned} & \|z_t(t)\|_2^2 + \|\Delta z_t(t)\|_2^2 + (\varpi \circ \Delta_x z_t)(t) + \|\Delta z(t)\|_2^2 \\ \leq \quad & \|z_1\|_2^2 + \|\Delta z_1\|_2^2 + \|\Delta z_0\|_2^2 + 2\int_0^t \int_\Omega |v(s)|^{r-2} v(s) z_t(t) dx ds, \end{aligned} \tag{59}$$

we estimate the last term in the right-hand side as follows: by Hölder's and Young's inequalities, we have

$$|\int_\Omega |v(s)|^{r-2} v(s) z_t(t) dx| \leq \Gamma \|z_t\|_2^2 + \frac{C}{\Gamma}[\|\Delta z\|_2^{2r^- - 2} + \|\Delta z\|_2^{2r^+ - 2}],$$

then (59) becomes

$$\|z_t(t)\|_2^2 + \|\Delta z_t(t)\|_2^2 + e\|\Delta z(t)\|_2^2 \leq \lambda_0 + 2\Gamma T \sup \|z_t\|_2^2 + \frac{TC}{2\Gamma} \sup[\|\Delta v v\|_2^{2r^- - 2} + \|\Delta v\|_2^{2r^+ - 2}],$$

Hence, we obtain

$$\begin{aligned} & \sup\|z_t(t)\|_2^2 + \sup\|\Delta z_t(t)\|_2^2 + e \sup\|\Delta z(t)\|_2^2 \\ \leq \quad & \lambda_0 + 2\Gamma T \sup \|z_t\|_2^2 + \frac{TC}{2\Gamma} \sup[\|\Delta v\|_2^{2r^- - 2} + \|\Delta v\|_2^{2r^+ - 2}], \end{aligned}$$

where $\lambda_0 = \|z_1(\|_2^2 + \|\Delta z_1\|_2^2 + \|\Delta z_0\|_2^2$, choose $\Gamma = \frac{1}{T}$ such that

$$\|z\|_V^2 \leq \lambda_0 + T^2 C \sup[\|\Delta v\|_2^{2r^- - 2} + \|\Delta v\|_2^{2r^+ - 2}],$$

for all $v \in M_T$, by choosing R large enough so

$$\|z\|_V^2 \leq \lambda_0 + 2T^2 C R^{2(r^+ - 1)} \leq R^2,$$

choosing T sufficiently small, we obtain $\|z\|_V \leq R$, which shows that

$$\Phi(M_T) \subseteq M_T.$$

Now, we prove that Φ is a contraction in M_T. Taking v_1 and v_2 in M_T, subtracting the two equations in (29), for $z_1 = \Phi(v_1)$ and $z_2 = \Phi(v_2)$, and setting $z = v_1 - v_2$, we obtain

$$\begin{cases} z_{tt} + \Delta_x^2 \left[z_{tt} + z - \int_0^t \varpi(t-s) z(s) ds \right] + |z_{1t}|^{\nu(x)-2} z_{1t} - |z_{2t}|^{\nu(x)-2} z_{2t} \\ = |v_1|^{r(x)-2} v_1 - |v_2|^{r(x)-2} v_2, \\ \\ z(x,t) = \frac{\partial z}{\partial \eta}(x,t) = 0, \\ z(x, t=0) = 0, z_t(x, t=0) = 0 \end{cases} \tag{60}$$

Multiplying by z_t and integrating over $\Omega \times (0, t)$, we obtain

$$\begin{aligned} & \frac{1}{2}\|z_t\|_2^2 + \frac{1}{2}(1 - \int_0^t \varpi(s) ds)\|\Delta z\|_2^2 + \frac{1}{2}\|\Delta z_t\|_2^2 + \frac{1}{2}(\varpi \circ \Delta z) dx ds \\ + \quad & \int_0^t \int_\Omega [|z_{1t}|^{\nu(x)-2} z_{1t} - |z_{2t}|^{\nu(x)-2} z_{2t}](z_{1t} - z_{2t}) \\ \leq \quad & \int_0^t \int_\Omega [|v_1|^{r(x)-2} v_1 - |v_2|^{r(x)-2} v_2] z_t dx ds, \end{aligned}$$

