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Robust Positively Invariant Conditions for Perturbed Linear Discrete-Time Systems Using Dual Optimization

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Abstract: This paper presents both sufficient and necessary conditions for polyhedral sets and symmetric polyhedral sets to be robust positively invariant sets within perturbed linear discrete-time systems. These conditions are derived through the application of optimization and dual optimization theory. By leveraging the definition of a robust positively invariant set and employing the Pontryagin difference, we have obtained robust positively invariant conditions in optimized forms. Through the use of dual optimization theory, various equivalent forms are introduced, offering additional tools for verifying that polyhedral sets are indeed robust positively invariant sets for perturbed linear discrete-time dynamic systems. The efficacy of these conclusions is further evidenced by numerical examples.

Keywords: perturbedlinear discrete-time dynamic system; robust positively invariant set; optimization; polyhedral set

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1. Introduction

The concept of a robust positively invariant set is pivotal in the research of dynamic system control theory. When the initial state of a system originates from this set, subsequent trajectories, influenced by external input disturbances, remain within it. This principle underpins the broad applicability of robust positively invariant set theory in areas such as the design of dynamic system regulators, the development of full order state observers, and model predictive control [1–7].

During explorations into dynamic system stability, Lyapunov function plays crucial theoretical backing [8,9]. A primary area of investigation focuses on deducing the positive invariance conditions for the robust positively invariant sets of dynamic systems.

In [10], Bitsoris firstly theoretically derived the necessary and sufficient conditions for a polyhedral set to be a positive invariant set of linear discrete systems. Reference [11] proposes a necessary and sufficient condition for a polyhedron set to be a positive invariant set of perturbed linear systems based on the positive invariance condition of the polyhedron set in [10]. The explicit characterization of the necessary and sufficient conditions for convex polyhedra to be an invariant set of (A, B) linear discrete systems is given in the form of linear matrix inequalities in [12]. In [13], the sufficient and necessary conditions for a polyhedron set to be a positive invariant set of the perturbed positive linear discrete system are derived theoretically using linear programming inequalities. In [14], perturbed linear discrete

systems are studied and the set of perturbations of the system as well as some properties of the positive invariance of polyhedra are given. Based on the theory of a positively invariant sets of systems, unknown but bounded systems with input perturbations are studied in [5], full-order state observers with estimation error limits are designed, and a sufficient necessary condition that the set of polyhedra is a positively invariant set of the system is given to a certain extent based on the results of the error test. In [15], the form of bilinear algebraic inequalities is applied to verify the sufficient conditions for a given polyhedron set to be a positive invariant set of a dynamically perturbed system, and corresponding characterizations of the positive invariant polyhedron set are given. The Minkowski set addition and Pontryagin set difference methods provide effective tools for deriving positive invariance conditions and designing controllers when studying invariant sets of perturbed linear discrete systems and their applications. In [16], a detailed study is provided on the definition of Pontryagin difference, and the conditions for a polyhedron set to be an invariant set of linear perturbation systems are given through Pontryagin difference and support functions. In [17], a type of pipe controller is proposed for bounded disturbance systems based on Pontryagin difference and invariant set theory. The minimum robust positive invariant set is studied in [18] using Pontryagin difference for linear discrete-time systems, and a robust positive invariant approximation is obtained by applying a standard Luenberger-type observer, which solves the robust positive invariance problem in some sense.

The linear properties of polyhedral sets and perturbed linear discrete-time systems make linear programming methods suitable for solving problems with positive invariance conditions [19]. These conditions, determined through linear programming, are simpler than those derived theoretically and can be solved using MATLAB (R2023b) and Lingo solvers to directly obtain the desired conclusions from the results [20]. A linear programming algorithm, based on Haar's Lemma and another source, tests if a set of polyhedra is a necessary and sufficient condition for a system's positive invariant set. The necessary and sufficient conditions for polyhedral and polyhedral cones to be positive invariant sets for linear and nonlinear systems are provided by optimization and dual optimization theories. This article uses these theories to derive positive invariance conditions for perturbed discrete systems. The main contribution is using the definitions of robust positively invariant set and Pontryagin differences to provide positive invariance conditions for the optimization form of polyhedral sets and symmetric polyhedral sets as robust positively invariant sets for perturbed linear discrete systems. It also provides their equivalent general form duality, Lagrange duality, and Wolfe duality robust positive invariance conditions. This robust positively invariant condition avoids calculating polyhedral vertices, reducing computational complexity. Additionally, the proposed dual optimization forms simplify the feasible domain of optimization problems and offer more options for verifying the robust positively invariant condition of polyhedral sets. The paper verifies the effectiveness of the proposed methods through numerical examples.

The remaining parts of this article are composed as follows: Section 2 presents the preliminary knowledge and definitions required for this article. Section 3 provides the robust positively invariance conditions for the polyhedral set in P_b form. Section 4 presents the robust positively invariance conditions for the polyhedral set in P_y form. Section 5 offers several numerical examples. Section 6 provides a summary of this article.

Notations. x_k, $x_{k+1} \in R^n$ denotes the state vector, where $k \in \mathbb{N} \cup \{0\}$. R and R^n represent sets of real numbers and $n \times 1$-dimensional column vectors, respectively. $R^{n\times n}$ represents a real square matrix of the $n \times n$ dimension. G represents an $m \times n$-dimensional matrix, and G_i^T represents the i-th row of matrix G. $I(m)$ represents the index set $\{1, 2, ..., m\}$.

