

Article

Trivial Homology Groups of Warped Product Semi-Slant Submanifolds in Kenmotsu Space Forms

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Abstract: This paper investigates the relationship between homology groups and warped product semi-slant submanifolds in Kenmotsu space forms. Some rigidity theorems for vanishing homology groups on warped product semi-slant submanifolds are obtained using the moving-frame method and the second fundamental form inequality. Our results are an extension of previous studies in this direction.

Keywords: Kenmotsu space form; homology groups; stable currents; second fundamental form; pinching theorems

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1. Introduction and Motivation

Originally developed in algebraic topology, homology groups are typically linked to a series of algebraic objects, like modules or abelian groups. This notion compelled us to identify the homology groups if the two shapes could be differentiated by looking at their holes. Originally, homology was a strict mathematical technique used to identify and classify holes in a manifold. The algebraic summary of a manifold can be obtained from its homology groups, which are the most constructive topological invariants. There are numerous uses for this homology, which aids in locating deep topological details about the manifold's dimensions, holes, tunnels, and related components. Indeed, the homology theory has applications in gene expression analysis, protein docking, image segmentation, and root construction. Similarly, Federer–Fleming [1] demonstrated the existence of integral homology classes by rectifiable currents of least mass. Under various geometric conditions, one may create deformations of these currents, which are mass-decreasing. Thus, homology is a useful tool for studying the topology of the manifold in the class of warped product submanifolds.

A new sequence of vanishing homology and non-existence of stable integral currents in warped product submanifolds is constructed, and it first appeared in [2]. By utilizing the results in [3–5] and considering several conditions on the warping function of contact CR-warped product submanifolds in Sasakian space forms with a constant curvature of one, it was proved that homology groups are zero and do not exist in stable currents [see [2] for more detail]. Later on, F. Sahin [6,7] proved similar results for CR-warped

product submanifolds in Euclidean spaces and nearly-Kaehler six spheres $\mathbb{S}^6$. Motivated by previous findings, Ali et al. [8] generalized the study of [2] to be considered a warped product pointwise semi-slant submanifold in the unit sphere. Following the study of Fu et al. [9], Ali et al. [10] derived the same results as [2] for CR-warped products in complex hyperbolic spaces. As applications, such a study took a flow in several types of warped products immersed in different structures [10–15].

In this work, a generalized version of a CR-warped product submanifold was considered as a class of non-trivial warped product semi-slant submanifolds in Kenmotsu space forms of types $\mathbb{H}^{2(\frac{t}{2}+s)+1}$ and $\mathbb{H}^{2(t+s)+1}$ with a holomorphic constant sectional curvature equal to -1 . We show that the stable current does not exist and that there are no homology groups in the warped product semi-slant submanifold in Kenmotsu space forms of types $\mathbb{H}^{2(\frac{t}{2}+s)+1}$ and $\mathbb{H}^{2(t+s)+1}$.

2. Preliminaries

An almost contact structure (φ, ξ, η, g) endowed with a $(2m+1)$ -dimensional manifold $\tilde{V}$ satisfies the following necessary compatibility conditions

$$\varphi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \quad \varphi(\xi) = 0, \quad \eta \circ \varphi = 0. \quad (1)$$

$$g(\varphi E_1, \varphi E_2) = g(E_1, E_2) - \eta(E_1)\eta(E_2), \quad \eta(E_1) = g(E_1, \xi), \quad (2)$$

for any E_1, E_2 tangent to $\tilde{V}$. Moreover, φ , ξ , and η stand for $(1,1)$ -tensor, a vector field, a dual 1-form, respectively. A contact manifold $\tilde{V}$ together with (1) and (2) meets the following conditions according to [16,17]:

$$(\tilde{\nabla}_{E_1}\varphi)E_2 = g(\varphi E_1, E_2)\xi - \eta(E_2)\varphi E_1 \quad (3)$$

$$\tilde{\nabla}_{E_1}\xi = E_1 - \eta(E_1)\xi, \quad (4)$$

which is represented as a Kenmotsu manifold provided that $\tilde{\nabla}$ is the Riemannian connection. Moreover, $\Gamma(T\tilde{V})$ stands for the Lie algebra of vector fields. The Kenmotsu manifold is a Kenmotsu space form of a constant φ -sectional curvature ϵ when the Riemannian curvature tensor can be written as (see [8])

$$\begin{aligned} \tilde{R}(E_1, E_2, E_3, E_4) = & \left(\frac{\epsilon - 3}{4} \right) \left\{ g(E_2, E_3)g(E_1, E_4) - g(E_1, E_3)g(E_2, E_4) \right\} \\ & + \left(\frac{\epsilon + 1}{4} \right) \left\{ \eta(E_1)\eta(E_3)g(E_2, E_4) + \eta(E_4)\eta(E_2)g(E_1, E_3) \right. \\ & \quad - \eta(E_2)\eta(E_3)g(E_1, E_4) - \eta(E_1)g(E_2, E_3)\eta(E_4) \\ & \quad + g(\varphi E_2, E_3)g(\varphi E_1, E_4) - g(\varphi E_1, E_3)g(\varphi E_2, E_4) \\ & \quad \left. + 2g(E_1, \varphi E_2)g(\varphi E_3, E_4) \right\} \end{aligned} \quad (5)$$

where $E_1, E_2, E_3, E_4 \in \Gamma(T\tilde{V}^{2m+1})$.

