*axioms*

MDPI

*Article*

# Nonlocal Conformable Differential Inclusions Generated by Semigroups of Linear Bounded Operators or by Sectorial Operators with Impulses in Banach Spaces

**Faryal Abdullah Al-Adsani [1] and Ahmed Gamal Ibrahim [1,2,*]**

[1] Department of Mathematics and Statistics, College of Science, King Faisal University, Al-Ahsa 31982, Saudi Arabia; faladsani@kfu.edu.sa
[2] Department of Mathematics, College of Science, Cairo University, Cairo 12613, Egypt
* Correspondence: agamal@kfu.edu.sa or agamal2000@yahoo.com

**Abstract:** This paper aims to explore sufficient conditions for the existence of mild solutions to two types of nonlocal, non-instantaneous, impulsive semilinear differential inclusions involving a conformable fractional derivative, where the linear part is the infinitesimal generator of a $C_0$-semigroup or a sectorial operator and the nonlinear part is a multi-valued function with convex or nonconvex values. We provide a definition of the mild solutions, and then, by using appropriate fixed-point theorems for multi-valued functions and the properties of both the conformable derivative and the measure of noncompactness, we achieve our findings. We did not assume that the semigroup generated by the linear part is compact, and this makes our work novel and interesting. We give examples of the application of our theoretical results.

**Keywords:** differential inclusions; infinitesimal generator of a $C_0$-semigroup; sectorial operator; conformable fractional derivative; instantaneous and non-instantaneous impulses; mild solutions; measure of noncompactness

**MSC:** 34A08; 26A33

check for updates

Academic Editor: Trushit Patel

Received: 5 February 2025
Revised: 13 March 2025
Accepted: 18 March 2025
Published: 21 March 2025

**Citation:** Al-Adsani, F.A.; Ibrahim, A.G. Nonlocal Conformable Differential Inclusions Generated by Semigroups of Linear Bounded Operators or by Sectorial Operators with Impulses in Banach Spaces. *Axioms* **2025**, *14*, 230. https://doi.org/10.3390/axioms14040230

**Copyright:** © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

## 1. Introduction

Differential inclusions take the following form:

$$\zeta(\theta) \in G(\theta, \zeta(\theta)), \ a.e.$$

where $G$ is a multi-valued function and is, therefore, a generalization of differential equations. Many authors have studied numerous differential inclusions [1–4]. Impulsive differential problems are proper models for expressing processes that, at certain times, change their situation speedily. These processes can not be described by classical differential equations. If the effect of this change is instantaneous, they are termed instantaneous impulses, but if it remains over a period of time, they are called non-instantaneous impulses. For an example of non-instantaneous impulses, we can look at the consequence of instituting medications into the bloodstream and their soaking up by the body. To equip the reader with applications of non-instantaneous impulses in physics, biology, population dynamics, ecology, and pharmacokinetics, we refer to [5–7]. In [8–10], there are many studies on differential inclusions with non-instantaneous impulses.

Differential equations and inclusions containing fractional derivatives have many applications in various branches of science, engineering, and medicine [11–14], which indi-

cates the importance of fractional derivatives. Therefore, many researchers pay attention to giving different concepts to fractional derivatives, such as Riemann-Liouville, Caputo, Hilfer, Katugampola, Hadamard, and Atangana–Baleanu. All known fractional derivatives, except conformable fractional derivatives introduced by Khalil et al. [15], do not satisfy many basic properties of a usual derivative, such as the product rule, quotient rule, mean value theorem, chain rule, and Taylor power series expansion. Therefore, conformable fractional derivatives represent the most natural fractional derivative. For this reason, many researchers have shown interest in exploring more properties of conformable fractional derivatives and studying differential equations involving them. In [16–19], conformable fractional derivative properties are given, and in [20–22], some of their applications are presented. Nonlocal telegraph equations with a conformable fractional derivative are considered in [23]. Meng et al. [24] looked for the existence of external iteration solutions to conformable fractional differential equations. Tajadodi et al. [25] used an exact solution to a nonlinear differential equation involving a conformable derivative.

In [26,27], there are other findings on differential equations with conformable derivatives.

Let $\alpha \in (0,1]$, $Z$ be a Banach space, $\mathrm{Y} = [0,b]$, $f : \mathrm{Y} \times Z \to Z$ be a single-valued function, $\psi : Z \to Z$, $A$ be the infinitesimal generator of a $C_0$-semigroup, $\{T(\theta) : \theta \geq 0\}$, on $Z$, and $\zeta_0 \in Z$ be a fixed point. Without assuming the compactness of the family $\{T(\theta) : \theta \geq 0\}$, Bouaouid et al. [28] proved, under the condition that $\psi$ is continuous and compact, the existence of mild solutions to the nonlocal conformable fractional semilinear differential equations:

$$\begin{cases} \frac{d^\alpha}{d\zeta^\alpha}\zeta(\theta) = A\zeta(\theta) + f(\theta, \zeta(\theta)), \ \theta \in \mathrm{Y}, \\ \zeta(0) = \zeta_0 + \psi(\zeta), \end{cases} \tag{1}$$

where $\frac{d^\alpha}{d\zeta^\alpha}\zeta(\theta)$ is the conformable derivative of the function $\zeta$ at point $\theta$.

Bouaouid et al. [29] studied the existence of mild solutions to the following nonlocal conformable fractional semilinear differential equations:

$$\begin{cases} \frac{d^\alpha}{d\zeta^\alpha}\zeta(\theta) = B\zeta(\theta) + f(\theta, \zeta(\theta)), \ \theta \in \mathrm{Y}, \\ \zeta(0) = \zeta_0 + \psi(\zeta), \end{cases} \tag{2}$$

where $B$ is a sectorial operator on $Z$ and generates a strongly analytic semigroup, $\{K(\theta) : \theta \geq 0\}$, such that the operators $K(\theta) : \theta > 0$ are compact.

Yousif et al. [30] presented a method that combines a conformable derivative, finite difference, and a non-polynomial curve to solve the non-homogeneous and nonlinear Burger-Huxley equation with fractional time. This method is called the non-polynomial logarithmic curve method. Zhang et al. [31] introduced the definition of interval-valued fractional conformable derivatives and provided some applications. Ahmad et al. [32] offered the existence of solutions to impulsive differential equations involving conformable fractional derivatives. Ahmed et al. [33] investigated the existence of solutions to a Langevin differential equation containing conformable fractional derivatives of different orders.

Motivated by the above works, especially that carried out in [28,29], in this paper, we present six results for the existence of mild solutions to two types of nonlocal semilinear differential inclusions containing a conformable fractional derivative in the presence of non-instantaneous impulses in Banach spaces. In fact, we generalize the work in [28,29] and prove three results for the existence of mild solutions to each of the problems ((1) and (2)): when $f$ is replaced by a multi-valued function $\Phi$, and in the presence of non-instantaneous impulses. Unlike [29], we do not assume the compactness of $K(\theta), \forall \theta > 0$.

In order to formulate the problems that we studied, let $0 = s_0 < b_1 \leq s_1 < b_2 \leq s_2 < ... < s_r < b_{r+1} = b$, $\mathrm{Y}_0 = [0, b_1]$, $\mathrm{Y}_i = (s_i, b_{i+1}]$, $J_i = (b_i, s_i]; i \in \Lambda_{1,r} = \{1, 2, ...r\}$,

$\Phi : \mathrm{Y} \to 2^{Z} - \{\phi\}$ be a multi-valued function, $g_i : [b_i, s_i] \times Z \to Z$ , and $g : PC(\mathrm{Y}, Z) \to Z$. The Banach space $PC(\mathrm{Y}, Z)$ will be defined in the next section.

Consider the following two nonlocal semilinear fractional differential inclusions involving a conformable derivative in the presence of non-instantaneous impulses:

$$\begin{cases} \frac{d^{\alpha}}{d\zeta^{\alpha}}\zeta(\theta) \in A\zeta(\theta) + \Phi(\theta, \zeta(\theta)), \ a.e.\, \theta \in \cup_{i=0}^{i=r}\mathrm{Y}_i, \\ \zeta(\theta) = g_i(\theta, \zeta(b_i^{-})), \theta \in J_i, i \in \Lambda_{1,r}, \\ \zeta(0) = \zeta_0 + g(\zeta), \end{cases} \tag{3}$$

and

$$\begin{cases} \frac{d^{\alpha}}{d\zeta^{\alpha}}\zeta(\theta) \in B\zeta(\theta) + \Phi(\theta, \zeta(\theta)), \ a.e.\, \theta \in \cup_{i=0}^{i=r}\mathrm{Y}_i, \\ \zeta(\theta) = g_i(\theta, \zeta(b_i^{-})), \ \theta \in J_i, i \in \Lambda_{1,r}, \\ \zeta(0) = \zeta_0 + g(\zeta). \end{cases} \tag{4}$$

We will explore the sufficient conditions for assuring that $S_3(A, \Phi)$ and $S_4(B, \Phi)$ are non-empty and compact in $PC(\mathrm{Y}, Z)$, where $S_3(A, \Phi)$ and $S_4(B, \Phi)$ are the set of mild solutions to problems (3) and (4) respectively.

**Remark 1.** *1. This work is novel because this is the first time that nonlocal semilinear fractional differential inclusions involving a conformable derivative in the presence of non-instantaneous impulses in infinite dimensional Banach spaces—where the linear part is the infinitesimal generator of a $C_0$ —semigroup (not necessarily compact) or a sectorial operator generates an analytic semigroup (not necessarily compact)—has been considered. Moreover, we consider the case where the values of $\Phi$ are convex and nonconvex.*

*2. This work is interesting because we studied problems involving a conformable derivative, which possesses many properties of a usual derivative and is not verified for all the other known fractional derivatives.*

The significant contributions are the following:

1. The representation of mild solutions to problems (3) and (4) is formulated (Definitions 4 and 6);
2. We extended Problem (1), studied by Bouaouid et al. [28], to a case where the single-valued $f$ is replaced with a multi-valued function, $\Phi$, in the presence of non-instantaneous impulsive effects (Problem 3). In fact, three results for the existence of mild solutions to Problem (3) are given (Theorems 1–3). In Theorem 1, the values of $\Phi$ are non-empty, convex, and compact, and $\Phi$ satisfies a compactness condition containing a measure of noncompactness. In Theorem 2, the values of $\Phi$ are non-empty, convex, and compact, and $\Phi$ satisfies a Lipschitz condition. In Theorem 3, the values of $\Phi$ are non-empty and compact (and not necessarily convex), and $\Phi$ satisfies a compactness condition containing a measure of noncompactness.
3. We have extended Problem (2), as studied in [29], to the case where the single-valued $f$ is replaced with a multi-valued function $\Phi$ and is in the attendance of non-instantaneous impulsive effects (Theorem 5). Moreover, we do not suppose that the semigroup generated by the operator $B$ is compact, as it is in [29]. In fact, three results for the existence of mild solutions to Problem (4) are given (Theorems 4–6).

