

Article

The Upper Bounds of the Numerical Radius on Hilbert C^* -Modules

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Abstract: In this paper, we give the generalized Cauchy–Schwarz inequality and an extension of Buzano inequality on quasi-Hilbert C^* -modules. Additionally, the upper bounds of the numerical radius of bounded adjointable operators on Hilbert C^* -modules are improved. Lastly, we obtain the upper bounds of the numerical radius for the product of bounded adjointable operators on Hilbert C^* -modules.

Keywords: numerical radius; Cauchy–Schwarz inequality; Buzano inequality; Hilbert C^* -module

MSC: 47A12; 46C05; 47C10

1. Introduction

The Hilbert C^* -module serves as a prevalent instrument within the realms of operator and operator algebra theory. Moreover, the theory of Hilbert C^* -modules is a captivating subject independently. This is because the Hilbert C^* -module is associated with the theory of operator algebra and assimilates various concepts. Hilbert C^* -modules initially emerged in the work of Kaplansky, I. [1]. Kaplansky, I. employed Hilbert C^* -modules to demonstrate that derivations of type I AW^* -algebras are internal. Subsequently, the theory of Hilbert C^* -modules was broadened to general C^* -algebras and Hilbert spaces by Paschke, W.L. [2] and Rieffel, M.A. [3]. Paschke, W.L. and Rieffel, M.A. utilized Hilbert C^* -modules to prove that induced representations of C^* -algebras are connected to Morita equivalence. Since then, Hilbert C^* -module theory has flourished. Hilbert C^* -modules are the main tools of noncommutative geometry [4], KK-theory [5], quantum group theory [6], generalized index theory [7], operator-valued free probability theory [8], and quantum probability theory [9–12]. Hilbert C^* -module theory is developed in the application process. Hilbert C^* -modules serve to generalize the structure of the Hilbert space. Hilbert C^* -modules achieve this generalization by redefining the inner product such that its values lie within a C^* -algebra. The values of the inner product within a Hilbert space belong to the complex number field. That is to say, Hilbert C^* -modules are the consequence of integrating the concepts of Hilbert spaces and C^* -algebras. Additionally, it can be stated that Hilbert spaces and C^* -algebras can be considered as special cases of Hilbert C^* -modules. Consequently, the theory that serves as the foundation for Hilbert C^* -modules arises from the intersection between the theoretical frameworks of Hilbert spaces and those of C^* -algebras.

The investigation into the numerical range and numerical radius has a long and remarkable history. The concept of the numerical range originated from the Rayleigh



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quotients utilized in the 19th century and has developed to be applied in numerous modern fields. These fields cover multiple aspects. For example, in functional analysis, it is used for norm estimation and the determination of bounds. In operator theory, especially in the context of differential operators, it plays a role. In numerical analysis, it is related to the convergence rate of algorithms. In quantum computing, it is applied to quantum error correction. In quantum information theory, it is involved in the study of quantum channels. And in quantum control, it is used for optimizing witnesses. The numerical radius represents the radius of the smallest disk centered at the origin that encloses the closure of the numerical range. The numerical radius also serves as a valuable tool for characterizing the numerical range. Furthermore, the numerical radius plays a highly significant role in the stability theory of the finite difference approximate solutions for the hyperbolic initial value problem. In the context of researching inequalities within the fields of operator theory and matrix analysis, the study of inequalities related to the numerical radius is one of the most crucial topics (see [13–17]). Recently, many scholars have studied properties of the numerical range and the numerical radius for bounded adjointable operators on Hilbert C^* -modules (see [1,2,18–25]).

Let us recall the definition of a Hilbert C^* -module over a C^* -algebra A (as shown in [1]).

Let A be a C^* -algebra. A semi-inner product A -module is a linear space E which is a right A -module with the compatible scalar multiplication of

$$\lambda(xa) = (\lambda x)a = x(\lambda a) \quad \text{for all } x \in E, a \in A, \text{ and } \lambda \in \mathbb{C}$$

together with a map $\langle \cdot, \cdot \rangle_E : E \times E \rightarrow A$, which has the following properties:

- (1) $\langle x, \alpha y + \beta z \rangle_E = \alpha \langle x, y \rangle_E + \beta \langle x, z \rangle_E$, $x, y, z \in E$, $\alpha, \beta \in \mathbb{C}$,
- (2) $\langle x, ya \rangle_E = \langle x, y \rangle_E a$, $x, y \in E$, $a \in A$,
- (3) $\langle x, y \rangle_E^* = \langle y, x \rangle_E$, $x, y \in E$.

For every $x \in E$, we put $\|x\|_E = \|\langle x, x \rangle_E\|_A^{\frac{1}{2}}$. A semi-inner product space E , which satisfies

$$\|x\|_E = 0 \Leftrightarrow x = 0, \quad (1)$$

is called an inner product A -module. A complete inner-product A -module is called a Hilbert C^* -module.

If A degenerates into the complex field $\mathbb{C}$, then the Hilbert C^* -module E becomes a Hilbert space. Thus, the Hilbert space can be regarded as a special case of the Hilbert C^* -module. If A degenerates into the complex field $\mathbb{C}$ and the inequality (1) is not satisfied, then the Hilbert C^* -module E is a semi-Hilbert space (see [26–28]).

Suppose that E and F are Hilbert C^* -modules. We define $\mathcal{L}(E, F)$ to be the set of all maps $t : E \rightarrow F$ for which there is a map $t^* : F \rightarrow E$ such that $\langle tx, y \rangle_E = \langle x, t^*y \rangle_E$, for all $x \in E$, $y \in F$. It is known that t must be a bounded A -linear map (that is, t is a bounded linear map and $t(xa) = (tx)a$ for all $x \in E$ and $a \in A$). If $E = F$, then $\mathcal{L}(E)$ is a C^* -algebra together with the operator norm. A state on a C^* -algebra A is a positive linear functional on A of norm one. We denote the state space of A by $S(A)$.

In [29], Buzano, M.L. proved the Buzano inequality

$$|\varphi(\langle x, e \rangle_E) \varphi(\langle e, y \rangle_E)| \leq \frac{1}{2} \left[(\varphi(\langle x, x \rangle_E) \varphi(\langle y, y \rangle_E))^{\frac{1}{2}} + |\varphi(\langle x, y \rangle_E)| \right], \quad (2)$$

where $x, y, e \in E$ and $\varphi \in S(A)$ with $\varphi(|e|^2) = 1$. If we take $e = \varphi^{-\frac{1}{2}}(\langle y, y \rangle_E)y$ in (2), we get a Cauchy–Schwarz inequality as follows:

$$|\varphi(\langle x, y \rangle_E)|^2 \leq \varphi(\langle x, x \rangle_E)\varphi(\langle y, y \rangle_E). \quad (3)$$

For $t \in \mathcal{L}(E)$, let $W_A(t)$, $\omega_A(t)$, $\|t\|$ denote the numerical range, numerical radius, and operator norm, respectively, namely

$$\begin{aligned} W_A(t) &= \{\varphi(\langle x, tx \rangle_E) : x \in E, \varphi \in S(A), \varphi(|x|^2) = 1\}, \\ \omega_A(t) &= \sup\{|\varphi(\langle x, tx \rangle_E)| : x \in E, \varphi \in S(A), \varphi(|x|^2) = 1\}, \\ \|t\| &= \sup\{|\varphi(\langle x, ty \rangle_E)| : x, y \in E, \varphi \in S(A), \varphi(|x|^2) = \varphi(|y|^2) = 1\}, \end{aligned}$$

where $|x| = \langle x, x \rangle_E^{\frac{1}{2}}$, $|y| = \langle y, y \rangle_E^{\frac{1}{2}}$.