2. Preliminaries

2.1. Perturbed Linear Discrete-Time Dynamic System

The main system under consideration in this paper is the perturbed linear discrete-time system. This system can be characterized as follows:

$$x_{k+1} = Ax_k + E\omega_k, \tag{1}$$

where x_k, $x_{k+1} \in R^n$ denote the state vector, $k \in \mathbb{N} \cup \{0\}$, and ω denotes the bounded disturbance, $\omega \in W$, where $W \subset R^n$, $A \in R^{n\times n}$, $E \in R^{n\times n}$.

2.2. Convex Set and Robust Positively Invariant Set

In this paper, we primarily explore the robust positively invariant conditions for two distinct types of polyhedral sets to ensure they remain robustly positively invariant within the system (1).

The set $P_b = \{x \in R^n \mid Gx \leq b\}$ is called a polyhedral set, where $G \in R^{m\times n}$, $b \in R^m$. The set $P_\eta = \{x \in R^n \mid -\eta \leq Gx \leq \eta\}$ represents a symmetric polyhedron set, where $\eta \in R^m$, $\eta > 0$.

Definition 1. *The polyhedron set P is the robust positively invariant set of a discrete-time linear perturbation system if $x_0 \in P$, and $Ax_k + E\omega_k \in P$ for all $x_k \in P$, and all $\omega_k \in W$. The perturbation set W is closed and bounded.*

Definition 2. *The polyhedral set P is a robust positively invariant set of perturbed linear discrete systems if and only if the perturbed set $W \subset W^*$, where W^* is the maximum perturbed set that makes the set P the robust positively invariant set of the system* (1).

2.3. Lagrange Function

$$L(x, \lambda, \mu) = f(x) + \sum_{i=1}^{m} \lambda_i g_i(x) + \sum_{l=1}^{n} \mu_l y_l(x), \tag{2}$$

where $f(x)$ represents the minimized objective function, and $g(x)$ and $y(x)$ represent inequality and equality constraints, respectively. And λ, μ is called the Lagrange operator, and it requires that $\lambda_i \geq 0$, $i = 1, 2, ..., m$.

2.4. Wolfe Dual Theory

Let the functions $f(x)$ and $g(x)$ be differentiable with respect to x, and (3) is a convex optimization problem. Then, the Wolfe duality transformation can be performed on them. Denote the gradient of the function $g(x)$ as $\nabla g(x)$. When the primal problem is of the following form:

$$\begin{aligned} min \quad & f(x) \\ s.t \quad & g_i(x) \leq 0, \quad i = 1, 2, ..., m. \end{aligned} \tag{3}$$

The dual Wolfe problem is:

$$\begin{aligned} max \quad & f(x) + \sum_{i=1}^{m} c_i g_i(x) \\ s.t \quad & \nabla f(x) = -\sum_{i=1}^{m} c_i \nabla g_i(x), \quad c_i \geq 0. \end{aligned} \tag{4}$$

2.5. Pontryagin Difference

Given two sets P and S, which are $P \subset R^n$ and and $S \subset R^n$, the Pontryagin set difference can be defined as:

$$P \ominus S = \{x \in R^n \mid x + s \in P, \forall s \in S\}.$$

3. Robust Positively Invariance Conditions for Polyhedral Set P_b

This section primarily investigates the necessary and sufficient conditions for the polyhedral set P_b to be the robust positively invariant set of perturbed linear discrete-time systems. The Pontryagin difference offers a convenience method for the proof process.

The optimization theory converts the procedure of confirming that a polyhedron set is a robust positively invariant set of perturbed linear systems from theoretical derivation to algebraic verification. The robust positively invariant conditions for four forms of optimization are presented below.

Lemma 1. *The polyhedron set P_b is the robust positively invariant set of a perturbed linear discrete-time system $x_{k+1} = Ax_k + E\omega_k$ (where $\omega \in W$, W is a polyhedral set) if and only if $W \subset \{\omega \mid G_i^T E\omega \leq \min_x b_i - G_i^T Ax\}$.*

Proof. The polyhedron set $P_b = \{x \in R^n \mid Gx \leq b\}$ is the robust positively invariant set of a perturbed linear discrete-time system (1) if $Ax + E\omega \in P_b$, for all $x \in P_b$ and $\omega \in W$, that is, $AP_b + EW \subset P_b$, $\forall \omega \in W$. It can be inferred from the Pontryagin difference, that is

$$EW \subset P_b \ominus AP_b. \tag{5}$$

In other words, if the polyhedron set P_b is the robust positively invariant set of system (1), then for $\forall x \in P_b$, $\forall \omega \in W$, $G(Ax + E\omega) \leq b$ holds, i.e., $GE\omega \leq b - GAx$. By combining it with (5), one can obtain

$$EW \subset \{x \mid G_i^T \omega \leq \min_x b_i - G_i^T Ax\}.$$

Then, we obtain

$$W \subset \{\omega \mid G_i^T E\omega \leq \min_x b_i - G_i^T Ax\}. \tag{6}$$

(6) is the maximum perturbation set that makes the polyhedron set a robust positive invariant set of system (1), because

$$G_i^T E\omega \leq \min_x b_i - G_i^T Ax \Leftrightarrow \max G_i^T E\omega \leq \min_x b_i - G_i^T Ax. \tag{7}$$

So when the perturbation set $W \subset \{\omega \mid G_i^T E\omega \leq \min_x b_i - G_i^T Ax\}$, the polyhedral set P_b is the robust positively invariant set of the given perturbation linear discrete-time system. □

Theorem 1. *The polyhedral set $P_b = \{x \in R^n \mid Gx \leq b\}$ system is given by $x_{k+1} = Ax_k + E\omega_k$ (where $\omega \in W$, W is a polyhedral set). If the optimal value of the optimization problem*

$$\begin{aligned} \min_x \quad & b_i - G_i^T Ax \\ s.t \quad & Gx - b \leq 0. \end{aligned} \tag{8}$$

is M^*. The set of polyhedra P_b is the robust positively invariant set of the system (1) if and only if $W \subset \{\omega \mid G_i^T E\omega \leq M^*\}$ holds.

