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Strong Stability of the Thermoelastic Bresse System with Second Sound and Fractional Delay

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Abstract: The thermoelastic Bresse system is a mathematical model that describes the dynamic behavior of elastic beams accounting for both mechanical deformations and thermal effects. Incorporating concepts such as second sound and fractional delay into this system enhances its ability to model complex physical phenomena. The paper studies a Bresse thermoelastic system with fractional delay and second sound. Firstly, we prove the existence and uniqueness of the solution for our system using semi-group theory. Additionally, we derive an exponential decay estimate for the associated semi-group utilizing suitable multiplier techniques.

Keywords: Bresse system; second-sound phenomenon; thermoelasticity; strong stability; fractional delay; multiplier techniques; fractional derivatives

MSC: 93D15; 74J30

1. Introduction and Problem Statement

Around 1867, Maxwell drew scientists' attention to a theoretical contradiction hidden in Fourier's law, which produces a contradiction in many scientific applications that have any kind of relationship to the subject: thermodynamics, machinists, thermal technicians, and so on. The core of the problem is very simple. The task of thermal diffusion described in Fourier's law leads to the equivalent differential equation, and as a result of the infinite velocity of diffusion. As a solution to this kind of contradiction, dozens of modified laws and methods of modification have been implemented, and the common feature of these efforts is that they all result in a hyperbolic differential equation and the propagation velocity becomes finite. This is the origin of the name "second sound", which has become the slogan of the entire phenomenon; see [1–3]. The extension of derivatives and integrals to fractional orders was discussed as soon as integral and differential calculus was introduced. However, even though the area has many applications, little attention has been paid to this area until recently. For instance, models of thermoelastic bodies, continuous media with memory, and temperature and humidity transformations in atmospheric layers and in diffusion equations make use of fractional calculus; see [4–6]. It is well known in the theory of thermoelasticity that the compound system that determines these phenomena includes differential equations and fractional equations. As the thermoelasticity theory is essentially a linear theory, these differential equations are also linear, so they can be solved using fairly simple methods. Similarly to integro-differential, systems of fractional-integral and fractional-differential equations usually do not have exact solutions. The presence of a fractional derivative with respect to time in the first equation of the Bresse system is interpreted as a reflection of a special property of the process being described,

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such as memory effects, or, in the case of a stochastic process, the influence of past states; see [7–9]. Fractional derivatives with respect to coordinates usually reflect the self-similar inhomogeneity of the structure or medium in which the process develops. Such structures are called fractals.

The new problem of interactions between the effects of the second-sound phenomenon and fractional delay on thermoelasticity has not been solved yet. The very complicated feature of the question has several signs; please see [10,11]. One of which is the impact of the delay and second sound on the stability. Another is related to thermoelasticity; see [12,13].

Let $x \in \Omega = (0,1)$ and $t \in (0,\infty)$. We consider a thermoelastic Bresse system

$$\begin{cases} \rho_1 \varrho_{tt} - k(\varphi + \varrho_x + l\vartheta)_x - lk_0(\vartheta_x - l\varrho) + a_0\varrho_t + \mu\partial_t^{\alpha,\alpha_0}\varrho(x,t-\tau_0) = 0 \\ \rho_2\varphi_{tt} - b\varphi_{xx} + k(\varphi + \varrho_x + l\vartheta) + \gamma\theta_x = 0 \\ \rho_1\vartheta_{tt} - k_0(\vartheta_x - l\varrho)_x + lk(\varphi + \varrho_x + l\vartheta) = 0 \\ \rho_3\theta_t + q_x + \gamma\varphi_{tx} = 0 \\ \tau q_t + \beta q + \theta_x = 0 \\ \\ \varrho(x,0) = \varrho_0(x), \varrho_t(x,0) = \varrho_1(x), \theta(x,0) = \theta_0(x) \\ \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), q(x,0) = q_0(x) \\ \vartheta(x,0) = \vartheta_0(x), \vartheta_t(x,0) = \vartheta_1(x), \varrho_t(x,-t) = f_0(x,t), t \in (0,\tau_0) \\ \varrho(0,t) = \varrho_x(1,t) = \varphi(1,t) = \varphi_x(0,t) = 0 \\ \vartheta(1,t) = \vartheta_x(0,t) = \theta(0,t) = q(1,t) = 0. \end{cases} \tag{1}$$

Here, the parameters $\rho_1, \rho_2, \rho_3, k, k_0, b, l, \alpha_0, a_0, \gamma, \tau, \beta$ are all positive, $\alpha \in (0,1)$, $\alpha_0 > 0$ and $\mu \in \mathbb{R}$, where the time delay is represented by $\tau_0 > 0$. Here, $\varrho = \varrho(x,t)$, $\varphi = \varphi(x,t)$, and $\vartheta = \vartheta(x,t)$ represent the vertical, longitudinal, and shear angle displacements, respectively, while the fourth and fifth coupled equations with variables θ and q represent the additional thermoelastic damping phenomenon. The system (1) is, of course, provided with specific boundary conditions (with respect to x) and initial conditions (with respect to t). The second sound physical properties come from the heat conduction, which is described by Fourier's law, which implies an infinite speed of heat propagation. However, in certain materials, especially at low temperatures, heat propagates at a finite speed, a phenomenon known as "second sound". This behavior is often modeled using Cattaneo's law, modifying the heat equation to account for finite thermal propagation speeds. In the context of the Bresse system, incorporating second sound leads to a more accurate representation of the thermoelastic interactions.