In [19], Mehrazin, M. obtained that if $t \in \mathcal{L}(E)$, then

$$\frac{1}{2}\|t\| \leq \omega_A(t) \leq \|t\|. \quad (4)$$

The inequality is sharp, $\omega_A(t) = \|t\|$, if t is a self-adjoint element of $\mathcal{L}(E)$.

Mehrazin, M. also improved the inequality (4). It has been shown that

$$\frac{1}{4}\|t^*t + tt^*\| \leq \omega_A^2(t) \leq \frac{1}{2}\|t^*t + tt^*\|. \quad (5)$$

In [18], Fakri, M.S. generalized the second inequality in (5). It is well known that if $t \in \mathcal{L}(E)$ and $r \geq 1$, then

$$\omega_A^{2r}(t) \leq \frac{1}{2}\| |t|^{2r} + |t^*|^{2r} \|. \quad (6)$$

And they also obtained the upper bound of the numerical radius for the product of two bounded adjointable operators on Hilbert C^* -modules.

$$\omega_A^{2r}(s^*t) \leq \frac{1}{2}\| |t|^{4r} + |s|^{4r} \| \quad r \geq 1, \quad (7)$$

where $|t| = (t^*t)^{\frac{1}{2}}$ is the absolute value of $t \in \mathcal{L}(E)$.

In Section 2, we give the generalized inequality (2) and the extension inequality (3). In Section 3, we present the upper bounds of the numerical radius of the bounded adjointable operators on Hilbert C^* -modules. And, our bounds refine and generalize the existing related upper bounds, as shown in Remark 6 in the text. In Section 4, we obtain the upper bounds of the numerical radius of the product of two bounded adjointable operators on Hilbert C^* -modules. And, our bounds refine inequality (7), as shown in Remark 8 in the text.

2. Preliminaries

To establish our principal results, we require the following series of lemmas.

Lemma 1 ([18]). *Let $t \in \mathcal{L}(E)$, $t \geq 0$ and $x \in E$. Then, for every $\varphi \in S(A)$ with $\varphi(|x|^2) = 1$,*

$$\begin{aligned} (a) \varphi^r(\langle x, tx \rangle_E) &\leq \varphi(\langle x, t^r x \rangle_E), \quad r \geq 1. \\ (b) \varphi^r(\langle x, tx \rangle_E) &\geq \varphi(\langle x, t^r x \rangle_E), \quad 0 < r \leq 1. \end{aligned}$$

Lemma 2 ([30]). If $a, b > 0$ and $0 \leq \alpha \leq 1$, then, for $m = 1, 2, 3, \dots$, we have

$$(a^\alpha b^{(1-\alpha)})^m + r_0^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq (\alpha a + (1-\alpha)b)^m,$$

where $r_0 = \min\{\alpha, 1-\alpha\}$. In particular, if $\alpha = \frac{1}{2}$, then,

$$(a^{\frac{1}{2}} b^{\frac{1}{2}})^m + (\frac{1}{2})^m (a^{\frac{m}{2}} - b^{\frac{m}{2}})^2 \leq (\frac{a+b}{2})^m.$$

Lemma 3 ([31]). Let f be a non-negative convex function on $[0, \infty)$ and let $t, s \in \mathcal{L}(E)$ be positive operators. Then,

$$\left\| f\left(\frac{t+s}{2}\right) \right\| \leq \left\| \frac{f(t) + f(s)}{2} \right\|.$$

In particular, if $r \geq 1$, then,

$$\left\| \left(\frac{t+s}{2}\right)^r \right\| \leq \left\| \frac{t^r + s^r}{2} \right\|.$$

Lemma 4 ([24]). Let $\mathbb{D}$ be a subset of $\mathbb{R}$ and let $f, g : \mathbb{D} \rightarrow [0, \infty)$ be a mapping such that $f(\beta) + g(\beta) = 1$ for all $\beta \in \mathbb{D}$. Let E be a pre-Hilbert C^* -module and $\varphi \in S(A)$. If $x, y, e \in E$ such that $\varphi(|e|^2) = 1$, then for any $r \geq 1$ and $\alpha \in \mathbb{C} \setminus \{0\}$, we have

$$\begin{aligned} & |\varphi(\langle x, e \rangle_E) \varphi(\langle e, y \rangle_E)|^{2r} \\ & \leq \frac{\max\{1, |\alpha - 1|^2\}}{|\alpha|^2} (\varphi(\langle x, x \rangle_E) \varphi(\langle y, y \rangle_E))^r \\ & + \frac{g(\beta)}{|\alpha|^2} \left| \varphi(\langle x, y \rangle_E) (\varphi(\langle x, e \rangle_E) \varphi(\langle e, y \rangle_E))^{r-1} \right|^2 \\ & + \frac{f(\beta) + 2 \max\{1, |\alpha - 1|\}}{|\alpha|^2} (\varphi(\langle x, x \rangle_E) \varphi(\langle y, y \rangle_E))^{\frac{r}{2}} \left| \varphi(\langle x, y \rangle_E) (\varphi(\langle x, e \rangle_E) \varphi(\langle e, y \rangle_E))^{r-1} \right|. \end{aligned}$$

Remark 1. If we take $r = 1$ and $\alpha = 2$ in Lemma 4, then,

$$\begin{aligned} |\varphi(\langle x, e \rangle_E) \varphi(\langle e, y \rangle_E)|^2 & \leq \frac{1}{4} \varphi(\langle x, x \rangle_E) \varphi(\langle y, y \rangle_E) + \frac{g(\beta)}{4} |\varphi(\langle x, y \rangle_E)|^2 \\ & + \frac{f(\beta) + 2}{4} (\varphi(\langle x, x \rangle_E) \varphi(\langle y, y \rangle_E))^{\frac{1}{2}} |\varphi(\langle x, y \rangle_E)|. \end{aligned}$$

Remark 2. When the Hilbert C^* -module space degenerates to a Hilbert space, let $r = 1$, $\alpha = 2$, $g(t) = \frac{h(t)}{2(h(t)+1)}$ and $f(t) = \frac{h(t)+2}{2(h(t)+1)}$ in Remark 1, where $h : (0, 1) \rightarrow \mathbb{R}^+$. Then,

$$\begin{aligned} & |\langle x, e \rangle \langle e, y \rangle|^2 \\ & \leq \frac{1}{4} \|x\|^2 \|y\|^2 + \frac{h(t)}{8(h(t)+1)} |\langle x, y \rangle|^2 + \frac{5h(t)+6}{8(h(t)+1)} \|x\| \|y\| |\langle x, y \rangle|. \end{aligned} \quad (8)$$

