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# On Coefficient Inequalities for Functions of Symmetric Starlike Related to a Petal-Shaped Domain

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**Abstract:** The research on coefficient inequalities in various classes of univalent holomorphic functions focuses on interpreting their coefficients through the coefficients associated with Carathéodory functions. Therefore, researchers can investigate the behavior of coefficient functionals by applying the known inequalities for Carathéodory functions. This study will explore various coefficient inequalities employing the techniques developed for the previously discussed family of functions. These coefficient inequalities include the Krushkal, Zalcman, and Fekete-Szegö inequalities, along with the second and third Hankel determinants. The class of symmetric starlike functions linked with a petal-shaped domain is the primary focus of our study.

**Keywords:** coefficient inequalities; symmetric starlik function; Zalcman and Krushkal inequalities; Carathéodory function

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## 1. Introduction and Definitions

To ensure a clear understanding of the key terminology utilized throughout our primary findings, we first present some preliminary concepts. First, let us consider $\mathbb{D} = \{\xi \in \mathbb{C} : |\xi| < 1\}$ to denote the open unit disk, and let $\mathcal{A}$ represent the family of holomorphic functions normalized by the conditions $g(0) = 0$ and $g'(0) - 1 = 0$. This indicates that $g \in \mathcal{A}$ possesses a form

$$g(\xi) = \xi + \sum_{u=1}^{\infty} d_u \xi^u, \quad \xi \in \mathbb{D}. \tag{1}$$

Also, keep in mind that a holomorphic function is univalent within the domain $\mathbb{D}$ if it assigns each value in $\mathbb{D}$ at most once. Specifically, for $g$ to be univalent in $\mathbb{D}$, it means that $g(\xi_1) = g(\xi_2)$ only when $\xi_1 = \xi_2$, where $\xi_1, \xi_2 \in \mathbb{D}$. We use the symbol $\mathcal{S}$ to represent the family of univalent functions defined by the series expansion (1). In 1907, Köebe established this family.

In 1916, Bieberbach [1] put forward a notable result in function theory, which became known as the "Bieberbach conjecture". The conjecture asserts that for $g \in \mathcal{S}$, the inequality $|d_u| \leq u$ holds for $u \geq 2$. He also demonstrated that this holds specifically for $u = 2$. It is apparent that numerous prominent scholars have utilized different techniques to approach

this result. Löwner [2] solved this result for $u = 3$, while Spence and Schaeffer [3] addressed it by applying the variational technique and the Löwner differential equation. Jenkins [4] later established the coefficient inequality $|d_3| \leq 3$ through the application of quadratic differentials. Schiffer and Garabedian [5] employed the variational technique to establish that $|d_4| \leq 4$. Meanwhile, Schiffer and Pederson [6] utilized the Schiffer–Garabedian inequality ([7], p. 108) to show that $|d_5| \leq 5$. Furthermore, Ozawa [8,9] and Pederson [10] independently proved that $|d_6| \leq 6$ by utilizing Grunsky's inequality ([7], p. 60). Numerous scholars have attempted to resolve this conjecture for years, but none have succeeded in proving it for $u \geq 7$. De Branges [11] successfully resolved the result for $u \geq 2$ in 1985 by employing hypergeometric functions.

From 1916 to 1985, researchers focused on determining the estimates of the $u$-th coefficient for subfamilies of univalent functions such as starlike $\mathcal{S}^*$, convex $\mathcal{C}$, and close-to-convex $\mathcal{K}$, trying to solve the problem. Some of the key classes are defined as:

$$\begin{aligned}
\mathcal{S}^* &= \left\{ g \in \mathcal{S} : \mathfrak{Re} \frac{\xi g'(\xi)}{g(\xi)} > 0, \quad \xi \in \mathbb{D} \right\}, \\
\mathcal{C} &= \left\{ g \in \mathcal{S} : \mathfrak{Re} \frac{(\xi g'(\xi))'}{g'(\xi)} > 0, \quad \xi \in \mathbb{D} \right\}, \\
\mathcal{K} &= \left\{ g \in \mathcal{S} : \mathfrak{Re} \frac{\xi g'(\xi)}{h(\xi)} > 0 \text{ with } h \in \mathcal{S}^*, \quad \xi \in \mathbb{D} \right\}.
\end{aligned}$$

By setting $h(\xi) = \xi$, the $\mathcal{K}$ family is transformed into the class of bounded turning functions, represented by the notation $\mathcal{BT}$. In 1992, researchers [12] examined a univalent function $\phi$ in $\mathbb{D}$ that fulfilled the conditions $\phi'(0) > 0$ and $\mathfrak{Re}\phi > 0$. They also noted that the region $\phi(\mathbb{D})$ is star-shaped around the fixed point $\phi(0) = 1$ and is symmetric across the real axis. By utilizing the concept of subordination, they established the following subclass within the family $\mathcal{S}$.

$$\mathcal{S}^*(\phi) = \left\{ g \in \mathcal{S} : \frac{\xi g'(\xi)}{g(\xi)} \prec \phi(\xi), \quad \xi \in \mathbb{D} \right\}.$$

They focused on several results, including the distortion theorem, growth properties and covering theorem. In recent years, scholars have explored various subclasses of the collection $\mathcal{S}$ as potential choices for the class $\mathcal{S}^*(\phi)$. A recent study identifies the following classes as particularly significant.

$(i)$. $\mathcal{S}^*_{(u-1)\mathcal{L}} \equiv \mathcal{S}^*(\Psi_{u-1}(\xi))$ [13] with $\Psi_{u-1}(\xi) = 1 + \frac{u}{u+1}\xi + \frac{1}{u+1}\xi^u$ for $u \geq 4$,
$(ii)$. $\mathcal{S}^*_{\sin} \equiv \mathcal{S}^*(1 + \sin(\xi))$ [14], $\mathcal{S}^*_{\cosh} \equiv \mathcal{S}^*(\cosh(\xi))$ [15],
$(iii)$. $\mathcal{S}^*_{car} \equiv \mathcal{S}^*\left(1 + \frac{2}{3}\xi + \frac{1}{3}\xi^2\right)$ [16], $\mathcal{S}^*_c \equiv \mathcal{S}^*(1 + \xi + \frac{1}{2}\xi^2)$ [17].

The determinant $\mathcal{D}_{\delta,\eta}(g)$, where $\eta, \delta \in \mathbb{N} = \{1, 2, \ldots\}$, is known as the Hankel determinant. It is formed from the coefficients of the function $g$ that resides in the class $\mathcal{S}$

$$\mathcal{D}_{\delta,\eta}(g) = \begin{vmatrix} d_\eta & d_{\eta+1} & \ldots & d_{\eta+\delta-1} \\ d_{\eta+1} & d_{\eta+2} & \ldots & d_{\eta+\delta} \\ \vdots & \vdots & \ldots & \vdots \\ d_{\eta+\delta-1} & d_{\eta+\delta} & \ldots & d_{\eta+2\delta-2} \end{vmatrix}.$$

Pommerenke [18,19] made this contribution. In particular, the second- and third-order Hankel determinants are recognized as the following determinants:

$$\begin{aligned}
\mathcal{D}_{2,1}(g) &= \begin{vmatrix} 1 & d_2 \\ d_2 & d_3 \end{vmatrix} = d_3 - d_2^2, \\
\mathcal{D}_{2,2}(g) &= \begin{vmatrix} d_2 & d_3 \\ d_3 & d_4 \end{vmatrix} = d_2 d_4 - d_3^2, \\
\mathcal{D}_{3,1}(g) &= \begin{vmatrix} 1 & d_2 & d_3 \\ d_2 & d_3 & d_4 \\ d_3 & d_4 & d_5 \end{vmatrix} = d_3\left(d_2 d_4 - d_3^2\right) - d_4(d_4 - d_2 d_3) + d_5\left(d_3 - d_2^2\right). \qquad (2)
\end{aligned}$$

The first two determinants have been the focus of detailed studies in the literature for numerous subclasses of univalent functions. The research by the authors in [20–22] is notable for establishing the estimates of the second determinant. For further details on the analysis of this determinant, refer to the articles [23–26].

The third-order determinant presented above is the most complex problem to analyze, particularly in terms of determining its sharp estimates. Babalola [27] became the first to analyze the estimates of the third-order determinant for $\mathcal{S}^*$, $\mathcal{BT}$ and $\mathcal{K}$ classes in 2010. Zaprawa [28] improved the Babalola's results in 2017 by utilizing an innovative approach. In 2019, Lecko et al. [29] established the sharp estimate of $|\mathcal{D}_{3,1}(g)|$ for functions within the family $\mathcal{S}^*(1/2)$. Additionally, In 2022, Arif et al. [30] presented the sharp estimate $|\mathcal{D}_{3,1}(g)| \leq 16/225$ for $g \in \mathcal{S}^*_{3l}$ in their publication. Readers may find the study of [31–35] noteworthy, as they determined sharp estimates for the third-order Hankel determinant in new subclasses of univalent functions.

Using the function $\phi$ with the constraints as given above, we now consider the following Ma–Minda type Sakaguchi starlike functions

$$\mathcal{S}^*_s(\phi) = \left\{ g \in \mathcal{S} : \frac{2g'(\xi)}{g(\xi) - g(-\xi)} \prec \phi(\xi), \quad \xi \in \mathbb{D} \right\}.$$

It is to be noted that class $\mathcal{S}^*_s(\phi)$ with $\phi(z) = \frac{1+z}{1-z}$ consists of starlike functions about symmetric points and was introduced by Sakaguchi [36] in 1959. Also, he proved that the family $\mathcal{S}^*_s$ is a subclass of the set $\mathcal{K}$ of close-to-convex functions and also includes both odd starlike functions and convex functions, considering the origin as a reference point. Several researchers studied numerous subclasses of univalent functions that are linked with the Sakaguchi function. In [37], Zaprawa investigated the estimates of initial logarithmic coefficients for the family $\mathcal{S}^*_s$. Later on, Zaprawa [38] estimated the logarithmic coefficients, Zalcman inequalities, and Hankel determinants for symmetric starlike functions related to the exponential function. Furthermore, Faisal et al. [39] studied the initial coefficients, Zalcman and Krushkal inequalities, along with the Hankel determinants for the family $\mathcal{S}^*_{SG}$. The authors also studied the inverse and logarithmic properties of the same family.