In Section 3 of this paper, we will demonstrate three results for the existence of mild solutions to Problem (3). Section 4 contains three results for the existence of mild solutions to Problem (4). In Section 5, we provide examples.

## 2. Preliminaries and Notations

We use the following notations:

1. $P_b(Z) = \{\Delta \subseteq Z : \Delta$ is non-empty and bounded$\}$;
2. $P_{cc}(Z) = \{\Delta \subseteq Z : \Delta$ is non-empty, convex, and closed$\}$;
3. $P_{bc}(Z) = \{\Delta \subseteq Z : \Delta$ is non-empty, bounded, and closed$\}$;
4. $P_{ck}(Z) = \{\Delta \subseteq Z : \Delta$ is non-empty, convex, and compact$\}$;
5. $S_3(A, \Phi)$ and $S_4(B, \Phi)$ are the set of mild solutions to problems (3) and (4), respectively;
6. $p$ is a positive real number with $p > \frac{1}{\alpha}$;
7. $PC(\mathrm{Y}, Z)\& = \&\{z : \mathrm{Y} \to Z,\ z$ is continuous on $[0, b_1]$ and on $(b_i, b_{i+1}]$, and $\lim_{\theta \to b_i^+} z(\theta)$ exists for all $i \in \&\Lambda_{1,r}\}$.

Note that the space $PC(\mathrm{Y}, Z)$ is a Banach space, where the norm is given by

$$||f||_{PC(\mathrm{Y},Z)} := \max\{||z(\theta)|| : \theta \in \mathrm{Y}\}.$$

The Hausdorff measure of noncompactness on $PC(\mathrm{Y}, Z)$ is given by $\chi_{PC(\mathrm{Y},\,Z)} : P_b(PC(\mathrm{Y}, Z)) \to [0, \infty]$,

$$\chi_{PC(\mathrm{Y},Z)}(D) := \max_{i \in L_{0,r}} \chi_{C(\overline{\mathrm{Y}_i},Z)}(D_{|\overline{\mathrm{Y}}_i}),$$

where $\chi_{C(\overline{\mathrm{Y}_i},Z)}$ is the Hausdorff measure of noncompactness on the Banach space $C(\overline{\mathrm{Y}}_i,\ Z)$.

As in [15], we give the following definitions:

**Definition 1.** *The conformable integral of order $\alpha$ for a function, $f \in L^1(\mathrm{Y}, Z)$, is given by*

$$I^\alpha f(\theta) := \int_0^\theta s^{\alpha-1} f(s) ds.$$

**Definition 2.** *The conformable fractional derivative of order $\alpha$ for a function, $f : [0, \infty) \to Z$, at a point, $\theta \in (0,\ \infty)$ , is defined by*

$$D^\alpha f(\theta) = \lim_{\varepsilon \to 0} \frac{f(\theta + \varepsilon\theta^{1-\alpha}) - f(\theta)}{\varepsilon},\ \theta \in (0,\ \infty).$$

The proof of the following Lemmas is exactly the same as in the scalar case treated in [15].

**Lemma 1.** *Suppose that $f : [0, \infty) \to Z,\ g : [0, \infty) \to \mathbb{R}$ are conformable fractional differentiables of order $\alpha$ at a point $\theta \in (0, \infty)$. Then,*

*1-$D^\alpha(f(\theta) + g(\theta)) = D^\alpha f(\theta) + D^\alpha g(\theta)$.*
*2-$D^\alpha(fg)(\theta) = gD^\alpha f(\theta) + fD^\alpha g(\theta)$.*
*3-$D^\alpha(\frac{f}{g})(\theta) = \frac{gD^\alpha f(\theta) - fD^\alpha g(\theta)}{[g(\theta)]^2}, g(\theta) \neq 0$.*

**Lemma 2.** *If $f : [0, \infty) \to Z$ is differentiable at a point $\theta \in (0, \infty)$, then it is a conformable fractional differentiable of order $\alpha$ at $\theta$ and $D^\alpha f(\theta) = \theta^{1-\alpha} \frac{df}{d\theta}$.*

**Remark 2.** *As a consequence of the previous Lemma, the physical meaning of the conformable fractional derived at point $t$, measures the rate at which a quantity changes at $t$, times by $t^{1-\alpha}$. Moreover, for more information about the physical meaning of the conformable fractional derivative, we refer to* [22]*, and for its applications in physics, see* [34].

## 3. The Compactness of the Solution Set to Problem (3)

In this section, we will present three theorems. In each one, we explore the conditions that make the set of mild solutions to Problem (3) non-empty and compact in the Banach

space, $PC([0,T],Z)$. In Theorem 1, the multi-valued function $\Phi$ has convex values and satisfies a compactness condition involving a measure of noncompactness. In Theorem 2, the multi-valued function $\Phi$ has convex values and verifies a Lipschitz condition, and in Theorem 3, the values of $\Phi$ are not necessarily convex.

**Definition 3** ([28] Definition 6)**.** *Let $A$ be the infinitesimal generator of a $C_0$- semigroup $\{T(\theta) : \theta \geq 0\}$ and $f : [0,T] \times Z \to Z$ be continuous. A continuous function, $\zeta : [0,T] \to Z$, is called a mild solution to the problem (1)*

*if*

$$\zeta(\theta) = T(\frac{\theta^\alpha}{\alpha})(\zeta_0 + g(\zeta)) + \int_0^\theta s^{\alpha-1}\, T(\frac{\theta^\alpha - s^\alpha}{\alpha}) f(s,\zeta(s)ds.$$

Based on this definition, we present the definition of mild solutions to Problem (3).

**Definition 4.** *A function, $\zeta \in PC(\mathrm{Y},Z)$, is called a mild solution to Problem (3) if*

$$\zeta(\theta) = \begin{cases} T(\frac{\theta^\alpha}{\alpha})(\zeta_0 + g(\zeta)) + \int_0^\theta s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha}) f(s)ds, \theta \in \mathrm{Y}_0, \\ g_i(\theta, \zeta(b_i^-)),\ \theta \in J_i, i \in \Lambda_{1,r}, \\ T(\frac{\theta^\alpha - s_i^\alpha}{\alpha}) g_i(s_i, \zeta(b_i^-)) - \int_0^{s_i} s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha}) f(s)ds \\ + \int_0^\theta s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha}) f(s)ds,\ \theta \in \mathrm{Y}_i,\ i \in \Lambda_{1,r}. \end{cases} \tag{5}$$

*where $f \in S^1_{\Phi(.,\zeta(.))} = \{z \in L^1([0,T],Z) : f(\theta) \in \Phi(\theta,\zeta(\theta)),\ a.e.\}$.*

**Remark 3.** *The solution function given by (5) agrees with definition (4) in [27] and is continuous at the points $s_i$, and hence, is continuous on $(b_i, b_{i+1}]$, $i \in \{0,1,..,r\}$.*

We consider the following assumptions:

$(HA)$ $A : D(A) \subseteq Z \to Z$ is the infinitesimal generator of a $C_0$-semigroup $\{T(\theta) : \theta \geq 0\}$ in $Z$.

$(H\Phi)_1$ $\Phi : \mathrm{Y} \times Z \to P_{ck}(Z)$ with the following:

(i) For any $z \in Z$, there is a strongly measurable function, $\xi : \mathrm{Y} \to Z$, satisfying $\xi(\theta) \in \Phi(\theta,z)$ *a.e.*, and for almost $\theta \in \mathrm{Y}$, $\zeta \to \Phi(\theta,\zeta)$ is upper semicontinuous from $Z$ to $Z$.

(ii) There is a function, $\varphi \in L^p(\mathrm{Y},\mathbb{R}^+)$, such that

$$\sup_{u \in \Phi(\theta,\, z)} ||u|| \leq \varphi(\theta)(1 + ||z||),\ for\ a.e. \theta \in \mathrm{Y} \text{ and } z \in Z. \tag{6}$$

(iii) There is $\beta \in L^p(\mathrm{Y},\mathbb{R}^+)$ such that for any bounded set, $D \subseteq Z$,

$$\chi(\Phi(\theta,D)) \leq \beta(\theta)\chi(D), \tag{7}$$

where $\chi$ is the Hausdorff measure of noncompactness on $Z$.

$(Hg)_1$ The function $g : PC(\mathrm{Y},Z) \to Z$ is compact and continuous, and there are two positive real numbers $a, d$ such that

$$||g(\zeta)| \leq a||\zeta|| + d,\ \forall \zeta \in PC(\mathrm{Y},Z).$$

$(H)_1$ For any $i \in \Lambda_{1,r}$, the function $g_i : [t_i,s_i] \times Z \to Z$ is uniformly continuous on bounded sets, $g_i(\theta,.)$ is compact, and there is $h_i > 0$ such that

$$||g_i(\theta,z)|| \leq h_i||z||,\ \theta \in [b_i,s_i],\ z \in Z. \tag{8}$$

We need the following Lemmas:

**Lemma 3** ([35]). *Let $\Phi : \Upsilon \times Z \to P_{ck}(Z)$ be a multifunction satisfying $(i)$ and $(ii)$ in $(H\Phi)_1$; then, for any $u \in PC(\Upsilon, Z)$, the set $S^1_{\Phi(.,u(.))} = \{z \in L^1(\Upsilon, Z) : z(\theta) = \Phi(\theta, u(\theta)),\ a.\ e.\}$ is non-empty and weakly closed.*

Let $\Re \in P_{cc}(Z)$, $\chi$ be a non-singular measure of noncompactness defined on subsets of $Z$, $\aleph : \Re \to P_{ck}(Z)$ be a closed multifunction, and $Fix(\aleph) = \{z \in Z : z \in \aleph(z)\}$.

**Lemma 4** (Kakutani-Glicksberg-Fan theorem) ([2], Corollary 3.3.1)). *If $\aleph : \Re \to P_{ck}(\Re)$ is $v-$condensing, then $Fix(\aleph) \neq \phi$.*

**Lemma 5** ([2], Proposition.3.5.1). *In a supplement to the hypothesis of Lemma (4), if $\chi$ is a monotone measure of noncompactness defined on $Z$ and $Fix(\aleph)$ is bounded, then it is compact.*

In the following theorem, we will obtain conditions that make $S_3(A,\ \Phi)$ non-empty and compact.

