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Rogue Waves in the Nonlinear Schrödinger, Kadomtsev–Petviashvili, Lakshmanan–Porsezian–Daniel and Hirota Equations

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Abstract: We give some of our results over the past few years about rogue waves concerning some partial differential equations, such as the focusing nonlinear Schrödinger equation (NLS), the Kadomtsev–Petviashvili equation (KPI), the Lakshmanan–Porsezian–Daniel equation (LPD) and the Hirota equation (H). For the NLS and KP equations, we give different types of representations of the solutions, in terms of Fredholm determinants, Wronskians and degenerate determinants of order $2N$. These solutions are called solutions of order N. In the case of the NLS equation, the solutions, explicitly constructed, appear as deformations of the Peregrine breathers P_N as the last one can be obtained when all parameters are equal to zero. At order N, these solutions are the product of a ratio of two polynomials of degree $N(N+1)$ in x and t by an exponential depending on time t and depending on $2N-2$ real parameters: they are called quasi-rational solutions. For the KPI equation, we explicitly obtain solutions at order N depending on $2N-2$ real parameters. We present different examples of rogue waves for the LPD and Hirota equations.

Keywords: Fredholm determinants; Wronskians; Peregrine breathers; lumps

MSC: 35C99; 35Q35; 35Q53

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Academic Editors: Yuting Ding, Feng Li and Patricia J. Y. Wong

Received: 10 December 2024
Revised: 8 January 2025
Accepted: 17 January 2025
Published: 27 January 2025

Citation: Gaillard, P. Rogue Waves in the Nonlinear Schrödinger, Kadomtsev–Petviashvili, Lakshmanan–Porsezian–Daniel and Hirota Equations. *Axioms* **2025**, *14*, 94. https://doi.org/10.3390/axioms14020094

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1. Introduction

This study concerns results obtained about rogue waves in some partial differential equations of mathematical physics.

Different types of representations of the solutions in terms of Fredholm determinants, Wronskians and degenerate determinants of order N are given.

The most important case is the case of the NLS equation, where a complete study has been realized and the explicit expressions of the solutions until order 13 depending on 24 real parameters have been constructed. It has to be stressed that these solutions can be seen as deformations of the Peregrine breather P_N because this last Peregrine breather is obtained when all parameters are chosen as equal to zero.

The term rogue wave (freak wave) was first introduced into the scientific community by Draper in 1964 [1]. The commonly accepted criterion for defining a rogue wave in the ocean is that the vertical distance between the trough and the crest is at least twice as large as the average wave height among one-third of the highest waves observed over a period of 10 to 30 min. The first rogue wave recorded was made on the oil platform of Draupner in 1995, in the North Sea between Norway and Scotland.

This phenomenon is far from being isolated, as evidenced by studies of the National Oceanic and Atmospheric Administration. In 2007, a list of 50 maritime incidents due to rogue waves was given.The phenomenon known as the "Three Sisters", in which three

large waves form at the same time, has been observed, as predicted in the case of the model of the NLS equation.

Rogue waves have caused many disasters, which is one of the reasons why the scientific community is taking a close look at this phenomenon in order to better understand its mechanism.

The subject of rogue waves far exceeds the strict framework of studying waves in the ocean and plays a significant role in other areas, such as in nonlinear optics [2], Bose–Einstein condensates [3], meteorology [4] and finance [5].

Different models have been created to take into account these phenomena. In 1968, Zakharov proved that, for certain type of waves, the amplitude of these waves approximately satisfies the following nonlinear equation [6], which is now called the one-dimensional nonlinear focusing Schrödinger equation:

$$iu_t + u_{xx} + 2|u|^2 u = 0. \tag{1}$$

This equation has two types of formulations. The previous one is given in the context of hydrodynamics. It can also be formulated within the framework of nonlinear optics. From the framework of hydrodynamics, we can move to the case of nonlinear optics with the transformations $t \to \frac{\tau}{2}$, $x \to \xi$. In this case, the nonlinear Schrödinger equation takes the form

$$iu_\tau + \frac{1}{2}u_{\xi\xi} + |u|^2 u = 0. \tag{2}$$

In the following, we consider the NLS equation only in the frame of hydrodynamics.

Different approaches were used to solve this NLS equation.

In 1972, Zakharov and Shabat solved this equation [7] using the inverse scattering method.

In particular, a certain type of solution has been given by Zakharov in the Formula (28) of [7] as a ratio of exponential functions of x and t, depending on two arbitrary parameters: ν and η.

$$u(x,t) = -\frac{4\eta\lambda^{*2}a_2^*}{1+|a_2|^2|\lambda|^4}\frac{-2+|\lambda|^4\mu a_2^*}{1+|\mu|^2|\lambda|^4}, \tag{3}$$

with

$$\mu = \frac{2\eta[a_1 + 2(x+1/2\eta)a_2]}{1+|a_2|^2|\lambda|^4}, \quad \lambda = \frac{e^{i\xi x}}{2\eta}, \quad a_1 = a_1(0)(1+8\nu\xi t)e^{4i\xi^2 t}$$
$$a_2 = a_2(0)e^{4i\xi^2 t},$$

ν, η being complex parameters, and ξ a real number.

Other types of solutions were found by Kuznetsov in 1977 [8] and Ma in 1979 [9]. They can be written as the quotient of combinations of elementary trigonometric functions of x and t by an exponential function of x, depending on two parameters a and b. The function u is defined by

$$u(x,t) = \left(1 + \frac{2(1-2a)\cosh(bx) + ib\sinh(bx)}{\sqrt{2a}\cos(2\omega t) - \cosh(bx)}\right)e^{ix}, \tag{4}$$

where $\omega = 2\sqrt{1-2a}$, and $b = \sqrt{8a(1-2a)}$ is a solution of the NLS equation.

These solutions are periodic in time and localized in space.

The function u defined by

$$u(x,t) = \frac{\cos(\omega t - 2i\varphi) - \cosh(\varphi)\cosh(px)}{\cos(\omega t) - \cosh(\varphi)\cosh(px)} e^{2it}, \tag{5}$$

where $\omega = 2\sinh(2\varphi)$ and $p = 2\sinh(\varphi)$, represents a solution of the NLS equation.

The first quasi-rational solution of the NLS equation was found in 1983 by Peregrine (Howell Peregrine (30 December 1938–20 March 2007) integrated Bristol University Mathematics in 1964, where he spent his entire career), performing a passage at the appropriate limit in the solutions previously found. This solution is commonly referred to today under the name of "Peregrine breather"; his expression takes the form (Formula 6.7 of the article [10]), in the frame of hydrodynamics, of a quotient of two polynomials of degree two in x and t by one exponential depending on t,

$$u(x,t) = \left(1 - \frac{4(1+4it)}{1+4x^2+16t^2}\right) e^{2it}. \tag{6}$$

This solution is localized in space and time. It corresponds to an *extreme event coming from nowhere and leaving no trace*.

Other solutions of the NLS equation were published in 1985 by Akhmediev, Eleonski and Kulagin using a system of ordinary differential equations whose associated solutions determine solutions to the NLS equation. They obtained families of solutions expressed in terms of elliptical Jacobi functions *dn* [11].

Akhmediev, Eleonski and Kulagin found another family of NLS solutions in 1986 that tends toward a flat wave when t tends toward infinity [12]. They found these solutions by a substitution technique; the solutions are written as a ratio of combinations of elementary trigonometric functions by a time-dependent exponential.

