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Duality and Some Links Between Riemannian Submersion, *F*-Harmonicity, and Cohomology

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Abstract: Fundamentally, duality gives two different points of view of looking at the same object. It appears in many subjects in mathematics (geometry, algebra, analysis, PDEs, Geometric Measure Theory, etc.) and in physics. For example, Connections on Fiber Bundles in mathematics, and Gauge Fields in physics are exactly the same. In n-dimensional geometry, a fundamental notion is the "duality" between chains and cochains, or domains of integration and the integrands. In this paper, we extend ideas given in our earlier articles and connect seemingly unrelated areas of F-harmonic maps, f-harmonic maps, and cohomology classes via duality. By studying cohomology classes that are related with p-harmonic morphisms, F-harmonic maps, and f-harmonic maps, we extend several of our previous results on Riemannian submersions and p-harmonic morphisms to F-harmonic maps and f-harmonic maps, which are Riemannian submersions.

Keywords: duality; F-harmonic maps; p-harmonic morphism; cohomology class; Riemannian submersion; minimal submanifold

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1. Introduction

In physics, a fundamental dual system contains two opposite elements, where two opposite elements form a fundamental dual pair and in which one element is dual to the other and vice versa. Such duality can be seen as a fundamental property in nature and it appears everywhere in our universe (see, e.g., [1,2]). For example, I. Newton advocated in the late 17th century that light was corpuscular, while on the other hand, C. Huygens took an opposing wave description. While Isaac Newton had favored a particle approach, he was the first to attempt to reconcile both wave and particle theories of light, and the only one in their time to consider both; therefore, he anticipated modern wave–particle duality (see, e.g., [3]).

In theoretical physics, the theory of fundamental duality is a concept that explores the idea of two opposing, yet complementary principles existing at the most fundamental level of reality with the quantum dualiton being a proposed particle-like excitation arising from this dual field, and topological dual invariance signifying a property where certain physical quantities remain unchanged under a duality transformation, often with connections to topological properties of the system involved. Fundamental duality is a concept which refers to two irreducible, heterogeneous principles which are in opposite and complementary of each other. The complementary principle in quantum mechanics is praised by the

well-known physicist Niels Bohr, a Nobel laureate in physics, according to physicist B. T. T. Wong [4].

In mathematics, a duality translates theorems, mathematical structures, or concepts into other theorems, structures, or concepts in a one-to-one fashion. Fundamentally, duality yields two different points of view of looking at the same object, which appears in many subjects in mathematics (e.g., geometry, algebra, analysis, PDEs, Geometric Measure Theory, etc.) and in physics (cf., [5–9]). For instance, Connections on Fiber Bundles in mathematics and Gauge Fields in physics are exactly the same (see, e.g., [10–14]). In n-dimensional geometry, a fundamental notion is the "duality" between chains and cochains or domains of integration and the integrands. A (paramount) notion which includes both chains and cochains is that of a "current". This was introduced by de Rham and was used effectively and developed extensively ever since. A zero-dimensional current is a distribution (in the sense of Laurent Schwartz), a fundamental and indispensable concept in mathematics.

In this paper, we extend the ideas in our earlier papers [15,16] and connect seemingly unrelated areas of Riemannian submersions, F-harmonic maps, f-harmonic maps, and cohomology classes via duality.

2. Preliminaries

We recall some basic facts, notations, definitions, and formulas for minimal submanifolds, submersions, p-harmonic morphisms, F-harmonic maps, and f-harmonic maps for later use (see, e.g., [15,17–22] for details).

2.1. Basic Formulas and Equations

Let $\widetilde{M}$ be a Riemannian manifold with Levi-Civita connection $\widetilde{\nabla}$. The tangent bundle of $\widetilde{M}$ is denoted by $T\widetilde{M}$, and the (infinite dimensional) vector space of smooth sections of a smooth vector bundle E is denoted by $\Gamma(E)$. Let M be a submanifold of dimension $n \geq 2$ in $\widetilde{M}$. Denote by ∇ and D the Levi-Civita connection and the normal connection of M, respectively. For each normal vector $\xi \in T_x^{\perp}M$, $x \in M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space T_xM at x. Then, the shape operator and the second fundamental form h are related by (cf., [17])

$$\langle \mathsf{h}(X,Y), \xi \rangle = \langle A_\xi X, Y \rangle \tag{1}$$

for X, Y tangent to M and ξ normal to M.

The formulas of Gauss and Weingarten are given, respectively, by (cf. [17])

$$\widetilde{\nabla}_X Y = \nabla_X Y + \mathsf{h}(X,Y), \tag{2}$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{3}$$

for tangent vector fields X, Y and normal vector field ξ on M.

The *mean curvature vector field* of a submanifold M is defined by

$$H = \frac{1}{n} \operatorname{trace} \mathsf{h}. \tag{4}$$

A submanifold M in $\widetilde{M}$ is called *totally geodesic* (respectively, *minimal*) if its second fundamental form h (respectively, its mean curvature vector field H) vanishes identically.