The operator $\partial_t^{\alpha,\alpha_0}$ is the generalized fractional derivative of order α, and it is expressed by:

$$\partial_t^{\alpha,\alpha_0} u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} e^{-\alpha_0(t-s)} \frac{du}{ds}(s) ds.$$

Fractional delay refers to the incorporation of memory effects into the system, acknowledging that the current state is influenced by its history. This is achieved by introducing terms that account for past states, often modeled using fractional calculus. In the Bresse system, the fractional delay can be represented through integral terms that capture the influence of past deformations and thermal states on the current behavior. In the following, we recall some works on the asymptotic behavior of solutions for thermoelastic Bresse systems. In [14], a Bresse system with thermal dissipation effective is considered

$$\begin{cases} \rho_1 \varrho_{tt} - k(\varphi + \varrho_x + l\vartheta)_x - lk_0(\vartheta_x - l\varrho) = 0 \\ \rho_2 \varphi_{tt} - b\varphi_{xx} + k(\varphi + \varrho_x + l\vartheta) + \gamma\theta_x = 0 \\ \rho_1 \vartheta_{tt} - k_0(\vartheta_x - l\varrho)_x + lk(\varphi + \varrho_x + l\vartheta) = 0 \\ \theta_t - k_1\theta_{xx} + m\varphi_{tx} = 0. \end{cases}$$

The exponential stability is shown if and only if the speeds of wave propagation are equal. A similar model was studied later in [15], where the authors proposed a Timoshenko system of thermoelasticity of type III with a delay

$$\begin{cases} \rho_1 \varrho_{tt} - k(\varrho_x + \varphi)_x + \mu_1 \varrho_t + \mu_2 \varrho_t(x, t-\tau) = 0 \\ \rho_2 \varphi_{tt} - b\varphi_{xx} + k(\varphi + \varrho_x) + \beta\theta_{tx} = 0 \\ \rho_3 \theta_{tt} - \delta\theta_{xx} + \gamma\varphi_{tx} - K\theta_{txx} = 0. \end{cases}$$

When certain conditions on the initial data were present, the authors proved the exponential decay of the solution when the wave prorogation speeds were equal, regardless of the presence of the delay term.

Motivated by the above articles, we study Problem (1). We structure our article as follows. In Section 2, some useful tools and results are listed, which will be used later. In the third section, we define a new system that is related to (1). Then, we give the existence and uniqueness result for the new system by using the semi-group theory. The strong stability for the new section is proved in Section 5 using the multiplier technique.

2. Preliminary

In this section, we state useful results for the model (1). We begin with the following theorem, and lemma are needed.

Theorem 1 ([16]). *Define the function ϑ as follows:*

$$\vartheta(\xi) = |\xi|^{(2\alpha-1)/2} \quad -\infty < \xi < +\infty, 0 < \alpha < 1.$$

Then, the 'input' U and the 'output' O of the system

$$\partial_t \varphi(\xi, t) + (\xi^2 + k)\varphi(\xi, t) - U(t)\vartheta(\xi) = 0,$$

$$\varphi(\xi, 0) = 0,$$

$$O(t) = (\pi)^{-1} \sin(\alpha\pi) \int_{-\infty}^{+\infty} \vartheta(\xi)\varphi(\xi, t)d\xi,$$

have a relationship that is given by

$$O = I^{1-\alpha,\alpha_0} U = D^{\alpha,\alpha_0} U,$$

where

$$[I^{\alpha,\alpha_0} f](t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} e^{-\alpha_0(t-s)} f(s)ds.$$

The following lemma is given by [12], and it is important for our reformulation.

Lemma 1. *Let*

$$D_{\alpha_0} = \{\lambda \in \mathbb{C} : Re\lambda + \alpha_0 > 0\} \cup \{\lambda \in \mathbb{C} : Im\lambda \neq 0\}$$

If $\lambda \in D_{\alpha_0}$, *then*

$$\int_{-\infty}^{+\infty} \frac{\vartheta^2(\xi)}{\lambda + \alpha_0 + \xi^2} d\xi = \frac{\pi}{\sin(\alpha\pi)} (\lambda + \alpha_0)^{\alpha - 1}.$$

For positive real numbers a and b, and conjugate exponents p and q satisfying

$$\frac{1}{p} + \frac{1}{q} = 1,$$

Young's inequality states:

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

This can be extended beyond products to convolutions. For functions f and g in appropriate L^p spaces, the convolution $f * g$ satisfies:

$$\|f * g\|_r \leq \|f\|_p \|g\|_q,$$

where

$$\frac{1}{r} = \frac{1}{p} + \frac{1}{q}.$$

3. The Major Results

The system (1) can be reformulated to a suitable problem. To do this, the following variable will be introduced, as in [4]:

$$z(x, \rho, t) = \varrho_t(x, t - \rho\tau_0),\ \rho \in (0, 1).$$

With a simple differentiation, we can show that this variable satisfies

$$\tau_0 z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0,\ \rho \in (0, 1).$$

Hypotheses on the weights of damping terms with/without delay are taken as

$$\mu\alpha_0^{\alpha - 1} < a_0.$$

Consequently, Theorem 1 enables us to derive an equivalent system:

$$\begin{cases} \rho_1 \varrho_{tt} - k(\varphi + \varrho_x + l\vartheta)_x - lk_0(\vartheta_x - l\varrho) + a_0\varrho_t + \zeta \int_{-\infty}^{+\infty} \phi(\nu)\Psi(x, \nu, t) d\nu = 0 \\ \rho_2 \varphi_{tt} - b\varphi_{xx} + k(\varphi + \varrho_x + l\vartheta) + \gamma\theta_x = 0 \\ \rho_1 \vartheta_{tt} - k_0(\vartheta_x - l\varrho)_x + lk(\varphi + \varrho_x + l\vartheta) = 0 \\ \rho_3 \theta_t + q_x + \gamma\varphi_{tx} = 0 \\ \tau q_t + \beta q + \theta_x = 0 \\ \tau_0 z_t(x, \rho, t) + z_\rho(x, \rho, t) = 0, \\ \partial_t \Psi(x, \nu, t) + (\nu^2 + \alpha_0)\Psi(x, \nu, t) - z(x, 1, t)\phi(\nu) = 0, \nu \in \mathbb{R} \\ \varrho(x, 0) = \varrho_0(x), \varrho_t(x, -t) = f_0(x, t), \theta(x, 0) = \theta_0(x), t \in (0, \tau_0) \\ \varphi(x, 0) = \varphi_0(x), \varphi_t(x, 0) = \varphi_1(x), q(x, 0) = q_0(x) \\ \vartheta(x, 0) = \vartheta_0(x), \vartheta_t(x, 0) = \vartheta_1(x), z(w, 0, t) = \varrho_t(x, t), \\ \varrho(0, t) = \varrho_x(1, t) = \varphi(1, t) = \varphi_x(0, t) = 0 \\ \vartheta(1, t) = \vartheta_x(0, t) = \theta(0, t) = q(1, t) = 0, \\ \Psi(x, \nu, 0) = 0, \nu \in \mathbb{R}, \end{cases} \tag{2}$$