Moreover,

$$\begin{aligned} & |\langle x, e \rangle \langle e, y \rangle|^2 \\ & \leq \frac{1}{4} \|x\|^2 \|y\|^2 + \frac{h(t)}{8(h(t)+1)} |\langle x, y \rangle|^2 + \frac{5h(t)+6}{8(h(t)+1)} \|x\| \|y\| |\langle x, y \rangle| \\ & \leq \frac{1}{4} \|x\|^2 \|y\|^2 + \frac{h(t)}{8(h(t)+1)} \|x\|^2 \|y\|^2 + \frac{5h(t)+6}{8(h(t)+1)} \|x\| \|y\| |\langle x, y \rangle| \\ & \leq \frac{3h(t)+2}{8(h(t)+1)} \|x\|^2 \|y\|^2 + \frac{5h(t)+6}{8(h(t)+1)} \|x\| \|y\| |\langle x, y \rangle|, \end{aligned} \quad (9)$$

which imply that inequality (8) refines Lemma 2.20 of [32], namely

$$|\langle x, e \rangle \langle e, y \rangle|^2 \leq \frac{3h(t) + 2}{8(h(t) + 1)} \|x\|^2 \|y\|^2 + \frac{5h(t) + 6}{8(h(t) + 1)} \|x\| \|y\| |\langle x, y \rangle|. \quad (10)$$

Lemma 5 ([24]). Let $\mathbb{D}$ be a subset of $\mathbb{R}$ and let $f, g : \mathbb{D} \rightarrow [0, \infty)$ be a mapping such that $f(\beta) + g(\beta) = 1$ for all $\beta \in \mathbb{D}$. Let E be a pre-Hilbert C^* -module and $\varphi \in S(A)$. If $x, y \in E$, then for any $r \geq 1$ and $\alpha \in \mathbb{C} \setminus \{0\}$, we have

$$\begin{aligned} & (1 - \frac{g(\beta)}{|\alpha|^2}) |\varphi(\langle x, y \rangle_E)|^{2r} \\ & \leq \frac{\max\{1, |\alpha - 1|^2\}}{|\alpha|^2} (\varphi(\langle x, x \rangle_E) \varphi(\langle y, y \rangle_E))^r \\ & + \frac{f(\beta) + 2 \max\{1, |\alpha - 1|\}}{|\alpha|^2} (\varphi(\langle x, x \rangle_E) \varphi(\langle y, y \rangle_E))^{\frac{r}{2}} |\varphi(\langle x, y \rangle_E)|^r. \end{aligned}$$

Remark 3. If we take $r = 1$ and $\alpha = 2$ in Lemma 5, then,

$$\begin{aligned} & |\varphi(\langle x, y \rangle_E)|^2 \\ & \leq \frac{1}{4 - g(\beta)} \varphi(\langle x, x \rangle_E) \varphi(\langle y, y \rangle_E) + \frac{f(\beta) + 2}{4 - g(\beta)} (\varphi(\langle x, x \rangle_E) \varphi(\langle y, y \rangle_E))^{\frac{1}{2}} |\varphi(\langle x, y \rangle_E)|. \end{aligned}$$

3. The Upper Bounds of the Numerical Radius on Hilbert C^* -Modules

In this section, we present a general upper bound for the numerical radius of bounded adjointable operators on Hilbert C^* -modules with different parameters. Furthermore, when the Hilbert C^* -module space degenerates to a Hilbert space and the parameter takes a special value, the inequality of the numerical radius of the bounded linear operator in the Hilbert space is improved.

Theorem 1. Let $t \in \mathcal{L}(E)$ and let $\mathbb{D}$ be a subset of $\mathbb{R}$. Let $f, g : \mathbb{D} \rightarrow [0, \infty)$ be a mapping such that $f(\beta) + g(\beta) = 1$ for all $\beta \in \mathbb{D}$. Then, for $m = 1, 2, 3, \dots$, we have

$$\begin{aligned} \omega_A^{4m}(t) & \leq \frac{1}{8} \| |t|^{4m} + |t^*|^{4m} \| + \frac{g(\beta)}{4} \omega_A^{2m}(t^2) + \frac{f(\beta) + 2}{8} \| |t|^{2m} + |t^*|^{2m} \| \omega_A^m(t^2) \\ & - \inf_{\varphi(|x|^2)=1} \gamma(x), \end{aligned}$$

where

$$\gamma(x) = (\frac{1}{2})^{m+2} (\varphi^m(\langle x, |t|^2 x \rangle_E) - \varphi^m(\langle x, |t^*|^2 x \rangle_E))^2.$$

Proof. If we take $x = t^* x$, $e = x$ and $y = tx$ in Remark 1, then,

$$\begin{aligned} & |\varphi(\langle x, tx \rangle_E)|^4 \\ & \leq \frac{1}{4} \varphi(\langle tx, tx \rangle_E) \varphi(\langle t^* x, t^* x \rangle_E) + \frac{g(\beta)}{4} |\varphi(\langle t^* x, tx \rangle_E)|^2 \\ & + \frac{f(\beta) + 2}{4} (\varphi(\langle tx, tx \rangle_E) \varphi(\langle t^* x, t^* x \rangle_E))^{\frac{1}{2}} |\varphi(\langle t^* x, tx \rangle_E)| \\ & \quad (\text{by Remark 1}) \\ & = \frac{1}{4} \varphi(\langle x, |t|^2 x \rangle_E) \varphi(\langle x, |t^*|^2 x \rangle_E) + \frac{g(\beta)}{4} |\varphi(\langle x, t^2 x \rangle_E)|^2 \\ & + \frac{f(\beta) + 2}{4} (\varphi(\langle x, |t|^2 x \rangle_E) \varphi(\langle x, |t^*|^2 x \rangle_E))^{\frac{1}{2}} |\varphi(\langle x, t^2 x \rangle_E)|. \end{aligned}$$

Further,

$$\begin{aligned}
 & |\varphi(\langle x, tx \rangle_E)|^{4m} \\
 & \leq \left[\frac{1}{4} \varphi(\langle x, |t|^2 x \rangle_E) \varphi(\langle x, |t^*|^2 x \rangle_E) + \frac{g(\beta)}{4} |\varphi(\langle x, t^2 x \rangle_E)|^2 \right. \\
 & \quad \left. + \frac{f(\beta) + 2}{4} (\varphi(\langle x, |t|^2 x \rangle_E) \varphi(\langle x, |t^*|^2 x \rangle_E))^{\frac{1}{2}} |\varphi(\langle x, t^2 x \rangle_E)|^m \right] \\
 & \leq \frac{1}{4} (\varphi(\langle x, |t|^2 x \rangle_E) \varphi(\langle x, |t^*|^2 x \rangle_E))^m + \frac{g(\beta)}{4} |\varphi(\langle x, t^2 x \rangle_E)|^{2m} \\
 & \quad + \frac{f(\beta) + 2}{4} (\varphi(\langle x, |t|^2 x \rangle_E) \varphi(\langle x, |t^*|^2 x \rangle_E))^{\frac{m}{2}} |\varphi(\langle x, t^2 x \rangle_E)|^m. \\
 & \quad \text{(by Lemma 3)}
 \end{aligned}$$