Let X and Y be smooth vector fields on M; we define a vector field $[X,Y]$ called the Lie bracket of X and Y by setting

$$[X,Y]_x f = X_x Y(f) - Y_x X(f), \quad x \in M.$$

Following [17,23], let M be a manifold of dimension $m = n + k$ and suppose that to each $x \in M$ is assigned an n-dimensional subspace Δ_x of $T_x(M)$. Suppose, moreover, that in a neighborhood U of each $x \in M$, there are n linearly independent C^∞ vector fields $X_1, \ldots, X_n$, which form a basis of Δ_y, for every $y \in U$. Then, we shall say that Δ is a C^∞ *distribution of dimension n on M* and $X_1, \ldots, X_n$ is a local basis of Δ. We shall say that the distribution Δ is *involutive* if there exists a local basis $X_1, \ldots, X_n$ in a neighborhood of each point such that

$$[X_i, X_j] = \Sigma_{i=1}^{n} c_{i,j}^{k} X_k, \quad 1 \leq I, \quad j \leq n,$$

where $c_{i,j}^{k}$ is in general a C^∞ function on a neighborhood and in general not a constant.

If Δ is a C^∞ distribution on M, N is a connected C^∞ manifold, and $i : N \to M$ is a one-to-one immersion such that for each $y \in N$ we have $di(T_y(N)) \subset \Delta_{i(y)}$, then we shall say that the immersed submanifold is an integral manifold of Δ.

We shall say that Δ is *completely integrable* if each point $x \in M$ has a coordinate neighborhood U, ϕ such that if $x^1, \ldots, x^m$ denote the local coordinates; then, the n vectors $E_i = d\phi_i^{-1} \frac{\partial}{\partial x^i}, i = 1, \ldots, n$, are a local basis on U for Δ.

Theorem 1 (FROBENIUS). *A distribution Δ on a manifold M is completely integrable if and only if it is involutive.*

Theorem 2 (HODGE). *Let M be a compact, oriented Riemannian manifold. Then, the space $\mathcal{H}^q$ of harmonic forms of degree q has finite dimension and is isomorphic to the cohomology space of dimension q. Moreover, the Hodge duality operator $* : \mathcal{H}^q \to \mathcal{H}^{n-q}$ is an isomorphism. In particular, $\mathcal{H}^q$ and $\mathcal{H}^{n-q}$ are dual finite dimensional spaces (Poincaré duality) and* $\dim \mathcal{H}^q = \dim \mathcal{H}^{n-q}$.

Furthermore, every cohomology class on M contains a unique harmonic representative and the q-th cohomology group. H^q is isomorphic to the vector space $\mathcal{H}^{n-q}$ of harmonic $n - q$-forms on M.

2.2. Energy and Harmonic Maps

Let u: $(M, g_M) \to (N, g_N)$ be a differential map between two Riemannian manifolds M and N. Denote with e_u the *energy density* of u, which is given by (see, e.g., [20,24–26])

$$e_u = \frac{1}{2} \sum_{i=1}^{m} g_N\big(du(e_i), du(e_i)\big) = \frac{1}{2}|du|^2, \tag{5}$$

where $\{e_1, \cdots, e_n\}$ is a local orthonormal frame field on M and $|du|$ is the Hilbert–Schmidt norm of du, determined by the metric g_M of M and the metric g_N of N. *The energy of u*, denoted by $E(u)$, is defined to be as follows

$$E(u) = \int_M e_u \, dv_g.$$

A smooth map u: $M \to N$ is called *harmonic* if u is a critical point of the energy functional E with respect to any compactly supported variation.

2.3. Riemannian Submersions

The concept of Riemannian submersions was independently developed by B. O'Neill [27] and A. Gray [28]. A differential map u: $(M, g_M) \to (N, g_N)$ between two Riemannian manifolds is called a submersion at a point $x \in M$ if its differential

$$du_x \colon T_x(M) \to T_{u(x)}(N)$$

is a surjective linear map. A differentiable map u that is a submersion at each point $x \in M$ is called a *submersion*. For each point $x \in N$, $u^{-1}(x)$ is called a *fiber*. For a submersion $u\colon M \to N$, let $\mathcal{H}_x$ denote the orthogonal complement of Kernal $\big(du_x\colon T_x(M) \to T_{u(x)}(N)\big)$ in $T_x(M)$ at $x \in M$.

Let $\mathcal{H} = \{\mathcal{H}_x : x \in M\}$ denote the horizontal distribution of u. A submersion $u\colon (M, g_M) \to (N, g_N)$ is called *horizontally weakly conformal* if the restriction of du_x to $\mathcal{H}_x$ is conformal, i.e., there exists a smooth function λ on M such that

$$u^* g_N = \lambda^2 g_{M|_{\mathcal{H}}} \quad \text{or} \quad g_N\big(du_x(X), du_x(Y)\big) = \lambda^2(x) g_M(X, Y) \tag{6}$$

for all $X, Y \in \mathcal{H}_x$ and $x \in M$. If the function λ in (6) is positive, then u is called *horizontally conformal* and λ is called the *dilation* of u.