where

$$\zeta = \mu(\pi)^{-1}\sin(\alpha\pi),$$

and

$$I = \int_{-\infty}^{+\infty} \frac{\phi^2(\nu)}{\nu^2+\alpha_0} d\nu.$$

Now, the energy of solutions of (2) can be defined by

$$\begin{aligned}\mathcal{E}(t) = \frac{1}{2}\int_0^1 \Big[\rho_1\varrho_t^2 + \rho_2\varphi_t^2 + b\varphi_x^2 + \rho_1\vartheta_t^2 + \rho_3\theta^2 + \tau q^2\Big]dx + \frac{\delta}{2}\int_0^1\int_0^1 z^2(x,\rho,t)d\rho dx \\ + \frac{\Lambda}{2}\int_0^1\int_{-\infty}^{+\infty}\Psi^2(x,\nu,t)d\nu dx + \frac{1}{2}\int_0^1 \Big[k_0(\vartheta_x - l\varrho)^2 + k(\varrho_x+\varphi+l\vartheta)^2\Big]dx,\end{aligned} \tag{3}$$

with

$$\Lambda I\tau_0 < \delta < \tau_0(2a_0 - \zeta I) \quad and \quad \zeta < \Lambda. \tag{4}$$

This energy equation satisfies the next estimate.

Lemma 2. *Let*

$$U = (\varrho, u, \varphi, v, \vartheta, \varpi, \theta, q, z, \Psi)^T,$$

be a solution to system (2)*, then the energy defined by* (3) *satisfies*

$$\mathcal{E}'(t) \le -C\left[\int_0^1 q^2dx + \int_0^1 \varrho_t^2 dx + \int_0^1\int_{-\infty}^{+\infty}\Big(\nu^2+\alpha_0\Big)\Psi^2(x,\nu,t)d\nu dx + \int_0^1 z^2(x,1,t)dx\right]. \tag{5}$$

Proof. By multiplying $(2)_1, (2)_2, (2)_3, (2)_4, (2)_5, (2)_6$, and $(2)_7$ by $\varrho_t, \varphi_t, \vartheta_t, \theta, q, \delta z$, and $\Lambda\Psi$, respectively, and then integrating over Ω and summing up, we obtain

$$\begin{aligned}\mathcal{E}'(t) &= -\zeta\int_0^1\int_{-\infty}^{+\infty}\Psi(x,\nu,t)\varrho_t\phi(\nu)d\nu dx - \beta\int_0^1 q^2dx \\ &- \Lambda\int_0^1\int_{-\infty}^{+\infty}\Psi^2(x,\nu,t)(\nu^2+\alpha_0)d\nu dx \\ &- \Lambda\int_0^1\int_{-\infty}^{+\infty}\Psi(x,\nu,t)z(x,1,t)\phi(\nu)d\nu dx \\ &- a_0\int_0^1\varrho_t^2dx - \frac{\delta}{\tau_0}\int_0^1\int_0^1 z(x,\rho,t)z_\rho(x,\rho,t)d\rho dx.\end{aligned} \tag{6}$$

We have

$$\begin{aligned}-\frac{\delta}{\tau_0}\int_0^1\int_0^1 z(x,\rho,t)z_\rho(x,\rho,t)d\rho dx &= -\frac{\delta}{2\tau_0}\int_0^1\int_0^1\frac{\partial}{\partial\rho}z^2(x,\rho,t)d\rho dx \\ &= \frac{\delta}{2\tau_0}\int_0^1\Big[\varrho_t^2 - z^2(x,1,t)\Big]dx.\end{aligned}$$

Now, Young's inequality yields

$$\begin{aligned}&-\zeta\int_0^1\int_{-\infty}^{+\infty}\Psi(x,\nu,t)z(x,1,t)\phi(\nu)d\nu dx\\&\leq\frac{\zeta I}{2}\left(\int_0^1 z^2(x,1,t)dx\right)+\frac{\zeta}{2}\int_0^1\int_{-\infty}^{+\infty}(\nu^2+\alpha_0)\Psi^2 d\nu dx,\end{aligned}$$

and

$$-\Lambda\int_0^1\int_{-\infty}^{+\infty}\phi(\nu)\Psi(x,\nu,t)\varrho_t d\nu dx\leq\frac{\Lambda I}{2}\left(\int_0^1\varrho_t^2 dx\right)+\frac{\Lambda}{2}\int_0^1\int_{-\infty}^{+\infty}(\nu^2+\alpha_0)\Psi^2 d\nu dx.$$

Inserting the last two inequalities in (6) will lead to

$$\begin{aligned}\mathcal{E}'(t)\leq&\left(-a_0+\frac{\zeta I}{2}+\frac{\delta}{2\tau_0}\right)\int_0^1\varrho_t^2 dx+\left(\frac{\Lambda I}{2}-\frac{\delta}{2\tau_0}\right)\int_0^1 z^2(x,1,t)dx\\&-\beta\int_0^1 q^2 dx+\left(\frac{\zeta I}{2}-\frac{\Lambda}{2}\right)\int_0^1\int_{-\infty}^{+\infty}\left(\nu^2+\alpha_0\right)\Psi^2(x,\nu,t)d\nu dx,\end{aligned}$$

from (4), the proof is completed. □

4. Existence and Uniqueness

The semi-group theory will be needed to prove that there exists a unique solution to the system (2). Define

$$U=(\varrho,u,\varphi,v,\vartheta,\varpi,\theta,q,z,\Psi)^T,$$

where $u=\varrho_t$, $v=\varphi_t$ and $\varpi=\vartheta_t$.