Remark 1. In [18], Chinea and Gonzales proved that a Kenmotsu space form with a φ -sectional curvature $\epsilon = -1$ stands for a hyperbolic space $\mathbb{H}^{2n+1} = \{x_1 \cdots x_{2n+1} \in \mathbb{R}^{2n+1} | x_i > 0\}$.

The Weingarten and Gauss formulas, where $\nabla(\nabla^\perp)$ represents induced connections on the tangent bundle TV (normal bundle $T^\perp V$) of submanifold $V \subseteq \tilde{V}$, are given by

$$(i) \quad \tilde{\nabla}_{E_1} E_3 = -A_{E_3} E_1 + \nabla_{E_1}^\perp E_3, \quad (ii) \quad \tilde{\nabla}_{E_1} E_2 = \nabla_{E_1} E_2 + B(E_1, E_2), \quad (6)$$

for each $E_1, E_2 \in \Gamma(TV)$ and $E_3 \in \Gamma(T^\perp V)$, where B and A_{E_3} are the second fundamental form and shape operator. They are written as

$$g(B(E_1, E_2), E_3) = g(A_{E_3} E_1, E_2). \quad (7)$$

For any vector field $E_1 \in \Gamma(TV)$, from (4) and (6) (i), we may write

$$(i) \quad \nabla_{E_1} \xi = \{E_1 - \eta(E_1)\xi\}, \quad (ii) \quad B(E_1, \xi) = 0. \quad (8)$$

If $E_1 \in \Gamma(TV)$ and $E_2 \in \Gamma(T^\perp V)$,

$$(i) \quad \varphi E_1 = TE_1 + FE_1, \quad (ii) \quad \varphi E_2 = tE_2 + fE_2, \quad (9)$$

where $TE_1(tE_2)$ and $FE_1(fE_2)$ stand for tangential and normal components of $\varphi E_1(\varphi E_2)$, respectively. The $(1, 1)$ -tensor T follows that

$$g(TE_1, E_2) = -g(E_1, TE_2), \quad (10)$$

for every $E_1, E_2 \in \Gamma(TV)$. If the following equation holds:

$$\begin{aligned} \tilde{R}(E_1, E_2, F_3, F_4) = & g(B(E_1, F_3), B(E_2, F_4)) - g(B(E_1, F_4), B(E_2, F_3)) \\ & + R(E_1, E_2, F_3, F_4), \end{aligned} \quad (11)$$

for any $E_1, E_2, F_3, F_4 \in \mathfrak{X}(TV)$, then Equation (11) is called a Gauss equation.

It should be noted that throughout this manuscript, we will use V as a submanifold of the *Kenmotsu manifold* $\tilde{V}$, i.e., $V \subseteq \tilde{V}$. Now, we have the following classification of the submanifold theory:

- (a) If the H mean curvature of V is equal to zero, then V is minimal in $\tilde{V}$.
- (b) If the second fundamental form B is absent everywhere on V , then V is totally geodesic in $\tilde{V}$.
- (c) A totally umbilical form of V in $\tilde{V}$ implies $B(E_1, E_2) = g(E_1, E_2)H$, for any $E_1, E_2 \in \Gamma(TV)$.
- (d) If the tangent bundle $T_p V$ of V is satisfied $\varphi(T_p V) \subseteq T_p V$ for $\xi \in V$, then V is an invariant submanifold. An anti-invariant submanifold is satisfied $\varphi(T_p V) \subset T_p^\perp V$ for a normal bundle $T_p^\perp V$ of V at point p .
- (e) If the angle $\lambda(E_1)$ between φE_1 and $T_p V$ is constant for any vector $0 \neq E_1 \in V$, then V is a slant submanifold of $\tilde{V}$ along slant angle $\lambda(E_1)$ (see [19] for more details). Moreover, V is a slant if it satisfies the following:

$$T^2 = \nu(-I + \eta \otimes \xi), \quad (12)$$

for $\nu \in [0, 1]$ and $\nu = \cos^2 \lambda$ [17,19].

- (f) If the tangent bundle TV is satisfied $TV = \mathfrak{D} \oplus \mathfrak{D}^\perp \oplus \langle \xi \rangle$ for an invariant distribution $\mathfrak{D}$, anti-invariant $\mathfrak{D}^\perp$, then V is a CR-submanifold. Moreover, $\langle \xi \rangle$ is the 1-dimensional distribution spanned by the structure vector field ξ .

Motivated by (12), we have a useful equality

$$g(TE_1, TE_2) = \cos^2 \lambda \left(g(E_1, E_2) - \eta(E_1)\eta(E_2) \right), \quad (13)$$

$$g(FE_1, FE_2) = \sin^2 \lambda \left(g(E_1, E_2) - \eta(E_1)\eta(E_2) \right), \quad (14)$$

for any $E_1, E_2 \in \Gamma(TV)$.