They also found Peregrine's solution in the context of nonlinear optics, in Formula (10) of the paper [12], by making one of the parameters tend toward 0, i.e., the spatial period to infinity. These solutions are qualitatively different from those built by Ma: they are spatially periodic and are localized in time. It has been possible experimentally to observe solutions of Peregrine as a limit case of Akhmediev breathers [13,14]. It has to be stressed that Peregrine's solution corresponds to the limit case of Ma and Akmediev when one of the parameters goes to 0, that is, when the periods temporally and spatially tend toward infinity, respectively.

Another approach uses the Zakharov–Shabat system and the so-called Darboux transformations. Akhmediev and his collaborators have used this method in a whole series of articles to build higher-order solutions from the years 1986 to 2013 [15–27] until order 4. Numerical studies have made it possible to carry out a detailed classification up to the order 6 [28].

In 1986, Eleonski, Krichever and Kulagin [29] built quasi-rational solutions to the modified NLS equation

$$iu_t + u_{xx} + 2(|u|^2 - M^2)u = 0,$$

for M, an arbitrary real number. These solutions are given as the quotient of two determinants of matrices composed of partial derivatives.

An algebro-geometric approach was created by the Saint-Petersburg school in the early 1970s. The initiators were Dubrovin, Matveev and Novikov, whose founding article [30] has been the starting point of many advances in the resolution of various nonlinear differential equations. Matveev's article [31] perfectly traces the development of this revolutionary new approach and its applications over the last thirty years. The book by Belokolos, Bobenko, Enol'skii, Its and Matveev Ref. [32] provides a broad overview of the different applications of this theory. Another more recent work by Gesztesy and Holden in several volumes [33,34]

also gives applications of this method to many types of nonlinear differential equations. As a reference to the applications of this new approach to integrable systems, we can cite the book of Babelon, Bernard and Talon [35].

This new method has proven to be very powerful for solving a considerable number of partial differential equations.

For the NLS equation, in 1988, Its, Rybin and Salle [36] proposed solutions as a quotient of truncated Riemann theta functions by an exponential depending on time.

2. Rogue Waves in the NLS Equation

This equation is fundamental for describing the origin of rogue waves. In particular, since 2010, a significant amount of work has been carried out in different domains. In optics, the NLS equation is used as a pattern of wave propagation in optical fibers; recent work has been carried out by Solli [2], or Kibler [13]. In hydrodynamics, the NLS equation describes the evolution of the wave envelope; phenomena have been highlighted recently with the experiences of Akhmediev and their collaborators [37].

2.1. New Approaches to Solving the NLS Equation

2.1.1. The Modified NLS Equation

The Wronskian representation

Over the past decade, there has been a renewed interest in the NLS equation. In 2010, V.B. Matveev proposed a new formulation of NLS solutions in [38] as a ratio of Wronskians of order $2N$ for N, an arbitrary integer; these last solutions depend on $2N-2$ parameters.

$$u(x,t) = -\frac{W(\partial_x f_1, \ldots, \partial_x f_{2N})}{B^{2N-1} W(f_1, \ldots, f_{2N})} e^{2iB^2 t}, \tag{7}$$

$W(f_1, \ldots, f_{2N})$ being the Wronskian of the functions $f_1, \ldots, f_{2N}$, the functions $f_j(x,t)$ being expressed as partial derivatives of the functions

$$f(k,x,t) := \frac{\exp(kx + ik^2 t + \Phi(k))}{q_{2N}(k)}$$

$$\begin{aligned}
f_j(x,t) &= \left(\frac{k^2}{k^2+B^2}\partial_k\right)^{2j-1} f(k,x,t)|_{k=B}, \\
f_{N+j}(x,t) &= \left(\frac{k^2}{k^2+B^2}\partial_k\right)^{2j-1} f(k,x,t)|_{k=-B}, \\
q_{2N}(k) &:= \prod_{j=1}^{N}\left(k^2 - \frac{\omega^{2m_j+1}+1}{\omega^{2m_j+1}-1}B^2\right), \\
\omega &:= \exp\left(\frac{i\pi}{2N+1}\right), \quad \Phi(k) := i\sum_{l=1}^{2N}\varphi_l (ik)^l,
\end{aligned}$$

the integers m_j verifying $0 \le m_j \le 2N-1, \quad m_l \ne 2N - m_j$, B and φ_j for $1 \le j \le 2N$ being real numbers.

We obtain the following statement [38]:

Theorem 1. *The functions*

$$s(x,t) = -\frac{W(\partial_x f_1, \ldots, \partial_x f_{2N})}{B^{2N-1} W(f_1, \ldots, f_{2N})} \tag{8}$$

are quasi-rational solutions of the modified NLS equation

$$is_t + s_{xx} + 2(|s|^2 - B^2)s = 0, \tag{9}$$

where $W(f_1, \ldots, f_{2N})$ *is the Wronskian of the functions* $f_1, \ldots, f_{2N}$, *the functions* f_i *being defined by*

$$f_j(x,t) = \left(\frac{k^2}{k^2+B^2}\partial_k\right)^{2j-1} f(k,x,t)|_{k=B}$$
$$f_{n+j}(x,t) = \left(\frac{k^2}{k^2+B^2}\partial_k\right)^{2j-1} f(k,x,t)|_{k=-B}$$

The function f *is defined by*

$$f(k,x,t) := \frac{\exp(kx + ik^2t + \Phi(k))}{q_{2N}(k)},$$

with

$$q_{2N}(k) := \prod_{j=1}^{N}\left(k^2 - \frac{\omega^{2m_j+1}+1}{\omega^{2m_j+1}-1}B^2\right), \quad \omega := \exp\left(\frac{i\pi}{2N+1}\right)$$
$$\Phi(k) := i\sum_{l=1}^{2N}\varphi_l(ik)^l,$$

the integers m_j *verifying*

$$0 \le m_j \le 2N-1, \quad m_l \ne 2N - m_j.$$

Then, the function u *expressed as* $u(x,t) = s(x,t)e^{2iB^2t}$ *is a solution to the NLS equation.*

In this paper, Kadomtsev–Petviashvili-I (KPI) solutions were deduced from NLS solutions by a judicious change in variables found by V.B. Matveev. A study of particular solutions of the KdV equation was also carried out.

In 2013, V.B. Matveev [39] completed the results on the solutions given by means of Wronskians of order $2N$ with asymptotic studies and videos in [39]. This article contains quasi-rational solutions of order 3 that depend on four parameters, as well as those to the order 4 that depend on six parameters. Even if these results only appeared in March 2013, they were the first to have given these solutions; they had already been highlighted and communicated to myself in 2012.

The IST method (in the case of this modified NLS equation)

For this modified NLS equation, it was still solved by the IST (inverse scattering transform) [40,41].

In the paper [40], Bilman and Miller studied this modified NLS equation for $B = 1$. They proposed a modification of the standard (IST) method for this modified NLS equation formulated with non-zero boundary conditions at infinity.

Biondini and Kovacic [41] used the (IST) method for the modified NLS equation with non-zero boundary conditions at infinity.

2.1.2. The Inverse Scattering Method

In 2007, Aktosun, Demantis and van der Mee [42] used a modified form of the (IST) method to construct globally analytic exact solutions to the NLS equation. Solutions are given in a compact form in terms of matrix exponentials.

In 2015, Prinari and Vitale [43] used the (IST) method to solve the initial-value problem for the NLS equation with a one-sided non-zero boundary value.