**Theorem 1.** *Assume that $(HA)$, $(H\Phi)_1$, $(Hg)_1$, and $(H)_1$ are satisfied.*
*Then, $S_3(A, \Phi)$ is non-empty and compact provided that*

$$Ma + 2M\ \Pi||\varphi||_{L^p(J,\ \mathbb{R}^+)} + Mh + h < 1, \tag{9}$$

*and*

$$4M\ \Pi||\beta||_{L^p(\Upsilon,\mathbb{R}^+)} < 1, \tag{10}$$

*where $M = \sup\{||T(\theta)||,\ \theta \geq 0\}, \Pi = b^{\alpha-\frac{1}{p}}(\frac{p-1}{\alpha p-1})^{\frac{1}{p-1}}$, and $h = \max\{h_i : i = 1, 2, ..., r\}$.*

**Proof.** Let $\zeta \in PC(\Upsilon, Z)$. In view of Lemma (4), there is $f \in S^1_{\Phi(.,\zeta(.))}$. Then, we can define a multi-valued function, $\aleph : PC(\Upsilon, Z) \to 2^{PC(\Upsilon,Z)} - \{\phi\}$, where $\phi$ is the empty set, as follows: $y \in \aleph(\zeta)$ if and only if

$$y(\theta) = \begin{cases} T(\frac{\theta^\alpha}{\alpha})(\zeta_0 + g(\zeta)) + \int_0^\theta s^{\alpha-1}T(\frac{\theta^\alpha - s^\alpha}{\alpha})f(s)ds,\ \theta \in \Upsilon_0, \\ g_i(\theta, \zeta(b_i^-)),\ \theta \in J_i, i \in \Lambda_{1,r}, \\ T(\frac{\theta^\alpha - s_i^\alpha}{\alpha})g_i(s_i, \zeta(b_i^-)) - \int_0^{s_i} s^{\alpha-1}T(\frac{\theta^\alpha - s^\alpha}{\alpha})f(s)ds \\ + \int_0^\theta s^{\alpha-1}T(\frac{\theta^\alpha - s^\alpha}{\alpha})f(s)ds,\ \theta \in \Upsilon_i,\ i \in \Lambda_{1,r}. \end{cases} \tag{11}$$

where $f \in S^1_{\Phi(.,\zeta(.))}$. Obviously, $Fix(\aleph) \subseteq S_3(A, \Phi)$. So, by applying Lemma (4), we show that $Fix(\aleph) \neq \phi$. This will proceed in the following steps:

Step 1. There is a natural number, $\wp$, such that $\aleph(D_\wp) \subseteq D_\wp$, , where $D_\wp = \{\zeta \in PC(\Upsilon, Z) : ||\zeta|| \leq \wp$.

Assume the contrary. Then, for any $n \in \mathbb{N}$, there are $\zeta_n,\ y_n \in PC(\Upsilon, Z)$ with $||y_n||_{PC(\Upsilon,Z)} > n$, $||\zeta_n||_{PC(\Upsilon,Z)} \leq n$ and $y_n \in \aleph(\zeta_n)$ such that

$$y_n(\theta) = \begin{cases} T(\frac{\theta^\alpha}{\alpha})(\zeta_0 + g(\zeta_n)) + \int_0^\theta s^{\alpha-1}T(\frac{\theta^\alpha - s^\alpha}{\alpha})f_n(s)ds,\ \theta \in \Upsilon_0, \\ g_i(\theta, \zeta_n(b_i^-)),\ \theta \in J_i, i \in \Lambda_{1,r}, \\ T(\frac{\theta^\alpha - s_i^\alpha}{\alpha})g_i(s_i, \zeta_n(b_i^-)) - \int_0^{s_i} s^{\alpha-1}T(\frac{\theta^\alpha - s^\alpha}{\alpha})f_n(s)ds \\ + \int_0^\theta s^{\alpha-1}T(\frac{\theta^\alpha - s^\alpha}{\alpha})f_n(s)ds,\ \theta \in \Upsilon_i,\ i \in \Lambda_{1,r}. \end{cases} \tag{12}$$

where $f_n \in S^1_{\Phi(.,\zeta_n(.))}$. By using (6), we obtain

$$||f_n(\theta)|| \leq \varphi(\theta)(1 + n), a.e. \tag{13}$$

Let $\theta \in [0, b_1]$. Since $p > \frac{1}{\alpha}$, then $s^{\alpha-1} \in L^q(\mathrm{Y}, \mathbb{R}^+)$, where $q = \frac{p}{p-1}$, and so, from (6), $(H_g)_1$, (12), (13), and Hölder's inequality; it results in

$$\begin{aligned} ||y_n(\theta)|| &\leq M(||\zeta_0|| + an + d) + M(1+n)\int_0^\theta s^{\alpha-1}\varphi(s)ds \\ &\leq M(||\zeta_0|| + an + d) + M(1+n)\Pi||\varphi||_{L^p(\mathrm{Y},\mathbb{R}^+)}. \end{aligned} \tag{14}$$

If $\theta \in [b_1, s_1]$, then from (8), we obtain

$$||g_i(\theta, \zeta_n(b_i^-))|| \leq nh_i, \ \forall n \geq 1. \tag{15}$$

Let $i \in \Lambda_{1,r}$ and $\theta \in (s_i, b_{i+1}]$. Then, as above, we obtain

$$||y_n(\theta)|| \leq Mnh_i + 2M(1+n)\Pi||\varphi||_{L^p(\mathrm{Y},\mathbb{R}^+)}. \tag{16}$$

Inequalities (14)–(16) lead to

$$n < M(||\zeta_0|| + an + d) + 2M\,\Pi(1+n)||\varphi||_{L^p(J,\mathbb{R}^+)} + Mnh_i + nh_i$$

Dividing this inequality by $n$ and letting $n \to \infty$ yields

$$1 < Ma + 2M\,\Pi||\varphi||_{L^p(J,\mathbb{R}^+)} + Mh + h,$$

which contradicts (9).

Step 2. We demonstrate that, if $y_n, \zeta_n \in D_\wp$ such that $\zeta_n \to \zeta$ and $y_n \in \aleph(\zeta_n), \forall n \in \mathbb{N}$, then $y_n \to y$ and $y \in \aleph(\zeta)$.

Because $y_n \in \aleph(\zeta_n), \forall n \in \mathbb{N}$, (12) is satisfied.

Since $p > 1$, then from (13), the set $\{f_n : n \geq 1\}$ is weakly compact in $L^p(\mathrm{Y}, Z)$. In the application of Mazur's Lemma, there is, without loss of generality, a subsequence $(f_n^*), n \in \mathbb{N}$, of convex combinations of $(f_n)$ , and this converges almost everywhere to a function $f \in L^p(\mathrm{Y}, Z) \subseteq L^1(\mathrm{Y}, Z)$. By the continuity of both $g$ and $g_i(\theta, .)$, it follows that by letting $n \to \infty$ in (12), $y_n \to y$, where

$$y(\theta) = \begin{cases} T(\frac{\theta^\alpha}{\alpha})(\zeta_0 + g(\zeta)) + \int_0^\theta s^{\alpha-1}T(\frac{\theta^\alpha - s^\alpha}{\alpha})f(s)ds, \ \theta \in \mathrm{Y}_0, \\ g_i(\theta, \zeta(b_i^-)), \ \theta \in J_i, i \in \Lambda_{1,r}, \\ T(\frac{\theta^\alpha - s_i^\alpha}{\alpha})g_i(s_i, \zeta(b_i^-)) - \int_0^{s_i} s^{\alpha-1}T(\frac{\theta^\alpha - s^\alpha}{\alpha})f(s)ds \\ + \int_0^\theta s^{\alpha-1}T(\frac{\theta^\alpha - s^\alpha}{\alpha})f(s)ds, \ \theta \in \mathrm{Y}_i, \ i \in \Lambda_{1,r}. \end{cases}$$

Since $\Phi(\theta, .)$ is upper semicontinuous, then this implies $f(\theta) \in \Phi(\theta, \zeta(\theta))$, *a.e.*, and therefore, $y \in \aleph(\mu)$.

Step 3. For any $\zeta \in D_\wp$, the set $\aleph(\zeta)$ is compact, in $PC(\mathrm{Y}, Z)$.

Let $y_n \in \aleph(\zeta); n \in \mathbb{N}$. Using the same arguments as in Step 2, one can show that there is a subsequence of $(y_{n_k})$ that converges to $y \in \aleph(\zeta)$. This shows that $\aleph(\zeta)$ is relatively compact, but Step (2) leads to the closedness of $\aleph(\zeta)$, and consequently, $\aleph(\zeta)$ is compact in $PC(\mathrm{Y}, Z)$.

Step 4. The family of functions

$$\begin{aligned} \aleph(D_\wp)|_{\overline{\mathrm{Y}_i}} &= \{\omega^* \in C(\overline{\mathrm{Y}_i}, Z) : \omega^*(\theta) = \omega(\theta), \ \theta \in (b_i, b_{i+1}], \\ \omega^*(b_i) &= \lim_{\theta \to b_i^+} \omega(\theta), \omega \in \aleph(D_\wp)\}, i \in \{0, 1, .., \ r\}, \end{aligned}$$

are equicontinuous in $C([s_i, b_{i+1}], Z)$. Assume that $\omega^* \in \aleph(D_\wp)|_{[s_i,b_{i+1}]}; i \in \{0, 1, .., r\}$. Then, there is $\zeta \in D_\wp$ with

$$\omega^*(\theta) = \begin{cases} T(\frac{\theta^\alpha}{\alpha})(\zeta_0 + g(\zeta)) + \int_0^\theta s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha}) f(s) ds, \ \theta \in \Upsilon_0, \\ g_i(\theta, \zeta(b_i^-)), \ \theta \in J_i, i \in \Lambda_{1,r}, \\ T(\frac{\theta^\alpha - s_i^\alpha}{\alpha}) g_i(s_i, \ \zeta(b_i^-)) - \int_0^{s_i} s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha}) f(s) ds \\ + \int_0^\theta s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha}) f(s) ds, \ \theta \in \Upsilon_i, i \in \Lambda_{1,r}. \end{cases} \tag{17}$$

and $\omega^*(b_i) = \lim_{\theta \to b_i^+} \omega(\theta)$. We consider the following cases:

Case 1. Let $\theta, \theta + d \in [0, b_1], d > 0$. Using (17) yields

$$\begin{aligned} & \lim_{d\to 0} ||\omega^*(\theta + d) - \omega^*(\theta)|| \\ \leq & \lim_{d\to 0} ||T(\frac{(\theta+d)^\alpha}{\alpha}) - T(\frac{\theta^\alpha}{\alpha})|| \ ||\zeta_0 + g(\zeta)|| \\ & + \lim_{d\to 0} || \int_0^{\theta+d} s^{\alpha-1} T(\frac{(\theta+d)^\alpha - s^\alpha}{\alpha}) f(s) ds - \int_0^\theta s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha}) f(s) ds|| \\ \leq & \lim_{d\to 0} ||T(\frac{(\theta+d)^\alpha}{\alpha}) - T(\frac{\theta^\alpha}{\alpha})|| \ ||\zeta_0 + g(\zeta)|| \\ & + \lim_{d\to 0} \int_\theta^{\theta+d} || \ s^{\alpha-1} T(\frac{(\theta+d)^\alpha - s^\alpha}{\alpha}) f(s)|| ds \\ & + \lim_{d\to 0} || \int_0^\theta s^{\alpha-1} T(\frac{(\theta+d)^\alpha - s^\alpha}{\alpha}) f(s) - s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha}) f(s) ds|| \\ = & \ I_1 + I_1 + I_3. \end{aligned}$$

In view of $(HA)$, $\lim_{d\to 0} I_1 = 0$. By making use of (13), we obtain

$$\lim_{d\to 0} I_2 \leq (1 + \wp) M \lim_{d\to 0} \int_\theta^{\theta+d} s^{\alpha-1} \varphi(s) ds = 0,$$

Next, the strong continuity of $\{T(\theta) : \theta \geq 0\}$ leads to

$$\lim_{d\to 0} I_3 \leq (1 + \wp) \lim_{d\to 0} \int_0^\theta s^{\alpha-1} ||T(\frac{(\theta+d)^\alpha - s^\alpha}{\alpha}) - T(\frac{\theta^\alpha - s^\alpha}{\alpha})|| \ \varphi(s) ds = 0,$$

Case 2. Let $\theta, \theta + d \in (b_1, s_1]$. The uniform continuity on bounded sets of $g_1$ results in

$$\begin{aligned} & \lim_{d\to 0} ||\omega^*(\theta + d) - \omega^*(\theta)|| \\ = & \lim_{d\to 0} ||g_1(\theta + d, \zeta(b_1^-)) - g_1(\theta, \zeta(b_1^-))|| = 0, \end{aligned}$$

independently of $\zeta$. Moreover,

$$\begin{aligned} & \lim_{d\to 0} ||\omega^*(\theta + d) - \omega^*(b_i)|| \\ = & \lim_{d\to 0} \lim_{\theta \to b_1^+} ||g_1(\theta + d, \zeta(b_1^-)) - g_1(\theta, \zeta(b_1^-))|| \\ = & \ 0, \end{aligned}$$

independently of $\zeta$. Similarly, one can show that $\aleph(D_\wp)|_{\overline{Y_i}}, i \in L_{1,r}$ is equicontinuous.

Step 5. The set $\pounds = \cap_{n=1} D_n$ is compact, where $D_1 = \aleph(B_\wp)$ and $D_{n+1} = \aleph(D_n), n \geq 1$.

Because $D_n \subseteq D_{n+1}$, then, as stated by Cantor's intersection property [36], it is enough to manifest that

$$\lim_{n\to\infty} \varkappa_{PC(\Upsilon,Z)}(D_n) = 0. \tag{18}$$

Let $\kappa > 0$, and let $n \geq 1$ be fixed. In view of Lemma 2.9 in [37], there is a sequence $(y_m)$ in $D_n$ with

$$\begin{aligned}\chi_{PC(\mathrm{Y},Z)}(D_n) &\leq 2\chi_{PC(\mathrm{Y},Z)}\{y_m : m \geq 1\} + \kappa \\ &= 2\max_{i\in L_{0,r}} \chi_{C(\overline{\mathrm{Y}_i},Z)}(D_{|\overline{\mathrm{Y}}_i}).\end{aligned}$$

As a result of Step 4, the sets $D_{|\overline{\mathrm{Y}}_i}$ are equicontinuous, and hence, the last inequality becomes

$$\chi_{PC(\mathrm{Y},Z)}(D_n) \leq 2\max_{\theta\in\mathrm{Y}} \chi(y_m(\theta) : m \geq 1\} + \kappa. \tag{19}$$

Now, since $D_n = \aleph(D_{n-1})$, there is $\zeta_m \in D_{n-1}$, with $y_m \in \aleph(\zeta_m)$, which means that

$$y_m(\theta) = \begin{cases} T(\frac{\theta^\alpha}{\alpha})(\zeta_0 + g(\zeta_m)) + \int_0^\theta s^{\alpha-1}T(\frac{\theta^\alpha - s^\alpha}{\alpha})f_m(s)ds,\ \theta \in \mathrm{Y}_0, \\ g_i(\theta, \zeta_m(b_i^-)),\ \theta \in J_i, i \in \Lambda_{1,r}, \\ T(\frac{\theta^\alpha - s_i^\alpha}{\alpha})g_i(s_i, \zeta_m(b_i^-)) - \int_0^{s_i} s^{\alpha-1}T(\frac{\theta^\alpha - s^\alpha}{\alpha})f_m(s)ds \\ + \int_0^\theta s^{\alpha-1}T(\frac{\theta^\alpha - s^\alpha}{\alpha})f_m(s)ds,\ \theta \in \mathrm{Y}_i, i \in \Lambda_{1,r}, \end{cases}$$

where $f_m \in S^1_{\Phi(.,\zeta_m)}$. If $\theta \in [0, b_1]$, then due to the continuity of $T(\frac{\theta^\alpha}{\alpha})$, the compactness of $g$, and (7), we obtain

$$\begin{aligned}\chi\{y_m(\theta) &: m \geq 1\} \\ &\leq \chi\{\int_0^\theta s^{\alpha-1}T(\frac{\theta^\alpha - s^\alpha}{\alpha})f_m(s)ds : m \geq 1\} \\ &\leq 2M\int_0^\theta s^{\alpha-1}\chi\{f_m(s) : m \geq 1\}ds \\ &\leq 2M\int_0^\theta s^{\alpha-1}\chi\{\zeta_m(s)) : m \geq 1\}\beta(s)ds \\ &\leq 2M\chi_{PC(\mathrm{Y},Z)}(D_{n-1})\int_0^\theta s^{\alpha-1}\beta(s)ds \\ &\leq 2M\,\chi_{PC(\mathrm{Y},Z)}(D_{n-1})\Pi||\beta||_{L^p(\mathrm{Y},\mathbb{R}^+)}. \end{aligned} \tag{20}$$

If $\theta \in (b_i, s_i]; i \in \Lambda_{1,r}$, then the compactness of $g_i(\theta, .)$ implies

$$\chi\{y_m(\theta) : m \geq 1\} = 0. \tag{21}$$

Finally, if $\theta \in (s_i, b_{i+1}], i \in \Lambda_{1,r}$, then as in (20),

$$\begin{aligned}\chi\{y_m(\theta) &: m \geq 1\} \\ &\leq 4M\,\chi_{PC(\mathrm{Y},Z)}(D_{n-1})\Pi||\beta||_{L^p(\mathrm{Y},\mathbb{R}^+)}. \end{aligned} \tag{22}$$

It yields from (19)–(22), that

$$\chi_{PC(\mathrm{Y},Z)}(D_n) \leq 4M\,\Pi||\beta||_{L^p(\mathrm{Y},\mathbb{R}^+)}\,\chi_{PC(\mathrm{Y},Z)}(D_{n-1}) + \kappa.$$

Since this relation verifies for any $n$, it follows that

$$\chi_{PC(\mathrm{Y},Z)}(D_n) \leq (4M\,\Pi||\beta||_{L^p(\mathrm{Y},\mathbb{R}^+)})^n\,\chi_{PC(\mathrm{Y},Z)}(D_1).$$

By using this inequality along (10), we obtain (18), and then $D$ is compact.

Step 6. When applying Lemma (4), the multi-valued function $\aleph : D \to P_{ck}(PC(\mathrm{Y}, Z))$ has a fixed point, which is the solution to Problem (3). Moreover, by using the same argu-

ments in Step 1, we can show that $Fix(\aleph)$ is bounded, and hence, according to Lemma (5), $S_3(A, \Phi)$ is compact. □

In the following theorem, we offer another result for the existence of mild solutions to Problem (3).

**Lemma 6** ([38]). *If $G : \Omega \to P_{bc}(\Omega)$ is a contraction, then $Fix(G)$ is non-empty, where $\Omega$ is a complete metric space.*

**Theorem 2.** *In addition to $(HA)$, suppose the following assumptions:*

$(H\Phi)_2$ $\Phi : \mathrm{Y} \times Z \to P_{ck}(Z)$ such that the following applies:

(i) For any $u \in Z$, the multifunction $\theta \to \Phi(\theta, u)$ has a strongly measurable selection;
(ii) There is a function, $\varphi \in L^1(J, \mathbb{R}^+)$, satisfying

$$h(\Phi(\theta, u),\ \Phi(\theta, v)) \leq \varphi(\theta)\|u - v\|,\ \forall u, v \in Z \text{ and for } a.e.\ \theta \in \mathrm{Y}, \tag{23}$$

and

$$Sup\{||u||\ :\ u \in \Phi(\theta, 0)\} \leq\ \varphi(\theta),\ \text{for } a.e.\ \theta \in \mathrm{Y}, \tag{24}$$

where $h$ is the Hausdorff distance between two closed convex bounded sets.

$(Hg)_2$ There is $a > 0$ with

$$\|g(\zeta_1) - g(\zeta_2)\| \leq a\|\zeta_1 - \zeta_2\|,\ \forall \zeta_1,\ \zeta_2 \in PC(\mathrm{Y}, Z). \tag{25}$$

$(H)_2$ For each $i \in \Lambda_{1,r}$, there is $h_i >\ 0$ such that for any $\theta \in \mathrm{Y}$,

$$||g_i(\theta,\ u) - g_i(\theta,\ v)\ || \leq\ h_i\ ||u - v\ ||,\ \text{for all}\ \ u, v \in Z. \tag{26}$$

Then, $S_3(A, \Phi)$ is non-empty if inequality (10) is verified.