Employing the generalization of Young inequality in Lemma 2, we have

$$\begin{aligned}
 & |\varphi(\langle x, tx \rangle_E)|^{4m} \\
 & \leq \frac{1}{4} \left[\left(\frac{\varphi^2(\langle x, |t|^2 x \rangle_E) + \varphi^2(\langle x, |t^*|^2 x \rangle_E)}{2} \right)^m \right. \\
 & \quad \left. - \left(\frac{1}{2} \right)^m (\varphi^m(\langle x, |t|^2 x \rangle_E) - \varphi^m(\langle x, |t^*|^2 x \rangle_E))^2 \right] + \frac{g(\beta)}{4} |\varphi(\langle x, t^2 x \rangle_E)|^{2m} \\
 & \quad + \frac{f(\beta) + 2}{4} \cdot \left(\frac{\varphi(\langle x, |t|^2 x \rangle_E) + \varphi(\langle x, |t^*|^2 x \rangle_E)}{2} \right)^m |\varphi(\langle x, t^2 x \rangle_E)|^m \\
 & \quad \text{(by Lemma 2, AM-GM inequality)} \\
 & \leq \frac{1}{4} \left[\frac{\varphi^{2m}(\langle x, |t|^2 x \rangle_E) + \varphi^{2m}(\langle x, |t^*|^2 x \rangle_E)}{2} \right. \\
 & \quad \left. - \left(\frac{1}{2} \right)^m (\varphi^m(\langle x, |t|^2 x \rangle_E) - \varphi^m(\langle x, |t^*|^2 x \rangle_E))^2 \right] + \frac{g(\beta)}{4} |\varphi(\langle x, t^2 x \rangle_E)|^{2m} \\
 & \quad + \frac{f(\beta) + 2}{4} \cdot \frac{\varphi^m(\langle x, |t|^2 x \rangle_E) + \varphi^m(\langle x, |t^*|^2 x \rangle_E)}{2} |\varphi(\langle x, t^2 x \rangle_E)|^m \\
 & \quad \text{(by Lemma 3)} \\
 & \leq \frac{1}{4} \left[\frac{\varphi(\langle x, (|t|^{4m} + |t^*|^{4m})x \rangle_E)}{2} - \left(\frac{1}{2} \right)^m (\varphi^m(\langle x, |t|^2 x \rangle_E) - \varphi^m(\langle x, |t^*|^2 x \rangle_E))^2 \right] \\
 & \quad + \frac{g(\beta)}{4} |\varphi(\langle x, t^2 x \rangle_E)|^{2m} + \frac{f(\beta) + 2}{8} \cdot \varphi(\langle x, (|t|^{2m} + |t^*|^{2m})x \rangle_E) |\varphi(\langle x, t^2 x \rangle_E)|^m. \\
 & \quad \text{(by Lemma 1)}
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 & |\varphi(\langle x, tx \rangle_E)|^{4m} \\
 & \leq \frac{1}{8} \| |t|^{4m} + |t^*|^{4m} \| + \frac{g(\beta)}{4} \omega_A^{2m}(t^2) + \frac{f(\beta) + 2}{8} \| |t|^{2m} + |t^*|^{2m} \| \omega_A^m(t^2) \\
 & \quad - \left(\frac{1}{2} \right)^{m+2} (\varphi^m(\langle x, |t|^2 x \rangle_E) - \varphi^m(\langle x, |t^*|^2 x \rangle_E))^2.
 \end{aligned}$$

Now, by taking supremum over all $x \in E$ and $\varphi \in S(A)$ with $\varphi(|x|^2) = 1$,

$$\begin{aligned}
 \omega_A^{4m}(t) & \leq \frac{1}{8} \| |t|^{4m} + |t^*|^{4m} \| + \frac{g(\beta)}{4} \omega_A^{2m}(t^2) + \frac{f(\beta) + 2}{8} \| |t|^{2m} + |t^*|^{2m} \| \omega_A^m(t^2) \\
 & \quad - \inf_{\varphi(|x|^2)=1} \gamma(x),
 \end{aligned}$$

where

$$\gamma(x) = \left(\frac{1}{2}\right)^{m+2} (\varphi^m(\langle x, |t|^2 x \rangle_E) - \varphi^m(\langle x, |t^*|^2 x \rangle_E))^2.$$

□

Remark 4. When the Hilbert C^* -module space degenerates to a Hilbert space, let $\alpha = 2$, $m = 1$, $g(\lambda) = \frac{\lambda}{\lambda+1}$ and $f(\lambda) = \frac{1}{\lambda+1}$ in Theorem 1, where $\lambda \geq 0$. Then,

$$\begin{aligned} & \omega^4(t) \\ & \leq \frac{1}{8} \| |t|^4 + |t^*|^4 \| + \frac{\lambda}{4(\lambda+1)} \omega^2(t^2) + \frac{2\lambda+3}{8(\lambda+1)} \| |t|^2 + |t^*|^2 \| \omega(t^2) - \inf_{\|x\|^2=1} \gamma(x), \end{aligned} \quad (11)$$

where

$$\gamma(x) = \frac{1}{8} (\langle x, |t|^2 x \rangle - \langle x, |t^*|^2 x \rangle)^2.$$

Moreover,

$$\begin{aligned} & \omega^4(t) \\ & \leq \frac{1}{8} \| |t|^4 + |t^*|^4 \| + \frac{\lambda}{4(\lambda+1)} \omega^2(t^2) + \frac{2\lambda+3}{8(\lambda+1)} \| |t|^2 + |t^*|^2 \| \omega(t^2) - \inf_{\|x\|^2=1} \gamma(x) \\ & \quad \text{(by inequality (7))} \\ & \leq \frac{1}{8} \| |t|^4 + |t^*|^4 \| + \frac{\lambda}{8(\lambda+1)} \| |t|^4 + |t^*|^4 \| + \frac{2\lambda+3}{8(\lambda+1)} \| |t|^2 + |t^*|^2 \| \omega(t^2) - \inf_{\|x\|^2=1} \gamma(x) \\ & = \frac{2\lambda+1}{8(\lambda+1)} \| |t|^4 + |t^*|^4 \| + \frac{2\lambda+3}{8(\lambda+1)} \| |t|^2 + |t^*|^2 \| \omega(t^2) - \inf_{\|x\|^2=1} \gamma(x), \end{aligned}$$

which imply that inequality (11) refines Theorem 2.9 of [33], namely

$$\omega^4(t) \leq \frac{2\lambda+1}{8(\lambda+1)} \| |t|^4 + |t^*|^4 \| + \frac{2\lambda+3}{8(\lambda+1)} \| |t|^2 + |t^*|^2 \| \omega(t^2), \quad \lambda \geq 0.$$

Next, we will present Example 1 and Example 2 to illustrate the effectiveness of the results in Remark 4.