Proof. Since (8) contains two variables x, ω is difficult to solve in practice. Therefore, we first use x as a variable to obtain the optimal value M^*, that is, the optimal value of the following optimization problem is M^*.

$$\begin{aligned} \min_{x} \quad & b_i - G_i^T Ax \\ s.t \quad & Gx - b \leq 0. \end{aligned}$$

That is, when the given disturbance set $W \subset \{\omega : G_i^T E\omega \leq M^*\}$ holds, the polyhedral set P_b is the robust positively invariant set of the system (1). □

Remark 1. *From $G_i^T E\omega \leq M^*$, it can be seen that the perturbation set that enables the polyhedron set to become a robust positively invariant set for a given perturbed discrete system (1) is also in the form of a polyhedron set.*

By leveraging Theorem 1, we transform the robust positively invariant set of system (1) to address the optimization challenge presented in Equation (8). However, tackling this optimization problem directly can be intricate. To circumvent this difficulty, one can resort to solving the dual optimization formulation of the problem. Consequently, this paper delineates three variants of dual optimization, thereby broadening the spectrum of solution methodologies.

The general dual form corresponding to (8) is elucidated in Theorem 2. Moreover, the feasible region associated with the primary optimization problem (8) can be streamlined through the application of dual optimization.

Theorem 2. *The polyhedral set $P_b = \{x \in R^n \mid Gx \leq b\}$ is the robust positively invariant set of the discrete linear system $x_{k+1} = Ax_k + E\omega(k)$ (where $\omega \in W$, W is a polyhedral set) if and only if $W \subset \{\omega \mid G_i^T E\omega \leq \{\max_y b_i - yb | y \geq 0, yG = GA\}\}$ holds.*

Proof. Taking (8) as the primal problem, then we have

$$\begin{aligned} \min_{x} \quad & b_i - G_i^T Ax \\ s.t \quad & Gx - b \leq 0. \end{aligned} \quad \Rightarrow \quad \begin{aligned} & b_i - \max_{x} G_i^T Ax \\ s.t \quad & Gx - b \leq 0. \end{aligned} \tag{9}$$

Perform dual processing on the optimized part, and obtain its general dual form as follows:

$$\begin{aligned} \max_{x} \quad & G_i^T Ax \\ s.t \quad & Gx - b \leq 0. \end{aligned} \quad \Rightarrow \quad \begin{aligned} \min_{y} \quad & yb \\ s.t \quad & y \geq 0, \\ & yG = G_i^T A. \end{aligned} \tag{10}$$

where $y \in R^{1\times m}$, $i \in I(m)$, $G_i^T \in R^{1\times n}$. From (9) and (10), it can be concluded that the form of (8) after the general duality is:

$$\begin{aligned} \max_{y} \quad & b_i - yb \\ s.t \quad & y \geq 0, \\ & yG = G_i^T A. \end{aligned} \tag{11}$$

Let the optimal value of the optimization problem (11) be denoted as M_1^*. Consequently, we can derive the disturbance set $\{\omega \mid G_i^T E\omega \leq M_1^*\}$. This implies that the polyhedron set P_η is the robust positively invariant set for the perturbed system (1). Hence, when the given perturbation set $W \subset \{\omega \mid G_i^T E\omega \leq M_1^*\}$, it can be deduced that the polyhedral set

P_η indeed serves as the robust positively invariant set for the specified perturbed linear discrete system exoressed in Equation (1). □

Theorem 3 converts constrained optimization problems into unconstrained optimization problems using Lagrange dual optimization.

Theorem 3. *The polyhedral set $P_b = \{x \in R^n \mid Gx \leq b\}$ is the robust positively invariant set of the discrete linear system $x_{k+1} = Ax_k + E\omega(k)$ (where $\omega \in W$, W is a polyhedral set) if and only if $\exists U \in R^{m\times m}$, $U \geq 0$, such that $W \subset \{\omega \mid G_i^T E\omega \leq \{\max\limits_{U_i \geq 0} \min\limits_{x} b_i - G_i^T Ax + U_i^T(Gx - b)\}\}$ holds.*

Proof. Firstly, the Lagrange function $L(x, U_i)$ of (8) is given, where $U \geq 0$ is the Lagrange operator, then

$$L(x, U_i) = b_i - G_i^T Ax + U_i^T(Gx - b), \quad i \in I(m). \tag{12}$$

Due to $Gx - b \leq 0$ and $U_i \geq 0$, when the objective function of (8) satisfies the constraint, then there is

$$\max_{U_i \geq 0} L(x, U_i) = b_i - G_i^T Ax, \quad i \in I(m). \tag{13}$$

Take the minimum value of x at both ends of the above equation to obtain

$$\min_{x} \max_{U_i \geq 0} L(x, U_i) = \{\min_{x} b_i - G_i^T Ax | Gx \leq b\}, \quad i \in I(m). \tag{14}$$

Finally, the weak duality theorem yields

$$\min_{x} \max_{U_i \geq 0} L(x, U_i) \geq \max_{U_i \geq 0} \min_{x} L(x, U_i), \quad i \in I(m). \tag{15}$$

Therefore, when $W \subset \{G_i^T E\omega \leq \{\max\limits_{U_i \geq 0} \min\limits_{x} b_i - G_i^T Ax + U_i^T(Gx - b)\}\}$ is established, the polyhedral set P_b is the robust positively invariant set of the system (1). □

Remark 2. *It is necessary to obtain the form of $\max\limits_{U_i \geq 0} \min\limits_{x} L(x, U_i)$ through the weak duality theorem, as in practical solving Lagrange dual optimization problems, the partial derivative of x is usually taken first.*

Theorem 4 presents a more special scenario where the Lagrange function $L(x, U_i)$ is a convex function about x. In this instance, the Wolfe duality theorem [21] can be utilized.