2.1.3. The Darboux Dressing Approach

The expression "Darboux transformation" was introduced by V.B. Matveev in the framework of the one-dimensional Schrödinger equation, who generalized it for the hier-

archy of partial linear differential equations of any order with matrix- or operator-valued coefficients [44,45].

In 2010, Aktosun and van der Mee [46] analyzed a particular class of integral equations associated with Marchenko equations and Gel'fand–Levitan equations. In particular, they provided the explicit Darboux transformation for the Zakharov–Shabat system and gave a one-parameter family of Darboux transformations for the NLS equation.

Guo, Ling and Liu [47] used in 2012 a generalization of the Darboux dressing method to construct solutions to the NLS equation as a quotient of determinants of order $2N$ [47].

The article [47] in particular proposes the study of solutions to order 3 through images, putting in an evidence ring and triangle structures for the modulus of solution, in the coordinate plane $(x;t)$.

In the paper [48], Wen, Yang and Yan proposed a generalized perturbation of the Darboux transformations for the modified nonlinear Schrödinger (MNLS) equation. They discuss the wave structures of the rogue wave solutions of the modified NLS equation and the dynamical behaviors of these multi-rogue wave solutions.

2.1.4. The Hirota Bilinear Approach

Ohta and Yang published in 2012 an article [49] that proposes solutions to the NLS equation also as a ratio of determinants using the Hirota bilinear method [50] introduced in 1973. They proved that the general N-th order rogue waves contain $N-1$ free irreducible complex parameters. Some rogue waves associated to certain choices of these parameters have been shown.

In 2021, the same author conducted a study of rogue wave patterns for the NLS equation. It has been shown that these waves have clear geometric structures, such as a triangle, pentagon, heptagon and nonagon, with a lower-order rogue wave at its center. These patterns are explicitly determined by the root structures of the Yablonskii–Vorob'ev polynomial hierarchy. This property was already mentioned by Gaillard in 2019 [51]

2.2. The Algebro-Geometric Approach and the Multi-parametric Deformations of the Peregrine Breather Solutions to the NLS Equation

This section summarizes the author's results for the NLS equation. The main result is the construction of multi-parametric solutions of the nonlinear Schrödinger equation as deformations of the Peregrine breather and its different representations.

V.B. Matveev carried out in 2010 a project to study the NLS equation, its quasi-rational solutions and its rogue waves. It was possible to construct in 2010 new representations of solutions of the NLS equation in terms of Wronskians [38] and deduced rational solutions at order N depending on $2N-2$ parameters with the explicit expressions of associated polynomials up to the order $N=5$.

2.2.1. The Choice of Parameters

During the months of July and August 2010, quasi-rational solutions in the case of determinants of order 2 were built and known results from Akhmediev, Matveev and others were recovered. A new representation of the solutions to the NLS equation was given in terms of Wronskians of order $2N$ different from the one previously given in 2010. The way was thus opened to the construction of solutions of higher orders.

By comparing the computation requested by the method of V.B. Matveev and the last one, there was no ambiguity: the last one proposed was much faster. So, it was possible to build quasi-rational solutions of order 3, 4, 5, 6 and 7 from July 2011 to November 2013. The solutions built were written as functions whose arguments depended on quantities κ_j, δ_j, γ_j and $x_{r,j}$, all depending on a parameter λ_j, as well as a parameter $e_j = ia_j + b_j$, for j between 1 and $2N$. In the article [52], the parameter λ_j was written as $\lambda_j = 1 - 2\epsilon^2 d_j^2$,

and e_j as $e_j = i\tilde{a}_j\epsilon^{2N-1} \pm \tilde{b}_j\epsilon^{2N-1}$, $2N$ being the order of determinant. V.B. Matveev was the first to indicate that the term d_j was inessential and therefore could be deleted from the expression of the solution.

Another fundamental remark on his part was to state that the quantities $e_j = \tilde{a}_j\epsilon^{2N-1}$, for j varying from 1 to $2N$, could be reduced to only two quantities. He demonstrated that the solutions obtained were the same as those cited in [38]. From 2011 until November 2013, all of these solutions were reduced to two parameters, although $e_j = \tilde{a}_j\epsilon^{2N-1}$ for j between 1 and $N/2$.

It was therefore necessary to look for a way to build solutions depending on a maximum number of parameters equal to $2N-2$, the maximum number proved in [38]. In the meantime, a new representation of solutions in terms of degenerate determinants depending on more than one parameter limit was built, and the results were published in 2013 in [53].

After various attempts and checks, it was possible to build the solutions with the maximum number of parameters equal to $2N-2$ defined by

$$\begin{array}{l} e_j = i\sum_{k=1}^{N-1} a_k\epsilon^{2k+1}j^{2k+1} - \sum_{k=1}^{N-1} b_k\epsilon^{2k+1}j^{2k+1}, \\ e_{N+j} = i\sum_{k=1}^{N-1} a_k\epsilon^{2k+1}j^{2k+1} + \sum_{k=1}^{N-1} b_k\epsilon^{2k+1}j^{2k+1}, a_i,\ b_i \text{ reals} \quad 1 \le j \le N-1. \end{array} \tag{10}$$

2.2.2. The New Results

The results can be given as follows [52] (this theorem is a continuation of a previous result giving NLS solutions without parameters published in the Journal Of Mathematical Sciences [54]).

Using the solutions given in [36] as truncated Riemann theta functions, it was possible to build a representation of solutions in terms of Wronskians depending on x and t.

Another important result given in [52] was to construct explicitly quasi-rational solutions of the NLS equation. The idea was to make a transition to the limit when a parameter (ϵ) tended toward 0. These quasi-rational solutions are given in [52] as a limit of a ratio of two Wronskians of order $2N$.

With this result, we have built explicitly quasi-rational solutions to the order 2, 3 and the Peregrine breather of 4 in the article [52]. Another formulation of quasi-rational solutions to the NLS equation without a limit, in terms of determinant quotients, was constructed in 2012. The solution takes the form of a ratio of two determinants of order $2N$.

2.2.3. Expression of the Solutions to the NLS Equation in Terms of Riemann Theta Functions

The solution of the NLS equation

$$iv_t + v_{xx} + 2|v|^2v = 0, \tag{11}$$

is given in terms of a truncated Riemann theta function by (see [36])

$$v(x,t) = \frac{\theta_3(x,t)}{\theta_1(x,t)}\exp(2it - i\varphi). \tag{12}$$

The functions $\theta_r(x,t)$, $(r = 1, 3)$ are the functions defined by

$$\theta_r(x,t) = \sum_{k\in\{0;1\}^{2N}} g_{r,k} \tag{13}$$

with $g_{r,k}$ given by

$$g_{r,k} = \exp\left\{ \sum_{\mu>\nu,\,\mu,\nu=1}^{2N} \ln\left(\frac{\gamma_\nu - \gamma_\mu}{\gamma_\nu + \gamma_\mu}\right)^2 k_\mu k_\nu \right. \tag{14}$$

$$\left. + \left(\sum_{\nu=1}^{2N} i\kappa_\nu x - 2\delta_\nu t + (r-1)\ln\frac{\gamma_\nu - i}{\gamma_\nu + i} + \sum_{\mu=1,\,\mu\neq\nu}^{2N} \ln\left|\frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu}\right| + e_\nu + \pi i \epsilon_\nu \right) k_\nu \right\}.$$

The solutions depend on the following parameters:
N parameters λ_j, satisfying the relations

$$0 < \lambda_j < 1, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \le j \le N; \tag{15}$$

where ϵ_ν, $1 \le \nu \le 2N$ are arbitrary numbers equal to 0 or 1.