In [40], Arora and Kumar study the petal-shaped domain $\Gamma_P$ specified by $w \in \mathbb{C} : |\sinh(w-1)| < 1$ based on the families presented in [13–17]. This domain is functionally defined through $P(\xi) = 1 + \sinh^{-1}\xi$, which plays a key role in establishing the family. $P(\xi)$ is clearly recognized as a Ma–Minda function. Remember that $\sinh^{-1}\xi$ is a multivalued function with branch cuts along the segments $(-\iota\infty, -\iota)$ and $(\iota, \iota\infty)$ on the imaginary axis. As a result, it remains analytic in $\mathbb{D}$. A petal-shaped domain finds applications in numerous disciplines, such as mathematics, biology, computer graphics, architecture,

botany, materials science, engineering, art, and geometry. Moreover, Arora and Kumar presented a unique subfamily of starlike functions that is defined as

$$\mathcal{S}_P^* := \left\{ g \in \mathcal{A} : \frac{\xi g'(\xi)}{g(\xi)} \prec 1 + \sinh^{-1} \xi, \quad \xi \in \mathbb{D} \right\}.$$

Based on the given definition, we deduce that $g \in \mathcal{S}_P^*$ if and only if there exists a holomorphic function $q(\xi) \prec P(\xi)$ such that

$$g(\xi) = \xi \exp\left( \int_0^{\xi} \frac{q(\xi) - 1}{t} dt \right).$$

Later, Barukab et al. [41] examined sharp estimates for coefficient inequalities and the third Hankel determinant in the $\mathcal{BT}_s$ family. In 2022, Shi et al. [42] calculated the second Hankel determinant of logarithmic coefficients for the family $\mathcal{BT}_s$. Based on this definition, we present the class $\mathcal{S}_{\sinh}^*$ of symmetric starlike functions linked with a petal-shaped domain and is given as

$$\mathcal{S}_{\sinh}^* = \left\{ g \in \mathcal{S} : \frac{2\xi g'(\xi)}{g(\xi) - g(-\xi)} \prec 1 + \frac{4}{5}\xi + \frac{1}{5}\xi^4, \quad \xi \in \mathbb{D} \right\}. \tag{3}$$

In this research, our aim is to investigate the sharp estimates of coefficient inequalities, Krushkal, Zalcman and Fekete-Szegö inequalities for the class $\mathcal{S}_{\sinh}^*$ of symmetric starlike function related to a petal-shaped domain. In addtion, we also study the estimates of $|\mathcal{D}_{2,2}(g)|$, $|\mathcal{D}_{2,3}(g)|$, and $|\mathcal{D}_{3,1}(g)|$ for the above-mentioned class.

## 2. A Set of Lemmas

Let us consider $\mathcal{P}$ to denote the Carathéodory class, which contains all holomorphic functions $p$ in $\mathbb{D}$ that satisfy $\mathfrak{Re}(p(\xi)) > 0$ and and follow the given normalization:

$$p(\xi) = 1 + \sum_{i=1}^{\infty} \varsigma_i \xi^i, \quad \xi \in \mathbb{D}. \tag{4}$$

**Lemma 1** ([43]). *Let $p \in \mathcal{P}$ be given by (4) and $K \in [0,1]$ with $K(2K-1) \leq T \leq K$, then*

$$\left| \varsigma_3 - 2K\varsigma_1\varsigma_2 + T\varsigma_1^3 \right| \leq 2. \tag{5}$$

**Lemma 2** ([44]). *Let $p \in \mathcal{P}$ be given by (4). Then*

$$|\varsigma_i| \leq 2, \quad i \geq 1. \tag{6}$$

**Lemma 3** ([45]). *If $\varrho$, $\nu$, $\rho$ and $\varpi$ satisfy $\varrho \in (0,1)$ and $\rho \in (0,1)$ with*

$$\Big( (-\varpi + \rho(\rho+\varrho))^2 + (-2\nu + \varpi\rho)^2 \Big) 8(1-\varrho)\varrho + (-2\varrho\rho + \varpi)^2 (1-\rho)\rho \leq 4\rho^2(1-\rho)^2(1-\varrho)\varrho. \tag{7}$$

*Let $p \in \mathcal{P}$, be given by (4). Then*

$$\left| \nu\varsigma_1^4 + \varrho\varsigma_2^2 + 2\rho\varsigma_1\varsigma_3 - \frac{3}{2}\varpi\varsigma_1^2\varsigma_2 - \varsigma_4 \right| \leq 2.$$

**Lemma 4** ([7]). *Let $p \in \mathcal{P}$ be given by (4), then*

$$\left|\varsigma_{q+p} - \vartheta\varsigma_q\varsigma_p\right| \leq 2\max\{1, |2\vartheta - 1|\} = \begin{cases} 2 & \text{for} \quad \vartheta \in [0,1]; \\ 2|2\vartheta - 1| & \text{otherwise.} \end{cases}. \tag{8}$$

**Lemma 5** ([46]). *If $p \in \mathcal{P}$ be given by (4). Then for $\tau, \sigma, \zeta \in \overline{\mathbb{D}} \cup \{1\}$.*

$$2\varsigma_2 = \left(4 - \varsigma_1^2\right)\tau + \varsigma_1^2, \tag{9}$$

$$4\varsigma_3 = 2\left(4 - \varsigma_1^2\right)\tau\varsigma_1 - \tau^2\left(4 - \varsigma_1^2\right)\varsigma_1 + 2\sigma\left(1 - |\tau|^2\right)\left(4 - \varsigma_1^2\right) + \varsigma_1^3, \tag{10}$$

$$8\varsigma_4 = \left[\varsigma_1^2\left(-3\tau + \tau^2 + 3\right) + 4\tau\right]\left(4 - \varsigma_1^2\right)\tau - 4\left(1 - |\tau|^2\right)\left(4 - \varsigma_1^2\right) \left[(\tau - 1)\sigma\varsigma + \sigma^2\overline{\tau} - \zeta\left(1 - |\sigma|^2\right)\right] + \varsigma_1^4. \tag{11}$$

## 3. Coefficient Inequalities

This section explores the calculation of the initial four coefficient inequalities for functions within the family $\mathcal{S}^*_{\sinh}$.

**Theorem 1.** *If $g \in \mathcal{S}^*_{\sinh}$ has the series expansion (1). Then*

$$|d_2| \leq \frac{1}{2},$$
$$|d_3| \leq \frac{1}{2},$$
$$|d_4| \leq \frac{1}{4},$$
$$|d_5| \leq \frac{1}{4}.$$

*These outcomes are sharp.*

**Proof.** Let $g \in \mathcal{S}^*_{\sinh}$. Then

$$\frac{2\xi g'(\xi)}{g(\xi) - g(-\xi)} = 1 + \sinh^{-1}(w(\xi)), \quad \xi \in \mathbb{D}.$$

If $p \in \mathcal{P}$, we can express it using the Schwarz function $w(\xi)$ as follows

$$p(\xi) = \frac{1 + w(\xi)}{1 - w(\xi)} = 1 + \varsigma_1\xi + \varsigma_2\xi^2 + \varsigma_3\xi^3 + \cdots,$$

or correspondingly,

$$w(\xi) = \frac{p(\xi) - 1}{p(\xi) + 1} = \frac{\varsigma_1\xi + \varsigma_2\xi^2 + \varsigma_3\xi^3 + \cdots}{2 + \varsigma_1\xi + \varsigma_2\xi^2 + \varsigma_3\xi^3 + \cdots}. \tag{12}$$

Using (1), it follows

$$\frac{2\xi g'(\xi)}{g(\xi) - g(-\xi)} = 1 + 2d_2\xi + 2d_3\xi^2 + (4d_4 - 2d_2d_3)\xi^3 + \left(4d_5 - 2d_3^2\right)\xi^4 + \cdots. \tag{13}$$

By simplification and using the series expansion of (12), it yields

$$1+\sinh^{-1}(w(\xi)) = 1+\left(\frac{1}{2}\varsigma_1\right)\xi+\left(\frac{1}{2}\varsigma_2-\frac{1}{4}\varsigma_1^2\right)\xi^2+\left(\frac{1}{2}\varsigma_3-\frac{1}{2}\varsigma_1\varsigma_2+\frac{5}{48}\varsigma_1^3\right)\xi^3 + \left(-\frac{1}{32}\varsigma_1^4-\frac{1}{4}\varsigma_2^2-\frac{1}{2}\varsigma_1\varsigma_3+\frac{5}{16}\varsigma_1^2\varsigma_2+\frac{1}{2}\varsigma_4\right)\xi^4+\cdots. \quad (14)$$

Comparing (13) and (14), we obtain

$$d_2 = \frac{1}{4}\varsigma_1, \quad (15)$$

$$d_3 = \frac{1}{4}\varsigma_2-\frac{1}{8}\varsigma_1^2, \quad (16)$$

$$d_4 = \frac{1}{96}\varsigma_1^3-\frac{3}{32}\varsigma_1\varsigma_2+\frac{1}{8}\varsigma_3, \quad (17)$$

$$d_5 = -\frac{1}{32}\varsigma_2^2+\frac{3}{64}\varsigma_1^2\varsigma_2-\frac{1}{8}\varsigma_1\varsigma_3+\frac{1}{8}\varsigma_4. \quad (18)$$

Implementing (6) in (15), we obtain

$$|d_2| \le \frac{1}{2}.$$

Reordering (16), we obtain

$$d_3 = \frac{1}{4}\left(\varsigma_2-\frac{1}{2}\varsigma_1^2\right).$$

Using (8), we have

$$|d_3| \le \frac{1}{2}.$$

We can write (17) as

$$|d_4| = \frac{1}{8}\left|\left(\varsigma_3-2\left(\frac{3}{8}\right)\varsigma_1\varsigma_2+\frac{1}{12}\varsigma_1^3\right)\right|.$$

From (5), let

$$K = \frac{3}{8} \quad \text{and} \quad T = \frac{1}{12}.$$

It is clear that $0 \le K \le 1$ and $K \ge T$ with

$$K(2K-1) = -\frac{3}{32} \le T.$$

Thus all the conditions of (5) are satisfied. Hence we have

$$|d_4| \le \frac{1}{4}.$$

We can rewrite (18) as

$$d_5 = -\frac{1}{8}\left(\frac{1}{4}\varsigma_2^2+2\left(\frac{1}{2}\right)\varsigma_1\varsigma_3-\frac{3}{2}\left(\frac{1}{4}\right)\varsigma_1^2\varsigma_2-\varsigma_4\right).$$
$$= -\frac{1}{8}\left(\nu\varsigma_1^4+d\varsigma_2^2+2\rho\varsigma_1\varsigma_3-\frac{3}{2}\varpi\varsigma_1^2\varsigma_2-\varsigma_4\right), \quad (19)$$

where

$$\nu = 0, \quad \varrho = \frac{1}{4}, \quad \rho = \frac{1}{2}, \quad \varpi = \frac{1}{4},$$

are such that

$$\left((-\varpi+\rho(\rho+\varrho))^2+(-2\nu+\varpi\rho)^2\right)8(1-\varrho)\varrho+(-2\varrho\rho+\varpi)^2(1-\rho)\rho \le 4\varrho\rho^2(1-\rho)^2(1-\varrho),$$

$\varrho \in (0,1)$, $\rho \in (0,1)$, therefore by (7) and (19), we have

$$|d_5| \leq \frac{1}{4}.$$

The inequalities are sharp, and the functions listed below attain equality

$$\begin{aligned}
\frac{2\xi g'(\xi)}{g(\xi)-g(-\xi)} &= 1+\xi-\frac{1}{6}\xi^3+\frac{3}{40}\xi^5+\cdots,\\
\frac{2\xi g'(\xi)}{g(\xi)-g(-\xi)} &= 1+\xi^2-\frac{1}{6}\xi^6+\frac{3}{40}\xi^{10}+\cdots,\\
\frac{2\xi g'(\xi)}{g(\xi)-g(-\xi)} &= 1+\xi^3-\frac{1}{6}\xi^9+\frac{3}{40}\xi^{15}+\cdots,\\
\frac{2\xi g'(\xi)}{g(\xi)-g(-\xi)} &= 1+\xi^4-\frac{1}{6}\xi^{12}+\frac{3}{40}\xi^{20}+\cdots.
\end{aligned}$$

The required proof is accomplished. □

## 4. Fekete-Szegö Inequality

The determinant $\mathcal{D}_{2,1}(g)$ corresponds to a specific Fekete-Szegö inequality $\left|d_3-\vartheta d_2^2\right|$, where $\vartheta$ is a complex parameter. The Fekete-Szegö inequality [47] is among the first problems about the coefficients of univalent holomorphic functions, established in 1933. We compute the Fekete-Szegö inequality bound for $g \in \mathcal{S}^*_{\sinh}$ in this second section.