**Proof.** From (i) and (ii) of $(H\Phi)_2$, we obtain

$$\begin{aligned} h(\Phi(\theta, z), \{0\}) &\leq h(\Phi(\theta, z), \Phi(\theta, 0)) + h(\Phi(\theta, 0), \{0\}) \\ &\leq \varphi(\theta)(||z|| + 1), \forall z \in Z \text{ and for } a.e.\ \theta \in \mathrm{Y}. \end{aligned}$$

Then, (i) and (ii) of $(H\Phi)_1$ are satisfied. According to Lemma (3), the set $S^1_{\Phi(.,\zeta(.))}$ is non-empty, and hence, we can define a multi-valued function, $\aleph : PC(\mathrm{Y}, Z) \to 2^{PC(\mathrm{Y},Z)} - \{\phi\}$, which is defined by (11). We will use Lemma (6) to demonstrate that $\aleph$ has a fixed point. So, we will show that $\aleph$ is a contraction. In order to do this, let $\zeta_1, \zeta_2 \in PC(\mathrm{Y}, Z)$ and $y_1 \in \aleph(\zeta_1)$. In view of the definition of $\aleph$, we have

$$y_1(\theta) = \begin{cases} T(\frac{\theta^\alpha}{\alpha})(\zeta_0 + g(\zeta_1)) + \int_0^\theta s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha}) f_1(s)ds, \theta \in \mathrm{Y}_0, \\ g_i(\theta, \zeta_1(b_i^-)), \theta \in J_i,\ i \in \Lambda_{1,r}, \\ T(\frac{\theta^\alpha - s_i^\alpha}{\alpha}) g_i(s_i, \zeta_1(b_i^-)) - \int_0^{s_i} s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha}) f_1(s)ds \\ + \int_0^\theta s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha}) f_1(s)ds, \theta \in \mathrm{Y}_i, i \in \Lambda_{1,r}, \end{cases} \tag{27}$$

where $f_1 \in L^1(\mathrm{Y}, Z)$ satisfies $f_1(\theta) \in \Phi(\theta, \zeta_1(\theta))$.*a.e.*

Next, we consider the multi-valued function $\Gamma : \mathrm{Y} \to 2^Z$, defined by

$$\Gamma(\theta) = \{z \in \Phi(\theta,\ \zeta_2(\theta)) : ||f_1(\theta) - z|| = d(f_1(\theta), \Phi(\theta, \zeta_2(\theta))\}.$$

Since the values of $\Phi$ are in $P_{ck}(Z)$, then the values of $\Gamma(\theta)$ are non-empty. Moreover, $(H\Phi)_2(i)$ implies the measurability of $\Gamma$. Thanks to Theorem III-41 in [39], there is a measurable function, $f_2 : \Upsilon \to Z$, with $f_2 \in \Gamma(\theta)$, *a.e.*, and consequently,

$$\begin{aligned} ||f_1(\theta) - f_2(\theta)|| &= d(f_1(\theta), \Phi(\theta, \zeta_2(\theta)) \\ &\leq h(\Phi(\theta, \zeta_1(\theta)), \Phi(\theta, \zeta_2(\theta))) \\ &\leq \varphi(\theta)\|\zeta_1(\theta) - \zeta_2(\theta)\| \leq \varphi(\theta)||\zeta_1 - \zeta_2||_{PC(\Upsilon,Z)}, \ a.e. \end{aligned} \tag{28}$$

Next, define $y_2 : \Upsilon \to Z$ by

$$y_2(\theta) = \begin{cases} T(\frac{\theta^\alpha}{\alpha})(\zeta_0 + g(\zeta_2)) + \int_0^\theta s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha}) f_2(s)ds, \theta \in \Upsilon_0, \\ g_i(\theta, \zeta_2(b_i^-)), \theta \in J_i, \ i \in \Lambda_{1,r}, \\ T(\frac{\theta^\alpha - s_i^\alpha}{\alpha}) g_i(s_i, \zeta_2(b_i^-)) - \int_0^{s_i} s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha}) f_2(s)ds \\ + \int_0^\theta s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha}) f_2(s)ds, \theta \in \Upsilon_i, i \in \Lambda_{1,r}, \end{cases} \tag{29}$$

Obviously, $y_2 \in \aleph(\zeta_2)$. Now, we compute $||y_1 - y_2||_{PC(\Upsilon,\ Z)}$. If $\theta \in [0, b_1]$, then from (27) and (29), we obtain

$$\begin{aligned} & ||y_1(\theta) - y_2(\theta)|| \\ \leq\ & M\, ||g(\zeta_1) - g(\zeta_2)|| + M \int_0^\theta s^{\alpha-1} ||f_1(s) - f_2(s)|| ds \\ \leq\ & M\, a\, ||\zeta_1 - \zeta_2||_{PC(\Upsilon,\ Z)} + M||\zeta_1 - \zeta_2||_{PC(\Upsilon,Z)} \int_0^\theta s^{\alpha-1} \varphi(s) ds \\ \leq\ & M\, a\, ||\zeta_1 - \zeta_2||_{PC(\Upsilon,Z)} + M||\zeta_1 - \zeta_2||_{PC(\Upsilon,Z)}\, \Pi||\varphi||_{L^p(\Upsilon,Z)}. \end{aligned} \tag{30}$$

If $\theta \in J_i, i \in \Lambda_{1,r}$, then

$$\begin{aligned} & ||y_1(\theta) - y_2(\theta)|| \\ \leq\ & ||g_i(\theta, \zeta_1(b_i^-)) - g_i(\theta, \zeta_2(b_i^-))|| \\ \leq\ & h_i\, ||\zeta_1(b_i^-) - \zeta_2(b_i^-)\, || \\ \leq\ & h\, ||\zeta_1 - \zeta_2||_{PC(\Upsilon,Z)}. \end{aligned} \tag{31}$$

If $\theta \in \Upsilon_i, i \in \Lambda_{1,r}$, then, as in the pervious cases,

$$\begin{aligned} & ||y_1(\theta) - y_2(\theta)|| \\ \leq\ & h\, ||\zeta_1 - \zeta_2||_{PC(\Upsilon,Z)} + 2M\Pi||\zeta_1 - \zeta_2||_{PC(\Upsilon,Z)}\, ||\varphi||_{L^p(\Upsilon,Z)}. \end{aligned} \tag{32}$$

By combining relations (30)–(32), we obtain

$$||y_1 - y_2|| \leq ||\zeta_1 - \zeta_2||_{PC(\Upsilon,Z)}[h + a + 2M\, \Pi||\varphi||_{L^p(\Upsilon,Z)}].$$

By interchanging the rules of $y_1$ and $y_2$, this results in

$$h(\aleph(\zeta_1), \aleph(\zeta_2)) \leq ||\zeta_1 - \zeta_2||_{PC(\Upsilon,Z)}[h + a + 2M\, \Pi||\varphi||_{L^p(\Upsilon,Z)}].$$

This inequality along condition (11) leads to $\aleph$ being a contraction, and therefore, according to Lemma (6), $\aleph$ has a fixed point, which is a solution to Problem (3). □

Now, we offer another set of conditions that make $S_3(A, \Phi)$ non-empty when the values of $\Phi$ are not necessarily convex.

**Theorem 3.** *In addition to* $(Hg)_1$, $(H)_1$, *suppose that the following conditions hold:*
$(H\Phi)_3$ *(i)* $\Phi$ *has a measurable graph, and for any* $\theta \in \mathrm{Y}, z \to \Phi(\theta, z)$ *is lower semicontinuous;*
*(ii) There exists a function,* $\varphi \in L^1(T, \mathbb{R}^+)$, *such that for any* $z \in Z$

$$\|\Phi(\theta, z)\| \leq \varphi(\theta), a.e.\ \theta \in T; \tag{33}$$

*(iii) There is* $\beta \in L^1(\mathrm{Y}, \mathbb{R}^+)$ *such that for any bounded set,* $D \subseteq Z$,

$$\chi\Phi(\theta, D) \leq \beta(\theta)\chi(D).$$

*Then,* $S_3(A, \Phi)$ *is non-empty if condition (10) and the following condition are satisfied.*

$$Ma + h < 1. \tag{34}$$

**Proof.** First, by using Theorem 3 in [40], we show the existence of a continuous selection to the multi-valued Nemitsky operator $\Re : PC(\mathrm{Y}, Z) \to 2^{L^1(T,Z)}$

$$\Re(\zeta) = S^1_{\Phi(.,\zeta(.))} = \{f \in L^1(T, Z) : f(\theta) \in \Phi(\theta, \zeta(\theta))), a.e.\ \theta \in \mathrm{Y}\}.$$

Obviously, $\Re(\zeta)$ is decomposable for any $\zeta \in PC(\mathrm{Y}, Z)$. Since $\Phi$ has a measurable graph and satisfies (33), then according to Theorem 3.2 in [41], $\Re(\zeta)$ is non-empty. Because $\Phi(\zeta)$ is closed, $\Re(\zeta)$ is closed. To prove the lower semicontinuity of $\Re$, it is sufficient to show that, for every $v \in L^1(T, Z)$, $\zeta \to d(v, \Re(\zeta))$ is upper semicontinuous (Proposition 1.2.26 in [42]). This is equivalent to showing that for any $\epsilon \geq 0$, the set

$$v_\epsilon = \{\zeta \in PC(\mathrm{Y}, Z) : d(v, \Re(\zeta)) \geq \epsilon\}.$$

is closed. Let $\{\zeta_n\} \subseteq v_\theta$ and $\zeta_n \to \zeta$ in $PC(\mathrm{Y}, Z)$. Then, for all $\theta \in \mathrm{Y}$, $\zeta_n(\theta) \to \zeta(\theta)$ in $Z$. According to Theorem 2.2 in [41],

$$\inf_{z \in \Re(\zeta_n)} \int_0^b \|v(\theta) - z(\theta)\| d\theta = \int_0^b \inf_{z \in \Re(\zeta_n)} \|v(\theta) - z(\theta)\| d\theta.$$

Then, for any $n \in \mathbb{N}$,

$$\begin{aligned} d(v, \Re(\zeta_n)) &= \inf_{z \in \Re(\zeta_n)} \|v - z\|_{L^1(\mathrm{Y},Z)} = \inf_{v \in \Re(\zeta_n)} \int_0^b \|v(\theta) - z(\theta)\| d\theta \\ &= \int_0^b \inf_{z \in \Re(\zeta_n)} \|v(\theta) - z(\theta)\| d\theta = \int_0^b d(v(\theta), \Phi(\theta, \zeta_n(\theta))) d\theta. \end{aligned}$$