There are some other classifications, such as the following:

- (i) A submanifold V stands for a semi-slant submanifold [20] if the tangent bundle TV is satisfied $TV = \mathfrak{D} \oplus \mathfrak{D}^\lambda \oplus \langle \xi \rangle$ for the φ -invariant $\mathfrak{D}$ and proper slant $\mathfrak{D}^\lambda$ with a slant angle λ .
- (ii) A submanifold V stands for a pseudo-slant submanifold if the tangent bundle TV satisfies $TV = \mathfrak{D}^\perp \oplus \mathfrak{D}^\lambda \oplus \langle \xi \rangle$ for the anti-invariant $\mathfrak{D}^\perp$ and slant distribution $\mathfrak{D}^\lambda$ with an angle λ .

Now, if the tangent space TV has an orthonormal basis $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ and a normal bundle $T^\perp V$ has the orthonormal basis $\alpha_r \in \{\alpha_{n+1}, \alpha_{n+2}, \dots, \alpha_m\}$, then we define

$$B_{cd}^r = g(B(\alpha_c, \alpha_d), \alpha_r), \quad \|B\|^2 = \sum_{c,d=1}^n g(B(\alpha_c, \alpha_d), B(\alpha_c, \alpha_d)). \quad (15)$$

Furthermore, consider the gradient $\nabla\psi$ of the function ψ on V . Then, for any $E_1 \in \Gamma(TV)$, we have the following equalities:

$$\|\nabla\psi\|^2 = \sum_{c=1}^n (\alpha_c(\psi))^2, \quad g(\nabla\psi, E_1) = E_1(\psi). \quad (16)$$

φ has a covariant derivative

$$(\tilde{\nabla}_{E_1}\varphi)E_2 = \tilde{\nabla}_{E_1}\varphi E_2 - \varphi\tilde{\nabla}_{E_1}E_2 \quad (17)$$

for any $E_1, E_2 \in \Gamma(TV)$.

A general warped product of type $V = V_1 \times_\mu V_2$ is defined in [21] such that μ is a warping function defined on the base V_1 , and it is a positive differentiable function. The following formula is defined for vector fields F_1 and G_1 on V_1 and V_2 , respectively, in Lemma 7.3 [21]:

$$\nabla_{F_1}G_1 = \nabla_{G_1}F_1 = F_1(\ln \mu)G_1, \quad (18)$$

$$R(G_1, G_2)F_1 = \frac{H^\mu(G_1, F_1)}{\mu}G_2, \quad (19)$$

where ∇ denotes the Levi-Civita connection on base V . It is defined that the base V_1 is totally geodesic in V and the fiber V_2 is totally umbilical in $V = V_1 \times_\mu V_2$. Motivated by these concepts,

CR-Warped Product Manifold

A warped product of type $V = V_T \times_\mu V_\perp$ is called a CR-warped product that includes invariant V_T and anti-invariant submanifolds $V_\perp$. Moreover, warped product semi-slant submanifolds in the Kenmotsu manifold are discussed [17,22]. In our study, we consider the warped product semi-slant submanifold $V = V_T \times_\mu V_\lambda$ that includes invariant, anti-invariant, and slant submanifolds $\mathfrak{D}$, $V_\perp$ and V^λ , respectively. For simplicity throughout this section, the tangent spaces of V_T and V_λ are represented by $\mathfrak{D}$ and $\mathfrak{D}^\lambda$, respectively.

It can be seen that ξ is tangent to V_T if the semi-slant warped product submanifold discussed in [23] exists. That is, we have

$$\nabla_{F_1} \xi = F_1,$$

for any vector field $F_1 \in \Gamma(\mathfrak{D}^\lambda)$. Using (18) and then taking the inner product with F_1 , we obtain

$$\xi(\ln f) = 1.$$

The optimization for the second fundamental form was obtained by Lawson–Simons [3], which resulted in the nonexistence of stable currents in compact submanifolds in a connected space form and vanishing homology in a range of intermediate dimensions. For our preliminary findings, we shall use Theorem 1.1 [9], which is

Theorem 1. [9] Let V^n be a compact submanifold of dimension n in a hyperbolic space $\mathbb{H}^m(c)$ with a negative constant curvature $c < 0$. If the strict inequality for the second fundamental form B

$$\sum_{\alpha=1}^p \sum_{\beta=p+1}^n \left(2||B(e_\alpha, e_\beta)||^2 - g(B(e_\alpha, e_\alpha), B(e_\beta, e_\beta)) \right) < p(n+p)c, \quad (20)$$

is satisfied, then V^n has no stable p -currents with a vanished p th homology group, i.e.,

$$H_p(V^n, \mathbb{Z}) = H_q(V^n, \mathbb{Z}) = 0,$$

where $p + q = n$.