**Theorem 2.** *If* $g \in \mathcal{S}^*_{\sinh}$, *then*

$$\left|d_3-\vartheta d_2^2\right| \leq \max\left\{\frac{1}{2}, \frac{|\vartheta|}{4}\right\}, \text{ for } \vartheta \in \mathbb{C}.$$

*This inequality is sharp.*

**Proof.** Putting (15) and (16), we obtain

$$\left|d_3-\vartheta d_2^2\right| = \left|\frac{1}{4}\varsigma_2-\frac{1}{8}\varsigma_1^2-\vartheta\frac{1}{16}\varsigma_1^2\right|.$$

Application of (8), leads us to

$$\left|d_3-\vartheta d_2^2\right| \leq \frac{1}{4}\max\left\{2, 2\left|\left(\frac{2+\vartheta}{2}\right)-1\right|\right\}.$$

After the simplification, we obtain

$$\left|d_3-\vartheta d_2^2\right| \leq \max\left\{\frac{1}{2}, \frac{|\vartheta|}{4}\right\}.$$

This outcome is best possible and is obtained by

$$\frac{2\xi g'(\xi)}{g(\xi)-g(-\xi)} = 1+\xi^2-\frac{1}{6}\xi^6+\frac{3}{40}\xi^{10}+\cdots.$$

□

## 5. Zalcman Inequalities

In 1960, Lawrence Zalcman presented a conjecture stating that the coefficients of a function in $\mathcal{S}$ adhere to the sharp inequality

$$|d_u^2 - d_{2u-1}| \leq (u-1)^2.$$

The Köebe function and its rotations provide equality. We can now calculate the Zalcman inequalities for $g \in \mathcal{S}^*_{\sinh}$.

**Theorem 3.** *If $g \in \mathcal{S}^*_{\sinh}$, be given by (1). Then*

$$\left|d_5 - d_3^2\right| \leq \frac{1}{4}.$$

*This is the finest possible inequality.*

**Proof.** Using (16) and (18), we obtain

$$\left|d_5 - d_3^2\right| = \left|-\frac{3}{32}\varsigma_2^2 + \frac{7}{64}\varsigma_1^2\varsigma_2 - \frac{1}{64}\varsigma_1^4 - \frac{1}{8}\varsigma_1\varsigma_3 + \frac{1}{8}\varsigma_4\right|.$$

After the simplifying we have

$$\begin{aligned}\left|d_5 - d_3^2\right| &= -\frac{1}{8}\left|\frac{1}{8}\varsigma_1^4 + \frac{3}{34}\varsigma_2^2 + 2\left(\frac{1}{2}\right)\varsigma_1\varsigma_3 - \frac{3}{2}\left(\frac{7}{12}\right)\varsigma_1^2\varsigma_2 - \varsigma_4\right|. \\ &= -\frac{1}{8}\left|\nu\varsigma_1^4 + \varrho\varsigma_2^2 + 2\rho\varsigma_1\varsigma_3 - \frac{3}{2}\varpi\varsigma_1^2\varsigma_2 - \varsigma_4\right|,\end{aligned} \tag{20}$$

where

$$\nu = \frac{1}{8}, \quad \varrho = \frac{3}{34}, \quad \rho = \frac{1}{2}, \quad \varpi = \frac{7}{12},$$

are such that

$$\Big((-\varpi + \rho(\rho+\varrho))^2 + (-2\nu + \varpi\rho)^2\Big)8(1-\varrho)\varrho + (-2\varrho\rho + \varpi)^2(1-\rho)\rho \leq 4\varrho\rho^2(1-\rho)^2(1-\varrho),$$

$\varrho \in (0,1)$, $\rho \in (0,1)$, therefore by (7) and (20), we have

$$\left|d_5 - d_3^2\right| \leq \frac{1}{4}.$$

This result is sharp and is achieved by

$$\frac{2\xi g'(\xi)}{g(\xi) - g(-\xi)} = 1 + \xi^4 - \frac{1}{6}\xi^{12} + \frac{3}{40}\xi^{20} + \cdots.$$

□

**Theorem 4.** *If $g$ belongs to $\mathcal{S}^*_{\sinh}$, and given by (1). Then*

$$|d_2 d_3 - d_4| \leq \frac{1}{4}.$$

*This inequality is best possible.*

**Proof.** Putting (15)–(17), we have

$$|d_2 d_3 - d_4| = \frac{1}{8}\left|\varsigma_3 - 2\left(\frac{5}{8}\right)\varsigma_1\varsigma_2 + \frac{1}{3}\varsigma_1^3\right|.$$

From (5), we have

$$0 \leq K = \frac{5}{8} \leq 1, \quad K = \frac{5}{8} \geq T = \frac{1}{3},$$

and

$$K(2K-1) = \frac{5}{32} \leq T = \frac{1}{3}.$$

Using (5), we obtain

$$|d_2 d_3 - d_4| \leq \frac{1}{4}.$$

This outcome is sharp. Equality is achieved from

$$\frac{2\xi g'(\xi)}{g(\xi) - g(-\xi)} = 1 + \xi^3 - \frac{1}{6}\xi^9 + \frac{3}{40}\xi^{15} + \cdots.$$

□

## 6. Krushkal Inequalities

In [48], Krushkal proved the Zalcman conjecture for $u \leq 6$ by applying the holomorphic homotopy of univalent functions, which was originally presented in an unpublished article [49] for $u \geq 2$. It was also proved that, for $g \in \mathcal{S}$:

$$\left| d_u^t - d_2^{t(u-1)} \right| \leq 2^{t(u-1)} - u^t, \text{ with } u,\ t \geq 2.$$

The estimation of Zalcman inequalities are now presented here in this section.

**Theorem 5.** *If $g \in \mathcal{S}^*_{\sinh}$ be given by (1). Then*

$$\left| d_5 - d_2^4 \right| \leq \frac{1}{4}.$$

*The outcome is sharp.*

**Proof.** From (15) and (18), we obtain

$$\left| d_5 - d_2^4 \right| = \left| -\frac{1}{256}\varsigma_1^4 + \frac{3}{64}\varsigma_1^2\varsigma_2 - \frac{1}{32}\varsigma_2^2 - \frac{1}{8}\varsigma_1\varsigma_3 + \frac{1}{8}\varsigma_4 \right|.$$

After the simplifying we have

$$\left| d_5 - d_2^4 \right| = -\frac{1}{8}\left| \frac{1}{32}\varsigma_1^4 + \frac{1}{4}\varsigma_2^2 + 2\left(\frac{1}{2}\right)\varsigma_1\varsigma_3 - \frac{3}{2}\left(\frac{1}{4}\right)\varsigma_1^2\varsigma_2 - \varsigma_4 \right|. \tag{21}$$

Comparing the right side of (21) with

$$\left| \nu\varsigma_1^4 + \varrho\varsigma_2^2 + 2\rho\varsigma_1\varsigma_3 - \frac{3}{2}\varpi\varsigma_1^2\varsigma_2 - \varsigma_4 \right|,$$

we obtain

$$\nu = \frac{1}{32}, \quad \varrho = \frac{1}{4}, \quad \rho = \frac{1}{2}, \quad \varpi = \frac{1}{4}.$$

It follows that

$$\left((-\varpi + \rho(\rho+\varrho))^2 + (-2\nu + \varpi\rho)^2\right)8(1-\varrho)\varrho + (-2\varrho\rho + \varpi)^2(1-\rho)\rho = 0.0292968,$$

and

$$4\varrho\rho^2(1-\rho)^2(1-\varrho) = 0.046875.$$

From (7), we deduce that

$$\left|d_5 - d_2^4\right| \leq \frac{1}{4}.$$

The equality is attained from

$$\frac{2\xi g'(\xi)}{g(\xi) - g(-\xi)} = 1 + \xi^4 - \frac{1}{6}\xi^{12} + \frac{3}{40}\xi^{20} + \cdots.$$

□

**Theorem 6.** *If $g \in \mathcal{S}^*_{\sinh}$, and given by (1). Then*

$$\left|d_4 - d_2^3\right| \leq \frac{1}{4}.$$

*The result is best possible.*

**Proof.** Putting (15) and (17), we have

$$\left|d_4 - d_2^3\right| = \frac{1}{8}\left|\varsigma_3 - 2\left(\frac{3}{8}\right)\varsigma_1\varsigma_2 + \left(-\frac{1}{24}\right)\varsigma_1^3\right|.$$

From (5), let

$$K = \frac{3}{8} \quad \text{and} \quad T = -\frac{1}{24},$$

and $0 \leq K \leq 1$, $K \geq T$ with

$$K(2K - 1) = -0.09375 \leq T.$$

Thus, all the conditions of (5) are satisfied. Hence, we have

$$\left|d_4 - d_2^3\right| \leq \frac{1}{4}.$$

The equality is attained from

$$\frac{2\xi g'(\xi)}{g(\xi) - g(-\xi)} = 1 + \xi^3 - \frac{1}{6}\xi^9 + \frac{3}{40}\xi^{15} + \cdots.$$

□

## 7. Second and Third Hankel Determinants

Finally, we study the estimates of the Hankel determinants $\mathcal{D}_{2,2}(g)$, $\mathcal{D}_{2,3}(g)$ and $\mathcal{D}_{3,1}(g)$ for $g \in \mathcal{S}^*_{\sinh}$.