This equality, along with Fatou's Lemma, yields

$$\begin{aligned} \epsilon &\leq \limsup_{n \to \infty} d(v, \Re(\zeta_n)) = \limsup_{n \to \infty} \int_0^b d(v(\theta), \Phi(\theta, \zeta_n(\theta)) d\theta \\ &\leq \int_0^b \limsup_{n \to \infty} d(v(\theta), \Phi(\theta, \zeta_n(\theta)) d\theta. \end{aligned} \tag{35}$$

Since for any $\theta \in \mathrm{Y}$, $\Phi(\theta, z)$ is lower semicontinuous, the function, $z \to d(v(\theta), \Phi(\theta, z))$ is upper semicontinuous [42], and hence, $\lim_{n \to \infty} \sup d(v(\theta), \Phi(\theta, \zeta_n(\theta)) = d(v(\theta), \Phi(\theta, \zeta(\theta))$. Therefore, inequality (35) implies

$$\epsilon \leq \int_0^b d(v(\theta), \Phi(\theta, \zeta(\theta)) d\theta = d(v, \Re(\zeta)),$$

which proves that $v_\epsilon$ is closed; so, $\Re$ is lower semicontinuous. By applying Theorem 3 in [40], $\Re$ has a continuous selection; that is, there is a continuous function $\eta : PC(\mathrm{Y}, Z) \to L^1(\mathrm{Y}, Z)$, $\eta(\zeta) \in \Re(\zeta); \zeta \in PC(T, Z)$. Then, $\eta(\zeta)(s) \in \Phi(s, \zeta(s))$, *a.e.* Now, let $\aleph^* : PC(\mathrm{Y}, Z) \to PC(\mathrm{Y}, Z)$ defined by

$$(\aleph^*\zeta)(t) = \begin{cases} T(\frac{\theta^\alpha}{\alpha})(\zeta_0 + g(\zeta)) + \int_0^\theta s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha})\eta(\zeta)(s), \theta \in \mathrm{Y}_0, \\ g_i(\theta, \zeta_1(b_i^-)), \theta \in J_i, i \in \Lambda_{1,r}, \\ T(\frac{\theta^\alpha - s_i^\alpha}{\alpha}) g_i(s_i, \zeta_1(b_i^-)) - \int_0^{s_i} s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha})\eta(\zeta)(s) ds \\ + \int_0^\theta s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha})\eta(\zeta)(s), \theta \in \mathrm{Y}_i, i \in \Lambda_{1,r}. \end{cases}$$

Notice that the assumption (ii) in $(H\Phi)_3$ leads to $\sup_{u \in \Phi(\theta, z)} ||u|| \leq \varphi(\theta)$, for *a.e.* $\theta \in \mathrm{Y}$ and $z \in Z$. Then, by following what we did in Step 1 in Theorem 1, we can show that relation (34) leads to the existence of $\varrho > 0$ such that $\aleph^*(D_\varrho) \subseteq D_\varrho$. Next, as in Steps 2, 3, 4, and 5 in the proof of Theorem 1, the set $D^* = \cap_{n=1} \aleph^*(D_n)$, $D_1 = \aleph^*(D_\varrho)$, $D_{n+1} = \aleph^*(D_n)$ is non-empty, convex, and compact in $PC(\mathrm{Y}, Z)$. By applying Schauder's fixed-point theorem to the function $\aleph^* : D^* \to D^*$, there is a function $\zeta \in D^*$ with

$$\zeta(\theta) = \begin{cases} T(\frac{\theta^\alpha}{\alpha})(\zeta_0 + g(\zeta)) + \int_0^\theta s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha})\eta(\zeta)(s) ds, \theta \in \mathrm{Y}_0, \\ g_i(\theta, \zeta_1(b_i^-)), \theta \in J_i, i \in \Lambda_{1,r}, \\ T(\frac{\theta^\alpha - s_i^\alpha}{\alpha}) g_i(s_i, \zeta_1(b_i^-)) - \int_0^{s_i} s^{\alpha-1} \theta(\frac{\theta^\alpha - s^\alpha}{\alpha})\eta(\zeta)(s) ds \\ + \int_0^\theta s^{\alpha-1} T(\frac{\theta^\alpha - s^\alpha}{\alpha})\eta(\zeta)(s) ds, \theta \in \mathrm{Y}_i, i \in \Lambda_{1,r}. \end{cases}$$

Since $Z(\zeta)(s) \in \Phi(s, \zeta(s))$, *a.e.*, $S_3(A, \Phi)$ is non-empty. □

## 4. Existence of Solutions to Problem (4)

This section provides three results for the existence of mild solutions to Problem (4). We start by presenting the concepts and facts that we need.

**Definition 5** ([43]). *A linear, closed, densely defined operator $B : D(B) \to Z$ is said to be sectorial of type $(M^*, \tau, \sigma)$, where $M^* > 0, \tau \in \mathbb{R}, \sigma \in (0, \frac{\pi}{2})$ if the following applies:*

(1) $\mathbb{C} - (\tau + S_\theta) \subseteq \rho(B)$;
(2) *For any $\lambda \notin \tau + S_\theta$, $\|R(\lambda, B)\| \leq \frac{M^*}{|\lambda - \omega|}$;*

*where $\tau + S_\sigma = \{\tau + \lambda \in \mathbb{C} : 0 < |\arg(-\lambda)| < \sigma\}$, $\rho(B) = \{\lambda \in \mathbb{C} : (\lambda - B)^{-1}$ exists$\}$ is the resolvent set, and $R(\lambda, B) = (\lambda - B)^{-1}$ is the $\lambda$-resolvent operator of $B$ for any $\lambda$ in $\rho(B)$.*

**Lemma 7** ([43]). *A linear, closed, densely defined sectorial operator, $B$ generates a strongly analytic semigroup, $\{K(\theta) : \theta \geq 0\}$. Moreover,*

$$K(\theta) = \frac{1}{2\pi i} \int_\gamma e^{\lambda\theta} R(\lambda, B) d\lambda, \tag{36}$$

*where $\gamma$ is a suitable path inside $\rho(B)$.*

**Lemma 8** ([29]). *Let $B$ be a linear, closed, densely defined sectorial operator on a Banach space, $Z$, of type $(M, \omega, \sigma)$, where $M^* > 0$, $\omega \in \mathbb{R}$, $\sigma \in (0, \frac{\pi}{2})$ and $f : [0, b] \times Z \to Z$ are continuous. The continuous function*

$$\zeta(\theta) = K(\frac{\theta^\alpha}{\alpha})(\zeta_0 + g(\zeta)) + \int_0^\theta s^{\alpha-1} f(s, \zeta(s)) ds. \tag{37}$$

*is the mild solution to the semilinear Cauchy problem:*

$$\begin{cases} D^{\alpha}\zeta(\theta) = B\zeta(\theta) + f(\theta,\zeta(\theta)),\ \theta \in \Upsilon, \\ \zeta(0) = \zeta_0 + g(\zeta), \end{cases} \tag{38}$$

*where* $K(\theta);\ \theta \geq 0$ *is given by (36).*

Based on this Lemma, we present the concept of mild solutions to Problem (4).

**Definition 6.** *A function,* $\zeta \in PC(\Upsilon, Z)$ *, is called a mild solution to the problem (4) if*

$$\zeta(\theta) = \begin{cases} K(\frac{\theta^{\alpha}}{\alpha})(\zeta_0 + g(\zeta)) + \int_0^{\theta} s^{\alpha-1}K(\frac{\theta^{\alpha}-s^{\alpha}}{\alpha})f(s)ds,\ \theta \in \Upsilon_0, \\ g_i(\theta, \zeta(b_i^{-})),\ \theta \in J_i, i \in \Lambda_{1,r}, \\ K(\frac{\theta^{\alpha}-s_i^{\alpha}}{\alpha})g_i(s_i, \zeta(b_i^{-})) - \int_0^{s_i} s^{\alpha-1}K(\frac{\theta^{\alpha}-s^{\alpha}}{\alpha})f(s)ds \\ + \int_0^{\theta} s^{\alpha-1}K(\frac{\theta^{\alpha}-s^{\alpha}}{\alpha})f(s)ds,\ \theta \in \Upsilon_i,\ i \in \Lambda_{1,r}. \end{cases} \tag{39}$$

*where* $f\ \in S^1_{\Phi(.,\zeta(.))} = \{z \in L^1([0,T], Z) : f(\theta) \in \Phi(\theta, \zeta(\theta)), a.e.\}$.

Now, by following the same arguments used in the proof of Theorems 1–3, we can demonstrate the following results for the existence of mild solutions to Problem (4).

**Theorem 4.** *In addition to the assumptions* $(H\Phi)_1$, $(Hg)_1$, *and* $(H)_1$, *we suppose the following condition:*

$(HB)$ *Let* $B$ *be a linear, closed, densely defined sectorial operator on a Banach space,* $Z$*, of type* $(M^*, \omega, \theta)$*, where* $M^* > 0$, $\omega \in \mathbb{R}$, $\sigma \in (0, \frac{\pi}{2})$.

*Then,* $S_4(B, \Phi)$ *is non-empty and compact in* $PC(\Upsilon, Z)$ *provided that*

$$M^*a + 2M^*\Pi||\varphi||_{L^p(J,\mathbb{R}^+)} + M^*h + h < 1, \tag{40}$$

*and*

$$4M^*\Pi||\beta||_{L^1(\Upsilon,\mathbb{R}^+)} < 1, \tag{41}$$

*where* $M^* = \sup\{||K(\theta)||,\ \theta \geq 0\}$, *and* $h = \sum_{1=1}^{i=r} h_i$.