The crucial choice explained in the previous section for complex numbers e_ν $1 \le \nu \le 2N$ was the following:

$$\begin{array}{l} e_j = i\sum_{k=1}^{N-1} a_k \epsilon^{2k+1} j^{2k+1} - \sum_{k=1}^{N-1} b_k \epsilon^{2k+1} j^{2k+1}, \\ e_{N+j} = i\sum_{k=1}^{N-1} a_k \epsilon^{2k+1} j^{2k+1} + \sum_{k=1}^{N-1} b_k \epsilon^{2k+1} j^{2k+1}, a_i,\, b_i \text{ reals} \quad 1 \le j \le N. \end{array} \tag{16}$$

In the preceding formula, the terms κ_ν, δ_ν, γ_ν are functions of the parameters λ_ν, $\nu = 1, \ldots, 2N$, and they are given by the following equations:

$$\kappa_\nu = 2\sqrt{1 - \lambda_\nu^2}, \qquad \delta_\nu = \kappa_\nu \lambda_\nu, \qquad \gamma_\nu = \sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}}. \tag{17}$$

We have also

$$\kappa_{N+j} = \kappa_j, \qquad \delta_{N+j} = -\delta_j, \qquad \gamma_{N+j} = 1/\gamma_j, \qquad j = 1 \ldots N. \tag{18}$$

2.2.4. Representation of the Solutions to the NLS Equation in Terms of Fredholm Determinants

The function θ_r defined in (13) can be rewritten with a sum in terms of subsets of $[1, \ldots, 2N]$:

$$\theta_r(x,t) = \sum_{J\subset\{1,\ldots,2N\}} \prod_{\nu\in J} (-1)^{\epsilon_\nu} \prod_{\nu\in J,\,\mu\notin J} \left|\frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu}\right| \times \exp\left(\sum_{\nu\in J} i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu \right), \tag{19}$$

with

$$x_{r,\nu} = (r-1)\ln\frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad r = 1, 3. \tag{20}$$

In (19), the symbol $\sum_{J\subset\{1,\ldots,2N\}}$ means a summation on all subsets J of indexes of the set $\{1, \ldots, 2N\}$.

Let I be the unit matrix and $C_r = (c_{jk})_{1\le j,k\le 2N}$ be the matrix defined by

$$c_{\nu\mu} = (-1)^{\epsilon_\nu} \frac{\prod_{\eta\neq\mu} |\gamma_\nu + \gamma_\eta|}{\prod_{\eta\neq\nu} |\gamma_\nu - \gamma_\eta|} \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu), \tag{21}$$

$$\epsilon_j = j \quad 1 \le j \le N, \quad \epsilon_j = N + j, \quad N + 1 \le j \le 2N. \tag{22}$$

Then, $\det(I + C_r)$ can be expressed as

$$\det(I + C_r) = \sum_{J \subset \{1,\dots,2N\}} \prod_{\nu \in J} (-1)^{\epsilon_\nu} \prod_{\nu \in J} \prod_{\mu \notin J} \left| \frac{\gamma_\nu + \gamma_\mu}{\gamma_\nu - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu). \tag{23}$$

By comparing (23) with (19), we obtain the equality

$$\theta_r = \det(I + C_r). \tag{24}$$

We can give another formulation of the solutions of the NLS equation. Let $D_r = (d_{jk})_{1 \le j,k \le 2N}$, the matrix defined by

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left| \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu). \tag{25}$$

We have $\det(I + D_r) = \det(I + C_r)$, and then the solution of the NLS equation takes the form [54]

$$v(x,t) = \frac{\det(I + D_3(x,t))}{\det(I + D_1(x,t))} \exp(2it - i\varphi). \tag{26}$$

Theorem 2. *The function v defined by*

$$v(x,t) = \frac{\det(I + D_3(x,t))}{\det(I + D_1(x,t))} \exp(2it - i\varphi) \tag{27}$$

is a solution of the NLS equation, with the matrix $D_r = (d_{jk})_{1 \le j,k \le 2N}$ *defined by*

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \neq \mu} \left| \frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right| \exp(i\kappa_\nu x - 2\delta_\nu t + x_{r,\nu} + e_\nu),$$

where κ_ν, δ_ν, $x_{r,\nu}$, γ_ν, e_ν *are defined by (16), (17) and (20).*

2.2.5. Wronskian Representation of the Solutions to the NLS Equation

We give a relation between the previous matrix D_r defined in (25) and the Wronskian of particular functions. We choose the following functions:

$$\phi_{r,\nu} = \sin \Theta_{r,\nu}, \quad 1 \le \nu \le N, \quad \phi_{r,\nu} = \cos \Theta_{r,\nu}, \quad N+1 \le \nu \le 2N, \quad r = 1, 3, \tag{28}$$

with

$$\Theta_{r,\nu} = \kappa_\nu x/2 + i\delta_\nu t - i x_{r,\nu}/2 + \gamma_\nu y - i e_\nu/2, \quad 1 \le \nu \le 2N. \tag{29}$$

Let $W_r(y)$ be the Wronskian of the functions $\phi_{r,1}, \dots, \phi_{r,2N}$ defined by
$W_r(y) = \det[(\partial_y^{\mu-1} \phi_{r,\nu})_{\nu,\, \mu \in [1,\dots,2N]}]$.
Let the matrix $D_r = (d_{\nu\mu})_{\nu,\, \mu \in [1,\dots,2N]}$, previously defined.
Then, we obtain the following statement [54]:

Theorem 3.

$$\det(I + D_r) = k_r(0) \times W_r(\phi_{r,1}, \dots, \phi_{r,2N})(0), \tag{30}$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_{r,\nu})}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}.$$

From the initial formulation (27), we have

$$v(x,t) = \frac{\det(I + D_3(x,t))}{\det(I + D_1(x,t))} \exp(2it - i\varphi).$$

By using (30), we obtain the following relation between Fredholm determinants and Wronskians:

$$\det(I + D_3) = k_3(0) \times W_3(\phi_{r,1}, \ldots, \phi_{r,2N})(0)$$

and

$$\det(I + D_1) = k_1(0) \times W_1(\phi_{r,1}, \ldots, \phi_{r,2N})(0).$$

Then, we obtain [54] the following:

Theorem 4. *The function v defined by*

$$v(x,t) = \frac{W_3(\phi_{3,1}, \ldots, \phi_{3,2N})(0)}{W_1(\phi_{1,1}, \ldots, \phi_{1,2N})(0)} \exp(2it - i\varphi), \tag{31}$$

is a solution to the NLS equation depending on $2N-2$ *parameters* a_j *and* b_j $1 \le j \le N-1$, *with* $\phi_{r,\nu}$ *defined by*

$$\begin{aligned}
&\phi_{r,\nu} = \sin(\kappa_\nu x/2 + i\delta_\nu t - i x_{r,\nu}/2 + \gamma_\nu y - i e_\nu/2), \quad 1 \le \nu \le N,\\
&\phi_{r,\nu} = \cos(\kappa_\nu x/2 + i\delta_\nu t - i x_{r,\nu}/2 + \gamma_\nu y - i e_\nu/2), \quad N+1 \le \nu \le 2N, \quad r = 1,3,
\end{aligned}$$