Based on the above theorem, we announce our result as follows:

Theorem 2. If the warping function μ of the compact warped product semi-slant submanifold $V^n = V_T^{t+1} \times_\mu V_\lambda^s$ in the Kentmotsu space form $\mathbb{H}^{2(\frac{t}{2}+s)+1}$ satisfies the following equality with $n = t + s + 1$:

$$\left\{ (1-s)||\nabla \ln \mu||^2 + (2 \csc^2 \lambda + 1) \left(||\nabla \ln \mu||_{D-sp\{\xi\}}^2 - 1 \right) \right\} < \frac{1}{s} \left\{ s \Delta \ln \mu + 2s - 2(t+1)^2 \right\}, \quad (21)$$

then the stable $(t+1)$ -current does not exist and there are no homology groups in V^n , i.e., $H_{t+1}(V^n, \mathbb{Z}) = H_s(V^n, \mathbb{Z}) = 0$, where $\Delta \ln \mu$ and $\nabla \ln \mu$ of μ are defined as the Laplacian and the gradient, respectively.

Proof. Suppose the $V^n = V_T^{t+1} \times_\mu V_\lambda^s$ is a $(n = t + s + 1)$ -dimensional warped product semi-slant submanifold. If V_λ^s and V_T^{t+1} stand for integrable manifolds of D^λ and D , such that $\dim(V_T^{t+1}) = t + 1 = 2p_1 + 1$ and $\dim(V_\lambda^s) = s = 2p_2$, respectively, then the orthonormal bases of TV_T and TV_λ are represented as $\{\alpha_1, \alpha_2, \dots, \alpha_{p_1}, \alpha_{p_1+1} = \varphi\alpha_1, \dots, \alpha_{2p_1} = \varphi\alpha_{p_1}, \alpha_{2p_1+1} = \xi\}$ and $\{\alpha_{2p_1+2} = \alpha_1^*, \dots, \alpha_{2p_1+p_2} = \alpha_{p_2}^*, \alpha_{2p_1+p_2+1} = \alpha_{p_2+1}^* = \sec \lambda T\alpha_1^*, \dots, \alpha_{t+s+1} = \alpha_s^* = \sec \lambda T\alpha_{p_2}^*\}$. On the other hand, the orthonormal basis of FD^λ is represented as $\{\alpha_{n+1} = \bar{\alpha}_1 = \csc \lambda F\alpha_1^*, \dots, \alpha_{n+p_2} = \bar{\alpha}_{p_2} = \csc \lambda F\alpha_1^*, \alpha_{n+p_2+1} = \bar{\alpha}_{p_2+1} = \csc \lambda \sec \lambda FT\alpha_1^*, \dots, \alpha_{n+2p_2} = \bar{\alpha}_{2p_2} = \csc \lambda \sec \lambda FT\alpha_{p_2}^*\}$. Thus, by combining the

equation Gauss (11) and the Kenmotsu manifold $\mathbb{H}^{2(\frac{t}{2}+s)+1}$ of constant section curvature $\epsilon = -1$ (5), we have

$$\sum_{p_1=1}^{t+1} \sum_{p_2=1}^s g(R(\alpha_{p_1}, \alpha_{p_2})\alpha_{p_1}, \alpha_{p_2}) = (t+1)s + ||B(\alpha_{p_1}, \alpha_{p_2})||^2 - \sum_{p_1=1}^{t+1} \sum_{p_2=1}^s g(B(\alpha_{p_2}, \alpha_{p_2}), B(\alpha_{p_1}, \alpha_{p_1})). \quad (22)$$

Through a rearrangement of the second fundamental term, we find that

$$\begin{aligned} \sum_{p_1=1}^{t+1} \sum_{p_2=1}^s g(R(\alpha_{p_1}, \alpha_{p_2})\alpha_{p_1}, \alpha_{p_2}) + ||B(\alpha_{p_1}, \alpha_{p_2})||^2 \\ = 2||B(\alpha_{p_1}, \alpha_{p_2})||^2 + (t+1)s \\ - \sum_{p_1=1}^{t+1} \sum_{p_2=1}^s g(B(\alpha_{p_2}, \alpha_{p_2}), B(\alpha_{p_1}, \alpha_{p_1})). \end{aligned} \quad (23)$$

From orthonormal bases $\{\alpha_i\}_{1 \leq p_1 \leq t+1}$ and $\{\alpha_p\}_{1 \leq p_2 \leq s}$ of V_T^{t+1} and V_λ^s , respectively, in (19), we derive

$$R(\alpha_{p_1}, \alpha_{p_2})\alpha_{p_1} = \frac{\alpha_{p_2}}{\mu} H^\mu(\alpha_{p_1}, \alpha_{p_1}).$$

Utilizing the inner product with $\{\alpha_{p_2}\}_{1 \leq b \leq s}$ leads to

$$\sum_{p_1=1}^{t+1} \sum_{p_2=1}^s g(R(\alpha_{p_1}, \alpha_{p_2})\alpha_{p_1}, \alpha_{p_2}) = \frac{s}{\mu} \sum_{p_1=1}^{t+1} g(\nabla_{\alpha_{p_1}} \nabla \mu, \alpha_{p_1}). \quad (24)$$

Thus, from Equations (23) and (24), we derive

$$\begin{aligned} \sum_{p_1=1}^{t+1} \sum_{p_2=1}^s \left(2||B(\alpha_{p_1}, \alpha_{p_2})||^2 - g(B(\alpha_{p_2}, \alpha_{p_2}), B(\alpha_{p_1}, \alpha_{p_1})) \right) + (t+1)s \\ = \frac{s}{\mu} \sum_{p_1=1}^{t+1} \sum_{p_2=1}^s g(\nabla_{\alpha_{p_1}} \nabla \mu, \alpha_{p_1}) \\ + \sum_{p_1=1}^{t+1} \sum_{p_2=1}^s g(B(\alpha_{p_1}, \alpha_{p_2}^*), \alpha_{p_2})^2. \end{aligned} \quad (25)$$