**Theorem 5.** *If the assumptions* $(HB)$, $(H\Phi)_2$, $(Hg)_2$, *and* $(H)_2$ *hold, then* $S_4(B, \Phi)$ *is non-empty in the Banach space,* $PC(\Upsilon, Z)$*, provided that (40) is satisfied.*

**Theorem 6.** *If the assumptions* $(HB)$, $(H\Phi)_1$, $(Hg)_1$, *and* $(H)_1$ *hold, then* $S_4(B, \Phi)$ *is non-empty in the Banach space,* $PC(\Upsilon, Z)$*, provided that*

$$M^*a + h < 1. \tag{42}$$

**Theorem 7.** *If the assumptions* $(Hg)_1$, $(H)_1$ *,and* $(H\Phi)_3$ *hold, then* $S_4(B, \Phi)$ *is non-empty in the Banach space,* $PC(\Upsilon, Z)$*, provided that (42) is verified.*

## 5. Examples

**Example 1.** *Let* $\alpha = \frac{1}{2}$, $p = 3$, $Z = L^2[0,\infty)$, $\Upsilon = [0,1]$, *and* $s_0 = 0$, $s_i = \frac{2i}{9}$, $t_i = \frac{2i-1}{9}$, $i \in \Lambda_{1,4} = \{1,2,3,4\}$, $t_5 = 1$, $\Delta : Z \to Z$ *be a linear, bounded, compact operator, and let* $\Lambda \in P_{ck}(Z)$ *with* $0 \in \Lambda$. *On* $E$*, define the translation* $C_0$*—semigroup:*$T(t)f(s) = f(t+s);\ f \in Z$. *If* $A$ *is the infinitesimal generator of this semigroup, then* $Af = f'$*, where*

$$D(A) = \{f \in L^2[0,\infty) :\ \textit{the weak derivative } f' \textit{ exists and } f' \in L^2[0,\infty)\}.$$

Define $\Phi : \mathrm{Y} \times Z \to P_{ck}(Z)$, and $g : PC(\mathrm{Y}, Z) \to Z$ and $g_i : [t_i, s_i] \times Z \to Z$ as follows:

$$\Phi(\theta, z) = \frac{v\ ||z|| \sin\theta}{\omega\ (1 + ||z||)} \Lambda\ ; (\theta, z) \in \mathrm{Y} \times Z, \tag{43}$$

$$g(x) = \sum_{i=1}^{i=4} \kappa_i \Delta(x(b_i)), \tag{44}$$

$$g_i(\theta,\ z) := i\kappa_5 \theta \Delta(z),\ \forall (\theta, z) \in [b_i, s_i] \times Z, i \in \Lambda_{1,4}, \tag{45}$$

where $v, \kappa_i (i \in \Lambda_{1,4})$ are positive real numbers and $\omega = \sup\{||u|| : u \in \Lambda\}$. Notice that, for any $z \in Z$, the function $f(\theta) = \frac{v\ ||z|| \sin\theta}{\omega\ (1+||z||)} u_0; u_0 \in Z$ is a strongly measurable selection for the multi-valued function $\theta \to \Phi(\theta, z)$. Additionally, for any $\theta \in \mathrm{Y}$ and any $z,\ y \in Z$, we have

$$\sup_{z \in \Phi(\theta, z)} ||y|| \leq \frac{v\ ||z||\ |\sin\theta|}{(1 + ||z||)} \leq v < v(1 + ||z||), \tag{46}$$

and

$$\begin{aligned} H(\Phi(\theta, z), \Phi(\theta, y)) &\leq v\ ||\frac{||z||}{(1 + ||z||)} - \frac{||y||}{(1 + ||y||)}|| \\ &\leq v||z - y||, \end{aligned} \tag{47}$$

Resulting from (46) and (47), $\Phi(\theta, .)$ is upper semicontinuous, and for any bounded subset, $D \subseteq Z$,

$$\chi(\Phi(\theta, D)) \leq v\ \chi(D),\ \text{for } \theta \in [0, 1].$$

Then, the assumption $(H\Phi)_1$ is verified with $\varphi(\theta) = \beta(\theta) = v$, for $\theta \in [0, 1]$. Furthermore, the compactness of the operator $\Delta$ implies the compactness of $g$ and for any $x \in PC(\mathrm{Y}, Z)$,

$$||g(x)|| \leq ||\sum_{i=1}^{i=4} \kappa_i \Delta(x(b_i))|| \leq \sum_{i=1}^{i=4} \kappa_i ||\Delta||\ ||x(b_i)|| \leq ||\Delta||||x||_{PC(\mathrm{Y},Z)} \sum_{i=1}^{i=4} \kappa_i.$$

and hence, $(Hg)_1$ is satisfied with $a = ||\Delta|| \sum_{i=1}^{i=4} \kappa_i$ and $d = 0$. Next, the compactness of the operator $\Delta$ implies the compactness of $g_i;\ i \in \Lambda_{1,4}$. Moreover, in view of (47),

$$||g_i(\theta, z)|| = i\kappa_5\ ||(\Delta(z))|| \leq i\kappa_5 ||\Delta||\ ||z||_Z, \forall (\theta, z) \in [b_i, s_i] \times Z.$$

Consequently, the assumption $(H)_1$ is verified, where $h_i = i\kappa_5 ||\Delta||,\ i \in \Lambda_{1,4}$. Note that, $\Pi = 2$. By applying Theorem 1, the set of mild solutions to the following nonlocal impulsive conformable fractional semilinear differential inclusion

$$\begin{cases} \frac{d^{\frac{1}{2}}}{d\zeta^{\frac{1}{2}}} \zeta(\theta) \in \zeta\prime(\theta) + \frac{v\ ||\zeta|| \sin\theta}{\omega\ (1+||\zeta||)} \Lambda, a.e.,\ \theta \in \cup_{i=0}^{i=4} (s_i,\ b_{i+1}], \\ \zeta(\theta) = i\kappa_5 \theta \Delta(\zeta),\ \theta \in (b_i, s_i], i \in \Lambda_{1,4}, \\ \zeta(0) = \zeta_0 + \sum_{i=1}^{i=4} \kappa_i \Delta(\zeta(b_i)), \end{cases} \tag{48}$$

is non-empty and compact in $PC(\mathrm{Y}, Z)$ provided that

$$M(||\Delta|| \sum_{i=1}^{i=4} \kappa_i + 4M\ v + 4\kappa_5 ||\Delta||) + 4\kappa_5 ||\Delta|| < 1, \tag{49}$$

and

$$4Mv < 1, \tag{50}$$

where $M = \sup\{||T(\theta)||,\ \theta \geq 0\}$. By choosing $\kappa_i, i = 1, .., 5$, and with $v$ being small enough, both (49) and (50) will be satisfied.

**Example 2.** *Let* $\alpha = \frac{2}{3}$, $p = 2$, $Z = L^2([0,\ \pi])$ *and* $\Upsilon$, $s_i,\ t_i\ , i \in \Lambda_{1,4},\ t_5 = 1, \Delta$, *and let* $\Lambda$ *be as it is in Example 1. On E, define* $A : D(A) \subseteq Z \to Z\ , Az = z''$ *with*

$$D(A) = \{z \in Z : z,\ z' \text{ are absolutely continuous}, z(0) = z(\pi) = 0\}.$$

The operator $A$ is sectorial and is the infinitesimal generator of an analytic semigroup $\{K(t) : t \geq 0\}$,

$$K(t)(x) = \sum_{k=1}^{\infty} \cos kt < x\ , x_k > x_k,\ x \in Z,$$

where $x_k(y) = \sqrt{2}\ \sin ky,\ k = 1, 2, ...,$ is the orthonormal set of eigen functions of $A$. Moreover, $M = 1$. Let $\Phi$, $g$, $g_i, i \in \Lambda_{1,4}$ be as it is in Example 1. Note that $\Pi = 3$. Then, by following the same arguments as in Example 1 and applying Theorem 4, the set of mild solutions to the following nonlocal impulsive conformable fractional semilinear differential inclusion

$$\begin{cases} \frac{d^{\frac{2}{3}}}{d\zeta^{\frac{3}{3}}}\zeta(\theta) \in \zeta''(\theta) + \frac{v\ ||\zeta|| \sin\theta}{\omega\ (1+||\zeta||)}\Lambda, a.e.\ \theta \in \cup_{i=0}^{i=4}(s_i, b_{i+1}], \\ \zeta(\theta) = i\kappa_5\theta\Delta(\zeta),\ \theta \in (b_i, s_i]; i \in \Lambda_{1,4}, \\ \zeta(0) = \zeta_0 + \sum_{i=1}^{i=4} \kappa_i\Delta(\zeta(b_i)), \end{cases} \tag{51}$$

is non-empty and compact in $PC(\Upsilon, Z)$ provided that

$$||\Delta|| \sum_{i=1}^{i=4} \kappa_i + 6\ v + 8\kappa_5||\Delta|| < 1, \tag{52}$$

and

$$4v < 1. \tag{53}$$

By choosing $\kappa_i,\ i \in \Lambda_{1,4}$, and with $v$ being small enough, both (52) and (53) will be satisfied.

## 6. Discussion and Conclusions

Unlike all other known fractional derivatives, the conformable fractional derivative, which was introduced by Khalil et al. [15], satisfies many basic properties of the usual derivative, such as the product rule, quotient rule, mean value theorem, chain rule, and Taylor power series expansion. Therefore, the conformable fractional derivative is the most natural fractional derivative. For this reason, many researchers have shown interest in exploring more of the properties of conformable fractional derivatives and studying differential equations involving them. In this work, six results for the existence of mild solutions are presented for two types of fractional differential inclusions with nonlocal conditions involving the conformable fractional derivative in infinite-dimensional Banach spaces and in the attendance of non-instantaneous impulses. In contrast to [29], we did not assume that the semigroup generated by the linear part is compact. We considered a case where the linear part is the infinitesimal generated by the semigroup of linear bounded operators and a sectorial operator. Additionally, we consider the case where the values of the nonlinear part, $\Phi$, are convex and nonconvex.

We propose the following future directions:

1. Study the controllability of Problems (3) and (4);
2. Prove that the set of mild solutions to Problems (3) and (4) are $R_\delta$- sets;

3. Generalize the results obtained in [44] to the case of replacing the single-valued function f with a multi-valued function;
4. Extend the findings in [32,33] to a case where the right-hand side is a multi-valued function instead of a single-valued function;
5. Use the properties of interval-valued fractional conformable derivatives, introduced by Zhang et al. [31], to study differential inclusions containing this type of fractional derivative.

**Author Contributions:** The authors share equally in all contributions except that the first author alone manages the project. All authors have read and agreed to the published version of the manuscript.

**Funding:** This work was funded by the Deanship of Scientific Research, Vice Presidency for Graduate Studies and Scientific Research, King Faisal University, Saudi Arabia [Grant No. KFU250545].

**Institutional Review Board Statement:** Not applicable.

**Informed Consent Statement:** Not applicable.

**Data Availability Statement:** Data are contained within the article.

**Acknowledgments:** The authors acknowledge the Deanship of Scientific Research at King Faisal University for financial support.