**Theorem 7.** *If $g \in \mathcal{S}^*_{\sinh}$ and has the form (1), then*

$$|\mathcal{D}_{2,2}(g)| = \left|d_2 d_4 - d_3^2\right| \leq \frac{1}{4}.$$

*This outcome is best possible.*

**Proof.** From (15)–(17), we have

$$\mathcal{D}_{2,2}(g) = -\frac{5}{384}\varsigma_1^4 + \frac{5}{128}\varsigma_1^2\varsigma_2 + \frac{1}{32}\varsigma_1\varsigma_3 - \frac{1}{16}\varsigma_2^2.$$

Applying (9) and (10) to write $\varsigma_2$ and $\varsigma_3$ in terms of $\varsigma_1$ and observing that we can write $\varsigma_1 = \varsigma$, we achieve

$$\begin{aligned} |\mathcal{D}_{2,2}(g)| &= \left| -\frac{1}{768}\varsigma_1^4 + \frac{1}{256}\left(4-\varsigma^2\right)\varsigma^2\tau - \frac{1}{128}\left(4-\varsigma^2\right)\varsigma^2\tau^2 - \frac{1}{64}\left(4-\varsigma^2\right)^2\tau^2 \right. \\ &\quad \left. + \frac{1}{64}\varsigma\left(4-\varsigma^2\right)\left(1-|\tau|^2\right)\sigma \right|, \end{aligned}$$

Invoking $|\tau| = l$, $|\sigma| \leq 1$ with $l \leq 1$ and applying the triangle inequality we have

$$\begin{aligned} |\mathcal{D}_{2,2}(g)| &\leq \left| \frac{1}{768}\varsigma^4 + \frac{1}{256}\left(4-\varsigma^2\right)\varsigma^2 l + \frac{1}{128}\left(4-\varsigma^2\right)\varsigma^2 l^2 + \frac{1}{64}\left(4-\varsigma^2\right)^2 l^2 \right. \\ &\quad \left. + \frac{1}{64}\varsigma\left(4-\varsigma^2\right)\left(1-l^2\right) \right| := \varphi(\varsigma, l). \end{aligned}$$

It is now a straightforward task to illustrates that $\varphi'(\varsigma, l) \geq 0$ on $[0,1]$, and hence $\varphi(\varsigma, l) \leq \varphi(\varsigma, 1)$. Thus

$$|\mathcal{D}_{2,2}(g)| \leq \frac{1}{768}\varsigma^4 + \frac{1}{256}\varsigma^2\left(4-\varsigma^2\right) + \frac{1}{64}\left(4-\varsigma^2\right)^2 + \frac{1}{128}\left(4-\varsigma^2\right)\varsigma^2 := \varphi(\varsigma, 1).$$

By simple computation, it follows that $\varphi(\varsigma, 1)$ gets its maxima at 0. Hence

$$|\mathcal{D}_{2,2}(g)| \leq \frac{1}{4}.$$

The required $\mathcal{D}_{2,2}(g)$ is sharp. Equality is achieved from

$$\frac{2\xi g'(\xi)}{g(\xi) - g(-\xi)} = 1 + \xi^2 - \frac{1}{6}\xi^6 + \frac{3}{40}\xi^{10} + \cdots.$$

□

**Theorem 8.** *If $g \in \mathcal{S}^*_{\sinh}$, and has the form (1). Then*

$$|\mathcal{D}_{2,3}(g)| = \left| d_3 d_5 - d_4^2 \right| \leq \frac{840959}{11215872}.$$

**Proof.** Plugging (16)–(18), with $\varsigma_1 = \varsigma$ we obtain

$$\begin{aligned} \mathcal{D}_{2,3}(g) &= \frac{1}{9216}\left(-\varsigma^6 - 36\varsigma^4\varsigma_2 + 120\varsigma^3\varsigma_3 + 63\varsigma^2\varsigma_2^2 - 144\varsigma^2\varsigma_4 - 72\varsigma\varsigma_2\varsigma_3 \right. \\ &\quad \left. - 72\varsigma_2^3 + 288\varsigma_2\varsigma_4 - 144\varsigma_3^2\right). \end{aligned} \tag{22}$$

Let $m = 4 - \varsigma^2$ in (9)–(11). Now using these lemmas, we obtain

$$\begin{aligned}
36\varsigma^4\varsigma_2 &= 18\Big(\varsigma^6 + \varsigma^4 s\tau\Big),\\
120\varsigma^3\varsigma_3 &= -30\varsigma^4 m\tau^2 + 60\varsigma^3 m\Big(1-|\tau|^2\Big)\sigma + 60\varsigma^4 m\tau + 30\varsigma^6,\\
63\varsigma^2\varsigma_2^2 &= \frac{63}{4}\varsigma^6 + \frac{63}{2}\varsigma^4 m\tau + \frac{63}{4}\varsigma^2 m^2\tau^2,\\
144\varsigma^2\varsigma_4 &= 18\varsigma^6 + 18\varsigma^4 m\tau^3 - 54\varsigma^4 m\tau^2 + 54\varsigma^4 m\tau + 72\varsigma^2 m\tau^2 - 72\varsigma^3 m\tau\sigma\\
&\quad \Big(1-|\tau|^2\Big) - 72\varsigma^2 m\overline{\tau}\Big(1-|\tau|^2\Big)\sigma^2 + 72\varsigma^2 m\Big(1-|\tau|^2\Big)\Big(1-|\sigma|^2\Big)\zeta\\
&\quad + 72\varsigma^3 m\Big(1-|\tau|^2\Big)\sigma,\\
72\varsigma\varsigma_2\varsigma_3 &= -9\varsigma^2 m^2\tau^3 - 9\varsigma^4 m\tau^2 + 18\varsigma m^2\tau\Big(1-|\tau|^2\Big)\sigma + 18\varsigma^2 m^2\tau^2 + 27\varsigma^4 m\tau\\
&\quad + 18\varsigma^3 m\Big(1-|\tau|^2\Big)\sigma + 9\varsigma^6,\\
72\varsigma_2^3 &= 9\varsigma^6 + 27\varsigma^4 m\tau + 27\varsigma^2 m^2\tau^2 + 9m^3\tau^3,\\
288\varsigma_2\varsigma_4 &= 18\varsigma^6 + 18\varsigma^4 m\tau^3 - 54\varsigma^4 m\tau^2 + 72\varsigma^4 m\tau + 72\varsigma^2 m\tau^2 - 72\varsigma^3 m\tau\sigma\\
&\quad \Big(1-|\tau|^2\Big) - 72\varsigma^2 m\overline{\tau}\Big(1-|\tau|^2\Big)\sigma^2 + 72\varsigma^2 m\Big(1-|\tau|^2\Big)\Big(1-|\sigma|^2\Big)\zeta\\
&\quad + 72\varsigma^3 m\Big(1-|\tau|^2\Big)\sigma + 18\varsigma^2 m^2\tau^4 - 54\varsigma^2 m^2\tau^3 + 54\varsigma^2 m^2\tau^2\\
&\quad + 72m^2\tau^3 - 72\varsigma m^2\tau^2\Big(1-|\tau|^2\Big)\sigma - 72m^2\tau\overline{\tau}\Big(1-|\tau|^2\Big)\sigma^2\\
&\quad + 72m^2\tau\Big(1-|\tau|^2\Big)\Big(1-|\sigma|^2\Big)\zeta + 72\varsigma m^2\tau\Big(1-|\tau|^2\Big)\sigma,\\
144\varsigma_3^2 &= 9\varsigma^2 m^2\tau^4 - 36\varsigma m^2\tau^2\Big(1-|\tau|^2\Big)\sigma - 36\varsigma^2 m^2\tau^3 - 18\varsigma^4 m\tau^2 + 36\varsigma^4 m\tau\\
&\quad + 36m^2\Big(1-|\tau|^2\Big)^2\sigma^2 + 72\varsigma m^2\tau\Big(1-|\tau|^2\Big)\sigma + 36\varsigma^2 m^2\tau^2\\
&\quad + 36\varsigma^3 m\Big(1-|\tau|^2\Big)\sigma + 9\varsigma^6.
\end{aligned}$$

Plugging the above expressions in (22), we obtain

$$\begin{aligned}
\mathcal{D}_{2,3}(g) &= \frac{1}{9216}\Big\{6\varsigma^3 m\Big(1-|\tau|^2\Big)\sigma - 9\tau^3 m^3 + 72\tau^3 m^2 - 3\varsigma^4 m\tau^2 - \frac{45}{4}\varsigma^2\tau^2 m^2 - 9\varsigma^2\tau^3 m^2\\
&\quad +9\varsigma^2 m^2\tau^4 - 36m^2\Big(1-|\tau|^2\Big)^2\sigma^2 + \frac{3}{2}\varsigma^4\tau m - 18\varsigma\tau m^2\Big(1-|\tau|^2\Big)\sigma - \frac{1}{4}\varsigma^6\\
&\quad -36\varsigma\tau^2 m^2\Big(1-|\tau|^2\Big)\sigma - 72\tau m^2\Big(1-|\tau|^2\Big)\overline{\tau}\sigma^2 + 72\tau m^2\Big(1-|\tau|^2\Big)\Big(1-|\sigma|^2\Big)\zeta\Big\}.
\end{aligned}$$

Since $m = 4 - \varsigma^2$,

$$\mathcal{D}_{2,3}(g) = \frac{1}{9216}\Big(L_0(\varsigma,\tau) + L_1(\varsigma,\tau)\sigma + L_2(\varsigma,\tau)\sigma^2 + L_3(\varsigma,\tau,\sigma)\zeta\Big),$$

where

$$\begin{aligned}
L_0(\varsigma,\tau) &= -\frac{1}{4}\varsigma^6 + \Big(4-\varsigma^2\Big)\Big[\Big(4-\varsigma^2\Big)\Big(36\tau^3 - \frac{45}{4}\varsigma^2\tau^2\Big) - 3\varsigma^4\tau^2 + \frac{3}{2}\varsigma^4\tau\Big],\\
L_1(\varsigma,\tau) &= \Big(4-\varsigma^2\Big)\Big(1-|\tau|^2\Big)\Big[\Big(4-\varsigma^2\Big)\Big(-18\varsigma\tau - 36\varsigma\tau^2\Big) + 6\varsigma^3\Big],\\
L_2(\varsigma,\tau) &= \Big(4-\varsigma^2\Big)\Big(1-|\tau|^2\Big)\Big[\Big(4-\varsigma^2\Big)\Big(-36|\tau|^2 - 36\Big)\Big],\\
L_3(\varsigma,\tau,\sigma) &= \Big(4-\varsigma^2\Big)\Big(1-|\tau|^2\Big)\Big(1-|\sigma|^2\Big)\Big[72\tau\Big(4-\varsigma^2\Big)\Big].
\end{aligned}$$

By replacing $|\tau|$ by $\tau$ and $|\sigma|$ by $\kappa$, if we apply the statement $|\zeta| \leq 1$, it follows

$$\begin{aligned} |\mathcal{D}_{2,3}(g)| &\leq \frac{1}{9216}\Big(|L_0(\varsigma,\tau)| + |L_1(\varsigma,\tau)|\kappa + |L_2(\varsigma,\tau)|\kappa^2 + |L_3(\varsigma,\tau,\sigma)|\Big). \\ &\leq \frac{1}{9216}(F(\varsigma,\tau,\kappa)), \end{aligned} \tag{23}$$

where

$$F(\varsigma,\tau,\kappa) = R_0(\varsigma,\tau) + R_1(\varsigma,\tau)\kappa + R_2(\varsigma,\tau)\kappa^2 + R_3(\varsigma,\tau)\Big(1-\kappa^2\Big),$$

with

$$\begin{aligned} R_0(\varsigma,\tau) &= \frac{1}{4}\varsigma^6 + \Big(4-\varsigma^2\Big)\left[\Big(4-\varsigma^2\Big)\left(36\tau^3 + \frac{45}{4}\varsigma^2\tau^2\right) + 3\varsigma^4\tau^2 + \frac{3}{2}\varsigma^4\tau\right], \\ R_1(\varsigma,\tau) &= \Big(4-\varsigma^2\Big)\Big(1-\tau^2\Big)\Big[\Big(4-\varsigma^2\Big)\Big(18\varsigma\tau + 36\varsigma\tau^2\Big) + 6\varsigma^3\Big], \\ R_2(\varsigma,\tau) &= \Big(4-\varsigma^2\Big)\Big(1-\tau^2\Big)\Big[\Big(4-\varsigma^2\Big)\Big(36\tau^2+36\Big)\Big], \\ R_3(\varsigma,\tau) &= \Big(4-\varsigma^2\Big)\Big(1-\tau^2\Big)\Big[72\tau\Big(4-\varsigma^2\Big)\Big]. \end{aligned}$$

Now, we are to maximize $F(\varsigma,\tau,\kappa)$ in the closed cuboid $\Xi : [0,2]\times[0,1]\times[0,1]$.