For a horizontally conformal submersion u with dilation λ, the *energy density* is $e_u = \frac{1}{2} n \lambda^2$. A horizontally conformal submersion with dilation $\lambda \equiv 1$ is called a *Riemannian submersion*.

Let $\pi : M \to B$ be a Riemannian submersion. We choose an orthonormal basis $e_1, \ldots, e_b, e_{b+1}, \ldots, e_n$ of $T_p M$, $p \in M$, such that $e_1, \ldots, e_b$ are horizontal and $e_{b+1}, \ldots, e_n$ are vertical. The first author defined an invariant in [22], nowadays known as a *submersion δ-invariant or (Chen invariant)*, by

$$\check{A}_\pi(p) = \sum_{i=1}^{b} \sum_{r=b+1}^{n} \langle A_{e_i} e_r, A_{e_i} e_r \rangle, \tag{7}$$

where A is O'Neill's integrability tensor (cf., [17,18]). It is direct to verify that $\check{A}_\pi(p)$ is well-defined invariant of the Riemannian submersion $\pi : M \to B$.

The following optimal inequality involving $\check{A}_\pi$ was established by the first author in [17,22].

Theorem 3. *Let $\pi : M \to B$ be a Riemannian submersion with totally geodesic fibers. Then, for an isometric immersion ψ of M into a Riemannian manifold $\tilde{M}$, we have the following*

$$\check{A}_\pi \leq \frac{n^2}{4} \|H\|^2 + b(n-b) \max \tilde{K}, \tag{8}$$

where $\|H\|^2$ denotes the squared mean curvature of M in $\tilde{M}$ and $\max \tilde{K}(p)$ is the maximum value of the sectional curvature of $\tilde{M}$ restricted to plane sections in $T_p M \subset T_{\psi(p)} \tilde{M}$, $p \in M$.

Let $\mathcal{K}$ be a distribution of a Riemannian manifold N and let $\mathcal{K}^\perp$ denote the orthogonal complementary distribution of $\mathcal{K}$. Put the following

$$\mathring{h}(X, Y) = (\nabla_X Y)^\perp \tag{9}$$

for vector fields X, Y in $\mathcal{K}$, where $(\nabla_X Y)^\perp$ is the $\mathcal{K}^\perp$-component of $\nabla_X Y$ on N. Then, $\mathring{h}$ is a well-defined $\mathcal{K}^\perp$-valued $(0,2)$-tensor field. Moreover, it follows from Frobenius' theorem that $\mathcal{K}$ is integrable if and only if $\mathring{h}$ is symmetric.

Let $e_1, \ldots, e_k$ be an orthonormal basis of $\mathcal{K}$. If we put the following

$$\mathring{H} = \frac{1}{k} \operatorname{trace} \mathring{h} = \frac{1}{k} \sum_{j=1}^{k} \mathring{h}(e_j, e_j), \tag{10}$$

then, up to sign, $\mathring{H}$ is a well-defined vector field, which is called the *mean curvature vector of* $\mathcal{K}$. If $\mathring{H} = 0$, $\mathcal{K}$ is called a *minimal distribution*. In particular, if $\mathring{h} = 0$, then $\mathcal{K}$ is called a *totally geodesic distribution* (see [17]).

For Riemannian submersions, we also have the following results.

Theorem 4 ([22], Theorem 2). *Let $\pi : (M, g_M) \to (B, g_B)$ be a Riemannian submersion from a closed manifold M onto an orientable base manifold B. Then, the pullback of the volume element of B is harmonic if and only if the horizontal distribution $\mathcal{H}$ is integrable and fibers are minimal.*

Theorem 5 ([16], Theorem 2.5). *Let $\pi : (M, g_M) \to (B, g_B)$ be a Riemannian submersion. Then, π is a p-harmonic map for every $p > 1$ if and only if all fibers $\pi^{-1}(y), y \in B$, are minimal submanifolds in M.*

2.4. Harmonic Morphisms

A C^2–map

$$u\colon (M, g_M) \to (N, g_N)$$

is said to be a *harmonic morphism* if for any harmonic function f defined on an open set V of N, the composition $f \circ u$ is harmonic on $u^{-1}(V)$.

P. Baird and J. Eells used the stress-energy tensor to establish the following.

Theorem 6 ([29]). *Let $u\colon (M, g_M) \to (N, g_N)$ be a harmonic morphism which is a submersion everywhere on M. Then, $\big($setting $n = \dim N$, and e_u as in (5)$\big)$*

(a) *If $n = 2$, the fibers are minimal submanifolds;*
(b) *If $n > 2$, then the following properties are equivalent:*
   (i) *The fibers are minimal submanifolds;*
   (ii) *$\operatorname{grad}(e_u)$ is a vertical field;*
   (iii) *The horizontal distribution has mean curvature vector $\frac{\operatorname{grad}(e_u)}{2e_u}$.*

2.5. p-Harmonic Morphisms

A C^2–map $u : (M, g_M) \to (N, g_N)$ is said to be a *p-harmonic morphism* if for any p-harmonic function f defined on an open set V of N, the composition $f \circ u$ is p-harmonic on $u^{-1}(V)$. We have the following link between p-harmonic morphisms for every $p > 1$ and minimal fibers.