Example 1. Consider $\gamma(x)$ in the above Remark 4, namely

$$\gamma(x) = \frac{1}{8} (\langle x, |t|^2 x \rangle - \langle x, |t^*|^2 x \rangle)^2.$$

$$\begin{aligned} \text{Let } t &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \text{ then } |t|^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}, |t^*|^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}, t^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \\ |t|^4 &= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix}, |t^*|^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \text{ Let } \lambda = 1, \text{ then,} \end{aligned}$$

$$\begin{aligned} 1.5625 &= \omega^4(t) \\ &\leq \frac{1}{8} \| |t|^4 + |t^*|^4 \| + \frac{1}{8} \omega^2(t^2) + \frac{5}{16} \| |t|^2 + |t^*|^2 \| \omega(t^2) - \inf_{\|x\|^2=1} \gamma(x) \\ &= 3.8125, \end{aligned}$$

where

$$\inf_{\|x\|=1} \gamma(x) = 0.$$

In [33], we have

$$1.5625 = \omega^4(t) \leq \frac{3}{16} \| |t|^4 + |t^*|^4 \| + \frac{5}{16} \| |t|^2 + |t^*|^2 \| \omega(t^2) = 4.75.$$

Example 2. Consider $\gamma(x)$ in the above Remark 4, namely

$$\gamma(x) = \frac{1}{8} (\langle x, |t|^2 x \rangle - \langle x, |t^*|^2 x \rangle)^2.$$

$$\text{Let } t = \begin{bmatrix} 0 & 0 & 0 \\ I & 0 & 0 \\ 0 & 2I & 0 \end{bmatrix}, \text{ then } t^* = \begin{bmatrix} 0 & I & 0 \\ 0 & 0 & 2I \\ 0 & 0 & 0 \end{bmatrix}, t^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 2I & 0 & 0 \end{bmatrix}, |t|^2 = \begin{bmatrix} I & 0 & 0 \\ 0 & 4I & 0 \\ 0 & 0 & 0 \end{bmatrix}, |t^*|^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 16I \end{bmatrix}, |t|^4 = \begin{bmatrix} I & 0 & 0 \\ 0 & 16I & 0 \\ 0 & 0 & 0 \end{bmatrix}, |t^*|^4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 16I \end{bmatrix}, \text{ where } I \text{ is the identity operator in an infinite dimensional Hilbert space. Let } \lambda = 1, \text{ then,}$$

$$\begin{aligned} 1 &= \omega^4(t) \\ &\leq \frac{1}{8} \| |t|^4 + |t^*|^4 \| + \frac{1}{8} \omega^2(t^2) + \frac{5}{16} \| |t|^2 + |t^*|^2 \| \omega(t^2) - \inf_{\|x\|^2=1} \gamma(x) \\ &= 3.8125, \end{aligned}$$

where

$$\inf_{\|x\|=1} \gamma(x) = 0.$$

In [33], we have

$$1 = \omega^4(t) \leq \frac{3}{16} \| |t|^4 + |t^*|^4 \| + \frac{5}{16} \| |t|^2 + |t^*|^2 \| \omega(t^2) = 4.75.$$

Remark 5. By the above example, when $\inf_{\|x\|=1} \gamma(x) = 0$ holds, the inequality of Theorem 1 also improves Theorem 2.9 of [33]. When $\inf_{\|x\|=1} \gamma(x) > 0$ holds, we take $m = 1$ and t as a uniformly hyponormal operator ($t^*t - tt^* \geq M > 0$) [14] in Theorem 1.

Theorem 2. Let $t \in \mathcal{L}(E)$ and let $\mathbb{D}$ be a subset of $\mathbb{R}$. Let $f, g : \mathbb{D} \rightarrow [0, \infty)$ be a mapping such that $f(\beta) + g(\beta) = 1$ for all $\beta \in \mathbb{D}$. Then, for $r \geq 1$ and $\alpha \in \mathbb{C} \setminus \{0\}$, we have

$$\begin{aligned} &\omega_A^{4r}(t) \\ &\leq \frac{\max\{1, |\alpha - 1|^2\}}{4|\alpha|^2} \| |t|^{4r} + |t^*|^{4r} \| + \frac{\max\{1, |\alpha - 1|^2\}}{2|\alpha|^2} \omega_A(|t|^{2r} |t^*|^{2r}) \\ &+ \frac{g(\beta)}{|\alpha|^2} \omega_A^2(t^2) \omega_A^{4(r-1)}(t) + \frac{f(\beta) + 2\max\{1, |\alpha - 1|\}}{2|\alpha|^2} \| |t|^{2r} + |t^*|^{2r} \| \omega_A(t^2) \omega_A^{2(r-1)}(t). \end{aligned}$$

Proof. If we take $x = t^*x$, $e = x$ and $y = tx$ in Lemma 4, then,

$$\begin{aligned}
& |\varphi(\langle x, tx \rangle_E)|^{4r} \\
& \leq \frac{\max\{1, |\alpha - 1|^2\}}{|\alpha|^2} (\varphi(\langle x, |t|^2 x \rangle_E) \varphi(\langle x, |t^*|^2 x \rangle_E))^r \\
& + \frac{g(\beta)}{|\alpha|^2} |\varphi(\langle x, t^2 x \rangle_E)|^2 |\varphi(\langle x, tx \rangle_E)|^{4(r-1)} + \\
& \frac{f(\beta) + 2 \max\{1, |\alpha - 1|\}}{|\alpha|^2} (\varphi(\langle x, |t|^2 x \rangle_E) \varphi(x, |t^*|^2 x))^{\frac{r}{2}} |\varphi(\langle x, t^2 x \rangle_E)| |\varphi(\langle x, tx \rangle_E)|^{2(r-1)} \\
& \quad \text{(by Lemma 4)} \\
& \leq \frac{\max\{1, |\alpha - 1|^2\}}{|\alpha|^2} \varphi(\langle x, |t|^{2r} x \rangle_E) \varphi(\langle x, |t^*|^{2r} x \rangle_E) \\
& + \frac{g(\beta)}{|\alpha|^2} |\varphi(\langle x, t^2 x \rangle_E)|^2 |\varphi(\langle x, tx \rangle_E)|^{4(r-1)} + \\
& \frac{f(\beta) + 2 \max\{1, |\alpha - 1|\}}{|\alpha|^2} (\varphi(\langle x, |t|^{2r} x \rangle_E) \varphi(\langle x, |t^*|^{2r} x \rangle_E))^{\frac{1}{2}} |\varphi(\langle x, t^2 x \rangle_E)| |\varphi(\langle x, tx \rangle_E)|^{2(r-1)} \\
& \quad \text{(by Lemma 1)} \\
& = \frac{\max\{1, |\alpha - 1|^2\}}{|\alpha|^2} |\varphi(\langle |t|^{2r} x, x \rangle_E) \varphi(\langle x, |t^*|^{2r} x \rangle_E)| \\
& + \frac{g(\beta)}{|\alpha|^2} |\varphi(\langle x, t^2 x \rangle_E)|^2 |\varphi(\langle x, tx \rangle_E)|^{4(r-1)} + \\
& \frac{f(\beta) + 2 \max\{1, |\alpha - 1|\}}{|\alpha|^2} (\varphi(\langle x, |t|^{2r} x \rangle_E) \varphi(\langle x, |t^*|^{2r} x \rangle_E))^{\frac{1}{2}} |\varphi(\langle x, t^2 x \rangle_E)| |\varphi(\langle x, tx \rangle_E)|^{2(r-1)}.
\end{aligned}$$