Theorem 4. *Assuming that $b_i - G_i^T Ax$ and $Gx - b$ are affine functions, the polyhedral set $P_b = \{x \in R^n \mid Gx \leq b\}$ is the robust positively invariant set of a perturbed linear discrete system $x_{k+1} = Ax_k + E\omega(k)$ (where $\omega \in W$, W is a polyhedral set) if and only if $\exists H \in R^{m\times m}$, $H \geq 0$, such that $W \subset \{\omega \mid G_i^T E\omega \leq \{\max\limits_{x} b_i - G_i^T Ax + H_i^T(Gx - b) \mid GA = HG\}\}$ holds.*

Proof. Since $b_i - G_i^T Ax$ and $Gx - b$ are both affine functions and $H \geq 0$, then $b_i - G_i^T Ax + H_i^T(Gx - b)$ is a convex function with respect to x. Therefore, the Wolfe duality theorem can be applied to write the Wolfe duality form of (8), i.e.,

$$\begin{aligned} \max_{x} \quad & b_i - G_i^T Ax + H_i^T(Gx - b) \\ s.t \quad & GA = HG. \end{aligned} \tag{16}$$

Therefore, when $W \subset \{G_i^T E\omega \leq \{\max\limits_{x} b_i - G_i^T Ax + H_i^T(Gx - b) \mid GA = HG\}\}$, the polyhedral set P_b is the robust positively invariant set of the system (1). □

4. Robust Positively Invariance Conditions for Symmetric Polyhedron Set P_η

This section investigates the robust positively invariance conditions for polyhedral sets represented by P_η. In comparison to the polyhedral set P_b, the structure P_η is more intricate. Nevertheless, both optimization and dual optimization theories can can effectively address the robust positively invariance issue pertaining to this category of polyhedral set.

Theorem 5. *Let* $M_1 = \{\min_x B_j + Q_i^T Ax \mid Qx \leq B\}$*; then, the polyhedral set* P_η *is a robust positively invariant set of the disturbance linear discrete system* $x_{k+1} = Ax_k + E\omega(k)$ *(where* $\omega \in W$*, W is a polyhedral set) if and only if* $W \subset \{\omega : -Q_j^T E\omega \leq |M_1|\}$ *holds, where* $B = [\beta; \beta]$*,* $Q = [G; -G]$*.*

Proof. The polyhedron set $P_\eta = \{x \in R^n \mid -\eta \geq Gx \leq \eta\}$ is the robust positively invariant set of a linear discrete-time system (1) if $Ax + E\omega \in P_b$, for all $x \in P_b$, $\omega \in W$. That is to say, the following two inequalities need to be met simultaneously:

$$\begin{aligned} G(Ax + E\omega) &\leq \beta, \\ -\beta &\leq G(Ax + E\omega). \end{aligned}$$

Simplification can lead to

$$\begin{aligned} -GE\omega &\leq \beta + GAx, \\ GE\omega &\leq \beta - GAx. \end{aligned} \tag{17}$$

According to Theorem 1, we can transform the robust positively invariance condition of the polyhedral set P_η into a constrained optimization problem, i.e., which first obtain the optimal value M_1 of the optimization problem (18).

$$\begin{aligned} \min_x \quad & \beta + GAx \\ & \beta - GAx, \\ s.t \quad & Gx \leq \beta, \\ & -Gx \leq \beta. \end{aligned} \tag{18}$$

To simplify the derivation process in this section, we introduce the following new symbolic notations:

$$B = \begin{bmatrix} \beta \\ \beta \end{bmatrix}, B \in R^{2m \times 1}, \quad Q = \begin{bmatrix} G \\ -G \end{bmatrix}, Q \in R^{2m \times n}.$$

So, (18) can be simplified as

$$\begin{aligned} \min_x \quad & B_j + G_j^T Ax, \qquad j \in I(2m), \\ s.t \quad & Qx \leq B. \end{aligned} \tag{19}$$

Finally, the polyhedron set P_η is the robust positively invariant set for the system (1) if and only if $W \subset \{\omega : -Q_j^T E\omega \leq |M_1|\}$. □

Similarly, the Lagrange dual and Wolfe dual forms of (19) can be derived using dual optimization theory. These different dual optimization forms can simplify the feasible region of the primal problem to some extent.

Theorem 6. *Let* $M_2 = \{\max\limits_{K_j \geq 0} \min\limits_{x} B_j + Q_j^T Ax + K_j^T (Qx - B)\}$*; then, the polyhedral set* P_η *is a robust positively invariant set of the disturbance linear discrete system* $x_{k+1} = Ax_k + E\omega(k)$ *(where* $\omega \in W$*, W is a polyhedral set) if and only if* $\exists K \geq 0$*, such that* $W \subset \{\omega : -Q_j^T E\omega \leq |M_2|\}$ *holds, where* $B = [\beta; \beta]$*,* $Q = [G; -G]$*.*

Proof. Let the Lagrange function of (19) be $L(x, K_j) = B_j + Q_j^T Ax + K_j^T (Qx - B)$, where $K_j > 0$ is the Lagrange multiplier. Use $F(x)$ to represent the function in (19). First, consider K_j as an independent variable, and at this point, x is considered a constant, resulting in

$$\max_{K_j \geq 0} L(x, K_j) = \begin{cases} F(x), & Qx - B \leq 0 \\ \infty, & otherwise \end{cases} \tag{20}$$

When the constraint $Qx - B \leq 0$ is satisfied, there is

$$\max_{K_j \geq 0} L(x, K_i) = F(x) \tag{21}$$

Take the minimum value at both ends of the equal sign of (21), i.e.,

$$\min_{x} \max_{K_j \geq 0} L(x, K_j) = \min_{x} F(x) \tag{22}$$

From the weak duality theorem, it can be concluded that $\min\limits_{x} \max\limits_{K_j \geq 0} L(x, K_j) \geq \max\limits_{K_j \geq 0} \min\limits_{x} L(x, K_j)$.