Theorem 7 ([16], Proposition 2.4). *If $\pi : (M, g_M) \to (N, g_N)$ is a Riemannian submersion, then π is a p-harmonic morphism for every $p > 1$ if and only if all fibers $\pi^{-1}(y)$, $y \in N$, are minimal submanifolds of M.*

E. Loubeau [30] and J. M. Burel and E. Loubeau [31] obtained the following characterization of p-harmonic morphisms.

Theorem 8 ([30,31]). *A C^2–map $u : (M, g_M) \to (N, g_N)$ is a p-harmonic morphism with $p \in (1, \infty)$ if and only if it is a p-harmonic, horizontally conformal map.*

Theorem 7 ([16], Proposition 2.4) links p-harmonic maps for every $p > 1$ with minimal fibers in the presence of Riemannian submersion. This is dual to a minimal submanifold occurs from a p-harmonic map for every $p > 1$ in the presence of isometric immersion (cf., [32]).

In [33], P. Baird and S. Gudmundsson linked n-harmonic morphisms with minimal fibers as follows.

Theorem 9 ([33]). *If $u : (M, g_M) \to (N, g_N)$ is a horizontally conformal submersion from a manifold M onto a manifold N with $n = \dim N$, then u is n-harmonic if and only if the fibers of u are minimal in M.*

2.6. F-Harmonic Maps

Let F be as in (14) and let $u : (M, g_M) \to (N, g_N)$ be a smooth map between two compact Riemannian manifolds. Then, the map $u : M \to N$ is called *F-harmonic* if it is a critical point of the F-energy functional: $E_F(u) = \int_M F\Big(\frac{|du|^2}{2}\Big) dv_g$. In particular, when

$$F(t) = t, \ \ \frac{1}{p}(2t)^{\frac{p}{2}}, \ \ (1+2t)^{\alpha} - 1 \ \ (\alpha > 1, \dim M = 2),$$
$$e^t - 1, \ \ \sqrt{1+2t} - 1, \ \ 1 - \sqrt{1-2t}, \ \ \text{, respectively,}$$

the F-energy $E_F(u)$ becomes energy, p-energy, (normalized) α-energy, (normalized) exponential energy, (normalized) area functional in Euclidean space, and (normalized) area functional in Minkowski space. Hence, its critical point u or its graph is harmonic, p-harmonic, α-harmonic, exponential harmonic, minimal hypersurface in Euclidean space $\mathbb{R}^{n+1}$, and maximal spacelike hypersurface in Minkowski space $\mathbb{R}^{n+1}_1$, respectively.

For horizontally conformal F-harmonic maps, M. Ara proved the following.

Theorem 10 (cf., [34]). *Let $u : (M, g_M) \to (N, g_N)$, $m > n$, be an F-harmonic map, which is horizontally conformal with dilation λ. Assume that the zeros of $(n-2)F'(t) - 2tF''(t)$ are isolated. Then, the following three properties are equivalent:*

(1) *The fibers of u are minimal submanifolds;*
(2) $\operatorname{grad}(\lambda^2)$ *is vertical;*
(3) *The horizontally distribution of u has mean curvature vector* $\frac{\operatorname{grad}(\lambda^2)}{2\lambda^2}$.

2.7. f-Harmonic Maps

Let $f : (M, g) \to (0, \infty)$ be a smooth function. The notion of f-harmonic maps was first introduced and studied by A. Lichnerowicz in 1970 (see [35]).

A map

$$u : (M, g_M) \to (N, g_N)$$

between two Riemannian manifolds is said to be *f-harmonic*, if u is a critical point of the *f-energy functional* E_f with respect to any compactly supported variation (cf., [21,35]), where

$$E_f(u) = \frac{1}{2} \int_M f |du|^2 \, dv_g \, . \tag{11}$$

Examples of f-harmonic maps include harmonic maps with $f \equiv C$, a positive constant, and *submersive p-harmonic maps* with $f = |du|^{p-2}$.

The next result is due to Y.-L. Ou.

Theorem 11 ([36]). *Let $u : (M, g_M) \to (N, g_N)$ be a horizontally weakly conformal map with dilation λ. Then, any two of the following conditions imply the other one:*

(a) *u is an f-harmonic map;*
(b) $\operatorname{grad}(f\lambda^{2-n})$ *is vertical;*
(c) *u has minimal fibers.*

3. Some Duality Results in Mathematics

In the study of p-harmonic maps between Riemannian manifolds, we note that the following duality occurs between (A) conformal immersions (see Theorem 12) and (B) horizontally conformal submersions (see Theorem 13).

Theorem 12 ([37]).