then,

$$\frac{1}{2}\|z_t\|_2^2+\frac{1}{2}\|\Delta z\|_2^2+\frac{1}{2}\|\Delta z_t\|_2^2+\frac{1}{2}(\varpi\circ\Delta z)\leq\int_0^t\int_\Omega[|v_1|^{r(x)-2}v_1-|v_2|^{r(x)-2}v_2]z_t dxds, \quad (61)$$

we evaluate

$$I=\int_\Omega||v_1|^{r(x)-2}v_1-|v_2|^{r(x)-2}v_2||z_t|dx,$$

where $v=v_1-v_2$ and $\kappa=\sigma v_1+(1-\sigma)v_2$, $0\leq\sigma\leq 1$, using Young's inequality and Hölder's inequality, we obtain

$$\begin{aligned}
I &\leq \frac{\varsigma}{2}\|z_t\|_2^2+\frac{(r^+-1)^2}{2\varsigma}\int_\Omega|\kappa|^{2(r-2)}|v|^2dx \quad (62)\\
&\leq \frac{\varsigma}{2}\|z_t\|_2^2+\frac{(r^+-1)^2}{2\varsigma}\Big(\int_\Omega|v|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}}\Big(\int_\Omega|\kappa|^{n(r-2)}dx\Big)^{\frac{2}{n}}\\
&\leq \frac{\varsigma}{2}\|z_t\|_2^2+\frac{(r^+-1)^2}{2\varsigma}\Big(\int_\Omega|v|^{\frac{2n}{n-2}}dx\Big)^{\frac{n-2}{n}}\Big[\Big(\int_\Omega|\kappa|^{n(r^+-2)}dx\Big)^{\frac{2}{n}}+\Big(\int_\Omega|\kappa|^{n(r^--2)}dx\Big)^{\frac{2}{n}}\Big]\\
&\leq \frac{\varsigma}{2}\|z_t\|_2^2+\frac{(r^+-1)^2C}{2\varsigma}\|\Delta v\|_2^2[\|\Delta\kappa\|_2^{2(r^+-2)}+\|\Delta\kappa\|_2^{2(r^--2)}]\\
&\leq \frac{\varsigma}{2}\|z_t\|_2^2+\frac{(r^+-1)^2C}{2\varsigma}\|\Delta v\|_2^2[R^{2(r^+-2)}+R^{2(r^--2)}],
\end{aligned}$$

inserting (62) into (61), choosing ς small enough, we obtain

$$\|z\|_V^2\leq\frac{(r^+-1)^2CT}{\varsigma}[R^{2(r^+-2)}+R^{2(r^--2)}]\|v\|_V^2,$$

It is easy to see that

$$\|z(t,.)\|_V^2=\|\Phi(v_1)-\Phi(v_2)\|_V^2\leq\alpha\|v_1-v_2\|_V^2. \quad (63)$$

Finally by the contraction mapping theorem together with (63), we obtain that there exists a unique weak solution $z\in V$ to our problem (1) defined on $[0,T]$. The main statement of Theorem 1 is proved. □

4. Exponential and Polynomial Stability

In this section, the Komornik's inequality [15] is used. Also, the next lemma is needed.

Lemma 12. *Let the assumptions of Theorem 1 hold and $\nu^-,\nu^+>r^+$, then $\exists C>0$ such that*

$$\int_\Omega|z|^{\nu(x)}dx\leq CE(t). \quad (64)$$

Proof.