Now, we can rewrite (2) to be

$$\begin{cases}U'(t)=\mathbf{A}U(t)\\U(t=0)=U_0=(\varrho_0,\varrho_1,\varphi_0,\varphi_0,\vartheta_0,\vartheta_1,\theta_0,q_0,z_0,0)^T,\end{cases}\tag{7}$$

where

$$\mathbf{A}:D(\mathbf{A})\subset\mathrm{H}\to\mathrm{H},$$

is the linear operator defined by

$$\mathbf{A}U=\begin{bmatrix}u\\\frac{k}{\rho_1}(\varphi+\varrho_x+l\vartheta)_x+\frac{lk_0}{\rho_1}(\vartheta_x-l\varrho)-\frac{a_0}{\rho_1}u-\frac{\zeta}{\rho_1}\int_{-\infty}^{+\infty}\phi(\nu)\Psi(x,\nu,t)d\nu\\v\\\frac{b}{\rho_2}\varphi_{xx}-\frac{k}{\rho_2}(\varphi+\varrho_x+l\vartheta)-\frac{\gamma}{\rho_2}\theta_x\\\varpi\\\frac{k_0}{\rho_1}(\vartheta_x-l\varrho)_x-\frac{lk}{\rho_1}(\varphi+\varrho_x+l\vartheta)\\-\frac{1}{\rho_3}q_x-\frac{\gamma}{\rho_3}v_x\\-\frac{\beta}{\tau}q-\frac{1}{\tau}\theta_x\\-\frac{1}{\tau_0}z_\rho\\-(\nu^2+\alpha_0)\Psi+z(x,1,t)\phi(\nu),\end{bmatrix}\tag{8}$$

and H is the energy space given by

$$\begin{aligned}\mathrm{H} =& \mathrm{H}^1_*(\Omega)\times L^2(\Omega)\times \tilde{\mathrm{H}}^1_*(\Omega)\times L^2(\Omega)\times \tilde{\mathrm{H}}^1_*(\Omega)\times L^2(\Omega)\\ &\times L^2(\Omega)\times L^2(\Omega)\times L^2(\Omega\times\Omega)\times L^2(\Omega\times\mathbb{R}),\end{aligned}$$

with

$$\begin{aligned}&\mathrm{H}^1_*(\Omega)=\left\{f\in \mathrm{H}^1(\Omega): f(0)=0\right\}\\ &\tilde{\mathrm{H}}^1_*(\Omega)=\left\{f\in \mathrm{H}^1(\Omega): f(1)=0\right\}\\ &\mathrm{H}^2_*(\Omega)=\mathrm{H}^2(\Omega)\cap \mathrm{H}^1_*(\Omega)\\ &\tilde{\mathrm{H}}^2_*(\Omega)=\mathrm{H}^2(\Omega)\cap \tilde{\mathrm{H}}^1_*(\Omega).\end{aligned}$$

The inner product in H is defined by

$$\begin{aligned}(U,\tilde{U})_{\mathrm{H}} =& k\int_0^1 (\varrho_x+\varphi+l\vartheta)\left(\tilde{\varrho}_x+\tilde{\varphi}+l\tilde{\vartheta}\right)dx+k_0\int_0^1(\vartheta_x-l\varrho)\left(\tilde{\vartheta}_x-l\tilde{\varrho}\right)dx\\ &+\rho_1\int_0^1 u\tilde{u}dx+b\int_0^1\varphi_x\tilde{\varphi}_x dx+\rho_2\int_0^1 v\tilde{v}dx+\rho_1\int_0^1\omega\tilde{\omega}dx+\rho_3\int_0^1\theta\tilde{\theta}dx+\tau\int_0^1 q\tilde{q}dx\\ &+\delta\int_0^1\int_0^1 z\tilde{z}d\rho dx+\Lambda\int_0^1\int_{-\infty}^{+\infty}\Psi(x,\nu,t)\tilde{\Psi}(x,\nu,t)d\nu dx.\end{aligned}$$

The domain of the linear operator $\mathbf{A}$ is

$$D(\mathbf{A})=\left\{\begin{array}{l} U\in \mathrm{H}/\varrho\in \mathrm{H}^2_*(\Omega);\varphi,\vartheta\in\tilde{\mathrm{H}}^2_*(\Omega);u,\theta\in\mathrm{H}^1_*(\Omega);v,\omega,q\in\tilde{\mathrm{H}}^1_*(\Omega),\\ \varrho_x(1)=\varphi_x(0)=\vartheta_x(0)=0; z,z_\rho\in L^2\big((0,1)\times L^2(\Omega)\big), z(x,0)=\varrho(x),\\ (\nu^2+\alpha_0)\Psi-z(x,1,t)\phi(\nu)\in L^2(\Omega\times\mathbb{R}),\\ |\nu|\Psi\in L^2(\Omega\times\mathbb{R})\end{array}\right\}. \quad (9)$$

We are going to show that the operator $\mathbf{A}$ generates a C_0 semi-group of contractions in H. To this end, we start by proving that $\mathbf{A}$ is monotone.