(A) *Let $u : (M, g_M) \to (N, g_N)$ be a conformal immersion with $\dim M = p$. Then, u is a p-harmonic map if and only if $u(M)$ is a (parametric) minimal submanifold in N.*

Theorem 13 ([33]).

(B) *Let $u : (M, g_M) \to (N, g_N)$ be a horizontally conformal submersion with $\dim N = p$. Then, u is a p-harmonic map if and only if all the fibers of u are (parametric) minimal submanifolds in M.*

This duality between (A) and (B) links p-harmonic maps and (parametric) minimal submanifolds.

On the other hand, there also exists a duality between (C) (nonparametric) minimal hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ and (D) maximal spacelike (nonparametric) hypersurface in the Minkowski space $\mathbb{R}^{n,1}$ equipped with the Lorentzian metric

$$ds^2 = \sum_{i=1}^{n} (dx^i)^2 - dt^2,$$

where

(C) The minimal hypersurface in the Euclidean space $\mathbb{R}^{n+1}$ is given by the graph of a function u on a Euclidean domain satisfying the differential equation

$$\operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0; \tag{12}$$

(D) The maximal spacelike hypersurface in the Minkowski space $\mathbb{R}^{n,1}$ is furnished by the graph of a function v on a Euclidean domain satisfying a dual differential equation

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) = 0, \quad |\nabla v|^2 < 1. \tag{13}$$

E. Calabi showed in [38] that Equations (12) and (13) are equivalent over any simply connected domain in $\mathbb{R}^2$. Furthermore, the notion of F-harmonic maps unifies minimal hypersurfaces and maximal spacelike hypersurfaces; namely, the solutions u and v are F-harmonic maps from a domain in $\mathbb{R}^{n+1}$ to $\mathbb{R}$ with

$$F(t) = \sqrt{1+2t} - 1 \quad \text{and} \quad F(t) = 1 - \sqrt{1-2t},$$

respectively (see [16,39]). Hence, one can unify harmonic maps, p-harmonic maps, α-harmonic maps, exponential harmonic maps, (nonparametric) minimal hypersurfaces in Euclidean space, and (nonparametric) maximal spacelike hypersurface in Minkowski space $\mathbb{R}^{n,1}$ by F-harmonic maps (see [16,34]).

Also, the "F"-idea can be extended to Gauge Theory (see [39]). Here,

$$F : [0, \infty) \to [0, \infty) \quad \text{with} \quad F(0) = 0 \quad \text{is } C^2 \text{ strictly increasing}. \tag{14}$$

We recall that in the study of topology on a compact Riemannian manifold M, it is well-known that nontrivial fundamental groups $\pi_1(M)$, homology groups, and cohomology classes can be represented by stable closed geodesics, stable minimal rectifiable currents, and harmonic forms on M, respectively, by Cartan's Theorem ([40]), Federer–Fleming's Theorem ([41,42]), and the Hodge Theorem ([43]).

In an analogous spirit, S. W. Wei shown in [32] that homotopy classes can be represented by p-harmonic maps (for a definition and examples of p-harmonic maps, see, e.g., [16]):

Theorem 14. *If N is a compact Riemannian manifold, then for any positive integer i, each class in the i-th homotopy group $\pi_i(N)$ can be represented by a $C^{1,\alpha}$ p-harmonic map u_0 from an i-dimensional sphere S^i into N minimizing p-energy in its homotopy class for any $p > i$.*

Further applications and homotopically vanishing theorems were explored in [32,44], using stable p-harmonic maps as catalysts, whereas a homologically vanishing theorem was given in [6,45,46], using stable rectifiable currents as catalysts.

On the other hand, B.-Y. Chen established in [22] the following result, involving Riemannian submersion, minimal immersion, and cohomology class.

Theorem 15 ([22])**.** *Let $b = \dim B$ and let $\pi : (M, g_M) \to (B, g_B)$ be a Riemannian submersion with minimal fibers and orientable base manifold B. If M is a closed manifold with cohomology class $H^b(M, \mathbf{R}) = 0$, then the horizontal distribution $\mathcal{H}$ of the Riemannian submersion is never integrable. Thus, the submersion π is never nontrivial.*

Whereas p-harmonic maps represent homotopy classes, B.-Y. Chen and S. W. Wei connected the two seemingly unrelated areas of p-harmonic morphisms and cohomology classes in the following.

Theorem 16 ([15,47])**.** *Let $u : (M, g_M) \to (N, g_N)$ be an n-harmonic morphism which is a submersion with $\dim N = n$. If N is orientable and M is a closed manifold with n-th cohomology class $H^n(M, \mathbf{R}) = 0$, then the horizontal distribution $\mathcal{H}$ of u is never integrable. Henc, the submersion u is always nontrivial.*

This recaptures Theorem 15 when $\pi : M \to B$ is a Riemannian submersion with minimal fibers and orientable base manifold B. While a horizontally weak conformal p-harmonic map is a p-harmonic morphism (see Theorem 8), p-harmonic morphism is also linked to cohomology class as follows.