$$\begin{aligned}
\int_\Omega|z|^{\nu(x)}dx &= \max\Big\{\|z\|_{\nu(.)}^{\nu^-},\|z\|_{\nu(.)}^{\nu^+}\Big\}\\
&\leq \max\Big\{c_*^{\nu^-}\|\Delta_x z\|_{r^+}^{\nu^-},c_*^{\nu^+}\|\Delta_x z\|_{r^+}^{\nu^+}\Big\} \quad (65)\\
&\leq \max\Big\{c_*^{\nu^-}\|\Delta_x z\|_{r^+}^{\nu^--r^+},c_*^{\nu^-}\|\Delta_x z\|_{r^+}^{\nu^+-r^+}\Big\}\times\|\Delta_x z\|_{r^+}^{r^+},
\end{aligned}$$

using (7) gives

$$\int_\Omega|z|^{\nu(x)}dx\leq CE(t).$$

□

The initial energy lies in the positive as well as nonpositive state. For this task, we define

$$E(t) = \frac{1}{2}\|z_t\|_2^2 + \frac{1}{2}\left(1 - \int_0^t \varpi(s)ds\right)\|\Delta_x z\|_2^2 + \frac{1}{2}\|\Delta_x z_t\|_2^2 + \frac{1}{2}(\varpi \circ \Delta_x z) - \int_\Omega \frac{1}{r(x)}|z|^{r(x)}dx, \tag{66}$$

By the definition of $E(t)$, we also have

$$E'(t) = -\int_\Omega |z_t|^{\nu(x)}dx + \frac{1}{2}(\varpi' \circ \Delta_x z) - \frac{1}{2}\varpi(t)\|\Delta_x z\|_2^2 \leq 0. \tag{67}$$

Theorem 2. *Suppose that assumptions of Theorem 1 hold, then* $\exists \mu, \xi > 0$ *positive constants, such that*

$$\begin{cases} E(t) \leq \dfrac{\mu}{(1+t)^{\frac{1}{q}}}, & \forall t \geq 0, \qquad q > 2. \\ E(t) \leq \mu e^{-\xi t}, & \forall t \geq 0, \qquad q = 2. \end{cases} \tag{68}$$

Proof. Multiplication of $(1)_1$ by $zE^q(t)$, where $q > 0$, and integration over $\Omega \times (\sigma, T)$, lead to

$$\begin{aligned} &\int_\sigma^T \int_\Omega E^q(t)[zz_{tt} + z\Delta_x^2 z + z\Delta_x^2 z_{tt} - z\int_0^t \varpi(t-s)\Delta_x^2 z(s)ds + z|z_t|^{\nu(x)-2}z_t]dxdt \\ &= \int_\sigma^T E^q(t)\int_\Omega |z|^{r(x)}dxdt, \end{aligned} \tag{69}$$

Therefore,

$$\begin{aligned} &\int_\sigma^T \int_\Omega E^q(t)[(zz_t)_t - z_t^2 + z\Delta_x^2 z(1 - \int_0^t \varpi(t-s)ds) + z\Delta_x^2 z_{tt} + z|z_t|^{\nu(x)-2}z_t] \\ = &\int_\sigma^T E^q(t)\int_\Omega |z|^{r(x)}dxdt. \end{aligned} \tag{70}$$

Applying (66) to the third term on the left hand side of (70), we obtain

$$\begin{aligned} &\int_\sigma^T E^q(t)\int_\Omega [(zz_t)_t - z_t^2 + 2E(t) - \|\Delta z_t\|_2^2 - \|z_t\|_2^2 - (\varpi \circ \Delta z) + z\Delta_x^2 z_{tt} + z|z_t|^{\nu(x)-2}z_t] \\ = &\int_\sigma^T E^q(t)\int_\Omega \frac{r-2}{2}|z|^{r(x)}dxdt. \end{aligned} \tag{71}$$

By definition of the energy $E(t)$ and the next statement

$$\frac{d}{dt}\left(E^q(t)\int_\Omega zz_t\right)dx = qE^{q-1}(t)E'(t)\int_\Omega zz_t dx + E^q(t)\int_\Omega (zz_t)_t dx, \tag{72}$$