If M_2 is the optimal value of $\max\limits_{K_j \geq 0} \min\limits_{x} L(x, K_j)$, then we can obtain a set of $\max\limits_{K_j \geq 0} \min\limits_{x} L(x, K_j)$, so that the polyhedral set P_η is the robust positively invariant set of system (1). Determine whether the given disturbance set W belongs to the set $\{\omega : -Q_i^T E\omega \leq \{|M_2|\}$. If so, then the polyhedral set P_η is the robust positively invariant set of system (1). □

Theorem 7. *Let* $M_3 = \{\max\limits_{x} B_j + Q_j^T Ax + K_j^T (Qx - B) \mid QA = KQ\}$*, and assuming that* $b_i - G_i^T Ax$ *and* $Gx - b$ *are affine functions, the polyhedral set* P_η *is a robust positively invariant set of the disturbance linear discrete system* $x_{k+1} = Ax_k + E\omega(k)$ *(where* $\omega \in W$*, W is a polyhedral set) if and only if* $\exists K \geq 0$*, such that* $W \subset \{\omega : -Q_i^T E\omega \leq \{|M_3|\}$ *holds, where* $B = [\beta; \beta]$*,* $Q = [G; -G]$*.*

Proof. Since $b_i - G_i^T Ax$ and $Gx - b$ are affine functions, the conditions for applying the Wolfe duality theorem are met. Therefore, the Wolfe dual form of (19) can be obtained as:

$$\begin{aligned} \max_{x} \quad & B_j + Q_j^T Ax + K_j^T (Qx - B) \qquad j \in I(2m), \\ s.t \quad & QA = KQ. \end{aligned} \tag{23}$$

Let the optimal value of the optimization problem (23) be denoted as M_3. Consequently, we can derive the disturbance set $\{\omega : -Q_j^T E\omega \leq |M_3|\}$, which ensures that the set of the Polyhedron P_η becomes the robust positively invariant set for the perturbed system (1). Therefore, if the given perturbation set W is a subset of $\{\omega : -Q_j^T E\omega \leq |M_3|\}$, it can be inferred that the polyhedral set P_η indeed represents the robust positively invariant set for the given perturbation discrete system (1). □

5. Numerical Examples

In this section, we present two examples to verify that the polyhedron set and the symmetric polyhedron set are robust positively invariant sets for the given perturbed linear discrete system, using the methods derived in Sections 3 and 4, respectively.

Example 1. *Let the perturbed linear discrete system be* $x_{k+1} = Ax_k + E\omega_k$, *where matrix* $A = \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{bmatrix}$, *matrix* $E = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$, *and perturbation* $\omega \in W = \{\omega : \omega_1^{(k)} + \omega_2^{(k)} \leq 1, \omega_1^{(k)} \geq 0, \omega_2^{(k)} \geq 0\}$. *The polyhedron set is* $P_b = \{x | Gx \leq b\}$, *where* $G = \begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix}, b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

For Method 1, use the method outlined in Theorem 1 to to verify if the polyhedron set is a robust positively invariant set for the specified perturbed linear discrete system. This involves determining whether the disturbance set W is contained within $\{\omega \mid G_i^T E\omega \leq M^*\}$. From the provided data, we can identify two distinct optimization problems:

$$\begin{aligned} \min_x \quad & 1 - \begin{bmatrix} 0.1 & 0 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix}, \\ s.t \quad & \begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned} \tag{24}$$

$$\begin{aligned} \min_x \quad & 1 - \begin{bmatrix} 0.05 & 0.1 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix}, \\ s.t \quad & \begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \end{aligned} \tag{25}$$

The optimal values for both optimization problems are 1. Then, there are

$$\begin{bmatrix} 0.1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \omega_1^{(k)} \\ \omega_2^{(k)} \end{bmatrix} \leq 1, \tag{26}$$

$$\begin{bmatrix} 0.05 & 0.1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \omega_1^{(k)} \\ \omega_2^{(k)} \end{bmatrix} \leq 1. \tag{27}$$

$$\omega_1^{(k)} \geq 0, \omega_2^{(k)} \geq 0. \tag{28}$$

That is,

$$W \subset \{\omega : \begin{bmatrix} 0.2 & 0.1 \\ 0.2 & 0.15 \end{bmatrix} \begin{bmatrix} \omega_1^{(k)} \\ \omega_2^{(k)} \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}. \tag{29}$$

The polyhedral set P_b is a robust positively invariant set for a given perturbed linear discrete system.