By employing the Buzano inequality (2), we have

$$\begin{aligned}
& \leq \frac{\max\{1, |\alpha - 1|^2\}}{2|\alpha|^2} ((\varphi(\langle |t|^{2r} x, |t|^{2r} x \rangle_E) \varphi(\langle |t^*|^{2r} x, |t^*|^{2r} x \rangle_E))^{\frac{1}{2}} + |\varphi(\langle |t|^{2r} x, |t^*|^{2r} x \rangle_E)|) \\
& + \frac{g(\beta)}{|\alpha|^2} |\varphi(\langle x, t^2 x \rangle_E)|^2 |\varphi(\langle x, tx \rangle_E)|^{4(r-1)} + \\
& \frac{f(\beta) + 2 \max\{1, |\alpha - 1|\}}{|\alpha|^2} (\varphi(\langle x, |t|^{2r} x \rangle_E) \varphi(\langle x, |t^*|^{2r} x \rangle_E))^{\frac{1}{2}} |\varphi(\langle x, t^2 x \rangle_E)| |\varphi(\langle x, tx \rangle_E)|^{2(r-1)} \\
& \quad \text{(by inequality (2))} \\
& \leq \frac{\max\{1, |\alpha - 1|^2\}}{4|\alpha|^2} \varphi(\langle x, (|t|^{4r} + |t^*|^{4r}) x \rangle_E) + \frac{\max\{1, |\alpha - 1|^2\}}{2|\alpha|^2} |\varphi(\langle x, |t|^{2r} |t^*|^{2r} x \rangle_E)| \\
& + \frac{g(\beta)}{|\alpha|^2} |\varphi(\langle x, t^2 x \rangle_E)|^2 |\varphi(\langle x, tx \rangle_E)|^{4(r-1)} \\
& + \frac{f(\beta) + 2 \max\{1, |\alpha - 1|\}}{2|\alpha|^2} \varphi(\langle x, (|t|^{2r} + |t^*|^{2r}) x \rangle_E) |\varphi(\langle x, t^2 x \rangle_E)| |\varphi(\langle x, tx \rangle_E)|^{2(r-1)}. \\
& \quad \text{(by AM-GM inequality)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
& |\varphi(\langle x, tx \rangle_E)|^{4r} \\
& \leq \frac{\max\{1, |\alpha - 1|^2\}}{4|\alpha|^2} \| |t|^{4r} + |t^*|^{4r} \| + \frac{\max\{1, |\alpha - 1|^2\}}{2|\alpha|^2} \omega_A(|t|^{2r} |t^*|^{2r}) \\
& + \frac{g(\beta)}{|\alpha|^2} \omega_A^2(t^2) \omega_A^{4(r-1)}(t) + \frac{f(\beta) + 2 \max\{1, |\alpha - 1|\}}{2|\alpha|^2} \| |t|^{2r} + |t^*|^{2r} \| \omega_A(t^2) \omega_A^{2(r-1)}(t).
\end{aligned}$$

Now, by taking supremum over all $x \in E$ and $\varphi \in S(A)$ with $\varphi(|x|^2) = 1$,

$$\begin{aligned} & \omega_A^{4r}(t) \\ & \leq \frac{\max\{1, |\alpha - 1|^2\}}{4|\alpha|^2} \left\| |t|^{4r} + |t^*|^{4r} \right\| + \frac{\max\{1, |\alpha - 1|^2\}}{2|\alpha|^2} \omega_A(|t|^{2r} |t^*|^{2r}) \\ & + \frac{g(\beta)}{|\alpha|^2} \omega_A^2(t^2) \omega_A^{4(r-1)}(t) + \frac{f(\beta) + 2\max\{1, |\alpha - 1|\}}{2|\alpha|^2} \left\| |t|^{2r} + |t^*|^{2r} \right\| \omega_A(t^2) \omega_A^{2(r-1)}(t). \end{aligned}$$

□

Remark 6. When the Hilbert C^* -module space degenerates to a Hilbert space, let $\alpha = 2$ in Theorem 2. Then, we will obtain

$$\begin{aligned} \omega^{4r}(t) & \leq \frac{1}{16} \left\| |t|^{4r} + |t^*|^{4r} \right\| + \frac{1}{8} \omega(|t|^{2r} |t^*|^{2r}) + \frac{g(\beta)}{4} \omega^2(t^2) \omega^{4(r-1)}(t) \\ & + \frac{f(\beta) + 2}{8} \left\| |t|^{2r} + |t^*|^{2r} \right\| \omega(t^2) \omega^{2(r-1)}(t). \end{aligned} \quad (12)$$

Moreover,

$$\begin{aligned} \omega^{4r}(t) & \leq \frac{1}{16} \left\| |t|^{4r} + |t^*|^{4r} \right\| + \frac{1}{8} \omega(|t|^{2r} |t^*|^{2r}) + \frac{g(\beta)}{4} \omega^2(t^2) \omega^{4(r-1)}(t) \\ & + \frac{f(\beta) + 2}{8} \left\| |t|^{2r} + |t^*|^{2r} \right\| \omega(t^2) \omega^{2(r-1)}(t) \\ & \leq \frac{1}{16} \left\| |t|^{4r} + |t^*|^{4r} \right\| + \frac{1}{8} \omega(|t^*|^{2r} |t|^{2r}) + \frac{g(\beta)}{4} \omega^{4r}(t) \\ & + \frac{f(\beta) + 2}{8} \left\| |t|^{2r} + |t^*|^{2r} \right\| \omega^{2r}(t) \\ & \quad (\text{by } \omega(t^n) \leq \omega^n(t), n \in \mathbb{N}^+) \\ & \leq \frac{1}{16} \left\| |t|^{4r} + |t^*|^{4r} \right\| + \frac{1}{16} \left\| |t|^{4r} + |t^*|^{4r} \right\| + \frac{g(\beta)}{8} \left\| |t|^{4r} + |t^*|^{4r} \right\| \\ & + \frac{f(\beta) + 2}{16} \left\| |t|^{2r} + |t^*|^{2r} \right\|^2 \\ & \quad (\text{by inequality (6), inequality (7)}) \\ & \leq \frac{g(\beta) + 1}{8} \left\| |t|^{4r} + |t^*|^{4r} \right\| + \frac{f(\beta) + 2}{8} \left\| |t|^{4r} + |t^*|^{4r} \right\| \\ & \quad (\text{by Lemma 3}) \\ & = \frac{1}{2} \left\| |t|^{4r} + |t^*|^{4r} \right\|, \end{aligned}$$

which imply that inequality (12) refines Theorem 2 of [34], namely

$$\omega^{2r}(t) \leq \frac{1}{2} \left\| |t|^{2r} + |t^*|^{2r} \right\|, \quad r \geq 1.$$

4. The Upper Bounds of the Numerical Radius for a Product of Bounded Adjointable Operators on Hilbert C^* -Modules

In this section, we provide the upper bounds of the numerical radius of the product of two adjointable operators on Hilbert C^* -modules. And, our bounds refine inequality (7).