Theorem 17 ([15,47])**.** *Let $u : (M, g_M) \to (N, g_N)$ be an n-harmonic morphism with $n = \dim N$, which is a submersion. Then, the pull back of the volume element of N is a harmonic n-form if and only if the horizontal distribution $\mathcal{H}$ of u is completely integrable.*

Following the proofs given in [15,47], and by applying a characterization theorem of a p-harmonic morphism (see Theorems 8 and 9), we obtain a dual version of Theorem 17. In particular, we have determined that p-harmonic maps and cohomology classes are interrelated in [16] as follows.

Theorem 18. *Let M be a closed manifold and $u : M \to N$ be an n-harmonic map with $n = \dim N$, which is a submersion. Assume that the horizontal distribution $\mathcal{H}$ of u is integrable and u is an n-harmonic morphism. Then, we have $H^n(M, \mathbf{R}) \neq 0$.*

Theorem 19 ([16]). *Let $u : M \to N$ be an n-harmonic map with $n = \dim N$, which is a submersion. Let the horizontal distribution $\mathcal{H}$ of u be integrable. If M is a closed manifold with cohomology class $H^n(M, \mathbf{R}) = 0$. Then, u is not an n-harmonic morphism. Thus, the submersion u is always nontrivial.*

The next result is also well-known.

Theorem 20 ([48]). *A Riemannian submersion $\pi : (M, g) \to (B, g_B)$ is harmonic if and only if each fiber of π is a minimal submanifold.*

4. Statements of Theorems of This Paper

In the following, we shall assume that $\dim M = m > n = \dim N$. The purpose of this article is to extend the ideas in [15,16] and to connect the seemingly unrelated areas of F-harmonic maps, f-harmonic maps, and cohomology classes. More precisely we prove the following.

Theorem 21. *Let $u : M \to N$ be an F-harmonic map which is a horizontally conformal submersion with dilation λ. Furthermore, let* grad λ^2 *be vertical, and the zeros of $(n-2)F'(t) - 2tF''(t)$ be isolated. If $H^n(M, \mathbf{R}) = 0$, then the horizontal distribution $\mathcal{H}$ of u is not integrable. Thus, the submersion is always nontrivial.*

Remark 1. *(i) Theorem 21 generalizes Theorem 15, in which assumption u is a horizontal conformal submersion with dilation $\lambda = 1$, and u is a p-harmonic map for every $p > 1$ according to Theorem 7. If we choose $p \neq n$, then* grad $\lambda^2 = 0$, *$F(t) = \frac{1}{p}(2t)^{\frac{p}{2}}$ so that F-harmonic map becomes p-harmonic map, $p \neq n$, and the zeros of $(n-2)F'(t) - 2tF''(t)$ are isolated.*

(ii) Theorem augments Theorem 16 but does not generalize Theorem 16, since when $F(t) = \frac{1}{n}(2t)^{\frac{n}{2}}$, F-harmonic map u becomes n-harmonic map, and the zeros of $(n-2)F'(t) - 2tF''(t)$ are not isolated. Thus, n-harmonic maps are not applicable to Theorem 21.

(iii) Theorem 21 is equivalent to the following Theorem 22, due to the equivalence in Theorem 10.

Theorem 22. *Let $u : M \to N$ be an F-harmonic map, which is a horizontally conformal submersion with dilation λ. Assume that the horizontal distribution has mean curvature vector $\frac{\text{grad}(\lambda^2)}{2\lambda^2}$, and the zeros of $(n-2)F'(t) - 2tF''(t)$ are isolated. If $H^n(M, \mathbf{R}) = 0$, then the horizontal distribution $\mathcal{H}$ of u is not integrable. Thus, the submersion is always nontrivial.*

Theorem 23. *Let $u : M \to N$ be an F-harmonic map which is a horizontally conformal submersion with dilation λ. Assume that the horizontal distribution has mean curvature vector $\frac{\text{grad}(\lambda^2)}{2\lambda^2}$, and the zeros of $(n-2)F'(t) - 2tF''(t)$ are isolated. Then, the pullback of the volume element of N is a harmonic n-form if and only if the horizontal distribution $\mathcal{H}$ of u is completely integrable.*

Remark 2. *By similar arguments as in Remark 1, we hav the following:*

(i) Theorem 22 generalizes Theorem 4;

(ii) Theorem 22 augments Theorem 17 but does not generalize Theorem 17;

(iii) Theorem 22 is equivalent to the following Theorem 24.

Theorem 24. *Let $u : M \to N$ be an F-harmonic map which is a horizontally conformal submersion with dilation λ. Assume that* grad λ^2 *is vertical, and the zeros of $(n-2)F'(t) - 2tF''(t)$ are isolated. Then, the pullback of the volume element of N is a harmonic n-form if and only if the horizontal distribution $\mathcal{H}$ of u is completely integrable.*

Now, let us study f-harmonic maps, which were first introduced and studied by A. Lichnerowicz in 1970 (see [35]), where $f : (M, g) \to (0, \infty)$ is a smooth function.