Lemma 3. *The linear operator $\mathbf{A}$ is monotone and for any $U\in D(\mathbf{A})$, the following inequality is satisfied:*

$$(\mathbf{A}U,U)<-C\left[\int_0^1 q^2dx+\int_0^1 u^2dx+\int_0^1\int_{-\infty}^{+\infty}\left(\nu^2+\alpha_0\right)\Psi^2(x,\nu,t)d\nu dx+\int_0^1 z^2(x,1,t)dx\right]. \quad (10)$$

Proof. We can easily show that $D(\mathbf{A})$ is dense in H. Besides, for any

$$U=(\varrho,u,\varphi,v,\vartheta,\omega,\theta,q,z,\Psi)^T\in D(\mathbf{A}),$$

we have

$$\mathcal{E}(t)=\frac{1}{2}\|U\|^2.$$

Then, from (7), we have

$$\begin{aligned}\mathcal{E}'(t) &= (U',U)\\ &= (\mathbf{A}U,U).\end{aligned}$$

Using (10), we conclude that the operator $\mathbf{A}$ is monotone. □

Next, we prove the following lemma with respect to the operator $(I-\mathbf{A})$.

Lemma 4. *The operator* $(I - A)$ *is surjective.*

Proof. For any

$$G = (g_1, g_2, g_3, g_4, g_5, g_6, g_7, g_8, g_9, g_{10}) \in \mathrm{H},$$

there exists $U \in D(\mathbf{A})$, which satisfies

$$(I - \mathbf{A})U = G. \tag{11}$$

It can be observed that Equation (11) is equivalent to

$$\begin{cases} -u + \varrho = g_1 \in \mathrm{H}^1_*(\Omega) \\ -k(\varphi + \varrho_x + l\vartheta)_x - lk_0(\vartheta_x - l\varrho) + (a_0 + \rho_1)u + \zeta \int_{-\infty}^{+\infty} \phi(\nu)\Psi(x,\nu,t)d\nu = \rho_1 g_2 \in L^2(\Omega) \\ -v + \varphi = g_3 \in \tilde{\mathrm{H}}^1_*(\Omega) \\ -b\varphi_{xx} + k(\varphi + \varrho_x + l\vartheta) + \gamma\theta_x + \rho_2 v = \rho_2 g_4 \in L^2(\Omega) \\ -\varpi + \vartheta = g_5 \in \tilde{\mathrm{H}}^1_*(\Omega) \\ -k_0(\vartheta_x - l\varrho)_x + lk(\varphi + \varrho_x + l\vartheta) + \rho_1\varpi = \rho_1 g_6 \in L^2(\Omega) \\ q_x + \gamma v_x + \rho_3\theta = \rho_3 g_7 \in L^2(\Omega) \\ (\tau + \beta)q + \theta_x = \tau g_8 \in L^2(\Omega) \\ z_\rho + \tau_0 z = \tau_0 g_9 \in L^2(\Omega \times (0,1)) \\ (\nu^2 + \alpha_0)\Psi - z(x,1,t)\phi(\nu) + \Psi = g_{10}(x,\nu) \in L^2(\Omega \times \mathbb{R}). \end{cases} \tag{12}$$

By $(12)_8$, we have

$$\tau \int_0^x g_8(y)dy - (\tau + \beta)\int_0^x q(y)dy = \theta,$$

which means that $\theta(0,t) = 0$. Given that

$$z(x,0) = u(x),$$

$$u = \varrho - g_1,$$

$$v = \varphi - g_3,$$

and

$$\varpi = \vartheta - g_5.$$

Integrating the Equation $(12)_9$ yields

$$z(x,\rho) = \varrho(x)e^{-\tau_0\rho} - e^{-\tau_0\rho}g_1(x) + \tau_0 e^{-\tau_0\rho}\int_0^\rho e^{\tau_0 s}g_9(x,s)ds,$$

then

$$z(x,1) = \varrho(x)e^{-\tau_0} - e^{-\tau_0}g_1(x) + \tau_0 e^{-\tau_0}\int_0^1 e^{\tau_0 s}g_9(x,s)ds.$$

From $(12)_{10}$, one has

$$\Psi = \frac{g_{10}(x,\nu) + z(x,1,t)\phi(\nu)}{\nu^2 + \alpha_0 + 1}.$$

Now, it can be easily shown that ϱ,φ,ϑ and q satisfy the following system

$$\begin{cases} -k(\varphi + \varrho_x + l\vartheta)_x - lk_0(\vartheta_x - l\varrho) + (a_0 + \rho_1 + \zeta M e^{-\tau_0})\varrho = h_1 \in L^2(\Omega) \\ -b\varphi_{xx} + k(\varphi + \varrho_x + l\vartheta) + \rho_2\varphi - \gamma(\tau + \beta)q = h_2 \in L^2(\Omega) \\ -k_0(\vartheta_x - l\varrho)_x + lk(\varphi + \varrho_x + l\vartheta) + \rho_1\vartheta = h_3 \in L^2(\Omega) \\ -q_x + \rho_3(\tau + \beta)\int_0^x q(y)dy - \gamma\varphi_x = h_4 \in L^2(\Omega), \end{cases} \tag{13}$$

with

$$M = \int_{-\infty}^{+\infty} \frac{\phi^2(\nu)}{\nu^2 + \alpha_0 + 1} d\nu,$$

and

$$\begin{cases} h_1 = \rho_1 g_2 + (a_0 + \rho_1 + \zeta M e^{-\tau_0}) g_1 - \zeta M \tau_0 e^{-\tau_0} \int_0^1 e^{\tau_0 s} g_9(x,s) ds - \zeta \int_{-\infty}^{+\infty} \frac{g_{10}(x,\nu)\phi(\nu)}{\nu^2 + \alpha_0 + 1} d\nu \\ h_2 = (g_3 + g_4)\rho_2 - \gamma \tau g_8 \\ h_3 = (g_5 + g_6)\rho_1 \\ h_4 = -\gamma g_{3x} - \rho_3 \left(g_7 - \tau \int_0^x g_8(y) dy \right). \end{cases} \tag{14}$$