For this purpose, we have to find the maxima of $F(\varsigma,\tau,\kappa)$ in the interior of $\Xi$, in the interior of its faces and on the edges.

**1. Interior points of cuboid $\Xi$:**

Suppose $(\varsigma,\tau,\kappa)\in(0,2)\times(0,1)\times(0,1)$. Then, differentiate $F(\varsigma,\tau,\kappa)$ partially about the parameter $\kappa$, and it yields

$$\frac{\partial F}{\partial \kappa} = 6\Big(4-\varsigma^2\Big)\Big(1-\tau^2\Big)\Big[\kappa(12\tau-12)\Big(4-\varsigma^2\Big)(\tau-1) + \varsigma\Big(3\tau\Big(4-\varsigma^2\Big)(2\tau+1)+\varsigma^2\Big)\Big].$$

Taking $\frac{\partial F}{\partial \kappa} = 0$, we obtain

$$\kappa = \frac{\varsigma\big(3\tau(4-\varsigma^2)(2\tau+1)+\varsigma^2\big)}{(12\tau-12)[(4-\varsigma^2)(1-\tau)]} = \kappa_1.$$

If $\kappa_1$ should belong to $(0,1)$, then it is possible only if

$$3\varsigma\tau\Big(4-\varsigma^2\Big)(2\tau+1) + \varsigma^3 < (12\tau-12)\Big(4-\varsigma^2\Big)(1-\tau). \tag{24}$$

and

$$\varsigma^2 > 4. \tag{25}$$

At this stage, we will recognize the solution that fulfills both inequalities (24) and (25) as an optimal point.

Therefore we establish that $\varsigma^2 > 4$, and a simple analysis reveals that (24) is not valid for all $\tau\in(0,1)$. As a result, no optimal point for $F$ exists within $(0,2)\times(0,1)\times(0,1)$.

**2. Interior of all the six faces of cuboid $\Xi$:**

(i) if $\varsigma = 0$, we acquire

$$r_1(\tau,\kappa) = 576\Big[\tau^3 + \Big(1-\tau^2\Big)\Big(\kappa^2(\tau-1)^2 + 2\tau\Big)\Big] = F(0,\tau,\kappa).$$

Differentiating $r_1(\tau,\kappa)$ about the parameter $\kappa$, we have

$$\frac{\partial r_1}{\partial \kappa} = 1152\kappa(1-\tau^2)(\tau-1)^2.$$

But, $\frac{\partial r_1}{\partial \kappa} \neq 0$ for $\tau,\kappa\in(0,1)$. As a result, no optimal point for $r_1$ exists within $(0,1)\times(0,1)$.

(ii) when $\varsigma = 2$, we achieve

$$F(2, \tau, \kappa) = 16.$$

(iii) substituting $\tau = 0$, it yields

$$r_2(\varsigma, \kappa) = \frac{1}{4}\varsigma^6 + 6\varsigma^3\kappa(4 - \varsigma^2) + 36\kappa^2(4 - \varsigma^2)^2 = F(\varsigma, 0, \kappa).$$

Differentiating $r_2(\varsigma, \kappa)$ about the parameter $\kappa$, then about the parameter $\varsigma$ we have

$$\frac{\partial r_2}{\partial \kappa} = 6\varsigma^3(4 - \varsigma^2) + 72\kappa(4 - \varsigma^2)^2.$$

and

$$\frac{\partial r_2}{\partial \varsigma} = \frac{3}{2}\varsigma^5 - 12\varsigma^4\kappa + 18\varsigma^2\kappa\left(4 - \varsigma^2\right) - 144\varsigma\kappa^2(4 - \varsigma^2).$$

A calculation shows that $r_2(\varsigma, \kappa)$ has no optimal solution in $(0, 2) \times (0, 1)$.

(iv) selecting $\tau = 1$, we have

$$r_3(\varsigma, \kappa) = \frac{1}{4}\varsigma^6 + (4 - \varsigma^2)\left[(4 - \varsigma^2)\left(\frac{45}{4}\varsigma^2 + 36\right) + \frac{9}{2}\varsigma^4\right] = F(\varsigma, 1, \kappa).$$

It is clear that

$$\frac{\partial r_3}{\partial \varsigma} = 42\varsigma^5 - 144\varsigma^3 - 216\varsigma.$$

We see that $r_3'(\varsigma) < 0$ for $[0, 2]$. This implies that $r_3(\varsigma)$ is a decreasing function of $\varsigma$, so

$$r_3(\varsigma) \leq 576.$$

(v) If we choose $\kappa = 0$, we find that

$$\begin{aligned} r_4(\varsigma, \tau) &= \frac{1}{4}\varsigma^6 - 36\varsigma^4\tau^3 + 288\varsigma^2\tau^3 - 576\tau^3 + \frac{33}{4}\varsigma^6\tau^2 - 78\varsigma^4\tau^2 + 180\varsigma^2\tau^2 - \frac{3}{2}\varsigma^6\tau \\ &\quad +78\varsigma^4\tau - 576\varsigma^2\tau + 1152\tau = F(\varsigma, \tau, 0). \end{aligned}$$

Now, differentiating partially about the parameter $\varsigma$, then about the parameter $\tau$ and simplifying we have

$$\frac{\partial r_4}{\partial \varsigma} = \frac{3}{2}\varsigma^5 - 144\varsigma^3\tau^3 + 576\varsigma\tau^3 + \frac{99}{2}\varsigma^5\tau^2 - 312\varsigma^3\tau^2 + 360\varsigma\tau^2 - 9\varsigma^5\tau + 312\varsigma^3\tau - 1152\varsigma\tau.$$

and

$$\frac{\partial r_4}{\partial \tau} = -108\varsigma^4\tau^2 + 864\varsigma^2\tau^2 - 1728\tau^2 + \frac{33}{2}\varsigma^6\tau - 156\varsigma^4\tau + 360\varsigma^2\tau - \frac{3}{2}\varsigma^6 + 78\varsigma^4 - 576\varsigma^2 + 1152.$$

A numerical calculation shows that the solution does not exists for the system of equations

$$\frac{\partial r_4}{\partial \varsigma} = 0 \quad \text{and} \quad \frac{\partial r_4}{\partial \tau} = 0$$

in $(0, 2) \times (0, 1)$.

(vi) Choosing $\kappa = 1$, we obtain

$$\begin{aligned} r_5(\varsigma, \tau) &= 576 - 6\varsigma^5 - 576\tau^4 + 36\varsigma^4 + \frac{1}{4}\varsigma^6 - 288\varsigma^2 + 576\tau^3 + 24\varsigma^3 - 288\varsigma^2\tau^3 \\ &\quad +36\varsigma^4\tau^3 + \frac{33}{4}\varsigma^6\tau^2 - \frac{3}{2}\varsigma^6\tau - 36\varsigma^5\tau^4 - 18\varsigma^5\tau^3 + 42\varsigma^5\tau^2 + 288\varsigma^3\tau^4 \\ &\quad +18\varsigma^5\tau + 144\varsigma^3\tau^3 - 312\varsigma^3\tau^2 - 576\varsigma\tau^4 - 144\varsigma^3\tau - 288\varsigma\tau^3 - 36\varsigma^4\tau^4 \\ &\quad +288\varsigma^2\tau^4 + 180\varsigma^2\tau^2 - 78\varsigma^4\tau^2 + 6\varsigma^4\tau + 288\varsigma\tau + 576\varsigma\tau^2 = F(\varsigma, \tau, 1). \end{aligned}$$

Partial derivative of $r_5(\varsigma, \tau)$ about the parameter $\varsigma$ and then about the parameter $\tau$, we have

$$\begin{aligned}\frac{\partial r_5}{\partial \varsigma} &= -432\varsigma^2\tau + \frac{3}{2}\varsigma^5 - 576\tau^4 - 30\varsigma^4 + 72\varsigma^2 - 288\tau^3 + 576\tau^2 + 144\varsigma^3 \\ &\quad + 432\varsigma^2\tau^3 - 90\varsigma^4\tau^3 + \frac{99}{2}\varsigma^5\tau^2 - 144\varsigma^3\tau^4 - 9\varsigma^5\tau + 144\varsigma^3\tau^3 \\ &\quad - 312\varsigma^3\tau^2 + 576\varsigma\tau^4 + 24\varsigma^3\tau - 576\varsigma\tau^3 - 180\varsigma^4\tau^4 + 864\varsigma^2\tau^4 \\ &\quad - 936\varsigma^2\tau^2 + 210\varsigma^4\tau^2 + 90\varsigma^4\tau + 360\varsigma\tau^2 - 576\varsigma + 288\tau.\end{aligned}$$

and

$$\begin{aligned}\frac{\partial r_5}{\partial \tau} &= -2304\tau^3 + 1728\tau^2 - 864\varsigma^2\tau^2 + 108\varsigma^4\tau^2 + \frac{33}{2}\varsigma^6\tau - \frac{3}{2}\varsigma^6 - 144\varsigma^5\tau^3 \\ &\quad - 54\varsigma^5\tau^2 + 84\varsigma^5\tau + 1152\varsigma^3\tau^3 + 18\varsigma^5 + 432\varsigma^3\tau^2 - 624\varsigma^3\tau \\ &\quad - 2304\varsigma\tau^3 - 144\varsigma^3 - 864\varsigma\tau^2 - 144\varsigma^4\tau^3 + 1152\varsigma^2\tau^3 + 360\varsigma^2\tau \\ &\quad - 156\varsigma^4\tau + 6\varsigma^4 + 288\varsigma + 1152\varsigma\tau.\end{aligned}$$