Examples of f-harmonic maps include harmonic maps with $f \equiv C$, a positive constant, and *submersive p-harmonic maps* with $f = |du|^{p-2}$ (cf., Section 2). We have the following results.

Theorem 25. *Let $u : M \to N$ be an f-harmonic map which is a horizontally conformal submersion with dilation λ such that* $\operatorname{grad}(f\lambda^{2-n})$ *is vertical. If M is a closed manifold with cohomology class $H^n(M, \mathbf{R}) = 0$, then the horizontal distribution $\mathcal{H}$ of u is never integrable. Thus, the submersion u is always nontrivial.*

This recaptures Theorem 15, when f-harmonic map is an n-harmonic map, then $f = |du|^{n-2}$.

Theorem 26. *Let $u : M \to N$ be an f-harmonic map which is a horizontally conformal submersion and with dilation λ such that* $\operatorname{grad}(f\lambda^{2-n})$ *is vertical. Then, the pullback of the volume element of N is a harmonic n-form if and only if horizontal distribution is completely integrable.*

This recaptures Theorem 17, when f-harmonic map is an n-harmonic map.

5. Proofs of Theorems 21–26

Proof of Theorem 23. Let $u : (M, g_M) \to (N, g_N)$ be an F-harmonic map, which is a horizontally conformal submersion with dilation λ and $n = \dim N$. □

Assume that the horizontal distribution has mean curvature vector $\frac{\operatorname{grad}(\lambda^2)}{2\lambda^2}$, and the zeros of

$$(n-2)F'(t) - 2tF''(t)$$

are isolated. Then, the submersion u has minimal fibers according to Theorem 10.

Let $\{\bar{e}_1, \ldots, \bar{e}_n\}$ be an oriented local orthonormal frame of the base manifold (N, g_N) and let $\bar{\omega}^1, \ldots, \bar{\omega}^n$ denote the dual 1-forms of $\{\bar{e}_1, \ldots, \bar{e}_n\}$ on N. Then,

$$\bar{\omega} = \bar{\omega}^1 \wedge \cdots \bar{\omega}^n$$

is the volume form of (N, g_N), which is a closed n-form on N.

Consider the pull back of the volume form $\bar{\omega}$ of N via u, which is denoted by $u^*(\bar{\omega})$. Then, $u^*(\bar{\omega})$ is a simple n-form on M satisfying the following

$$d\big(u^*(\bar{\omega})\big) = u^*(d\bar{\omega}) = 0, \tag{15}$$

since the exterior differentiation d and the pullback u^* commute.

Assume that $\dim M = m = n + k$ and let $e_1, \ldots, e_{n+k}$ be a local orthonormal frame field with $\omega^1, \ldots, \omega^{n+k}$ being its dual coframe fields on M such that:

(i) $e_1, \ldots, e_n$ are basic horizontal vector fields satisfying $du(e_i) = \lambda \bar{e}_i$, $i = 1, \ldots, n$, and $du(e_1), \ldots, du(e_n)$ give a positive orientation of N and
(ii) $e_{n+1}, \ldots, e_{n+k}$ are vertical vector fields.

Then, we have

$$\omega^j(e_s) = 0, \quad \omega^i(e_j) = \delta_{ij}, \ \ 1 \le i, j \le n; \ n+1 \le s \le n+k\,. \tag{16}$$

Also, it follows from (i) that

$$u^*\bar{\omega}^i = \frac{1}{\lambda}\omega^i, \quad i = 1, \dots, n. \tag{17}$$

If we put the following

$$\omega = \omega^1 \wedge \cdots \wedge \omega^n \quad \text{and} \quad \omega^\perp = \omega^{n+1} \wedge \cdots \wedge \omega^{n+k}, \tag{18}$$

then

$$d\omega^\perp = \sum_{i=1}^{k} (-1)^i \omega^{n+1} \wedge \cdots \wedge d\omega^{n+i} \wedge \cdots \wedge \omega^{n+k}. \tag{19}$$

It follows from (16) and (19) that $d\omega^\perp = 0$ holds identically if and only if the following two conditions are satisfied:

$$d\omega^\perp(e_i, e_{n+1}, \dots, e_{n+k}) = 0 \tag{20}$$

for $i = 1, \dots, n$; and

$$d\omega^\perp(X, Y, V_1, \dots, V_{k-1}) = 0. \tag{21}$$

for any horizontal vector fields X, Y and vertical vector fields $V_1, \dots, V_{k-1}$.

Since the fibers of u are minimal in M, for each $1 \leq i \leq n$, we find the following

$$\begin{aligned} & d\omega^\perp(e_i, e_{n+1}, \dots, e_{n+k}) \\ & \quad = \sum_{j=1}^{k} (-1)^{j+1} \omega^\perp([e_i, e_{n+j}], e_{n+1}, \dots, \hat{e}_{n+j}, \dots, e_{n+k}) \\ & \quad = \sum_{j=1}^{k} (-1)^{j+1} \left(\omega^{n+j}(\nabla_{e_i} e_{n+j}) - \omega^{n+j}(\nabla_{e_{n+j}} e_i) \right) \\ & \quad = \sum_{j=1}^{k} -\langle \nabla_{e_{n+j}} e_i, e_{n+j} \rangle \\ & \quad = \sum_{j=1}^{k} \langle h(e_{n+j}, e_{n+j}), e_i \rangle \\ & \quad = 0, \end{aligned} \tag{22}$$

where "$\hat{\ }$" denotes the missing term, which proves that condition (20) holds.