Multiplying $(13)_1, (13)_2, (13)_3$ and $(13)_4$ by $\tilde{\varrho}, \tilde{\varphi}, \tilde{\vartheta}$ and

$$(\tau + \beta) \int_0^x \tilde{q}(y) dy,$$

respectively. Then, integrating over Ω and summing up, we obtsain the variational formulation of (13), as

$$\mathcal{B}\big((\varrho, \varphi, \vartheta, q), (\tilde{\varrho}, \tilde{\varphi}, \tilde{\vartheta}, \tilde{q})\big) = L(\varrho, \varphi, \vartheta, q), \tag{15}$$

where

$$\mathcal{B} : \left[\mathrm{H}_*^1(\Omega) \times \tilde{\mathrm{H}}_*^1(\Omega) \times \tilde{\mathrm{H}}_*^1(\Omega) \times L^2(\Omega) \right]^2 \to \mathbb{R},$$

is the bilinear form defined by

$$\begin{aligned} \mathcal{B}\big((\varrho, \varphi, \vartheta, q), (\tilde{\varrho}, \tilde{\varphi}, \tilde{\vartheta}, \tilde{q})\big) &= k \int_0^1 (\varrho_x + \varphi + l\vartheta)(\tilde{\varrho}_x + \tilde{\varphi} + l\tilde{\vartheta}) dx + k_0 \int_0^1 (\vartheta_x - l\varrho)(\tilde{\vartheta}_x - l\tilde{\varrho}) dx \\ &+ \rho_3 (\tau + \beta)^2 \int_0^1 \left(\int_0^x q(y) dy \right) \left(\int_0^x \tilde{q}(y) dy \right) dx + \gamma(\tau + \beta) \int_0^1 \tilde{q}\varphi dx \\ &+ \left(a_0 + \rho_1 + \zeta M e^{-\tau_0} \right) \int_0^1 \varrho\tilde{\varrho} dx \\ &+ b \int_0^1 \varphi_x \tilde{\varphi}_x dx + \rho_2 \int_0^1 \varphi\tilde{\varphi} dx - \gamma(\tau + \beta) \int_0^1 q\tilde{\varphi} dx + \rho_1 \int_0^1 \vartheta\tilde{\vartheta} dx + \gamma(\tau + \beta) \int_0^1 q\tilde{q} dx, \end{aligned}$$

and

$$L : \left[\mathrm{H}_*^1(\Omega) \times \tilde{\mathrm{H}}_*^1(\Omega) \times \tilde{\mathrm{H}}_*^1(\Omega) \times L^2(\Omega) \right] \to \mathbb{R},$$

is the linear functional given by

$$L(\varrho, \varphi, \vartheta, q) = \int_0^1 h_1 \tilde{\varrho} dx + \int_0^1 h_2 \tilde{\varphi} dx + \int_0^1 h_3 \tilde{\vartheta} dx + (\tau + \beta)^2 \int_0^1 h_4 \left(\int_0^x \tilde{q}(y) dy \right) dx.$$

Let us now define a new space V by

$$V = \mathrm{H}_*^1(\Omega) \times \tilde{\mathrm{H}}_*^1(\Omega) \times \tilde{\mathrm{H}}_*^1(\Omega) \times L^2(\Omega),$$

equipped with the norm

$$\|(\varrho, \varphi, \vartheta, q)\|_V = \|(\varrho_x + \varphi + l\vartheta)\|_2^2 + \|(\vartheta_x - l\varrho)\|_2^2 + \|\varphi\|_2^2 + \|q\|_2^2.$$

It is easy to see, if l is small enough, that

$$\int_0^1 \left(\varrho_x^2 + \varphi_x^2 + \vartheta_x^2 \right) dx \le c \int_0^1 \left[(\varrho_x + \varphi + l\vartheta)^2 + (\vartheta_x - l\varrho)^2 + \varphi_x^2 \right] dx,$$

which shows us that $\mathcal{B}$ and L are bounded in $V \times V$ and V, respectively. Moreover, the definition of $\mathcal{B}$ will lead to

$$\mathcal{B}((\varrho, \varphi, \vartheta, q), (\varrho, \varphi, \vartheta, q)) \geq c\|(\varrho, \varphi, \vartheta, q)\|_V^2,$$

which means that $\mathcal{B}$ is coercive. As a result, from the Lax–Milgram theorem, we conclude that the system (13) has a unique solution

$$(\varrho, \varphi, \vartheta, q) \in \mathrm{H}_*^1(\Omega) \times \tilde{\mathrm{H}}_*^1(\Omega) \times \tilde{\mathrm{H}}_*^1(\Omega) \times L^2(\Omega).$$

Now, by substituting $\varrho, \varphi, \vartheta$ and q into $(12)_1, (12)_3, (12)_5$ and $(12)_8$, respectively, we find that

$$(u, v, \varpi, \theta) \in \mathrm{H}_*^1(\Omega) \times \tilde{\mathrm{H}}_*^1(\Omega) \times \tilde{\mathrm{H}}_*^1(\Omega) \times L^2(\Omega).$$

If

$$(\tilde{\varphi}, \tilde{\vartheta}, \tilde{q}) = (0, 0, 0) \in \tilde{\mathrm{H}}_*^1(\Omega) \times \tilde{\mathrm{H}}_*^1(\Omega) \times L^2(\Omega),$$

then, (15) provides us

$$k\int_0^1 (\varrho_x + \varphi + l\vartheta)\big(\tilde{\varrho}_x + \tilde{\varphi} + l\tilde{\vartheta}\big)dx + k_0\int_0^1 (\vartheta_x - l\varrho)\big(\tilde{\vartheta}_x - l\tilde{\varrho}\big)dx = \int_0^1 h_1\tilde{\varrho}dx, \tag{16}$$

for all $\varrho \in \mathrm{H}_*^1(\Omega)$. The latter implies that

$$-k\varrho_{xx} = k\varrho_x + l(k_0 + k)\vartheta_x - \Big(k_0 l^2 + \rho_1 + a_0 + \zeta M e^{-\tau_0}\Big)\varrho + h_1 \in L^2(\Omega). \tag{17}$$