For Method 2, to check whether the given perturbation set W belongs to the set $\{\omega \mid G_i^T E\omega \leq \{\max_y b_i - yb | y \geq 0, yG = GA\}\}$, we can apply the method described in Theorem 2. This involves solving two optimization problems based on the provided data.

$$\begin{aligned} \max_y \quad & 1 - y_1 - y_2, \\ s.t \quad & y_1 \geq 0, y_2 \geq 0, \\ & \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.1 \\ 0 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{bmatrix}. \end{aligned} \tag{30}$$

$$\begin{aligned} \max_y \quad & 1 - y_1 - y_2, \\ s.t \quad & y_1 \geq 0, y_2 \geq 0, \\ & \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.05 \\ 0.1 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{bmatrix}. \end{aligned} \tag{31}$$

$$\omega_1^{(k)} \geq 0, \omega_2^{(k)} \geq 0. \tag{32}$$

The optimal values obtained are 0.65 and 0.675, respectively. Therefore, there are

$$W \subset \{ \begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \omega_1^{(k)} \\ \omega_2^{(k)} \end{bmatrix} \leq \begin{bmatrix} 0.65 \\ 0.675 \end{bmatrix} \}, \tag{33}$$

So, the polyhedral set P_b is a robust positively invariant set of the given perturbed linear discrete system above.

For Method 3, to check whether the given perturbation set W belongs to the set $\{\omega \mid G_i^T E\omega \leq \{\max_{U_i \geq 0} \min_x b_i - G_i^T Ax + U_i^T (Gx - b)\}\}$, we apply the method described in Theorem 3 by defining the matrix $U = \begin{bmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{bmatrix}$. According to Theorem 3 and the given data, the following two optimization problems can be obtained:

$$\max_{U_i \geq 0} \min_x 1 - \begin{bmatrix} 0.1 & 0 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} + \begin{bmatrix} U_{11} & U_{12} \end{bmatrix} [\begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}] \tag{34}$$

$$\max_{U_i \geq 0} \min_x 1 - \begin{bmatrix} 0.05 & 0.1 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} + \begin{bmatrix} U_{21} & U_{22} \end{bmatrix} [\begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}] \tag{35}$$

Let the functions in Equations (34) and (35) be g_1 and g_2. Take the partial derivatives of $x_1^{(k)}$ and $x_2^{(k)}$ in functions g_1 and g_2, respectively. That is,

$$\begin{aligned} \frac{\partial g_1}{x_1^{(k)}} &= -0.02 + 0.1U_{11} + 0.05U_{12} = 0, \\ \frac{\partial g_1}{x_2^{(k)}} &= -0.03 + 0.1U_{12} = 0. \end{aligned}$$

$$\begin{aligned} \frac{\partial g_2}{x_1^{(k)}} &= -0.02 + 0.1U_{21} + 0.05U_{22} = 0, \\ \frac{\partial g_2}{x_2^{(k)}} &= -0.025 + 0.1U_{22} = 0. \end{aligned}$$

So, matrix $U = \begin{bmatrix} 0.05 & 0.3 \\ 0.075 & 0.25 \end{bmatrix}$. The optimal values obtained by incorporating Equations (34) and (35) are 0.65 and 0.675, respectively.

$$W \subset \{ \begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \omega_1^{(k)} \\ \omega_2^{(k)} \end{bmatrix} \leq \begin{bmatrix} 0.65 \\ 0.675 \end{bmatrix} \}, \tag{36}$$

So, the polyhedral set P_b is a robust positively invariant set of the given perturbed linear discrete system above.

For Method 4, to check whether the polyhedral set P_b is a robust positively invariant for the given system, that is, we use the method outlined in Theorem 4. Specifically, we need to verify if $W \subset \{\omega \mid G_i^T E\omega \leq \{\max_x b_i - G_i^T Ax + H_i^T(Gx - b) \mid GA = HG\}\}$ holds. Firstly, let the matrix $H = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$. According to Theorem 4 and the given data, the following two optimization problems can be obtained:

$$\begin{aligned} \max_x \quad & 1 - \begin{bmatrix} 0.1 & 0 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} + \begin{bmatrix} H_{11} & H_{12} \end{bmatrix} [\begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}], \\ s.t \quad & \begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix} \end{aligned} \tag{37}$$

$$\begin{aligned} \max_x \quad & 1 - \begin{bmatrix} 0.05 & 0.1 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} + \begin{bmatrix} H_{21} & H_{22} \end{bmatrix} [\begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}], \\ s.t \quad & \begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix} \begin{bmatrix} 0.2 & 0.3 \\ 0.1 & 0.1 \end{bmatrix} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} 0.1 & 0 \\ 0.05 & 0.1 \end{bmatrix} \end{aligned} \tag{38}$$

Solve the matrix $H = \begin{bmatrix} 0.05 & 0.3 \\ 0.075 & 0.25 \end{bmatrix}$, $H \geq 0$. And the optimal values for the optimization problem are 0.65 and 0.675, respectively, so the polyhedral set P_b is a robust positively invariant set of a given perturbed linear discrete system.

Remark 3. *From Example 1, it can be seen that the method proposed in this article is also applicable to positive systems.*

Example 2. *Let the linear perturbed discrete system be* $x_{k+1} = Ax_k + E\omega_k$, *where matrix* $A = \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix}$, $E = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, *and perturbation set* $W = \{\omega : -0.1 \leq \omega \leq 0.1\}$. *The symmetric polyhedron set* $P_\eta = \{ \begin{bmatrix} -1 \\ -1 \end{bmatrix} \leq Gx \leq \begin{bmatrix} 1 \\ 1 \end{bmatrix} \}$, *where* $G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

For Method 1, to proceed with the verification of whether the symmetric polyhedron set P_η is a robust positively invariant set for the given system, we apply the method outlined in Theorem 5. We start by defining the matrix as follows $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}$,

$B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. The next step involves checking if the perturbation set W belongs to set $\{\omega : -Q_i^T E\omega \leq |M_1|\}$. To do this, we need to solve four optimization problems to find the optimal values.