the Equation (71) becomes

$$\begin{aligned} &2\int_\sigma^T E^{q+1}(t) = q\int_\sigma^T E^{q-1}(t)E'(t)\int_\Omega z_t z dxdt - \int_\sigma^T \frac{d}{dt}[E^q \int_\Omega zz_t dx]dt + \int_\sigma^T E^q(t)\int_\Omega z_t^2 dxdt \\ + &\int_\sigma^T E^q(t)\int_\Omega \|z_t\|_2^2 dxdt + \int_\sigma^T E^q(t)\int_\Omega \|\Delta z_t\|_2^2 dxdt + \int_\sigma^T E^q(t)\int_\Omega (\varpi \circ \Delta z)dxdt \\ - &\int_\sigma^T E^q(t)\int_\Omega z\Delta_x^2 z_{tt}dxdt - \int_\sigma^T E^q(t)\int_\Omega z|z_t|^{\nu(x)-2}z_t dxdt \\ + &\int_\sigma^T E^q(t)\int_\Omega \frac{r-2}{2}|z|^{r(x)}dxdt. \end{aligned} \tag{73}$$

We will estimate the terms in the right-hand side of (73) as follows by using Young's inequality (18) and (24); we have

$$\begin{aligned} I_1 &= \int_\sigma^T E^{q-1}(t)E'(t)\int_\Omega z_t z dx dt \le \int_\sigma^T E^{q-1}(t)(-E'(t))[\eta\|z\|_r^r + c_\eta c_r\|z_t\|_2^2]dt \\ &\le (\eta + c_\eta)q_1\int_\sigma^T E^q(t)(-E'(t))dt \le q_1[E^{q+1}(\sigma) - E^{q+1}(T)] \le q_1 E^q(0)E(\sigma). \end{aligned}$$

and

$$\begin{aligned} I_2 &= -\int_\sigma^T \frac{d}{dt}[E^q\int_\Omega z z_t dx]dt = E^q(\sigma)\int_\Omega z(\sigma)z_t(\sigma)dx - E^q(T)\int_\Omega z(T)z_t(T)dx \\ &\le E^q(\sigma)\int_\Omega |z(\sigma)z_t(\sigma)|dx + E^q(T)\int_\Omega |z(T)z_t(T)|dx \\ &\le q_2[E^{q+1}(\sigma) + E^{q+1}(T)] \le q_2 E^{q+1}(\sigma) \le q_2 E^q(0)E(\sigma). \end{aligned}$$

Similarly to (74), we have

$$\begin{aligned} I_3 &= \int_\sigma^T E^q(t)\|z_t\|_2^2 dt \le c_2^2\int_\sigma^T E^q(t)\|\nabla z_t\|_2^2 dt \\ &\le q_3 E^q(0)E(\sigma). \end{aligned} \tag{74}$$

Using Young's inequality

$$\begin{aligned} I_4 &= \int_\sigma^T E^q(t)\int_\Omega \|\Delta z_t\|_2^2 dt \le \int_\sigma^T E^q(t)[\eta\|\Delta z_t\|_2 + \frac{1}{4\eta}\|\Delta z_t\|_2]dt \\ &\le c_\eta^1 E^q(0)E(\sigma) \le q_4 E^q(0)E(\sigma). \end{aligned} \tag{75}$$

and

$$\begin{aligned} I_5 &= \int_\sigma^T E^q(t)\int_\Omega (\varpi \circ \Delta z)dx dt \\ &\le \int_\sigma^T E^q(t)\int_\Omega |\int_0^t \varpi(t-s)|ds\|\Delta z(t) - \Delta z(\sigma)\|_2^2 dx dt \\ &\le c\int_\sigma^T E^q(t)|(\varpi \circ \Delta z)|dt \le q_5 E^q(0)E(\sigma). \end{aligned} \tag{76}$$

Using Young's inequality

$$\begin{aligned} I_6 &= \int_\sigma^T E^q(t)\int_\Omega z\Delta^2 z_{tt} dx dt \le \int_\sigma^T E^q(t)[\eta\|z\|_2^2 + \frac{1}{4\eta}\|\Delta z_{tt}\|_2^2]dt \\ &\le C_\eta E^q(0)E(\sigma) \le q_6 E^q(0)E(\sigma). \end{aligned} \tag{77}$$