Then, from the regularity theory for the linear elliptic equations, we obtain $\varrho \in \mathrm{H}_*^2(\Omega)$. Moreover, (16) is also true for any ϕ satisfying $\phi \in C^1([0,1])$ and $\phi(0) = 0$, which is in $\mathrm{H}_*^1(\Omega)$. Therefore, for any $\phi \in C^1([0,1])$, we obtain

$$k\int_0^1 \varrho_x\phi_x dx - \int_0^1 \Big[k\varrho_x + l(k_0 + k)\vartheta_x - \Big(k_0 l^2 + \rho_1 + a_0 + \zeta M e^{-\tau_0}\Big)\varrho + h_1\Big]\phi dx = 0.$$

When the last equation is integrated in parts, using (17) will lead to

$$\phi(1)\varrho_x(1) = 0 \quad \forall \phi \in C^1([0,1]),$$

and, thus, $\varrho_x(1) = 0$. In the same way, we can verify that

$$\begin{cases} -b\varphi_{xx} = -k\varrho_x - (k + \rho_2)\varphi - kl\vartheta - \gamma(\tau + \beta)q + h_2 \in L^2(\Omega) \\ -b\vartheta_{xx} = -l(k_0 + k)\varrho_x - kl\varphi + (\rho_1 + l^2 k_0)\vartheta + h_3 \in L^2(\Omega) \\ -q_x = \gamma\varphi_x - (\tau + \beta)\rho_3\displaystyle\int_0^x q(y)dy + h_4 \in L^2(\Omega). \end{cases}$$

Then, we have

$$\varphi, \vartheta \in \tilde{H}_*^2(\Omega), q \in \tilde{H}_*^1(\Omega),$$

and

$$\vartheta_x(0) = \varphi_x(0) = 0,$$

which leads to the existence of a unique solution $U \in D(\mathbf{A})$ to (11). This completes the proof. □

The previous two lemmas imply that the operator $\mathbf{A}$ is a maximal monotone operator. Then, $\mathbf{A}$ is the infinitesimal generator of a linear contraction C_0-semi-group on H. In other words, this can be considered as the proof of the following theorem (see [17]).

Theorem 2. *(Existence and uniqueness)*

1. *If $U_0 \in D(\boldsymbol{A})$, then there exists a unique strong solution to system (7), and it is given by*

$$U \in C\left(\mathbb{R}^+, D(\boldsymbol{A})\right) \cap C^1\left(\mathbb{R}^+, \mathrm{H}\right).$$

2. *If $U_0 \in \mathrm{H}$, then there exists a unique weak solution to system (7), and it is given by*

$$U \in C\left(\mathbb{R}^+, \mathrm{H}\right).$$

5. Exponential Stability

The exponential stability of the semi-group $S(t)$ in the Hilbert space H will be demonstrated in this section, then there exists a strong unique solution to the system (7). We will need to employ the necessary and sufficient conditions for C_0-semi-groups to be exponentially stable in a Hilbert space, which was achieved by Gearhart [18].

Theorem 3. *Let*

$$S(t) = e^{\mathrm{A}t},$$

be a C_0-semi-group of contractions on Hilbert space H. Then, $S(t)$ is exponentially stable if and only if

$$\rho(\mathrm{A}) \supseteq \{i\chi : \chi \in \mathbb{R}\} \equiv i\mathbb{R}, \tag{18}$$

$$\overline{\lim_{|\chi|\to\infty}} \left\| (i\chi I - \mathrm{A})^{-1} \right\| < \infty, \tag{19}$$

hold, where $\rho(\mathrm{A})$ denotes the resolvent set of A.

Theorem 4. *The C_0-semi-group of contractions $e^{\mathrm{A}t}$, $t > 0$, generated by* A*, is exponentially stable.*

Proof. Proving the exponential stability of $e^{\mathrm{A}t}$ requires demonstrating the validity of Theorem 3. By verifying the properties (18) and (19). To begin with, we establish the following

$$\rho(\mathrm{A}) \supseteq \{i\lambda : \lambda \in \mathbb{R}\} \equiv i\mathbb{R}. \tag{20}$$

This can be proved in contradiction. Assume that $\tilde{\lambda} \in \mathbb{R}$ with $\tilde{\lambda} \neq 0$ and $U \neq 0$, such that

$$\mathrm{A}U = i\tilde{\lambda}U.$$

Then

$$\begin{cases} i\tilde{\lambda}\varrho - u = 0 \\ i\tilde{\lambda}\rho_1 u - k(\varphi + \varrho_x + l\vartheta)_x - lk_0(\vartheta_x - l\varrho) + a_0 u + \zeta \int_{-\infty}^{+\infty} \phi(\nu)\Psi(x,\nu,t)d\nu = 0 \\ i\tilde{\lambda}\varphi - v = 0 \\ i\tilde{\lambda}\rho_2 v - b\varphi_{xx} + k(\varphi + \varrho_x + l\vartheta) + \gamma\theta_x = 0 \\ i\tilde{\lambda}\vartheta - \varpi = 0 \\ i\tilde{\lambda}\rho_1\varpi - k_0(\vartheta_x - l\varrho)_x + lk(\varphi + \varrho_x + l\vartheta) = 0 \\ i\tilde{\lambda}\rho_3\theta + q_x + \gamma v_x = 0 \\ i\tilde{\lambda}\tau q + \beta q + \theta_x = 0 \\ i\tilde{\lambda}\tau_0 z + z_\rho = 0 \\ i\tilde{\lambda}\Psi + (\nu^2 + \alpha_0)\Psi - z(x,1,t)\phi(\nu) = 0. \end{cases} \tag{21}$$

From (10), we obtain

$$u = 0, \quad q = 0, \quad z(x,1) = 0.$$

From $(21)_{10}$ we obtain $\Psi = 0$, from $(21)_8$ we obtain $\theta = 0$, and $(21)_1$ gives us $\varrho = 0$. From $(21)_9$, we can write the solution

$$z(x,\rho) = Ce^{-i\tilde{\lambda}\tau_0\rho},$$

as the only one. With the fact $z(x,1) = 0$, we obtain $z(x,\rho) = 0$. We continue, using $(21)_7$ with the initial conditions to obtain $v = 0$, then $(21)_3$ gives us $\varphi = 0$. From $(21)_4$, we can conclude that $\vartheta = 0$. Finally, from $(21)_5$, we deal with $\varpi = 0$. In other words, $\|U\|_H = 0$, but this is a contradiction with $U \neq 0$; therefore, there are no imaginary eigenvalues.