$$\begin{aligned} &\min_x \quad 1 - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix}, \\ &s.t \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \end{aligned} \tag{39}$$

$$\begin{aligned} &\min_x \quad 1 - \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix}, \\ &s.t \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \end{aligned} \tag{40}$$

$$\begin{aligned} &\min_x \quad 1 - \begin{bmatrix} -1 & 0 \end{bmatrix} \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix}, \\ &s.t \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \end{aligned} \tag{41}$$

$$\begin{aligned} &\min_x \quad 1 - \begin{bmatrix} 0 & -1 \end{bmatrix} \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix}, \\ &s.t \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} \leq \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \end{aligned} \tag{42}$$

Find the optimal values for the four optimization problems mentioned above, and take the minimum absolute value to obtain the optimal value of 0.3. That is, $\{\omega : -Q_j^T E\omega \leq 0.3\} \Rightarrow \{\omega : -0.3 \leq \omega \leq 0.3\}$. Due to $W = \{\omega : -0.1 \leq \omega \leq 0.1\} \subset \{\omega : -0.3 \leq \omega \leq 0.3\}$, the symmetric polyhedron set is a robust positively invariant set of the given system.

For Method 2, use the method in Theorem 6 to verify whether the symmetric polyhedron set P_η is a robust positively invariant set of the given system. That is, check whether the given perturbation set W belongs to the set $\{\omega : -Q_i^T E\omega \leq |M_2|\}$. Firstly, let the matrix $K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix}$. And calculate the optimal values for the following four optimization problems.

$$\max_{K_j\geq 0}\min_{x} 1+\begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix}0.7 & 0.7\\ -0.7 & 0.7\end{bmatrix}\begin{bmatrix}x_1^{(k)}\\ x_2^{(k)}\end{bmatrix}+\begin{bmatrix}K_{11} & K_{12} & K_{13} & K_{14}\end{bmatrix}\left\{\begin{bmatrix}1 & 0\\ 0 & 1\\ -1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1^{(k)}\\ x_2^{(k)}\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}$$

That is,

$$\max_{K_j\geq 0}\min_{x} 1+0.7x_1^{(k)}+0.7x_2^{(k)}+K_{11}[x_1^{(k)}-1]+K_{12}[x_2^{(k)}-1]+K_{13}[-x_1^{(k)}-1]+K_{14}[-x_2^{(k)}-1] \tag{43}$$

$$\max_{K_j\geq 0}\min_{x} 1+\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}0.7 & 0.7\\ -0.7 & 0.7\end{bmatrix}\begin{bmatrix}x_1^{(k)}\\ x_2^{(k)}\end{bmatrix}+\begin{bmatrix}K_{21} & K_{22} & K_{23} & K_{24}\end{bmatrix}\left\{\begin{bmatrix}1 & 0\\ 0 & 1\\ -1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1^{(k)}\\ x_2^{(k)}\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}$$

That is,

$$\max_{K_j\geq 0}\min_{x} 1-0.7x_1^{(k)}+0.7x_2^{(k)}+K_{21}[x_1^{(k)}-1]+K_{22}[x_2^{(k)}-1]+K_{23}[-x_1^{(k)}-1]+K_{24}[-x_2^{(k)}-1] \tag{44}$$

$$\max_{K_j\geq 0}\min_{x} 1+\begin{bmatrix}-1 & 0\end{bmatrix}\begin{bmatrix}0.7 & 0.7\\ -0.7 & 0.7\end{bmatrix}\begin{bmatrix}x_1^{(k)}\\ x_2^{(k)}\end{bmatrix}+\begin{bmatrix}K_{31} & K_{32} & K_{33} & K_{34}\end{bmatrix}\left\{\begin{bmatrix}1 & 0\\ 0 & 1\\ -1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1^{(k)}\\ x_2^{(k)}\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}$$

That is,

$$\max_{K_j\geq 0}\min_{x} 1-0.7x_1^{(k)}-0.7x_2^{(k)}+K_{31}[x_1^{(k)}-1]+K_{32}[x_2^{(k)}-1]+K_{33}[-x_1^{(k)}-1]+K_{34}[-x_2^{(k)}-1] \tag{45}$$

$$\max_{K_j\geq 0}\min_{x} 1+\begin{bmatrix}0 & -1\end{bmatrix}\begin{bmatrix}0.7 & 0.7\\ -0.7 & 0.7\end{bmatrix}\begin{bmatrix}x_1^{(k)}\\ x_2^{(k)}\end{bmatrix}+\begin{bmatrix}K_{11} & K_{12} & K_{13} & K_{14}\end{bmatrix}\left\{\begin{bmatrix}1 & 0\\ 0 & 1\\ -1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1^{(k)}\\ x_2^{(k)}\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\right\}$$

That is,

$$\max_{K_j\geq 0}\min_{x} 1-0.7x_1^{(k)}+0.7x_2^{(k)}+K_{41}[x_1^{(k)}-1]+K_{42}[x_2^{(k)}-1]+K_{43}[-x_1^{(k)}-1]+K_{44}[-x_2^{(k)}-1] \tag{46}$$

Assuming that the functions in Equations (43), (44), (45), and (46) are g_1, g_2, g_3, and g_4, and then take partial derivatives for $x_1^{(k)}$ and $x_2^{(k)}$ in functions g_1, g_2, g_3, and g_4, respectively.