In order to work with μ , we must first calculate its Laplacian $\Delta \mu$ for V^n , allowing one to obtain

$$\begin{aligned} \Delta \mu &= - \sum_{i=1}^n g(\nabla_{\alpha_i} \text{grad} \mu, \alpha_i) \\ &= - \sum_{p_1=1}^{t+1} g(\nabla_{\alpha_{p_1}} \text{grad} \mu, \alpha_{p_1}) - \sum_{p_2=1}^s g(\nabla_{\alpha_{p_2}} \text{grad} \mu, \alpha_{p_2}). \end{aligned}$$

Then, one obtains

$$\begin{aligned}\Delta\mu = & -\sum_{p_1=1}^{p+1} g(\nabla_{\alpha_{p_1}} \text{grad}\mu, \alpha_{p_1}) - \sum_{j=1}^b g(\nabla_{\alpha_j} \text{grad}\mu, \alpha_j) \\ & - \sec^2 \lambda \sum_{j=1}^b g(\nabla_{T\alpha_j} \text{grad}\mu, T\alpha_j).\end{aligned}$$

After simplification, we arrive at

$$\begin{aligned}\Delta\mu = & -\sum_{p_1=1}^{t+1} g(\nabla_{\alpha_{p_1}} \text{grad}\mu, \alpha_{p_1}) - \sum_{j=1}^b \left(\alpha_j g(\text{grad}\mu, \alpha_j) - g(\nabla_{\alpha_j} \alpha_j, \text{grad}\mu) \right) \\ & - \sec^2 \lambda \sum_{j=1}^b \left(T\alpha_j g(\text{grad}\mu, T\alpha_j) - g(\nabla_{T\alpha_j} T\alpha_j, \text{grad}\mu) \right).\end{aligned}$$

When the gradient function property is used, then it can be determined that

$$\begin{aligned}\Delta\mu = & -\sum_{p_1=1}^{t+1} g(\nabla_{\alpha_{p_1}} \text{grad}\mu, \alpha_{p_1}) - \sum_{j=1}^b \left(\alpha_j (\alpha_j \mu) - (\nabla_{\alpha_j} \alpha_j \mu) \right) \\ & - \sec^2 \lambda \sum_{j=1}^b \left(T\alpha_j (T\alpha_j (\mu)) - (\nabla_{T\alpha_j} T\alpha_j \mu) \right).\end{aligned}$$

The result of some computation is as follows:

$$\begin{aligned}\Delta\mu = & -\sum_{p_1=1}^{t+1} g(\nabla_{\alpha_{p_1}} \text{grad}\mu, \alpha_{p_1}) - \sum_{j=1}^b \left(\alpha_j (g(\text{grad}\mu, \alpha_j)) - g(\nabla_{\alpha_j} \alpha_j, \text{grad}\mu) \right) \\ & - \sec^2 \lambda \sum_{j=1}^b \left(T\alpha_j (g(\text{grad}\mu, T\alpha_j)) - g(\nabla_{T\alpha_j} T\alpha_j, \text{grad}\mu) \right).\end{aligned}$$

It is observed that V_T^{t+1} is totally geodesic in V^n . It indicates that $\nabla\mu \in \mathfrak{X}(TV_T)$. Thus, we obtain

$$\begin{aligned}\Delta\mu = & -\frac{1}{\mu} \sum_{j=1}^b \left(g(\alpha_j, \alpha_j) \|\nabla(\mu)\|^2 + \sec^2 \lambda g(T\alpha_j, T\alpha_j) \|\nabla(\mu)\|^2 \right) \\ & - \sum_{p_1=1}^{t+1} g(\nabla_{\alpha_{p_1}} \text{grad}\mu, \alpha_{p_1}).\end{aligned}$$

Finally, from (13), we obtain

$$\frac{\Delta\mu}{\mu} = -\frac{1}{\mu} \sum_{p_1=1}^{t+1} g(\nabla_{\alpha_{p_1}} \text{grad}\mu, \alpha_{p_1}) - s \|\nabla(\ln \mu)\|^2.$$

By using the relation $\frac{\Delta\mu}{\mu} = \Delta(\ln \mu) - \|\nabla(\ln \mu)\|^2$, the previous equation gives

$$\frac{1}{\mu} \sum_{p_1=1}^{t+1} g(\nabla_{\alpha_{p_1}} \text{grad}\mu, \alpha_{p_1}) = -\Delta(\ln \mu) + (1-s) \|\nabla \ln \mu\|^2. \quad (26)$$