We have also

$$\begin{aligned} I_7 &= \int_\sigma^T E^q(t)\int_\Omega \frac{r-2}{r}|z|^r|dx dt \le c_r\int_\sigma^T E^q(t)\|z\|_r dt \\ &\le q_7 E^q(0)E(\sigma). \end{aligned} \tag{78}$$

The following Young's inequality is used for the third term

$$XY \le \frac{\xi}{\varrho_1}X^{\varrho_1} + \frac{1}{\varrho_2\xi^{\frac{\varrho_2}{\varrho_1}}}Y^{\varrho_2}, X, Y \ge 0, \xi > 0, \frac{1}{\varrho_1} + \frac{1}{\varrho_2} = 1, \tag{79}$$

with $\varrho_1(x) = \nu(x), \varrho_2 = \frac{\nu(x)}{\nu(x)-1}$. Then, by Lemma 7 and Lemma 12, it can be observed that

$$\begin{aligned} I_8 &= -\int_\sigma^T E^q(t) \int_\Omega z z_t |z_t|^{\nu(x)-2} dx dt \\ &\leq \int_\sigma^T E^q(t) \left(\eta c \int_\Omega |z|^{\nu(x)} dx + c_\eta \int_\Omega |z_t|^{\nu(x)} dx \right) dt \\ &\leq \eta c \int_\sigma^T E^q(t) \int_\Omega |z|^{\nu(x)} dx dt + c_\eta \int_\sigma^T E^q(-E'(t)) dt \\ &\leq \eta c \int_\sigma^T E^{q+1}(t) dt + q_8 E^q(0) E(\sigma). \end{aligned} \tag{80}$$

Regarding the third term of (73), we obtain

$$\begin{aligned} I_9 &= \int_\sigma^T E^q(t) \int_\Omega |z_t|^2 dx dt \\ &\leq \int_\sigma^T E^q(t) \left(\int_{\Omega^-} |z_t|^2 dx + \int_{\Omega^+} |z_t|^2 dx \right) dt \\ &\leq c \int_\sigma^T E^q(t) \left[\left(\int_{\Omega^-} |z_t|^{\nu^+} dx \right)^{\frac{2}{\nu^+}} + \left(\int_{\Omega^+} |z_t|^{\nu^-} dx \right)^{\frac{2}{\nu^-}} \right] dt \\ &\leq c \int_\sigma^T E^q(t) \left[\left(\int_{\Omega^-} |z_t|^{\nu(x)} dx \right)^{\frac{2}{\nu^+}} + \left(\int_{\Omega^+} |z_t|^{\nu(x)} dx \right)^{\frac{2}{\nu^-}} \right] dt. \end{aligned} \tag{81}$$

This implies

$$\begin{aligned} &\int_\sigma^T E^q(t) \int_\Omega |z_t|^2 dx dt \\ &\leq c \int_\sigma^T E^q(-E'(t))^{\frac{2}{\nu^+}} dt + c \int_\sigma^T E^q(-E'(t))^{\frac{2}{\nu^-}} dt. \end{aligned} \tag{82}$$

Using the Young's inequality with

$$\varrho_1 = (q+1)/q,$$

and

$$\varrho_2 = q+1,$$

to obtain

$$\int_\sigma^T E^q(-E'(t))^{\frac{2}{\nu^+}}(t) dt \leq \eta c \int_\sigma^T E^{q+1}(t) dt + c_\eta \int_\sigma^T (-E'(t))^{\frac{2(q+1)}{\nu^+}}(t) dt. \tag{83}$$

Taking $q = \frac{\nu^+}{2} - 1$, will result in

$$\int_\sigma^T E^q(-E'(t))^{\frac{2}{\nu^+}} dt \leq \eta c \int_\sigma^T E^{q+1}(t) dt + c_\eta \int_\sigma^T (-E'(t)) dt,$$

which implies

$$\int_\sigma^T E^q(-E'(t))^{\frac{2}{\nu^+}} dt \leq \eta c \int_\sigma^T E^{q+1}(t) dt + c_\eta E(\sigma). \tag{84}$$

Otherwise, we have

$$\int_\sigma^T E^q(-E'(t))^{\frac{2}{\nu^-}} dt \leq \eta c \int_\sigma^T E^{q+1} dt + c_\eta E(\sigma). \tag{85}$$