λ_ν, κ_ν, δ_ν, $x_{r,\nu}$, γ_ν, e_ν, $1 \le \nu \le 2N$ *defined by (15)–(20),*

$$\begin{aligned}
&-1 < \lambda_{N+1} < \lambda_{N+2} < \ldots < \lambda_{2N} < 0 < \lambda_N < \lambda_{N-1} < \ldots < \lambda_1 < 1, \quad \lambda_{N+j} = -\lambda_j,\\
&\kappa_j = 2\sqrt{1-\lambda_j^2}, \quad \delta_j = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\frac{1-\lambda_\nu}{1+\lambda_\nu}}, \quad x_{r,\nu} = (r-1)\ln\frac{\gamma_\nu - i}{\gamma_\nu + i}, \quad r = 1,3,\\
&\kappa_{j+N} = \kappa_j, \quad \delta_{j+N} = -\delta_{j+N}, \quad \gamma_{j+N} = \gamma_j^{-1}, \quad x_{r,j+N} = x_{r,j},\\
&e_j = i\sum_{k=1}^{N-1} a_k \epsilon^{2k+1} j^{2k+1} - \sum_{k=1}^{N-1} b_k \epsilon^{2k+1} j^{2k+1},\\
&e_{N+j} = i\sum_{k=1}^{N-1} a_k \epsilon^{2k+1} j^{2k+1} + \sum_{k=1}^{N-1} b_k \epsilon^{2k+1} j^{2k+1}, a_i, b_i \text{ reals} \quad 1 \le j \le N-1.
\end{aligned}$$

2.2.6. Quasi-Rational Solutions to the NLS Equation

A little later, at the end of the year 2012, it was possible to give a simpler formulation for the quasi-rational solutions to the NLS equation, i.e., solutions given in terms of a product of a rational function depending on x and t by an exponential depending on t. This was the purpose of the article [53]. To realize this representation, one takes the limit $\lambda_j \to 1$ for $1 \le j \le N$ and $\lambda_j \to -1$ for $N+1 \le j \le 2N$.

To keep it simple, we consider the parameters λ_j written in the form

$$\lambda_j = 1 - 2\epsilon^2 j^2, \quad 1 \le j \le N. \tag{32}$$

When ϵ tends toward 0, we realize the limited expansions at order p, for $1 \le j \le N$, of the terms

$$\begin{aligned}
&\kappa_j = 4j\epsilon(1-\epsilon^2 j^2)^{1/2}, \delta_j = 4j\epsilon(1-2\epsilon^2 j^2)(1-\epsilon^2 j^2)^{1/2},\\
&\gamma_j = j\epsilon(1-\epsilon^2 j^2)^{-1/2}, x_{r,j} = (r-1)\ln\frac{1 + i\epsilon j(1-\epsilon^2 j^2)^{-1/2}}{1 - i\epsilon j(1-\epsilon^2 j^2)^{-1/2}},\\
&\kappa_{N+j} = 4j\epsilon(1-\epsilon^2 j^2)^{1/2}, \delta_{N+j} = -4j\epsilon(1-2\epsilon^2 j^2)(1-\epsilon^2 j^2)^{1/2},\\
&\gamma_{N+j} = 1/(j\epsilon)(1-\epsilon^2 j^2)^{1/2}, x_{r,N+j} = (r-1)\ln\frac{1 - i\epsilon j(1-\epsilon^2 j^2)^{-1/2}}{1 + i\epsilon j(1-\epsilon^2 j^2)^{-1/2}}.
\end{aligned}$$

Then, we obtain the following result [53]:

Theorem 5. *With parameters λ_j defined by (32), $\lambda_j = 1 - 2\epsilon^2 j^2$, $1 \le j \le N$, the function v defined by*

$$v(x,t) = \lim_{\epsilon\to 0} \frac{W_3(\phi_{3,1},\dots,\phi_{3,2N})(0)}{W_1(\phi_{1,1},\dots,\phi_{1,2N})(0)} \exp(2it - i\varphi), \tag{33}$$

is a quasi-rational solution to the NLS eqation depending on $2N-2$ parameters
$e_j = i\sum_{k=1}^{N-1} a_k \epsilon^{2k+1} j^{2k+1} - \sum_{k=1}^{N-1} b_k \epsilon^{2k+1} j^{2k+1}$,
$e_{N+j} = i\sum_{k=1}^{N-1} a_k \epsilon^{2k+1} j^{2k+1} + \sum_{k=1}^{N-1} b_k \epsilon^{2k+1} j^{2k+1}, a_i, b_i$ *reals* $\quad 1 \le j \le N-1$,
$\phi_{r,\nu} = \sin(\kappa_\nu x/2 + i\delta_\nu t - i x_{r,\nu}/2 + \gamma_\nu y - i e_\nu/2), \quad 1 \le \nu \le N$,
$\phi_{r,\nu} = \cos(\kappa_\nu x/2 + i\delta_\nu t - i x_{r,\nu}/2 + \gamma_\nu y - i e_\nu/2), \quad N+1 \le \nu \le 2N, \quad r = 1,3$,
λ_ν, κ_ν, δ_ν, $x_{r,\nu}$, γ_ν, e_ν, $1 \le \nu \le 2N$ *defined in (15)–(20)*,
$-1 < \lambda_{N+1} < \lambda_{N+2} < \ldots < \lambda_{2N} < 0 < \lambda_N < \lambda_{N-1} < \ldots < \lambda_1 < 1, \quad \lambda_{N+j} = -\lambda_j$,
$\kappa_j = 2\sqrt{1-\lambda_j^2}, \quad \delta_j = \kappa_\nu \lambda_\nu, \quad \gamma_\nu = \sqrt{\dfrac{1-\lambda_\nu}{1+\lambda_\nu}}, \quad x_{r,\nu} = (r-1)\ln\dfrac{\gamma_\nu - i}{\gamma_\nu + i}, \quad r = 1,3$,
$\kappa_{j+N} = \kappa_j, \quad \delta_{j+N} = -\delta_{j+N}, \quad \gamma_{j+N} = \gamma_j^{-1}, \quad x_{r,j+N} = x_{r,j}$.

2.2.7. Degenerated Representation of the Quasi-Rational Solutions to the NLS Solutions

The previous study allows us to easily obtain this new representation.
We use the functions $f_{\nu,\mu}^{(r)}$, $1 \le \nu\,\mu \le 2N$, $r = 1,2$ defined by the following relations:

$f_{4j+1,1}^{(1)} \gamma_k^{4j-1} \sin A_1, \quad f_{4j+2,1}^{(1)} = \gamma_k^{4j} \cos A_1, \quad f_{4j+3,1}^{(1)} = -\gamma_k^{4j+1} \sin A_1$,
$f_{4j+4,1}^{(1)} = -\gamma_k^{4j+2} \cos A_1, \quad f_{4j+1,N+1}^{(1)} = \gamma_k^{2N-4j-2} \cos A_{N+1}$,
$f_{4j+2,N+1}^{(1)} = -\gamma_k^{2N-4j-3} \sin A_{N+1}, \quad f_{4j+3,N+1}^{(1)} = -\gamma_k^{2N-4j-4} \cos A_{N+1}$,
$f_{4j+4,N+1}^{(1)} = \gamma_k^{2N-4j-5} \sin A_{N+1}, \quad f_{4j+1,1}^{(2)} = \gamma_k^{4j-1} \sin B_1$,
$f_{4j+2,1}^{(2)} = \gamma_k^{4j} \cos B_1, \quad f_{4j+3,1}^{(2)} = -\gamma_k^{4j+1} \sin B_1, \quad f_{4j+4,1}^{(2)} = -\gamma_k^{4j+2} \cos B_1$,
$f_{4j+1,N+1}^{(2)} = \gamma_k^{2N-4j-2} \cos B_{N+1}, \quad f_{4j+2,N+1}^{(2)} = -\gamma_k^{2N-4j-3} \sin B_{N+1}$,
$f_{4j+3,N+1}^{(2)} = -\gamma_k^{2N-4j-4} \cos B_{N+1}, \quad f_{4j+4,N+1}^{(2)} = \gamma_k^{2N-4j-5} \sin B_{N+1}$, with
$A_\nu = \kappa_\nu x/2 + i\delta_\nu t - i x_{3,\nu}/2 - i e_\nu/2, \quad B_\nu = \kappa_\nu x/2 + i\delta_\nu t - i x_{1,\nu}/2 - i e_\nu/2$,

We denote $q(x,t)$, the quotient

$$q(x,t) = \frac{W_3(0)}{W_1(0)}$$

Then, it is clear that

$$q(x,t) = \frac{\det(f_{j,k}^{(1)})_{j,k\in[1,2N]}}{\det(f_{j,k}^{(2)})_{j,k\in[1,2N]}}. \tag{34}$$

The functions $f_{j,k}(1)$ and $f_{j,k}(2)$ and their derivatives depend on ϵ and can be extended by continuity when $\epsilon = 0$. According to the previous paragraph, the NLS solution is $v(x,t) = \exp(2it - it) \times Q(x,t)$, with $Q(x,t)$ defined by

$$Q(x,t)=\frac{\begin{vmatrix} \partial_\epsilon^0 f_{1,1}^{(1)} & \dots & \partial_\epsilon^{2(N-1)} f_{1,1}^{(1)} & \partial_\epsilon^0 f_{1,N+1}^{(1)} & \dots & \partial_\epsilon^{2(N-1)} f_{1,N+1}^{(1)} \\ \partial_\epsilon^0 f_{2,1}^{(1)} & \dots & \partial_\epsilon^{2(N-1)} f_{2,1}^{(1)} & \partial_\epsilon^0 f_{2,N+1}^{(1)} & \dots & \partial_\epsilon^{2(N-1)} f_{2,N+1}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \partial_\epsilon^0 f_{2N,1}^{(1)} & \dots & \partial_\epsilon^{2(N-1)} f_{2N,1}^{(1)} & \partial_\epsilon^0 f_{2N,N+1}^{(1)} & \dots & \partial_\epsilon^{2(N-1)} f_{2N,N+1}^{(1)} \end{vmatrix}}{\begin{vmatrix} \partial_\epsilon^0 f_{1,1}^{(2)} & \dots & \partial_\epsilon^{2(N-1)} f_{1,1}^{(2)} & \partial_\epsilon^0 f_{1,N+1}^{(2)} & \dots & \partial_\epsilon^{2(N-1)} f_{1,N+1}^{(2)} \\ \partial_\epsilon^0 f_{2,1}^{(2)} & \dots & \partial_\epsilon^{2(N-1)} f_{2,1}^{(2)} & \partial_\epsilon^0 f_{2,N+1}^{(2)} & \dots & \partial_\epsilon^{2(N-1)} f_{2,N+1}^{(2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \partial_\epsilon^0 f_{2N,1}^{(2)} & \dots & \partial_\epsilon^{2(N-1)} f_{2N,1}^{(2)} & \partial_\epsilon^0 f_{2N,N+1}^{(2)} & \dots & \partial_\epsilon^{2(N-1)} f_{2N,N+1}^{(2)} \end{vmatrix}} \tag{35}$$

using the following notations:

$$f_{j,1}^{(r)}[0]=f_{j,1}^{(r)}(x,t,\epsilon)|_{\epsilon=0},\quad f_{j,1}^{(r)}[k]=\frac{\partial^{2k-2}f_{j,1}^{(r)}}{\partial\epsilon^{2k-2}}(x,t,\epsilon)|_{\epsilon=0},\quad r=1,2 \tag{36}$$

We thus obtain the result [55].

Theorem 6.

$\lambda_j=1-j^2\epsilon^2$, $\lambda_{N+j}=-\lambda_j$, $0<\lambda_j<1$, $\kappa_j=2\sqrt{1-\lambda_j^2}$, $\delta_j=\kappa_j\lambda_j$,
$\gamma_j=\sqrt{\frac{1-\lambda_j}{1+\lambda_j}}$, $\kappa_{N+j}=\kappa_j$, $\delta_{N+j}=-\delta_j$, $\gamma_{N+j}=1/\gamma_j$, $x_{r,j}=(r-1)\ln\frac{\gamma_j-i}{\gamma_j+i}$,
$x_{r,N+j}=(r-1)\ln\frac{\gamma_{N+j}-i}{\gamma_{N+j}+i}$, $e_j=ia_j-b_j$, $e_{N+j}=ia_j+b_j$,
$e_j=i\sum_{k=1}^{N-1}a_k\epsilon^{2k+1}j^{2k+1}-\sum_{k=1}^{N-1}b_k\epsilon^{2k+1}j^{2k+1}$,
$e_{N+j}=i\sum_{k=1}^{N-1}a_k\epsilon^{2k+1}j^{2k+1}+\sum_{k=1}^{N-1}b_k\epsilon^{2k+1}j^{2k+1}, a_i,\ b_i$ *reals* $\quad 1\le j\le N-1$,
$f_{4j+1,1}^{(1)}\gamma_k^{4j-1}\sin A_1$, $f_{4j+2,1}^{(1)}=\gamma_k^{4j}\cos A_1$, $f_{4j+3,1}^{(1)}=-\gamma_k^{4j+1}\sin A_1$,
$f_{4j+4,1}^{(1)}=-\gamma_k^{4j+2}\cos A_1$, $f_{4j+1,N+1}^{(1)}=\gamma_k^{2N-4j-2}\cos A_{N+1}$,
$f_{4j+2,N+1}^{(1)}=-\gamma_k^{2N-4j-3}\sin A_{N+1}$, $f_{4j+3,N+1}^{(1)}=-\gamma_k^{2N-4j-4}\cos A_{N+1}$,
$f_{4j+4,N+1}^{(1)}=\gamma_k^{2N-4j-5}\sin A_{N+1}$, $f_{4j+1,1}^{(2)}=\gamma_k^{4j-1}\sin B_1$,
$f_{4j+2,1}^{(2)}=\gamma_k^{4j}\cos B_1$, $f_{4j+3,1}^{(2)}=-\gamma_k^{4j+1}\sin B_1$, $f_{4j+4,1}^{(2)}=-\gamma_k^{4j+2}\cos B_1$,
$f_{4j+1,N+1}^{(2)}=\gamma_k^{2N-4j-2}\cos B_{N+1}$, $\quad f_{4j+2,N+1}^{(2)}=-\gamma_k^{2N-4j-3}\sin B_{N+1}$,
$f_{4j+3,N+1}^{(2)}=-\gamma_k^{2N-4j-4}\cos B_{N+1}$, $\quad f_{4j+4,N+1}^{(2)}=\gamma_k^{2N-4j-5}\sin B_{N+1}$, *with*

$$\Delta^{(r)}=\begin{vmatrix} f_{1,1}^{(r)}[0] & \dots & f_{1,1}^{(r)}[N-1] & f_{1,N+1}^{(r)}[0] & \dots & f_{1,N+1}^{(r)}[N-1] \\ f_{2,1}^{(r)}[0] & \dots & f_{2,1}^{(r)}[N-1] & f_{2,N+1}^{(r)}[0] & \dots & f_{2,N+1}^{(r)}[N-1] \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{2N,1}^{(r)}[0] & \dots & f_{2N,1}^{(r)}[N-1] & f_{2N,N+1}^{(r)}[0] & \dots & f_{2N,N+1}^{(r)}[N-1] \end{vmatrix}$$

Then, the functions u defined by $u(x,t)=\frac{\Delta^{(1)}}{\Delta^{(2)}}e^{2it-if}$ *are quasi-rational solutions of the NLS equation depending on* $2N-2$ *parameters* a_j, b_j, $1\le j\le N-1$.