Thus, from (25) and (26), we compute that

$$\begin{aligned} \sum_{p_1=1}^{t+1} \sum_{p_2=1}^s \left\{ 2||B(\alpha_{p_1}, \alpha_{p_2})||^2 - g(B(\alpha_{p_2}, \alpha_{p_2}), B(\alpha_{p_1}, \alpha_{p_1})) \right\} \\ = -s\Delta(\ln f) + s(1-s)||\nabla(\ln \mu)||^2 \\ + \sum_{p_2=1}^s \sum_{p_1=1}^{t+1} g(B(\alpha_{p_1}, \alpha_{p_2}^*), \alpha_{p_2})^2 - (t+1)s. \end{aligned} \quad (27)$$

Now, the challenge is to derive the third term on the right-hand side of the proceeding equation, as follows:

$$\sum_{p_1=1}^{t+1} \sum_{p_2=1}^s g(B(\alpha_{p_1}, \alpha_{p_2}^*), \alpha_{p_2})^2 = \sum_{p_1=1}^t \sum_{p_2=1}^s g(B(\alpha_{p_1}, \alpha_{p_2}^*), \alpha_{p_2})^2 + \sum_{p_2=1}^s g(B(\nu, \alpha_{p_2}^*), \alpha_{p_2})^2.$$

By considering the dimension of the Kenmotsu manifold $\mathbb{H}^{2(\frac{t}{2}+s)+1}$, the ν -component in the second part on the right side with FD^λ -component is identical to zero [17]. From the definition of orthonormal basis of V_T^{t+1} and V_λ^s , such that $1 \leq p_1 \leq t+1$ and $1 \leq p_2 \leq s$, we define

$$\begin{aligned} \sum_{p_1=1}^{t+1} \sum_{p_2=1}^s g(B(\alpha_{p_1}, \alpha_{p_2}^*), \alpha_{p_2})^2 &= \csc^2 \lambda \sum_{l_1=1}^a \sum_{l_2,k=1}^b g(B(\alpha_{l_1}, \alpha_{l_2}^*), F\alpha_k^*)^2 \\ &+ \csc^2 \lambda \sec^2 \lambda \sum_{l_1=1}^a \sum_{l_2,k=1}^b g(B(\alpha_{l_1}, P\alpha_{l_2}^*), F\alpha_k^*)^2 \\ &+ \csc^2 \lambda \sec^2 \lambda \sum_{l_1=1}^a \sum_{l_2,k=1}^b g(B(\varphi\alpha_{l_1}, \alpha_{l_2}^*), FP\alpha_k^*)^2 \\ &+ \csc^2 \lambda \sec^2 \lambda \sum_{l_1=1}^a \sum_{l_2,k=1}^b g(B(\varphi\alpha_{l_1}, \alpha_{l_2}^*), FT\alpha_k^*)^2 \\ &+ \csc^2 \lambda \sec^4 \lambda \sum_{l_1=1}^a \sum_{l_2,k=1}^b g(B(\varphi\alpha_{l_1}, T\alpha_{l_2}^*), FT\alpha_k^*)^2. \\ &+ \csc^2 \lambda \sec^2 \lambda \sum_{l_1=1}^a \sum_{l_2,k=1}^b g(B(\varphi\alpha_{l_1}, T\alpha_{l_2}^*), F\alpha_k^*)^2 \\ &+ \csc^2 \lambda \sum_{l_1=1}^a \sum_{l_2,k=1}^b g(B(\varphi\alpha_{l_1}, \alpha_{l_2}^*), F\alpha_k^*)^2 \\ &+ \csc^2 \lambda \sec^4 \lambda \sum_{l_1=1}^a \sum_{l_2,k=1}^b g(B(\alpha_{l_1}, T\alpha_{l_2}^*), FT\alpha_k^*)^2. \end{aligned}$$

By referring back to all the relations of Lemma 4 [17], it is derived that

$$\begin{aligned} \sum_{p_1=1}^{t+1} \sum_{p_2=1}^s g(B(\alpha_{p_1}, \alpha_{p_2}^*), \alpha_{p_2})^2 &= 2 \left(\csc^2 \lambda + \cot^2 \lambda \right) \sum_{l_1=1}^a \sum_{l_2=1}^b (\alpha_{l_1} \ln \mu)^2 g(\alpha_{l_2}^*, \alpha_{l_2}^*)^2 \\ &+ 2 \left(\csc^2 \lambda + \cot^2 \lambda \right) \sum_{l_1=1}^a \sum_{l_2=1}^b (\varphi\alpha_{l_1} \ln \mu)^2 g(\alpha_{l_2}^*, \alpha_{l_2}^*)^2 \\ &- 2 \left(\csc^2 \lambda + \cot^2 \lambda \right) \sum_{l_1=1}^{t+1} \sum_{l_2=1}^s \eta(\alpha_{l_1}) g(\alpha_{l_2}, \alpha_{l_2}). \end{aligned}$$

This relation will be written as

$$\sum_{p_2=1}^s \sum_{p_1=1}^{t+1} g(B(\alpha_{p_1}, \alpha_{p_2}^*), \alpha_{p_2})^2 = 2 \left(\csc^2 \lambda + \cot^2 \lambda \right) \|\nabla \ln \mu\|_{D-sp\{\xi\}}^2 \sum_{p_2=1}^s g(\alpha_{p_2}^*, \alpha_{p_2}^*)^2 - 2b \left(\csc^2 \lambda + \cot^2 \lambda \right),$$

which implies

$$\sum_{p_2=1}^s \sum_{p_1=1}^{t+1} g(B(\alpha_{p_1}, \alpha_{p_2}^*), \alpha_{p_2})^2 = s \left(\csc^2 \lambda + \cot^2 \lambda \right) \left(\|\nabla \ln \mu\|_{D-sp\{\xi\}}^2 - 1 \right). \quad (28)$$