**Conflicts of Interest:** The authors declare no conflict of interest.

## References

1. Ntouyas, S.K. *Fractional Differential Equations, Inclusions and Inequalities with Applications*; MDPI: Basel, Switzerland, 2020. [CrossRef]
2. Kamenskii, M.; Obukhowskii, V.V.; Zecca, P. *Condensing Multivalued Maps and Semilinear Differential Inclusions in Banach Spaces*; Walter de Gruyter: Berlin, Germany, 2001. [CrossRef]
3. Alsheekhhussain, Z.; Ibrahim, A.G.; Al-Sawalha, M.; Jawarneh, Y. The existence of solutions for $w$-weighted $\psi$-Hilfer fractional differential inclusions of order $\mu \in (1,2)$ with non-instantaneous impulses in Banach Spaces. *Fractal Fract.* **2024**, *8*, 144. [CrossRef]
4. Al Nuwairan, M.; Ibrahim, A.G. The weighted generalized Atangana-Baleanu fractional derivative in Banach Spaces-definition and applications. *AIMS Math.* **2024**, *9*, 36293–36335. [CrossRef]
5. Agarwal, R.R.; Hristova, S.; O'Regan, D. *Non-Instantaneous Impulses in Differential Equations*; Springer: Berlin/Heidelberg, Germany, 2017. [CrossRef]
6. Ballinger, G.; Liu, X. Boundedness for impulsive delay differential equations and applications in population growth models. *Nonlinear Anal. Theor. Methods Appl.* **2003**, *53*, 1041–1062. [CrossRef]
7. Church, K.E.M. Applications of Impulsive Differential Equations to the Control of Malaria Outbreaks and Introduction to Impulse Extension Equations: A General Framework to Study the Validity of Ordinary Differential Equation Models with Discontinuities in State. Ph.D Thesis, University of Ottawa, Ottawa, ON, Canada, 2014. [CrossRef]
8. Al Nuwairan, M.; Ibrahim, A.G. Solutions and anti-periodic solutions for impulsive differential equations and inclusions containing Atangana-Baleanu fractional derivative of order in infinite dimensional Banach spaces. *AIMS Math.* **2024**, *9*, 10386–10415. [CrossRef]
9. Alsheekhhussain, Z.; Ibrahim, A.G.; Al-Sawalha, M.M.; Ababneh, O.U. Antiperiodic solutions for impulsive $w$-weighted $\psi$—Hilfer fractional differential inclusions in Banach Spaces. *Fractal Fract.* **2024**, *8*, 376. [CrossRef]
10. Aladsani, F.; Ibrahim, A.G. Existence and Stability of Solutions for $p$-proportional $w$-weighted $\kappa$-Hilfer fractional differential inclusions in the presence of non-instantaneous impulses in Banach Spaces. *Fractal Fract.* **2024**, *8*, 475. [CrossRef]
11. Tarasov, V.E. (Ed.) *Handbook of Fractional Calculus with Applications: Applications in Physics*; Part A. De Gruyter Reference; De Gruyter: Berlin, Germany, 2019; Volume 4. [CrossRef]
12. Baleanu, D.; Lopes, A.M. *Applications in Engineering, Life and Social Sciences, Part B*; De Gruyter: Berlin, Germany, 2019; Volume 8. [CrossRef]
13. Sudsutad, W.; Thaiprayoon, C.; Kongson, J.; Sae-dan, W. A mathematical model for fractal-fractional monkeypox disease and its application to real data. *AIMS Math.* **2024**, *9*, 8516–8563. [CrossRef]
14. Butt, A.I.K.; Imran, M.; Batool, S.; AL Nuwairan, M. Theoretical Analysis of a COVID-19 Fractional Model to Optimally Control the Spread of Pandemics. *Symmetry* **2023**, *15*, 380. [CrossRef]

15. Khalil, R.; Al Horani, A.; Yousef, M.; Sababheh, M. A new definition of fractional derivative. *J. Comput. Appl. Math.* **2014**, *264*, 65–70. [CrossRef]
16. Abdeljawad, T. On conformable fractional calculus. *J. Comput. Appl. Math.* **2015**, *279*, 57–66. [CrossRef]
17. Zhong, W.; Wang, L. Basic theory of initial value problems of conformable fractional differential equations. *Adv. Differ. Equ.* **2018**, *2018*, 321. [CrossRef]
18. Jarad, F.; Uğurlu, E.; Abdeljawad, T.; Băleanu, D. On a new class of fractional operators. *Adv. Differ. Equ.* **2017**, *2017*, 247.
19. Eltayeb, H.; Bachar, I.; Gad-Allah, M. Solution of singular one-dimensional Boussinesq equation using the double conformable Laplace decomposition method. *Adv. Differ. Equ.* **2019**, *2019*, 293. [CrossRef]
20. Chung, W.S. Fractional newton mechanics with conformable fractional derivative. *J. Comput. Appl. Math.* **2015**, *290*, 150–158. [CrossRef]
21. Martinez, L.; Rosales, J.J.; Carreño, C.A.; Lozano, J.M. Electrical circuits described by fractional conformable derivative. *Int. J. Circuit Theory Appl.* **2018**, *46*, 1091–1100. [CrossRef]
22. Zhao, D.; Luo, M. General conformable fractional derivative and its physical interpretation. *Calcolo* **2017**, *54*, 903–917. [CrossRef]
23. Bouaouid, M.; Hilal, K.; Melliani, S. Nonlocal telegraph equation in frame of the conformable time-fractional derivative. *Adv. Math. Phys.* **2019**, *2019*, 7528937. [CrossRef]
24. Meng, S.; Cui, Y. The extremal solution to conformable fractional differential equations involving integral boundary condition. *Mathematics* **2019**, *7*, 186. [CrossRef]
25. Tajadodi, H.; Khan, Z.A.; Irshad, A.R.; Gómez-Aguilar, J.F.; Khan, A.; Khan, H. Exact solutions of conformable fractional differential equations. *Results Phys.* **2021**, *22*, 103916. [CrossRef]
26. Bouaouid, M.; Hilal, K.; Melliani, S. Sequential evolution of conformable differential equations of second order with nonlocal condition. *Adv. Differ. Equ.* **2019**, *2019*, 21. [CrossRef]
27. Chalishajar, D.; Kasinathan, R.; Kasinathan, R.; Sandraseka, V. Trajectory Controllability of Clarke Subdifferential-Type Conformable Fractional Stochastic Differential Inclusions with Non-Instantaneous Impulsive Effects and Deviated Arguments. *Fractal Fract.* **2023**, *7*, 541. [CrossRef]
28. Bouaouid, M.; Hannabou, M.; Hilal, K. Nonlocal Conformable-Fractional Differential Equations with a measure of noncompactness in Banach Spaces. *J. Math.* **2020**, *2020*, 5615080. [CrossRef]
29. Bouaouid, M.; Hilal, K.; Melliani, S. Nonlocal conformable fractional Cauchy problem with sectorial operator. *Indian J. Pure Appl. Math.* **2019**, *50*, 999–1010. [CrossRef]
30. Yousif, M.A.; Agarwal, R.P.; Mohammed, P.O.; Lupas, A.A.; Jan, R.; Chorfi, N. Advanced Methods for Conformable Time-Fractional Differential Equations: Logarithmic Non-Polynomial Splines. *Axioms* **2024**, *13*, 551. [CrossRef]
31. Zhang, L.; Feng, M.; Agarwal, R.P.; Wang, G. Concept and application of interval-valued fractional conformable calculus. *Alex. Eng. J.* **2022**, *61*, 11959–11977. [CrossRef]
32. Ahmad, B.; Alghanmi, M.; Alsaedi, A.; Agarwal, R.P. On an impulsive hybrid system of conformable fractional differential equations with boundary conditions. *Int. J. Syst. Sci.* **2020**, *51*, 958–970. [CrossRef]
33. Ahmad, B.; Agarwal, R.P.; Alghanmi, M.; Alsaedi, A. Langevin equation in terms of conformable differential operators. *Analele ştiinţifice ale Univ. Ovidius Constanţa Ser. Mat.* **2020**, *28*, 514. [CrossRef]
34. Haouam, I. On the conformable fractional derivative and its applications in physics. *J. Theor. Appl. Phys.* **2024**, *18*. Available online: https://oiccpress.com/jtap/article/view/8330 (accessed on 17 March 2025).
35. Wang, J.R.; Ibrahim, A.G.; O'Regan, D.; Elmandouh, A.A. Nonlocal fractional semilinear differential inclusions with noninstantaneous impulses and of order $\alpha \in (1,2)$. *Int. J. Nonlinear Sci. Numer. Simul.* **2021**, *22*, 593–605. [CrossRef]
36. Cardinali, T.; Rubbioni, P. Impulsive mild solution for semilinear differential inclusions with nonlocal conditions in Banach spaces. *Nonlinear Anal. Theor.* **2012**, *75*, 871–879. [CrossRef]
37. Bothe, D. Multivalued perturbation of m-accerative differential inclusions. *Isr. J. Math.* **1998**, *108*, 109–138. [CrossRef]
38. Covitz, H.; Nadler, S.B. Multivalued contraction mapping in generalized metric space. *Isr. J. Math.* **1970**, *8*, 5–11. [CrossRef]
39. Castaing, C.; Valadier, M. *Convex Analysis and Measurable Multifunctions*; Lecture Notes in Mathematics; Springer: Berlin/Heidelberg, Germany, 1977; Volume 580. [CrossRef]
40. Bressan, A.; Colombo, G. Extensions and selections of maps with decomposable values. *Stud. Math.* **1988**, *90*, 69–86. Available online: https://api.semanticscholar.org/CorpusID:118899539 (accessed on 17 March 2025). [CrossRef]
41. Hiai, F.; Umegaki, H. Integrals, conditional expectation, and martingales of multivalued functions, *J. Multi. Anal.* **1977**, *7*, 149–182. [CrossRef]
42. Hu, S.; Papageorgiou, N.S. *Handbook of Multivalued Analysis*; Theory in Mathematics and Applications; Springer: Berlin/Heidelberg, Germany, 1979; Volume 1. Available online: https://link.springer.com/book/9780792346821 (accessed on 17 March 2025).

43. Pazy, A. *Semigropus of Linear Operators and Applications to Partial Differential Equations*; Springer: New York, NY, USA, 1983. Available online: https://link.springer.com/book/10.1007/978-1-4612-5561-1 (accessed on 17 March 2025).
44. Atraoui, M.; Bouaouid, M. On the existence of mild solutions for nonlocal differential equations of the second order with conformable fractional derivative. *Adv. Differ. Equ.* **2021**, *2021*, 447. [CrossRef]

**Disclaimer/Publisher's Note:** The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.