Theorem 3. Let $t, s \in \mathcal{L}(E)$ and let $\mathbb{D}$ be a subset of $\mathbb{R}$. Let $f, g : \mathbb{D} \rightarrow [0, \infty)$ be a mapping such that $f(\beta) + g(\beta) = 1$ for all $\beta \in \mathbb{D}$. Then, for $m = 1, 2, 3, \dots$, we have

$$\omega_A^{2m}(s^*t) \leq \frac{1}{2(4-g(\beta))} \| |t|^{4m} + |s|^{4m} \| + \frac{f(\beta) + 2}{2(4-g(\beta))} \| |t|^{2m} + |s|^{2m} \| \omega_A^m(s^*t) - \inf_{\varphi(|x|^2)=1} \gamma(x),$$

where

$$\gamma(x) = \frac{1}{(4-g(\beta))} \left(\frac{1}{2} \right)^m (\varphi^m(\langle x, |t|^2 x \rangle_E) - \varphi^m(\langle x, |s|^2 x \rangle_E))^2.$$

Proof. If we take $x = sx$ and $y = tx$ in Remark 3, then,

$$\begin{aligned} & |\varphi(\langle x, s^*tx \rangle_E)|^2 \\ & \leq \frac{1}{4-g(\beta)} \varphi(\langle x, |t|^2 x \rangle_E) \varphi(\langle x, |s|^2 x \rangle_E) \\ & + \frac{f(\beta) + 2}{4-g(\beta)} (\varphi(\langle x, |t|^2 x \rangle_E) \varphi(\langle x, |s|^2 x \rangle_E))^{\frac{1}{2}} |\varphi(\langle x, s^*tx \rangle_E)|. \end{aligned}$$

(by Remark 3)

Further,

$$\begin{aligned} & |\varphi(\langle x, s^*tx \rangle_E)|^{2m} \\ & \leq \left[\frac{1}{4-g(\beta)} \varphi(\langle x, |t|^2 x \rangle_E) \varphi(\langle x, |s|^2 x \rangle_E) \right. \\ & + \left. \frac{f(\beta) + 2}{4-g(\beta)} (\varphi(\langle x, |t|^2 x \rangle_E) \varphi(\langle x, |s|^2 x \rangle_E))^{\frac{1}{2}} |\varphi(\langle x, s^*tx \rangle_E)| \right]^m \\ & \leq \frac{1}{4-g(\beta)} (\varphi(\langle x, |t|^2 x \rangle_E) \varphi(\langle x, |s|^2 x \rangle_E))^m \\ & + \frac{f(\beta) + 2}{4-g(\beta)} (\varphi(\langle x, |t|^2 x \rangle_E) \varphi(\langle x, |s|^2 x \rangle_E))^{\frac{m}{2}} |\varphi(\langle x, s^*tx \rangle_E)|^m. \end{aligned}$$

(by Lemma 3)

By employing the generalization of Young inequality in Lemma 2, we have

$$\begin{aligned} & |\varphi(\langle x, s^*tx \rangle_E)|^{2m} \\ & \leq \frac{1}{4-g(\beta)} \left[\left(\frac{\varphi^2(\langle x, |t|^2 x \rangle_E) + \varphi^2(\langle x, |s|^2 x \rangle_E)}{2} \right)^m \right. \\ & - \left. \left(\frac{1}{2} \right)^m (\varphi^m(\langle x, |t|^2 x \rangle_E) - \varphi^m(\langle x, |s|^2 x \rangle_E))^2 \right] \\ & + \frac{f(\beta) + 2}{4-g(\beta)} \cdot \left(\frac{\varphi(\langle x, |t|^2 x \rangle_E) + \varphi(\langle x, |s|^2 x \rangle_E)}{2} \right)^m |\varphi(\langle x, s^*tx \rangle_E)|^m \end{aligned}$$

(by Lemma 2, AM-GM inequality)

$$\begin{aligned}
&\leq \frac{1}{4-g(\beta)} \left[\frac{\varphi^{2m}(\langle x, |t|^2 x \rangle_E) + \varphi^{2m}(\langle x, |s|^2 x \rangle_E)}{2} \right. \\
&\quad \left. - \left(\frac{1}{2}\right)^m (\varphi^m(\langle x, |t|^2 x \rangle_E) - \varphi^m(\langle x, |s|^2 x \rangle_E))^2 \right] \\
&\quad + \frac{f(\beta) + 2}{4-g(\beta)} \cdot \frac{\varphi^m(\langle x, |t|^2 x \rangle_E) + \varphi^m(\langle x, |s|^2 x \rangle_E)}{2} |\varphi(\langle x, s^* t x \rangle_E)|^m \\
&\quad \text{(by Lemma 3)} \\
&\leq \frac{1}{4-g(\beta)} \left[\frac{\varphi(\langle x, (|t|^{4m} + |s|^{4m})x \rangle_E)}{2} - \left(\frac{1}{2}\right)^m (\varphi^m(\langle x, |t|^2 x \rangle_E) - \varphi^m(\langle x, |s|^2 x \rangle_E))^2 \right] \\
&\quad + \frac{f(\beta) + 2}{4-g(\beta)} \cdot \frac{\varphi(\langle x, (|t|^{2m} + |s|^{2m})x \rangle_E)}{2} |\varphi(\langle x, s^* t x \rangle_E)|^m. \\
&\quad \text{(by Lemma 1)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
&|\varphi(\langle x, s^* t x \rangle_E)|^{2m} \\
&\leq \frac{1}{2(4-g(\beta))} \left\| |t|^{4m} + |s|^{4m} \right\| + \frac{f(\beta) + 2}{2(4-g(\beta))} \left\| |t|^{2m} + |s|^{2m} \right\| \omega_A^m(s^* t) \\
&\quad - \frac{1}{(4-g(\beta))} \left(\frac{1}{2}\right)^m (\varphi^m(\langle x, |t|^2 x \rangle_E) - \varphi^m(\langle x, |s|^2 x \rangle_E))^2.
\end{aligned}$$

Now, by taking supremum over all $x \in E$ and $\varphi \in S(A)$ with $\varphi(|x|^2) = 1$,

$$\begin{aligned}
\omega_A^{2m}(s^* t) &\leq \frac{1}{2(4-g(\beta))} \left\| |t|^{4m} + |s|^{4m} \right\| + \frac{f(\beta) + 2}{2(4-g(\beta))} \left\| |t|^{2m} + |s|^{2m} \right\| \omega_A^m(s^* t) \\
&\quad - \inf_{\varphi(|x|^2)=1} \gamma(x),
\end{aligned}$$

where

$$\gamma(x) = \frac{1}{(4-g(\beta))} \left(\frac{1}{2}\right)^m (\varphi^m(\langle x, |t|^2 x \rangle_E) - \varphi^m(\langle x, |s|^2 x \rangle_E))^2.$$

□

Remark 7. When the Hilbert C^* -module space degenerates to a Hilbert space, let $\alpha = 2$, $m = 1$, $g(\beta) = 1$ and $f(\beta) = 0$ in Theorem 3. Then, we observe that

$$\omega^2(s^* t) \leq \frac{1}{6} \left\| |t|^4 + |s|^4 \right\| + \frac{1}{3} \left\| |t|^2 + |s|^2 \right\| \omega(s^* t) - \inf_{\|x\|=1} \gamma(x),$$

where

$$\gamma(x) = \frac{1}{6} ((x, |t|^2 x) - (x, |s|^2 x))^2.$$

And this inequality refines Theorem 1 of [35], namely

$$\omega^2(s^* t) \leq \frac{1}{6} \left\| |t|^4 + |s|^4 \right\| + \frac{1}{3} \left\| |t|^2 + |s|^2 \right\| \omega(s^* t).$$

Next, we will give an example to show that there exists $t, s \in \mathcal{L}(E)$ such that $\inf_{\|x\|=1} \gamma(x) \neq 0$.