Now, the proof of (19) will be given.

Assume (19) is false, that is

$$\lim_{|\lambda|\to\infty} \sup \left\| (i\lambda I - \mathrm{A})^{-1} \right\| = \infty.$$

Then, there exists a sequence $(V_n) \in H$ and $\lambda_n \in \mathbb{R}$, such that

$$\left\| (i\lambda_n I - \mathrm{A})^{-1} V_n \right\| > n\|V_n\|,$$

for all $n > 0$. Given

$$i\lambda_n \in \rho(\mathrm{A}),$$

it can be observed that there exists a unique sequence $(U_n) \in D(\mathrm{A})$, such that

$$i\lambda_n U_n - \mathrm{A}U_n = V_n, \quad \|U_n\| = 1,$$

i.e.,

$$U_n = (i\lambda I - \mathrm{A})^{-1} V_n,$$

and

$$\|U_n\| > n\|i\lambda_n U_n - \mathrm{A}U_n\|.$$

Now, define

$$F_n = i\lambda_n U_n - \mathrm{A}U_n,$$

which results in $\|F_n\| \leq \frac{1}{n}$, and then $F_n \to 0$ (strong) in H and $n \to \infty$.

Let

$$u = \varrho_t, v = \varphi_t, \varpi = \vartheta_t,$$

and

$$U_n = (\varrho^n, u^n, \varphi^n, v^n, \vartheta^n, \varpi^n, \theta^n, q^n, z^n, \Psi^n),$$

and

$$F_n = (f_1^n, f_2^n, f_3^n, f_4^n, f_5^n, f_6^n, f_7^n, f_8^n, f_9^n, f_{10}^n).$$

From

$$F_n = i\lambda_n U_n - \mathrm{A}U_n,$$

the following equations can be obtained

$$\begin{cases} i\lambda_n \varrho^n - u^n = f_1^n \\ i\lambda_n \rho_1 u^n - k(\varphi^n + \varrho_x^n + l\vartheta^n)_x - lk_0(\vartheta_x^n - l\varrho^n) + a_0 u^n + \zeta \int_{-\infty}^{+\infty} \phi(\nu)\Psi^n(x,\nu,t)d\nu = \rho_1 f_2^n \\ i\lambda_n \varphi^n - v^n = f_3^n \\ i\lambda_n \rho_2 v^n - b\varphi_{xx}^n + k(\varphi^n + \varrho_x^n + l\vartheta^n) + \gamma\theta_x^n = \rho_2 f_4^n \\ i\lambda_n \vartheta^n - \varpi^n = f_5^n \\ i\lambda_n \rho_1 \varpi^n - k_0(\vartheta_x^n - l\varrho^n)_x + lk(\varphi^n + \varrho_x^n + l\vartheta^n) = \rho_1 f_6^n \\ i\lambda_n \rho_3 \theta^n + q_x^n + \gamma v_x^n = \rho_3 f_7^n \\ i\lambda_n \tau q^n + \beta q^n + \theta_x^n = \tau f_8^n \\ i\lambda_n \tau_0 z^n + z_\rho^n = \tau_0 f_9^n \\ i\lambda_n \Psi^n + (\nu^2 + \alpha_0)\Psi^n - z^n(x,1,t)\phi(\nu) = f_{10}^n. \end{cases} \tag{22}$$

Taking the real part of the inner product of $(i\lambda_n I - \mathrm{A})U_n$ and U_n in H, noting that U_n is bounded and $F_n \to 0$ and using (10) will lead to

$$a_0 \int_0^1 (u^n)^2 dx + \beta \int_0^1 (q^n)^2 dx + \int_0^1 (z^n(x,1))^2 dx \to 0, \tag{23}$$

where

$$u^n \to 0, \quad q^n \to 0, \quad z^n(x,1) \to 0, \tag{24}$$

then, from $(22)_{10}$ we obtain $\Psi^n \to 0$.

Now, with the fact that

$$\frac{k}{\rho_1} = \frac{b}{\rho_2},$$

and $k = k_0$, after some calculations, we obtain

$$\begin{aligned} \int_0^1 \Big[& \rho_2 (v^n)^2 + b(v_x^n)^2 + \rho_1 (\varpi^n)^2 + \rho_3 (\theta^n)^2 + (z^n(x,1))^2 \\ & + k(\varphi^n + \varrho_x^n + l\vartheta^n)^2 + k(\vartheta_x^n - l\varrho^n)^2 \Big] dx \to 0, \end{aligned} \tag{25}$$

so that

$$i\lambda_n \|U_n\|^2 - (\mathrm{A}U_n, U_n) \to 0.$$

Then, $\lambda_n \|U_n\| \to 0$, which is true only if $\|U_n\| \to 0$. This contradicts $\|U_n\| = 1$; therefore, the proof of the theorem is now completed. □

6. Conclusions

The integration of second sound and fractional delay into the thermoelastic Bresse system provides a more comprehensive framework to model the dynamic behavior of thermoelastic beams, particularly in scenarios where the speed of propagation of the heat and historical effects are significant, and this is the case in the present work. An analysis of the exponential stability is conducted. The study establishes both the existence and uniqueness of solutions for the system through semi-group theory, deriving an exponential decay estimate for the associated semi-group via appropriate multiplier techniques.

Our system models the interplay between mechanical and thermal waves in materials, incorporating advanced concepts from fractional calculus and non-local effects.

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