A simple computation illustrates that there exists a unique solution $(\varsigma, \tau) \approx (0.4196, 0.7198)$ for the system of equations

$$\frac{\partial r_5}{\partial \varsigma} = 0 \quad \text{and} \quad \frac{\partial r_5}{\partial \tau} = 0,$$

in $(0, 2) \times (0, 1)$. Thus, we have

$$F(\varsigma, \tau, 1) \leq 691.009.$$

**3. On the Edges of Cuboid Ξ:**

(i) By setting $\tau = 0$ and $\kappa = 0$, it yields

$$F(\varsigma, 0, 0) = \frac{1}{4}\varsigma^6 \leq 16.$$

(ii) By choosing $\tau = 0$ and $\kappa = 1$, we find that

$$r_6(\varsigma) = \frac{1}{4}\varsigma^6 - 6\varsigma^5 + 24\varsigma^3 + 36\varsigma^4 - 288\varsigma^2 + 576 = F(\varsigma, 0, 1).$$

Differentiating $r_6(\varsigma)$ about the parameter $\varsigma$, we have

$$r_6'(\varsigma) = \frac{3}{2}\varsigma^5 - 30\varsigma^4 + 72\varsigma^2 + 144\varsigma^3 - 576\varsigma.$$

By simple computation, we indicates that $r_6(\varsigma)$ achieves its maxima at 0. Thus

$$r_6(\varsigma) \leq 576.$$

(iii) By selecting $\varsigma = 0$ and $\tau = 0$, we obtain

$$r_7(\kappa) = 576\kappa^2 = F(0, 0, \kappa).$$

Since, $r_7'(\kappa) > 0$ for $[0, 1]$. This shows that $r_7(\kappa)$ is a increasing function of $\kappa$, therefore

$$F(0, 0, \kappa) = r_7(\kappa) \leq 576.$$

(iv) We note that $F(\varsigma, 1, \kappa)$ is free of $\kappa$, and we obtain

$$r_8(\varsigma) = F(\varsigma, 1, 1) = F(\varsigma, 1, 0).$$

$$r_8(\varsigma) = 7\varsigma^6 - 36\varsigma^4 - 108\varsigma^2 + 576.$$

By simple computation, we indicates that $r_8(\varsigma)$ achieves its maxima at 0. Thus

$$r_8(\varsigma) \leq 576.$$

(v) By taking $\varsigma = 0$ and $\tau = 1$, it yields

$$F(0, 1, \kappa) \leq 576.$$

(vi) By choosing $\varsigma = 2$, it becomes

$$F(2, \tau, \kappa) = 16.$$

We see that $F(2, \tau, \kappa)$ is free of $\kappa$, $\tau$, $\varsigma$. It follows

$$F(2, 1, \kappa) = F(2, \tau, 1) = F(2, \tau, 0) = F(2, 0, \kappa) = 16.$$

(vii) By setting $\varsigma = 0$ and $\kappa = 1$, we achieve

$$F(0, \tau, 1) = -576\tau^4 + 576\tau^3 + 576 = r_9(\tau).$$

It is clear that

$$r_9'(\tau) = -2304\tau^3 + 1728\tau^2.$$

Now, $\frac{\partial r_9}{\partial \tau} = 0$, gives $\tau \approx 0.750$ at which maxima is acquired for $r_9(\tau)$. Hence

$$F(0, \tau, 1) \leq \frac{2547}{4}.$$

(viii) By substituting $\varsigma = 0$ and $\kappa = 0$, we find that

$$F(0, \tau, 0) = -576\tau^3 + 1152\tau = r_{10}(\tau).$$

Differentiating $r_{10}(\tau)$ about the parameter $\tau$, we have

$$r_{10}'(\tau) = -1728\tau^2 + 1152.$$

Now $\frac{\partial r_{10}}{\partial \tau} = 0$, we obtain $\tau \approx 0.8164$ at which maxima is acquired for $r_{10}(\tau)$. Thus

$$F(0, \tau, 0) \leq 256\sqrt{2}\sqrt{3}.$$

Hence, from the above situations, we achieve

$$F(\varsigma, \tau, \kappa) \leq \frac{840959}{1217} \quad \text{on} \quad [0, 2] \times [0, 1] \times [0, 1].$$

By using Equation (23), it follows

$$|\mathcal{D}_{2,3}(g)| \leq \frac{1}{9216}(F(\varsigma, \tau, \kappa)) \leq \frac{840959}{11215872}.$$

The required proof is completed. □

**Conjecture 1.** *If $g \in \mathcal{S}^*_{\sinh}$, and has the form (1). Then*

$$|\mathcal{D}_{2,3}(g)| \leq \frac{1}{16}.$$

*This inequality is sharp and equality will be attained from*

$$\frac{2\xi g'(\xi)}{g(\xi) - g(-\xi)} = 1 + \xi^3 - \frac{1}{6}\xi^9 + \frac{3}{40}\xi^{15} + \cdots.$$

**Theorem 9.** *If $g \in \mathcal{S}^*_{\sinh}$. Then*

$$|\mathcal{D}_{3,1}(g)| \leq \frac{387691}{4939776}.$$

**Proof.** The determinant $\mathcal{D}_{3,1}(g)$ is described as follows:

$$\mathcal{D}_{3,1}(g) = 2d_2d_3d_4 - d_3^3 - d_4^2 + d_3d_5 - d_2^2d_5.$$

Plugging (15)–(18), with $\varsigma_1 = \varsigma$ we obtain

$$\begin{aligned}\mathcal{D}_{3,1}(g) &= \frac{1}{9216}\Big(11\varsigma^6 - 105\varsigma^4\varsigma_2 + 120\varsigma^3\varsigma_3 + 189\varsigma^2\varsigma_2^2 - 216\varsigma^2\varsigma_4 + 72\varsigma\varsigma_2\varsigma_3 \\ &\quad -216\varsigma_2^3 + 288\varsigma_2\varsigma_4 - 144\varsigma_3^2\Big). \end{aligned} \tag{26}$$

Let $m = 4 - \varsigma^2$ in (9)–(11). Now, using these lemmas, we obtain

$$\begin{aligned}
105\varsigma^4\varsigma_2 &= \frac{105}{2}\Big(\varsigma^6 + \varsigma^4 m\tau\Big),\\
120\varsigma^3\varsigma_3 &= -30\varsigma^4 m\tau^2 + 60\varsigma^3 m\Big(1 - |\tau|^2\Big)\sigma + 60\varsigma^4 m\tau + 30\varsigma^6,\\
189\varsigma^2\varsigma_2^2 &= \frac{189}{4}\varsigma^6 + \frac{189}{2}\varsigma^4 m\tau + \frac{189}{4}\varsigma^2 m^2\tau^2,\\
216\varsigma^2\varsigma_4 &= -108\Big(1 - |\tau|^2\Big)\varsigma^2\overline{\tau}\sigma^2 m - 81\varsigma^4 m\tau^2 + 108\Big(1 - |\tau|^2\Big)\varsigma^3\sigma m + 108\varsigma^2 m\tau^2\\
&\quad + 27\varsigma^4 m\tau^3 + 27\varsigma^6 + 108\Big(1 - |\sigma|^2\Big)\Big(1 - |\tau|^2\Big)\varsigma^2\zeta m + 81\varsigma^4 m\tau\\
&\quad - 108\Big(1 - |\tau|^2\Big)\varsigma^3 m\sigma\tau,\\
72\varsigma\varsigma_2\varsigma_3 &= -9\varsigma^2 m^2\tau^3 - 9\varsigma^4 m\tau^2 + 18\varsigma\tau m^2\Big(1 - |\tau|^2\Big)\sigma + 18\varsigma^2\tau^2 m^2 + 18\varsigma^3 m\sigma\\
&\quad \Big(1 - |\tau|^2\Big) + 27\varsigma^4 m\tau + 9\varsigma^6,\\
216\varsigma_2^3 &= 27\varsigma^6 + 81\varsigma^4 m\tau + 81\varsigma^2 m^2\tau^2 + 27m^3\tau^3,\\
288\varsigma_2\varsigma_4 &= 18\varsigma^6 + 18\varsigma^4 m\tau^3 + 72\varsigma^4 m\tau + 72\varsigma^2 m\tau^2 - 72\Big(1 - |\tau|^2\Big)m\overline{\tau}\sigma^2\varsigma^2\\
&\quad -72\Big(1 - |\tau|^2\Big)\varsigma^3 m\sigma\tau - 54\varsigma^4 m\tau^2 + 72\Big(1 - |\sigma|^2\Big)\Big(1 - |\tau|^2\Big)\varsigma^2\zeta m\\
&\quad + 18\varsigma^2 m^2\tau^4 - 54\varsigma^2 m^2\tau^3 + 72\Big(1 - |\tau|^2\Big)\varsigma^3\sigma m + 54\varsigma^2 m^2\tau^2\\
&\quad + 72m^2\tau^3 - 72\varsigma m^2\tau^2\Big(1 - |\tau|^2\Big)\sigma - 72\tau m^2\overline{\tau}\Big(1 - |\tau|^2\Big)\sigma^2\\
&\quad + 72\Big(1 - |\sigma|^2\Big)\Big(1 - |\tau|^2\Big)m^2\zeta\tau + 72\Big(1 - |\tau|^2\Big)m^2\tau\sigma\varsigma,\\
144\varsigma_3^2 &= 36\Big(1 - |\tau|^2\Big)^2 m^2\sigma^2 + 9\varsigma^2 m^2\tau^4 - 36\Big(1 - |\tau|^2\Big)m^2\tau^2\sigma\varsigma - 18\varsigma^4 m\tau^2\\
&\quad + 36\varsigma^2 m^2\tau^2 - 36\varsigma^2 m^2\tau^3 + 72\Big(1 - |\tau|^2\Big)m^2\tau\sigma\varsigma + 36\varsigma^4 m\tau + 9\varsigma^6\\
&\quad + 36\Big(1 - |\tau|^2\Big)\varsigma^3\sigma m.
\end{aligned}$$

Plugging the above expressions in (26), we obtain

$$\begin{aligned}\mathcal{D}_{3,1}(g) &= \frac{1}{9216}\Big\{6\varsigma^4\tau^2 m + 3\varsigma^4\tau m - 36\varsigma^2\tau^2 m - 9\varsigma^4\tau^3 m - 27\varsigma^2\tau^3 m^2 + 9\varsigma^2 m^2\tau^4 \\ &\quad - 36m^2\Big(1-|\tau|^2\Big)^2\sigma^2 - 27\tau^3 m^3 + 72\tau^3 m^2 + 36\varsigma^3\tau m\Big(1-|\tau|^2\Big)\sigma \\ &\quad + 36\Big(1-|\tau|^2\Big)m\overline{\tau}\sigma^2\varsigma^2 - 36\Big(1-|\sigma|^2\Big)\Big(1-|\tau|^2\Big)\varsigma^2\zeta m + 18\varsigma\tau m^2 \\ &\quad \Big(1-|\tau|^2\Big)\sigma - 36\Big(1-|\tau|^2\Big)\tau^2 m^2\varsigma\sigma - 72\Big(1-|\tau|^2\Big)\tau m^2\overline{\tau}\sigma^2 - \frac{1}{4}\varsigma^6 \\ &\quad + 72\Big(1-|\sigma|^2\Big)\Big(1-|\tau|^2\Big)m^2\tau\zeta + 6\Big(1-|\tau|^2\Big)\varsigma^3 m\sigma + \frac{9}{4}\varsigma^2\tau^2 m^2\Big\}.\end{aligned}$$