Next, we suppose that the horizontal distribution $\mathcal{H}$ is integrable. If X, Y are horizontal vector fields, then $[X, Y]$ is also horizontal by Frobenius theorem. So, for vertical vector fields $V_1, \dots, V_k$, we find (cf., [22], Formula (6.7) or [16], Formula (3.5))

$$\begin{aligned} d\omega^\perp(X, Y, V_1, \dots, V_{k-1}) &= \omega^\perp([X, Y], V_1, \dots, V_{k-1}) \\ &= 0. \end{aligned} \tag{23}$$

Consequently, from (22) and (23), we get the following

$$d\omega^\perp = 0. \tag{24}$$

Next, we show that if $\mathcal{H}$ is integrable, then we have the following

$$d(u^*\bar{\omega})^\perp = 0\,.$$

Since u is a horizontally conformal submersion with dilation λ, it preserves orthogonality, which is crucial to horizontal and vertical distributions, and the pullback u^* expands the length of 1-form by $\frac{1}{\lambda}$ in every direction. This, via (17) and (24), leads to

$$\begin{aligned} d(u^*\bar{\omega})^{\perp} &= d(u^*\bar{\omega}^1 \wedge \cdots \wedge u^*\bar{\omega}^n)^{\perp} \\ &= d\left(\frac{1}{\lambda}\omega^1 \wedge \cdots \wedge \frac{1}{\lambda}\omega^n\right)^{\perp} \\ &= d\left(\frac{1}{\lambda}\omega^{n+1} \wedge \cdots \wedge \frac{1}{\lambda}\omega^{n+k}\right) \\ &= d\left(\frac{1}{\lambda^k}\omega^{\perp}\right) \\ &= -\frac{k}{2}\lambda^{-k-2}d(\lambda^2) \wedge \omega^{\perp} + \frac{1}{\lambda^k}d\omega^{\perp} \\ &= -\frac{k}{2}\lambda^{-k-2}d(\lambda^2) \wedge \omega^{\perp} \\ &= 0. \end{aligned} \tag{25}$$

The last equality of (25) is due to the fact that $\operatorname{grad}(\lambda^2)$ is vertical and Theorem 10. Therefore, $\operatorname{grad}(\lambda^2), e_{n+1}, \cdots, e_{n+k}$ are linearly dependent. Hence, we obtain the following

$$d(\lambda^2) \wedge \omega^{\perp} = 0.$$

Because $d(u^*\omega)^{\perp} = 0$ is equivalent to $u^*\omega$ being co-closed, it follows from $d(u^*\omega) = 0$ and $d(u^*\omega)^{\perp} = 0$ that the pullback of the volume form, $u^*\omega$, is a harmonic n-form on M whenever $\mathcal{H}$ is completely integrable.

Conversely, it follows from the proof given above that if $u^*\omega$ is a harmonic n-form on M, then the horizontal distribution $\mathcal{H}$ is completely integrable. This proves Theorem 3.

Proof of Theorem 21. Since each nonzero harmonic form represents a nontrivial cohomology class by Hodge theorem [43], and since on a closed manifold a differential form is harmonic if and only if it is closed and co-closed, Theorem 21 follows from Theorem 23. □

Proof of Theorems 22 and 24. Theorems 22 and 24 follow from Theorem 21 together with Remarks 1 and 2, respectively. □

Proof of Theorems 25 and 26. By virtue of Theorem 11, the submersion u in Theorems 5 and 6 has minimal fibers; thus, we proceed as before. Thus u is an n-harmonic map by virtue of Theorem 9, and hence, u is an n-harmonic morphism according to Theorem 8.

We are then ready to apply Theorem 16 and Theorem 17 to complete the proof of Theorem 5 and Theorem 6, respectively. □

6. Epilogue

In physics, fundamental dual systems contain two opposite elements, and the two opposite elements form a fundamental dual pair. One element is dual to the other, and vice versa. Such duality can be seen as a fundamental property in nature and it appears everywhere in our universe.

In mathematics, a duality translates theorems, mathematical structures, or concepts, into other theorems, structures, or concepts in a one-to-one fashion. In n-dimensional geometry a fundamental notion is the "duality" between chains and cochains, or domains of integration and the integrands.

In this paper, by extending the ideas in our earlier papers [15,16] and by studying cohomology classes, we establish several relations between Riemannian submersions and F-harmonic maps or f-harmonic maps. It seems to be natural to us to search for some

relationships between p-harmonic morphisms, F-harmonic maps, or f-harmonic maps with Riemannian maps instead of Riemannian submersions.

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