At this stage, we distinguish two cases:

- If $\nu^- = 2$, then

$$\int_\sigma^T E^q(-E'(t))^{\frac{2}{\nu^-}}dt \leq cE(\sigma) \leq \eta c \int_\sigma^T E^{q+1}(t)dt + c_\eta E(\sigma).$$

- If $\nu^- > 2$, thanks to the Young's inequality, we have

$$\begin{aligned}\int_\sigma^T E^q(-E'(t))^{\frac{2}{\nu^-}}dt &\leq \eta c \int_\sigma^T E^{q\frac{\nu^-}{\nu^- -2}}(t)dt + c_\eta \int_\sigma^T (-E'(t))dt \\ &\leq \eta c \int_\sigma^T E^{q\frac{\nu^-}{\nu^- -2}}(t)dt + c_\eta E(\sigma). \end{aligned} \tag{86}$$

We denote that

$$q\frac{\nu^-}{\nu^- - 2} = q + 1 + \frac{\nu^- - \nu^+}{\nu^- - 2},$$

then,

$$\begin{aligned}\int_\sigma^T E^q(-E'(t))^{\frac{2}{\nu^-}}dt &\leq \eta c E((\sigma))^{\frac{\nu^- - \nu^+}{\nu^- - 2}} \int_\sigma^T E^{q+1}(t)dt + c_\eta E(\sigma) \\ &\leq \eta c (E(0))^{\frac{\nu^- - \nu^+}{\nu^- - 2}} \int_\sigma^T E^{q+1}(t)dt + c_\eta E(\sigma) \\ &\leq \eta c \int_\sigma^T E^{q+1}(t)dt + c_\eta E(\sigma). \end{aligned} \tag{87}$$

By substituting (82) and (84) in (85), we obtain

$$\int_\sigma^T E^q(t) \int_\Omega |z_t|^2 dxdt \leq \eta c \int_\sigma^T E^{q+1}(t)dt + c_\eta E(\sigma). \tag{88}$$

By inserting (74)–(88) in (73), we obtain

$$(2 - 2\eta c) \int_\sigma^T E^{q+1}(t)dt \leq (q_1 + q_2 + q_3 + q_4 + q_5 + q_6 + q_7 + q_8 + c_\eta) E^q(0)E(\sigma).$$

Choosing η small enough yields such that $2 - 2c\eta > 0$

$$\int_\sigma^T E^{q+1}(t)dt \leq cE(\sigma).$$

When T goes to ∞, we obtain

$$\int_\sigma^\infty E^{q+1}(t)dt \leq cE(\sigma).$$

Therefore, Komornik's Lemma provides the desired result. □

5. Conclusions

The article is devoted to the expansion of classes of nonlinear boundary value problems, for which it is possible to obtain qualitative results by methods known and well tested since the works of Lions and his predecessors. We considered a certain system of nonlinear wave equations with variable exponents, which models processes in viscoelastic media. The theorem on the unique solvability of the problem was proved, and an estimate of the stability of the solution to the kernel determination problem was obtained. The study is based on the Faedo–Galerkin method with the application of the Komornik inequality. Here, we see that the kernel function depends on time and enters Equation $(1)_1$ through the functions ϖ. To determine it, additional conditions (21) and (22) are set with respect

to the solution to the problem. The theorem on the unique solvability of the problem is proved by using the Faedo–Galerkin method combined with the contraction mapping principle, where compactness arguments are employed to handle variable exponents. In addition, the conditions on the variable exponents $\nu(.)$ and $r(.)$ are given in (2) and (3). An estimate of the stability of the solution to the problem is obtained. The study is based on Komornik's inequality.

Challenges and Open Problems

1. It will be very interesting to consider and study the same model with nonlinear averaged damping $\|z_t\|^{\nu(x)-2}z_t$ instead of classical nonlinear damping $|z_t|^{\nu(x)-2}z_t$.
2. The qualitative properties of the stability can be improved according to the rate of the kernel ϖ as in [1].

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