Since $m = 4-\varsigma^2$,

$$\mathcal{D}_{3,1}(g) = \frac{1}{9216}\Big(N_0(\varsigma,\tau) + N_1(\varsigma,\tau)\sigma + N_2(\varsigma,\tau)\sigma^2 + \chi(\varsigma,\tau,\sigma)\zeta\Big),$$

where $\sigma, \tau, \zeta \in \overline{\mathbb{D}}$, and

$$\begin{aligned}N_0(\varsigma,\tau) &= -\frac{1}{4}\varsigma^6 + \Big(4-\varsigma^2\Big)\Big[\Big(4-\varsigma^2\Big)\Big(9\varsigma^2\tau^4 - 36\tau^3 + \frac{9}{4}\varsigma^2\tau^2\Big) \\ &\qquad + 6\varsigma^4\tau^2 + 3\varsigma^4\tau - 36\varsigma^2\tau^2 - 9\varsigma^4\tau^3\Big], \\ N_1(\varsigma,\tau) &= \Big(4-\varsigma^2\Big)\Big(1-|\tau|^2\Big)\Big[\Big(4-\varsigma^2\Big)\Big(18\varsigma\tau - 36\varsigma\tau^2\Big) + 36\varsigma^3\tau + 6\varsigma^3\Big], \\ N_2(\varsigma,\tau) &= \Big(4-\varsigma^2\Big)\Big(1-|\tau|^2\Big)\Big[\Big(4-\varsigma^2\Big)\Big(-36|\tau|^2 - 36\Big) + 36\varsigma^2\overline{\tau}\Big], \\ \chi(\varsigma,\tau,\sigma) &= \Big(4-\varsigma^2\Big)\Big(1-|\tau|^2\Big)\Big(1-|\sigma|^2\Big)\Big[-36\varsigma^2 + 72\tau\Big(4-\varsigma^2\Big)\Big].\end{aligned}$$

By replacing $|\tau|$ by $\tau$ and $|\sigma|$ by $\kappa$, if we apply the statement $|\zeta| \leq 1$, and it follows

$$\begin{aligned}|\mathcal{D}_{3,1}(g)| &\leq \frac{1}{9216}\Big(|N_0(\varsigma,\tau)| + |N_1(\varsigma,\tau)|\kappa + |N_2(\varsigma,\tau)|\kappa^2 + |\chi(\varsigma,\tau,\sigma)|\Big). \\ &\leq \frac{1}{9216}(E(\varsigma,\tau,\kappa)),\end{aligned} \tag{27}$$

where

$$E(\varsigma,\tau,\kappa) = c_0(\varsigma,\tau) + c_1(\varsigma,\tau)\kappa + c_2(\varsigma,\tau)\kappa^2 + c_3(\varsigma,\tau)\Big(1-\kappa^2\Big),$$

with

$$\begin{aligned}c_0(\varsigma,\tau) &= \frac{1}{4}\varsigma^6 + \Big(4-\varsigma^2\Big)\Big[\Big(4-\varsigma^2\Big)\Big(9\varsigma^2\tau^4 + 36\tau^3 + \frac{9}{4}\varsigma^2\tau^2\Big) \\ &\qquad + 6\varsigma^4\tau^2 + 3\varsigma^4\tau + 36\varsigma^2\tau^2 + 9\varsigma^4\tau^3\Big], \\ c_1(\varsigma,\tau) &= \Big(4-\varsigma^2\Big)\Big(1-\tau^2\Big)\Big[\Big(4-\varsigma^2\Big)\Big(18\varsigma\tau + 36\varsigma\tau^2\Big) + 36\varsigma^3\tau + 6\varsigma^3\Big], \\ c_2(\varsigma,\tau) &= \Big(4-\varsigma^2\Big)\Big(1-\tau^2\Big)\Big[\Big(4-\varsigma^2\Big)\Big(36\tau^2 + 36\Big) + 36\varsigma^2\tau\Big], \\ c_3(\varsigma,\tau) &= \Big(4-\varsigma^2\Big)\Big(1-\tau^2\Big)\Big[36\varsigma^2 + 72\Big(4-\varsigma^2\Big)\tau\Big].\end{aligned}$$

Now, we are to maximize $E(\varsigma,\tau,\kappa)$ in the closed cuboid $\Xi : [0,2]\times[0,1]\times[0,1]$.

For this purpose, we have to find the maxima of $E(\varsigma,\tau,\kappa)$ in the interior of $\Xi$, in the interior of its faces and on the edges.

**1. Interior points of cuboid Ξ:**

Suppose $(\varsigma, \tau, \kappa) \in (0,2) \times (0,1) \times (0,1)$. Then, on differentiating $E(\varsigma, \tau, \kappa)$ partially about the parameter $\kappa$, it yields

$$\begin{aligned}\frac{\partial E}{\partial \kappa} &= 6\left(4-\varsigma^2\right)(1-\tau^2)\Big[12\kappa(\tau-1)\Big(\left(4-\varsigma^2\right)(\tau-1)+\varsigma^2\Big)\\ &\quad +\varsigma\Big(3\tau\left(4-\varsigma^2\right)(2\tau+1)+\varsigma^2(6\tau+1)\Big)\Big].\end{aligned}$$

Taking $\frac{\partial E}{\partial \kappa} = 0$, gives

$$\kappa = \frac{\varsigma\left(3\tau\left(4-\varsigma^2\right)(2\tau+1)+\varsigma^2(6\tau+1)\right)}{12(\tau-1)((4-\varsigma^2)(1-\tau)-\varsigma^2)} = \kappa^*.$$

If $\kappa^*$ should belong to $(0,1)$, then it is possible only if

$$\varsigma^3(6\tau+1)+3\varsigma\tau\left(4-\varsigma^2\right)(2\tau+1)+12(1-\tau)^2\left(4-\varsigma^2\right) < 12(1-\tau)\varsigma^2. \tag{28}$$

and

$$\varsigma^2 > \frac{4(1-\tau)}{2-\tau}. \tag{29}$$

Now, only that solution which meets both the inequalities (28) and (29) will be accepted as a critical point.

Suppose $U(\tau) = \frac{4(1-\tau)}{2-\tau}$. Thus, $U(\tau)$ is decreasing over $(0,1)$. Thus $\varsigma^2 > 0$, and a straightforward task illustrates that (28) is not hold for all values of $\tau \in (0,1)$. As a result, no optimal point for $E$ exists within $(0,2) \times (0,1) \times (0,1)$.

**2. Interior of all the six faces of cuboid Ξ:**

(i) Choosing $\varsigma = 0$, we achieve

$$e_1(\tau, \kappa) = 576\Big[\tau^3 + (1-\tau^2)(\kappa^2(\tau-1)^2+2\tau)\Big] = E(0, \tau, \kappa).$$

Differentiate $e_1(\tau, \kappa)$ partially about the parameter $\kappa$, and we obtain

$$\frac{\partial e_1}{\partial \kappa} = 1152\kappa(1-\tau^2)(\tau-1)^2.$$

But, $\frac{\partial e_1}{\partial \kappa} \neq 0$ for $\tau, \kappa \in (0,1)$. Hence, we have found no maxima for $E(0, \tau, \kappa)$ in $(0,1) \times (0,1)$.

(ii) Setting $\varsigma = 2$, we have

$$E(2, \tau, \kappa) = 16.$$

(iii) Taking $\tau = 0$, we obtain

$$E(\varsigma, 0, \kappa) = e_2(\varsigma, \kappa) = \frac{1}{4}\varsigma^6 + (4-\varsigma^2)\Big(6\varsigma^3\kappa - 72\varsigma^2\kappa^2 + 36\varsigma^2 + 144\kappa^2\Big).$$

Putting $\frac{\partial e_2}{\partial \kappa} = 0$, we achieve

$$\kappa = \frac{\varsigma^3}{24(\varsigma^2-2)} = \kappa_0.$$

$\kappa_0$ must lie within $(0,1)$ for the provided range of $\kappa$, that is only achievable if $\varsigma > \varsigma_0$, $\varsigma_0 \approx 1.4592733$. Also, substituting the value of $\kappa$ in $\frac{\partial e_2}{\partial \varsigma} = 0$ and simplifying we obtain

$$\frac{\partial e_2}{\partial \varsigma} = 3\varsigma^9 - 584\varsigma^7 + 3456\varsigma^5 - 6912\varsigma^3 + 4608\varsigma = 0. \tag{30}$$

The solution to (30) can be found using computation, which is $\varsigma \approx 1.3069302$ in the interval $(0,2)$. As a result, there is no optimal point for $e_2(\varsigma,\kappa)$ in $(0,2)\times(0,1)$.

(iv) Considering $\tau = 1$, we have

$$e_3(\varsigma,\kappa) = \frac{1}{4}\varsigma^6 + (4-\varsigma^2)\left((4-\varsigma^2)(36+\frac{45}{4}\varsigma^2) + 36\varsigma^2 + 18\varsigma^4\right) = E(\varsigma,1,\kappa).$$

Then

$$\frac{\partial e_3}{\partial \varsigma} = -39\varsigma^5 - 72\varsigma^3 + 72\varsigma.$$

By taking $\frac{\partial e_3}{\partial \varsigma} = 0$, we achieve $\varsigma \approx 0.848263$ at which $e_3(\varsigma,\kappa)$ attains its maxima, that is

$$e_3(\varsigma,\kappa) \leq 590.162.$$

(v) If we choose $\kappa = 0$, we find that

$$\begin{aligned} e_4(\varsigma,\tau) &= 9\varsigma^6\tau^4 - 72\varsigma^4\tau^4 + 144\varsigma^2\tau^4 + 288\varsigma^2\tau^3 - 576\tau^3 - \frac{15}{4}\varsigma^6\tau^2 + 6\varsigma^4\tau^2 + 36\varsigma^2\tau^2 - 36\varsigma^4 \\ &\quad + \frac{1}{4}\varsigma^6 - 9\varsigma^6\tau^3 - 3\varsigma^6\tau + 84\varsigma^4\tau - 576\varsigma^2\tau + 144\varsigma^2 + 1152\tau = E(\varsigma,\tau,0). \end{aligned}$$

Now, differentiating partially about the parameter $\varsigma$, then about the parameter $\tau$ and simplifying, we have

$$\begin{aligned} \frac{\partial e_4}{\partial \varsigma} &= \frac{3}{2}\varsigma^5 + 54\varsigma^5\tau^4 - 288\varsigma^3\tau^4 + 288\varsigma\tau^4 + 576\varsigma\tau^3 - \frac{45}{2}\varsigma^5\tau^2 + 24\varsigma^3\tau^2 + 72\varsigma\tau^2 - 54\varsigma^5\tau^3 \\ &\quad - 18\varsigma^5\tau + 336\varsigma^3\tau - 144\varsigma^3 - 1152\varsigma\tau + 288\varsigma. \end{aligned}$$

and

$$\begin{aligned} \frac{\partial e_4}{\partial \tau} &= 36\varsigma^6\tau^3 - 288\varsigma^4\tau^3 + 576\varsigma^2\tau^3 + 864\varsigma^2\tau^2 - 1728\tau^2 - \frac{15}{2}\varsigma^6\tau + 12\varsigma^4\tau + 72\varsigma^2\tau \\ &\quad - 27\varsigma^6\tau^2 - 3\varsigma^6 + 84\varsigma^4 - 576\varsigma^2 + 1152. \end{aligned}$$

From computation, we find that the solution does not exists for the system of equations

$$\frac{\partial e_4}{\partial \varsigma} = 0 \text{ and } \frac{\partial e_4}{\partial \tau} = 0$$

in $(0,2)\times(0,1)$.