Following on from (27) and (28), we arrive at

$$\begin{aligned} \sum_{p_2=1}^s \sum_{p_1=1}^{t+1} \left\{ 2\|B(\alpha_{p_1}, \alpha_{p_2})\|^2 - g(B(\alpha_{p_2}, \alpha_{p_2}), B(\alpha_{p_1}, \alpha_{p_1})) \right\} \\ = -(t+1)s - s\Delta(\ln \mu) + s \left(\csc^2 \lambda + \cot^2 \lambda \right) \|\nabla(\ln \mu)\|_{D-sp\{\xi\}}^2 \\ + s(1-s) \|\nabla(\ln \mu)\|^2 - s \left(\csc^2 \lambda + \cot^2 \lambda \right). \end{aligned}$$

Equation (21) in Theorem 2 must be satisfied for the following equation to be true:

$$\sum_{p_2=1}^s \sum_{p_1=1}^{t+1} \left\{ 2\|B(\alpha_{p_1}, \alpha_{p_2})\|^2 - g(B(\alpha_{p_2}, \alpha_{p_2}), B(\alpha_{p_1}, \alpha_{p_1})) \right\} < -(t+1)(n+t+1). \quad (29)$$

We may invoke Theorem 1.1 from [9]. This completes the proof of the theorem. $\square$

Theorem 3. Let $V^n = V_T^{2t+1} \times_\mu V_\lambda^s$ be a compact warped product semi-slant submanifold of the Kenmotsu manifold $\mathbb{H}^{2(t+s)+1}$ when $n = 2t + s + 1$ and the following restriction holds:

$$\|\nabla \ln \mu\|_{D-sp\{\xi\}}^2 < \left\{ 1 - \frac{(2t+1)(n+2t+1)}{2s(\csc^2 \lambda + \cot^2 \lambda)} \right\}. \quad (30)$$

Then, the stable $(2t+1)$ -current does not exist and there are no homology groups in V^n , i.e., $H_{2t+1}(V^n, \mathbb{Z}) = H_s(V^n, \mathbb{Z}) = 0$.

Proof. We will start the equality from Lemma 4.1 (i) of [11] as follows:

$$g(B(X_1, X_2), FZ_1) = 0, \quad (31)$$

for any $X_1, X_2 \in \mathfrak{X}(TV_T)$ and $Z_1 \in \mathfrak{X}(TV_\lambda)$. Then, we obtain the following conditions:

$$\sum_{p_2=1}^s \sum_{p_1=1}^{2t+1} \left(2\|B(\alpha_{p_1}, \alpha_{p_2})\|^2 - g(B(\alpha_{p_2}, \alpha_{p_2}), B(\alpha_{p_1}, \alpha_{p_1})) \right) = 2 \sum_{p_1=1}^{2t+1} \sum_{p_2=1}^s \|B(\alpha_{p_1}, \alpha_{p_2})\|^2.$$

Thus, following on from Equation (28), we obtain

$$\begin{aligned} \sum_{p_2=1}^s \sum_{p_1=1}^{2t+1} \left(2\|B(\alpha_{p_1}, \alpha_{p_2})\|^2 - g(B(\alpha_{p_2}, \alpha_{p_2}), B(\alpha_{p_1}, \alpha_{p_1})) \right) \\ = 2s \left\{ \left(\csc^2 \lambda + \cot^2 \lambda \right) \left(\|\nabla \ln \mu\|_{D-sp\{\xi\}}^2 - 1 \right) \right\}. \quad (32) \end{aligned}$$

If (30) is satisfied, then, from (32), we obtain

$$\sum_{p_2=1}^s \sum_{p_1=1}^{2t+1} \left(2\|B(\alpha_{p_1}, \alpha_{p_2})\|^2 - g(B(\alpha_{p_2}, \alpha_{p_2}), B(\alpha_{p_1}, \alpha_{p_1})) \right) < -(2t+1)(n+2t+1).$$

Therefore, using Theorem 1.1 [9], we obtain the required proof. $\square$

An application of Theorem 2 and Lawson–Simon (Theorem 4 of [8], p. 441) is given in the theorem.

Theorem 4. *If the warping function μ of the compact warped product semi-slant submanifold $V^n = V_T^{t+1} \times_{\mu} V_{\lambda}^s$ in a Kentmotsu space form $\mathbb{H}^{2(\frac{t}{2}+s)+1}$ satisfies the following with $n = t + s + 1$:*

$$\left\{ (1-s)\|\nabla \ln \mu\|^2 + (2\csc^2 \lambda + 1) \left(\|\nabla \ln \mu\|_{D-sp\{\xi\}}^2 - 1 \right) \right\} < \frac{1}{s} \left\{ s\Delta \ln \mu + 2s - 2(t+1)^2 \right\},$$

then V^n is homeomorphic to S^n , provided that $n = 4$, and it is homotopic to S^n if $n = 3$, where S^n is a standard unit sphere.