$$\frac{\partial g_1}{x_1^{(k)}}=0.7+K_{11}-K_{13}=0,$$
$$\frac{\partial g_1}{x_2^{(k)}}=0.7+K_{12}-K_{14}=0.$$

$$\frac{\partial g_2}{x_1^{(k)}}=-0.7+K_{21}-K_{23}=0,$$
$$\frac{\partial g_2}{x_2^{(k)}}=0.7+K_{22}-K_{24}=0.$$

$$\frac{\partial g_3}{x_1^{(k)}} = -0.7 + K_{31} - K_{33} = 0,$$
$$\frac{\partial g_3}{x_2^{(k)}} = -0.7 + K_{32} - K_{34} = 0.$$
$$\frac{\partial g_4}{x_1^{(k)}} = 0.7 + K_{41} - K_{43} = 0,$$
$$\frac{\partial g_4}{x_2^{(k)}} = -0.7 + K_{42} - K_{44} = 0.$$

So, matrix $K = \begin{bmatrix} 0 & 0 & 0.7 & 0.7 \\ 0.7 & 0 & 0 & 0.7 \\ 0.7 & 0.7 & 0 & 0 \\ 0 & 0.7 & 0.7 & 0 \end{bmatrix}$.

Substituting into Equations (43)–(46) yields $|M_2| = 0.4$. Therefore, $W \subset \{\omega : -0.4 \leq \omega \leq 0.4\}$. So, the symmetric polyhedron set P_η is a robust positively invariant set of a given system.

For Method 3, use the method in Theorem 7 to verify whether the symmetric polyhedron set P_η is a robust positively invariant set of the given system. That is, check whether the given perturbation set W belongs to the set $\{\omega : -Q_i^T E\omega \leq |M_3|\}$. Firstly, let the matrix $K = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix}$. And calculate the optimal values for the following four optimization problems.

$$\begin{aligned} \max_x \quad & 1 + \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} + \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \end{bmatrix} \{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \}, \\ s.t \quad & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned} \tag{47}$$

$$\begin{aligned} \max_x \quad & 1 + \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} + \begin{bmatrix} K_{21} & K_{22} & K_{23} & K_{24} \end{bmatrix} \{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \}, \\ s.t \quad & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0.7 & 0.7 \\ -0.7 & 0.7 \end{bmatrix} = \begin{bmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \end{aligned} \tag{48}$$

$$\begin{aligned}
&\max_{x} \quad 1+\begin{bmatrix}-1 & 0\end{bmatrix}\begin{bmatrix}0.7 & 0.7\\ -0.7 & 0.7\end{bmatrix}\begin{bmatrix}x_1^{(k)}\\ x_2^{(k)}\end{bmatrix}+\begin{bmatrix}K_{31} & K_{32} & K_{33} & K_{34}\end{bmatrix}\{\begin{bmatrix}1 & 0\\ 0 & 1\\ -1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1^{(k)}\\ x_2^{(k)}\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\},\\
&s.t \quad \begin{bmatrix}1 & 0\\ 0 & 1\\ -1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0.7 & 0.7\\ -0.7 & 0.7\end{bmatrix}=\begin{bmatrix}K_{11} & K_{12} & K_{13} & K_{14}\\ K_{21} & K_{22} & K_{23} & K_{24}\\ K_{31} & K_{32} & K_{33} & K_{34}\\ K_{41} & K_{42} & K_{43} & K_{44}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\\ -1 & 0\\ 0 & -1\end{bmatrix}
\end{aligned} \tag{49}$$

$$\begin{aligned}
&\max_{x} \quad 1+\begin{bmatrix}0 & -1\end{bmatrix}\begin{bmatrix}0.7 & 0.7\\ -0.7 & 0.7\end{bmatrix}\begin{bmatrix}x_1^{(k)}\\ x_2^{(k)}\end{bmatrix}+\begin{bmatrix}K_{41} & K_{42} & K_{43} & K_{44}\end{bmatrix}\{\begin{bmatrix}1 & 0\\ 0 & 1\\ -1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1^{(k)}\\ x_2^{(k)}\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\end{bmatrix}\},\\
&s.t \quad \begin{bmatrix}1 & 0\\ 0 & 1\\ -1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0.7 & 0.7\\ -0.7 & 0.7\end{bmatrix}=\begin{bmatrix}K_{11} & K_{12} & K_{13} & K_{14}\\ K_{21} & K_{22} & K_{23} & K_{24}\\ K_{31} & K_{32} & K_{33} & K_{34}\\ K_{41} & K_{42} & K_{43} & K_{44}\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & 1\\ -1 & 0\\ 0 & -1\end{bmatrix}
\end{aligned} \tag{50}$$

The matrix $K=\begin{bmatrix}0.7 & 0.7 & 0 & 0\\ 0 & 0.7 & 0.7 & 0\\ 0 & 0 & 0.7 & 0.7\\ 0.7 & 0 & 0 & 0.7\end{bmatrix}$ can be obtained, and the optimal value $|M_3|=0.4$. Therefore, $W\subset\{\omega:-0.4\leq\omega\leq 0.4\}$. So, the symmetric polyhedron set P_η is a robust positively invariant set of the given system.

6. Conclusions

This article employs the definitions of robust positively invariant sets and Pontryagin difference to establish necessary and sufficient conditions for polyhedral sets and symmetric polyhedral sets to qualify as robust positively invariant sets for perturbed linear discrete time systems. It then reformulates the robust positively invariance condition into a format suitable for solving a series of optimization and dual optimization problems. The introduction of robust positively invariant conditions in an optimization format circumvents the complexity associated with calculating polyhedral vertices during theoretical derivation. This optimized format is universally applicable to various perturbed linear discrete systems, presenting a unified approach. Crucially, the introduction of multiple dual optimization formulations offers numerous alternatives for verifying robust positively invariant sets. The efficacy of this method is substantiated through numerical examples.

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