(vi) Taking $\kappa = 1$, we obtain

$$\begin{aligned} e_5(\varsigma,\tau) &= 576 + 9\varsigma^6\tau^4 - \frac{15}{4}\varsigma^6\tau^2 - 9\varsigma^6\tau^3 - 3\varsigma^6\tau - 36\varsigma^5\tau^4 + 18\varsigma^5\tau^3 + 42\varsigma^5\tau^2 + 288\varsigma^3\tau^4 \\ &\quad - 18\varsigma^5\tau - 312\varsigma^3\tau^2 - 576\varsigma\tau^4 - 288\varsigma\tau^3 - 6\varsigma^5 - 108\varsigma^4\tau^4 - 432\varsigma^2\tau^3 - 576\tau^4 \\ &\quad + 36\varsigma^4 + \frac{1}{4}\varsigma^6 - 288\varsigma^2 + 576\tau^3 + 24\varsigma^3 + 432\varsigma^2\tau^4 + 180\varsigma^2\tau^2 - 30\varsigma^4\tau^2 \\ &\quad + 108\varsigma^4\tau^3 - 24\varsigma^4\tau + 288\varsigma\tau + 576\varsigma\tau^2 + 144\varsigma^2\tau = E(\varsigma,\tau,1). \end{aligned}$$

Partial derivative of $e_5(\varsigma, \tau)$ about the parameter $\varsigma$ and then about the parameter $\tau$, and we have

$$\begin{aligned}\frac{\partial e_5}{\partial \varsigma} &= 432\varsigma^3\tau^3 + 54\varsigma^5\tau^4 - 54\varsigma^5\tau^3 - \frac{45}{2}\varsigma^5\tau^2 - 432\varsigma^3\tau^4 - 18\varsigma^5\tau - 120\varsigma^3\tau^2 + 864\varsigma\tau^4 \\ &\quad - 864\varsigma\tau^3 + \frac{3}{2}\varsigma^5 - 180\varsigma^4\tau^4 - 576\tau^4 - 30\varsigma^4 + 72\varsigma^2 - 288\tau^3 + 576\tau^2 \\ &\quad + 144\varsigma^3 + 864\varsigma^2\tau^4 - 936\varsigma^2\tau^2 + 210\varsigma^4\tau^2 + 90\varsigma^4\tau^3 - 90\varsigma^4\tau + 288\varsigma\tau \\ &\quad + 360\varsigma\tau^2 - 96\varsigma^3\tau - 576\varsigma + 288\tau.\end{aligned}$$

and

$$\begin{aligned}\frac{\partial e_5}{\partial \tau} &= 36\varsigma^6\tau^3 - \frac{15}{2}\varsigma^6\tau - 27\varsigma^6\tau^2 - 3\varsigma^6 - 144\varsigma^5\tau^3 + 54\varsigma^5\tau^2 + 84\varsigma^5\tau + 1152\varsigma^3\tau^3 - 18\varsigma^5 \\ &\quad - 624\varsigma^3\tau - 2304\varsigma\tau^3 - 864\varsigma\tau^2 - 432\varsigma^4\tau^3 - 1296\varsigma^2\tau^2 - 2304\tau^3 + 1728\tau^2 \\ &\quad + 1728\varsigma^2\tau^3 + 360\varsigma^2\tau - 60\varsigma^4\tau + 324\varsigma^4\tau^2 - 24\varsigma^4 + 288\varsigma + 1152\varsigma\tau + 144\varsigma^2.\end{aligned}$$

A simple computation illustrates that there exists a unique solution $(\varsigma, \tau) \approx (0.6652, 0.7409)$ for the system of equations

$$\frac{\partial e_5}{\partial \varsigma} = 0 \quad \text{and} \quad \frac{\partial e_5}{\partial \tau} = 0,$$

in $(0, 2) \times (0, 1)$. Hence

$$E(\varsigma, \tau, 1) = e_5(\varsigma, \tau) \leq 723.304.$$

**3. On the Edges of Cuboid Ξ:**

(i) By selecting $\tau = 0$ and $\kappa = 0$, we find that

$$E(\varsigma, 0, 0) = \frac{1}{4}\varsigma^6 - 36\varsigma^4 + 144\varsigma^2 = e_6(\varsigma).$$

Differentiating $e_6(\varsigma)$ about the parameter $\varsigma$, we have

$$e_6'(\varsigma) = \frac{3}{2}\varsigma^5 - 144\varsigma^3 + 288\varsigma.$$

We note that $e_6'(\varsigma) = 0$ for the critical point $\varsigma \approx 1.4295$ at which $e_6(\varsigma)$ obtains its maxima. Thus

$$e_6(\varsigma) \leq 146.06.$$

(ii) By substituting $\tau = 0$ and $\kappa = 1$, it yields

$$E(\varsigma, 0, 1) = \frac{1}{4}\varsigma^6 - 6\varsigma^5 + 24\varsigma^3 + 36\varsigma^4 - 288\varsigma^2 + 576 = e_7(\varsigma).$$

Differentiating $e_7(\varsigma)$ about the parameter $\varsigma$, we have

$$e_7'(\varsigma) = \frac{3}{2}\varsigma^5 - 30\varsigma^4 + 72\varsigma^2 + 144\varsigma^3 - 576\varsigma.$$

As $e_7'(\varsigma) < 0$ for $[0, 2]$. This implies that $e_7(\varsigma)$ is a decreasing function of $\varsigma$, so

$$E(\varsigma, 0, 1) = e_7(\varsigma) \leq 576.$$

(iii) By choosing $\varsigma = 0$ and $\tau = 0$, we obtain

$$e_8(\kappa) = 576\kappa^2 = E(0, 0, \kappa).$$

Since $e_8'(\kappa) > 0$ for $[0, 1]$. This reveals that $e_8(\kappa)$ is increasing and maxima is acquired when $\kappa = 1$. Then

$$e_8(\kappa) \leq 576.$$

(iv) We note that $E(\varsigma, 1, \kappa)$ is free of $\kappa$. It follows

$$e_9(\varsigma) = E(\varsigma, 1, 1) = E(\varsigma, 1, 0).$$

$$e_9(\varsigma) = -\frac{13}{2}\varsigma^6 - 18\varsigma^4 + 36\varsigma^2 + 576.$$

Differentiate $e_9(\varsigma)$ partially about the parameter $\varsigma$, and we obtain

$$e_9'(\varsigma) = -39\varsigma^5 - 72\varsigma^3 + 72\varsigma.$$

By taking $e_9'(\varsigma) = 0$ we achieve $\varsigma \approx 0.8482$ at which $e_9(\varsigma)$ achieves its maxima. Thus

$$e_9(\varsigma) \leq 590.16.$$

(v) By selecting $\varsigma = 0$ and $\tau = 1$, we achieve

$$E(0, 1, \kappa) = 576.$$

(vi) By taking $\varsigma = 2$, it yields

$$E(2, \tau, \kappa) = 16.$$

We see that $E(2, \tau, \kappa)$ is free of $\kappa$, $\tau$, $\varsigma$. It follows

$$E(2, 1, \kappa) = E(2, \tau, 1) = E(2, \tau, 0) = E(2, 0, \kappa) = 16.$$

(vii) By substituting $\varsigma = 0$ and $\kappa = 1$, we find that

$$e_{10}(\tau) = -576\tau^4 + 576\tau^3 + 576 = E(0, \tau, 1).$$

It follows that

$$e_{10}'(\tau) = -2304\tau^3 + 1728\tau^2.$$

Now, $\frac{\partial e_{10}}{\partial \tau} = 0$, we achieve $\tau \approx 0.75$ at which $e_{10}(\tau)$ achieves its maxima. Hence

$$e_{10}(\tau) \leq \frac{2574}{4}.$$

(viii) By taking $\varsigma = 0$ and $\kappa = 0$, we have

$$E(0, \tau, 0) = -576\tau^3 + 1152\tau = e_{11}(\tau).$$

Clearly,

$$e_{11}'(\tau) = -1728\tau^2 + 1152.$$

We know that $e_{11}'(\tau) = 0$ gives $\tau \approx 0.816$ at which $e_{11}(\tau)$ obtain its maximum value, which is given by

$$E(0, \tau, 0) = e_{11}(\tau) \leq 256\sqrt{2}\sqrt{3}.$$

Hence, from the above situations, we achieve

$$E(\varsigma, \tau, \kappa) \leq \frac{387691}{536} \quad \text{on} \quad [0,2] \times [0,1] \times [0,1].$$

By using Equation (27), it follows

$$|\mathcal{D}_{3,1}(g)| \leq \frac{1}{9216}(E(\varsigma, \tau, \kappa)) \leq \frac{387691}{4939776}.$$

Thus, we have completed the proof. □

**Conjecture 2.** *If* $g \in \mathcal{S}^*_{\sinh}$, *and has the form* (1). *Then*

$$|\mathcal{D}_{3,1}(g)| \leq \frac{1}{16}.$$

*This inequality is sharp and equality will be attained from*

$$\frac{2\xi g'(\xi)}{g(\xi) - g(-\xi)} = 1 + \xi^3 - \frac{1}{6}\xi^9 + \frac{3}{40}\xi^{15} + \cdots.$$

## 8. Conclusions

In this work, we explore symmetric starlike functions linked to a petal-shaped domain. We have established the sharp coefficient inequalities for the above-mentioned class. The discussed coefficient inequalities include the Krushkal, Zalcman, and Fekete-Szegö inequalities, as well as the second and third Hankel determinants. We present several conjectures derived from our estimated problems, which are well supported by our findings. The sharpness of the results and the proposed conjectures distinguish this work from existing studies. This research tries to increase focus on coefficient-related problems involving inverse and logarithmic inverse functions for specific classes of analytic functions. For example, researchers can showcase their skills in various directions [50–52], including Toeplitz determinants, majorization results, convolution properties, and partial sum inequalities.

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