2.2.8. Another Representation of the Solutions to the NLS Equation

We give here a new representation of the solutions of the equation NLS depending only on the terms γ_ν, $1\le\nu\le 2N$. We can express each of the terms κ_ν, δ_ν and $x_{r,\nu}$ in the function of γ_ν for $1\le\nu\le 2N$, and we obtain

$$\begin{aligned}
&\kappa_j = \frac{4\gamma_j}{(1+\gamma_j^2)}, \quad \delta_j = \frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}, \quad x_{r,j} = (r-1)\ln\frac{\gamma_j - i}{\gamma_j + i}, \quad 1 \le j \le N,\\
&\kappa_j = \frac{4\gamma_j}{(1+\gamma_j^2)}, \quad \delta_j = -\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}, \quad x_{r,j} = (r-1)\ln\frac{\gamma_j + i}{\gamma_j - i}, \quad N+1 \le j \le 2N.
\end{aligned} \tag{37}$$

So, we obtain another representation [56]:

Theorem 7. *The function v defined by*

$$v(x,t) = \frac{\det[(\partial_y^{\mu-1}\tilde{\phi}_{3,\nu}(0))_{\nu,\,\mu\in[1,\ldots,2N]}]}{\det[(\partial_y^{\mu-1}\tilde{\phi}_{1,\nu}(0))_{\nu,\,\mu\in[1,\ldots,2N]}]}\exp(2it - i\varphi) \tag{38}$$

is a solution to the NLS equation

$$iv_t + v_{xx} + 2|v|^2 v = 0.$$

The functions $\tilde{\phi}_{r,\nu}$ are defined by

$$\begin{aligned}
&\tilde{\phi}_{r,j}(y) = \sin\left(\frac{2\gamma_j}{(1+\gamma_j^2)}x + i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t - i\frac{(r-1)}{2}\ln\frac{\gamma_j - i}{\gamma_j + i} + \gamma_j y - ie_j\right),\\
&\tilde{\phi}_{r,N+j}(y) = \cos\left(\frac{2\gamma_j}{(1+\gamma_j^2)}x - i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t + i\frac{(r-1)}{2}\ln\frac{\gamma_j - i}{\gamma_j + i} + \frac{1}{\gamma_j}y - ie_{N+j}\right),\\
&\text{where } \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}}, \qquad 1 \le j \le N.\\
&\lambda_j \text{ is a real parameter such that } 0 < \lambda_j < 1,\ \lambda_{N+j} = -\lambda_j,\ 1 \le j \le N.\\
&\text{The terms } e_\nu \text{ are defined by}\\
&e_j = i\sum_{k=1}^{N-1} a_k \epsilon^{2k+1} j^{2k+1} - \sum_{k=1}^{N-1} b_k \epsilon^{2k+1} j^{2k+1},\\
&e_{N+j} = i\sum_{k=1}^{N-1} a_k \epsilon^{2k+1} j^{2k+1} + \sum_{k=1}^{N-1} b_k \epsilon^{2k+1} j^{2k+1}, a_i,\ b_i \text{ reals} \quad 1 \le j \le N-1,
\end{aligned} \tag{39}$$

Remark 1. *In (38), the determinants* $\det[(\partial_y^{\mu-1} f_\nu(0))_{\nu,\,\mu\in[1,\ldots,2N]}]$ *are the Wronskians of the functions* $f_1, \ldots, f_{2N}$ *evaluated in* $y = 0$. *The expression* $\partial_y^0 f_\nu$ *means* f_ν.

To obtain quasi-rational solutions to the NLS equation, we take the limits $\lambda_j \to 1$ for $1 \le j \le N$ and $\lambda_j \to -1$ for $N+1 \le j \le 2N$. For this, we choose $\lambda_j = 1 - 2j\epsilon^2$. When ϵ tends to 0, we realize limited expansions at a certain order of the functions $\Phi_{r,\nu}$.

We choose the following notations:

$$\begin{aligned}
&X_j = \frac{2\gamma_j}{(1+\gamma_j^2)}x + i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t - i\ln\frac{\gamma_j - i}{\gamma_j + i} - ie_j,\\
&X_{N+j} = \frac{2\gamma_j}{(1+\gamma_j^2)}x - i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t + i\ln\frac{\gamma_j - i}{\gamma_j + i} - ie_{N+j},\\
&\text{for } 1 \le j \le N.\\
&Y_j = \frac{2\gamma_j}{(1+\gamma_j^2)}x + i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t - ie_j,\\
&Y_{N+j} = \frac{2\gamma_j}{(1+\gamma_j^2)}x - i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t - ie_{N+j},\\
&\text{for } 1 \le j \le N.
\end{aligned} \tag{40}$$

The expressions γ_ν and e_ν are defined by (75). To rewrite the ratio of Wronskians defined in (75), we consider the following functions:

$$\begin{array}{l}\varphi_{4j+1,k}=\gamma_k^{4j-1}\sin X_k,\quad \varphi_{4j+2,k}=\gamma_k^{4j}\cos X_k,\\ \varphi_{4j+3,k}=-\gamma_k^{4j+1}\sin X_k,\quad \varphi_{4j+4,k}=-\gamma_k^{4j+2}\cos X_k,\\ \varphi_{4j+1,N+k}=\gamma_k^{2N-4j-2}\cos X_{N+k},\quad \varphi_{4j+2,N+k}=-\gamma_k^{2N-4j-3}\sin X_{N+k},\\ \varphi_{4j+3,N+k}=-\gamma_k^{2N-4j-4}\cos X_{N+k},\quad \varphi_{4j+4,N+k}=\gamma_k^{2N-4j-5}\sin X_{N+k},\quad 1\le k\le N.\end{array} \tag{41}$$

Let $g_{j,k}$ be the function for $1\le j\le 2N$, $1\le k\le 2N$; in the same way, we replace only the term X_k by Y_k.

$$\begin{array}{l}\psi_{4j+1,k}=\gamma_k^{4j-1}\sin Y_k,\quad \psi_{4j+2,k}=\gamma_k^{4j}\cos Y_k,\\ \psi_{4j+3,k}=-\gamma_k^{4j+1}\sin Y_k,\quad \psi_{4j+4,k}=-\gamma_k^{4j+2}\cos Y_k,\\ \psi_{4j+1,N+k}=\gamma_k^{2N-4j-2}\cos Y_{N+k},\quad \psi_{4j+2,N+k}=-\gamma_k^{2N-4j-3}\sin Y_{N+k},\\ \psi_{4j+3,N+k}=-\gamma_k^{2N-4j-4}\cos Y_{N+k},\quad \psi_{4j+4,N+k}=\gamma_k^{2N-4j-5}\sin Y_{N+k},\quad 1\le k\le N\end{array} \tag{42}$$