Proof. The conclusion and explanation are given in Theorem 1.2 [9]. This completes the proof of the theorem. $\square$

The following corollaries are a direct consequence of Theorems 2 and 3 by substituting $\lambda = \frac{\pi}{2}$ to a generalized CR-warped product as

Corollary 1. *Let $V^n = V_T^{t+1} \times_{\mu} V_{\perp}^s$ be a compact contact CR-warped product submanifold of $\mathbb{H}^{2(\frac{t}{2}+s)+1}$ having the inequality*

$$\left\{ (1-s)\|\nabla \ln \mu\|^2 + 3 \left(\|\nabla \ln \mu\|_{D-sp\{\xi\}}^2 - 1 \right) \right\} < \frac{1}{s} \left\{ s\Delta \ln \mu + 2s - 2(t+1)^2 \right\},$$

then the stable $(t+1)$ -current does not exist and there are no homology groups in V^n , i.e., $H_{t+1}(V^n, \mathbb{Z}) = H_s(V^n, \mathbb{Z}) = 0$.

From Theorem 3, we have

Corollary 2. *Let $V^n = V_T^{2t+1} \times_{\mu} V_{\perp}^s$ be a compact contact CR-warped product submanifold of the Kenmotsu manifold $\mathbb{H}^{2(t+s)+1}$ that is compact. Thus, the following assumption is satisfied:*

$$\|\nabla \ln \mu\|_{D-sp\{\xi\}}^2 < \left\{ 1 - \frac{(2t+1)(n+2t+1)}{2s} \right\}.$$

Therefore, the stable $(2t+1)$ -current does not exist and there are no homology groups in V^n , i.e., $H_{2t+1}(V^n, \mathbb{Z}) = H_s(V^n, \mathbb{Z}) = 0$.

It is easy to prove the next theorem after proving Theorem 3.

Theorem 5. Let $V^n = V_T^{2t+1} \times_\mu V_\lambda^s$ be a compact warped product semi-slant submanifold of the Kenmotsu manifold $\mathbb{H}^{2(t+s)+1}$ when $n = 2t + s + 1$ and when preserving the following restriction:

$$\|\nabla \ln \mu\|_{D-sp\{\xi\}}^2 < \left\{ 1 - \frac{(2t+1)(n+2t+1)}{2s(\csc^2 \lambda + \cot^2 \lambda)} \right\}.$$

Thus, V^n is homeomorphic to $\mathbb{S}^n$, provided that $n = 4$, and it is homotopic to $\mathbb{S}^n$ if $n = 3$, where $\mathbb{S}^n$ is a standard unit sphere.

Proof. Due to Theorem 3 and Lawson–Simon (Theorem 4 of [8], p. 441), we obtain the required conclusion and explanation given in [6–9]. This completes the proof. $\square$

Using Theorem 5, the next corollary is given by putting $\lambda = \frac{\pi}{2}$, as follows:

Corollary 3. Let $V^n = V_T^{2t+1} \times_\mu V_\perp^s$ be a contact CR-warped product submanifold of $\mathbb{H}^{2(t+s)+1}$ that is compact, and the following restriction holds:

$$\|\nabla \ln \mu\|_{D-sp\{\xi\}}^2 < \left\{ 1 - \frac{(2t+1)(n+2t+1)}{2s} \right\}.$$

Thus, V^n is homeomorphic to $\mathbb{S}^n$, provided that $n = 4$, and it is homotopic to $\mathbb{S}^n$ if $n = 3$, where $\mathbb{S}^n$ is a standard unit sphere.

Follow Theorem 4 and Corollary 1, we find that

Corollary 4. Let $V^n = V_T^{t+1} \times_\mu V_\perp^s$ be a compact contact CR-warped product submanifold of $\mathbb{H}^{2(\frac{t}{2}+s)+1}$ having the inequality

$$\left\{ (1-s)\|\nabla \ln \mu\|^2 + 3\left(\|\nabla \ln \mu\|_{D-sp\{\xi\}}^2 - 1\right) \right\} < \frac{1}{s} \left\{ s\Delta \ln \mu + 2s - 2(t+1)^2 \right\}.$$

Thus, V^n is homeomorphic to $\mathbb{S}^n$ if $n = 4$ and it is homotopic to $\mathbb{S}^n$ if $n = 3$, where $\mathbb{S}^n$ is a standard sphere.

3. Conclusion Remark

Physical phase transitions, low-dimensional statistical mechanics, and liquid crystals stand among the various areas that singularity structures can be utilized for (see [21]). Furthermore, warped product manifolds are a model of spacetime that are found in general relativity. Two well-known warped product spaces exist. There are two types of spacetimes: conventional static spacetimes and the generalization of Robertson–Walker spacetimes [21]. Differential topological techniques are essential to general relativity, particularly in mathematical physics. In particular, quantum gravity makes use of the spacetime homology ([24–27]). The results of this research have physical applicability since they are related to warped product manifolds and the homotopy/homology theory.

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