Example 3. Consider $\gamma(x)$ in the above Remark 7, namely

$$\gamma(x) = \frac{1}{6}((x, |t|^2 x) - (x, |s|^2 x))^2.$$

$$\text{Let } t = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{2} \end{bmatrix}, s = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \text{ Then,}$$

$$2 = \omega^2(s^*t) \leq \frac{1}{6} \| |t|^4 + |s|^4 \| + \frac{1}{3} \| |t|^2 + |s|^2 \| \omega(s^*t) - \inf_{\|x\|=1} \gamma(x) \approx 2.0808,$$

where

$$\inf_{\|x\|=1} \gamma(x) = \frac{1}{6} \neq 0.$$

In [35], we have

$$2 = \omega^2(s^*t) \leq \frac{1}{6} \| |t|^4 + |s|^4 \| + \frac{1}{3} \| |t|^2 + |s|^2 \| \omega(s^*t) \approx 2.2475.$$

Theorem 4. Let $t, s \in \mathcal{L}(E)$ and let $\mathbb{D}$ be a subset of $\mathbb{R}$. Let $f, g : \mathbb{D} \rightarrow [0, \infty)$ be a mapping such that $f(\beta) + g(\beta) = 1$ for all $\beta \in \mathbb{D}$. Then, for $r \geq 1$, $\alpha \in \mathbb{C} \setminus \{0\}$ and $|\alpha|^2 \neq g(\beta)$, we have

$$\begin{aligned} & \omega_A^{2r}(s^*t) \\ & \leq \frac{\max\{1, |\alpha - 1|^2\}}{2(|\alpha|^2 - g(\beta))} \| |t|^{4r} + |s|^{4r} \| + \frac{f(\beta) + 2\max\{1, |\alpha - 1|\}}{2(|\alpha|^2 - g(\beta))} \| |t|^{2r} + |s|^{2r} \| \omega_A^r(s^*t). \end{aligned}$$

Proof. If we take $x = sx$ and $y = tx$ in Lemma 5, then,

$$\begin{aligned} & (1 - \frac{g(\beta)}{|\alpha|^2}) |\varphi(\langle x, s^*tx \rangle_E)|^{2r} \\ & \leq \frac{\max\{1, |\alpha - 1|^2\}}{|\alpha|^2} (\varphi(\langle x, |t|^2x \rangle_E) \varphi(\langle x, |s|^2x \rangle_E))^r \\ & + \frac{f(\beta) + 2\max\{1, |\alpha - 1|\}}{|\alpha|^2} (\varphi(\langle x, |t|^2x \rangle_E) \varphi(\langle x, |s|^2x \rangle_E))^{\frac{r}{2}} |\varphi(\langle x, s^*tx \rangle_E)|^r \\ & \quad (\text{by Lemma 5}) \\ & \leq \frac{\max\{1, |\alpha - 1|^2\}}{|\alpha|^2} \varphi(\langle x, |t|^{2r}x \rangle_E) \varphi(\langle x, |s|^{2r}x \rangle_E) \\ & + \frac{f(\beta) + 2\max\{1, |\alpha - 1|\}}{|\alpha|^2} (\varphi(\langle x, |t|^{2r}x \rangle_E) \varphi(\langle x, |s|^{2r}x \rangle_E))^{\frac{1}{2}} |\varphi(\langle x, s^*tx \rangle_E)|^r \\ & \quad (\text{by Lemma 1}) \\ & \leq \frac{\max\{1, |\alpha - 1|^2\}}{2|\alpha|^2} (\varphi^2(\langle x, |t|^{2r}x \rangle_E) + \varphi^2(\langle x, |s|^{2r}x \rangle_E)) \\ & + \frac{f(\beta) + 2\max\{1, |\alpha - 1|\}}{2|\alpha|^2} (\varphi(\langle x, |t|^{2r}x \rangle_E) + \varphi(\langle x, |s|^{2r}x \rangle_E)) |\varphi(\langle x, s^*tx \rangle_E)|^r \\ & \quad (\text{by AM-GM inequality}) \\ & \leq \frac{\max\{1, |\alpha - 1|^2\}}{2|\alpha|^2} \varphi(\langle x, (|t|^{4r} + |s|^{4r})x \rangle_E) \\ & + \frac{f(\beta) + 2\max\{1, |\alpha - 1|\}}{2|\alpha|^2} \varphi(\langle x, (|t|^{2r} + |s|^{2r})x \rangle_E) |\varphi(\langle x, s^*tx \rangle_E)|^r. \\ & \quad (\text{by Lemma 1}) \end{aligned}$$

Thus,

$$\begin{aligned} & \left(1 - \frac{g(\beta)}{|\alpha|^2}\right) |\varphi(\langle x, s^*tx \rangle_E)|^{2r} \\ & \leq \frac{\max\{1, |\alpha - 1|^2\}}{2|\alpha|^2} \| |t|^{4r} + |s|^{4r} \| + \frac{f(\beta) + 2\max\{1, |\alpha - 1|\}}{2|\alpha|^2} \| |t|^{2r} + |s|^{2r} \| \omega_A^r(s^*t). \end{aligned}$$

Now, by taking supremum over all $x \in E$ and $\varphi \in S(A)$ with $\varphi(|x|^2) = 1$,

$$\begin{aligned} & \omega_A^{2r}(s^*t) \\ & \leq \frac{\max\{1, |\alpha - 1|^2\}}{2(|\alpha|^2 - g(\beta))} \| |t|^{4r} + |s|^{4r} \| + \frac{f(\beta) + 2\max\{1, |\alpha - 1|\}}{2(|\alpha|^2 - g(\beta))} \| |t|^{2r} + |s|^{2r} \| \omega_A^r(s^*t). \end{aligned}$$

□

Remark 8. Let $\alpha = 2$ in Theorem 4. Then, we observe that

$$\omega_A^{2r}(s^*t) \leq \frac{1}{2(4 - g(\beta))} \| |t|^{4r} + |s|^{4r} \| + \frac{f(\beta) + 2}{2(4 - g(\beta))} \| |t|^{2r} + |s|^{2r} \| \omega_A^r(s^*t). \quad (13)$$

Moreover,

$$\begin{aligned} \omega_A^{2r}(s^*t) & \leq \frac{1}{2(4 - g(\beta))} \| |t|^{4r} + |s|^{4r} \| + \frac{f(\beta) + 2}{2(4 - g(\beta))} \| |t|^{2r} + |s|^{2r} \| \omega_A^r(s^*t) \\ & \quad (\text{by inequality (7)}) \\ & \leq \frac{1}{2(4 - g(\beta))} \| |t|^{4r} + |s|^{4r} \| + \frac{f(\beta) + 2}{4(4 - g(\beta))} \| |t|^{2r} + |s|^{2r} \|^2 \\ & \quad (\text{by Lemma 3}) \\ & \leq \frac{1}{2(4 - g(\beta))} \| |t|^{4r} + |s|^{4r} \| + \frac{f(\beta) + 2}{2(4 - g(\beta))} \| |t|^{4r} + |s|^{4r} \| \\ & = \frac{1}{2} \| |t|^{4r} + |s|^{4r} \|, \end{aligned}$$

which shows that inequality (13) refines inequality (7), namely

$$\omega_A^{2r}(s^*t) \leq \frac{1}{2} \| |t|^{4r} + |s|^{4r} \|, \quad r \geq 1.$$

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