The ratio of the Wronskians $q(x,t)$ defined by

$$q(x,t):=\frac{\det[(\partial_y^{\mu-1}\tilde{\phi}_{3,\nu}(0))_{\nu,\,\mu\in[1,\ldots,2N]}]}{\det[(\partial_y^{\mu-1}\tilde{\phi}_{1,\nu}(0))_{\nu,\,\mu\in[1,\ldots,2N]}]}$$

can be written as

$$q(x,t)=\frac{\Delta_3}{\Delta_1}=\frac{\det(\varphi_{j,k})_{j,k\in[1,2N]}}{\det(\psi_{j,k})_{j,k\in[1,2N]}}. \tag{43}$$

The functions $\varphi_{j,k}$ and $\psi_{j,k}$ and their derivatives depend on ϵ and can be extended by continuity in $\epsilon=0$.

We use the following expansion:

$$\begin{array}{l}\varphi_{j,k}(x,t,\epsilon)=\sum_{l=0}^{N-1}\frac{1}{(2l)!}\varphi_{j,1}[l]k^{2l}\epsilon^{2l}+O(\epsilon^{2N}),\quad \varphi_{j,1}[l]=\frac{\partial^{2l}\varphi_{j,1}}{\partial\epsilon^{2l}}(x,t,0),\\ \varphi_{j,1}[0]=\varphi_{j,1}(x,t,0),\quad 1\le j\le 2N,\quad 1\le k\le N,\quad 1\le l\le N-1,\\ \varphi_{j,N+k}(x,t,\epsilon)=\sum_{l=0}^{N-1}\frac{1}{(2l)!}\varphi_{j,N+1}[l]k^{2l}\epsilon^{2l}+O(\epsilon^{2N}),\quad \varphi_{j,N+1}[l]=\frac{\partial^{2l}\varphi_{j,N+1}}{\partial\epsilon^{2l}}(x,t,0),\\ \varphi_{j,N+1}[0]=\varphi_{j,N+1}(x,t,0),\quad 1\le j\le 2N,\quad 1\le k\le N,\quad 1\le l\le N-1.\end{array} \tag{44}$$

We have the same expansions for the functions $\psi_{j,k}$.

$$\begin{array}{l}\psi_{j,k}(x,t,\epsilon)=\sum_{l=0}^{N-1}\frac{1}{(2l)!}\psi_{j,1}[l]k^{2l}\epsilon^{2l}+O(\epsilon^{2N}),\quad \psi_{j,1}[l]=\frac{\partial^{2l}\psi_{j,1}}{\partial\epsilon^{2l}}(x,t,0),\\ \psi_{j,1}[0]=\psi_{j,1}(x,t,0),\quad 1\le j\le 2N,\quad 1\le k\le N,\quad 1\le l\le N-1,\\ \psi_{j,N+k}(x,t,\epsilon)=\sum_{l=0}^{N-1}\frac{1}{(2l)!}\psi_{j,N+1}[l]k^{2l}\epsilon^{2l}+O(\epsilon^{2N}),\quad \psi_{j,N+1}[l]=\frac{\partial^{2l}\psi_{j,N+1}}{\partial\epsilon^{2l}}(x,t,0),\\ \psi_{j,N+1}[0]=\psi_{j,N+1}(x,t,0),\quad 1\le j\le 2N,\quad 1\le k\le N,\quad N+1\le k\le 2N.\end{array} \tag{45}$$

Then, we obtain the following statement [53]:

Theorem 8. *The function v defined by*

$$v(x,t)=\exp(2it-i\varphi)\times\frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} \tag{46}$$

is a quasi-rational solution of the NLS quation $iv_t + v_{xx} + 2|v|^2 v = 0$ *depending on* $2N-2$ *real parameters* a_j, b_j, $1 \leq j \leq N-1$, *where*

$$\begin{aligned}
&n_{j1} = \varphi_{j,1}(x,t,0),\ 1 \leq j \leq 2N \quad n_{jk} = \frac{\partial^{2k-2}\varphi_{j,1}}{\partial\epsilon^{2k-2}}(x,t,0),\\
&n_{jN+1} = \varphi_{j,N+1}(x,t,0),\ 1 \leq j \leq 2N \quad n_{jN+k} = \frac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\epsilon^{2k-2}}(x,t,0),\\
&d_{j1} = \psi_{j,1}(x,t,0),\ 1 \leq j \leq 2N \quad d_{jk} = \frac{\partial^{2k-2}\psi_{j,1}}{\partial\epsilon^{2k-2}}(x,t,0),\\
&d_{jN+1} = \psi_{j,N+1}(x,t,0),\ 1 \leq j \leq 2N \quad d_{jN+k} = \frac{\partial^{2k-2}\psi_{j,N+1}}{\partial\epsilon^{2k-2}}(x,t,0),\\
&2 \leq k \leq N,\ 1 \leq j \leq 2N
\end{aligned}$$

The functions φ *and* ψ *are defined in (41) and (42).*

2.2.9. The P_N Breather and Its Maximum Modulus Equal to $2N+1$

In all the following, to simplify, we can designate, by the P_N breather, the Peregrine breather of order N. There is all latitude to choose γ_j in such a way that the conditions on λ_j be verified. From the work [52,53], we know that the Peregrine breathers of order N P_N are obtained when all parameters a_j and b_j are equal to 0. To obtain the simplest expressions in the determinants, we have to choose particular solutions in the previous families. We choose $\gamma_j = j\epsilon$ to be as simple as possible in such a way that the conditions on λ_j are verified.

Then, we obtain [56]:

Theorem 9. *The function* v_0 *defined by*

$$v_0(x,t) = \exp(2it - i\varphi) \times \left(\frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})}\right)_{(\tilde{a_j}=\tilde{b_j}=0,\,1\leq j\leq N-1)} \tag{47}$$

is the Peregrine breather of order N solution of the NLS equation whose maximum modulus is equal to $2N+1$.

2.2.10. Structure of the Solutions to the NLS Equation

In this section, we give the structure of the quasi-rational solutions of the NLS equation. We obtain families of solutions depending on $2N-2$ parameters of the order N, including families of 2 parameters built in previous works.

The functions φ and ψ are defined by

$$\begin{aligned}
&\varphi_{4j+1,k} = \gamma_k^{4j-1}\sin X_k, \quad \varphi_{4j+2,k} = \gamma_k^{4j}\cos X_k,\\
&\varphi_{4j+3,k} = -\gamma_k^{4j+1}\sin X_k, \quad \varphi_{4j+4,k} = -\gamma_k^{4j+2}\cos X_k,
\end{aligned} \tag{48}$$

for $1 \leq k \leq N$, and

$$\begin{aligned}
&\varphi_{4j+1,N+k} = \gamma_k^{2N-4j-2}\cos X_{N+k}, \quad \varphi_{4j+2,N+k} = -\gamma_k^{2N-4j-3}\sin X_{N+k},\\
&\varphi_{4j+3,N+k} = -\gamma_k^{2N-4j-4}\cos X_{N+k}, \quad \varphi_{4j+4,N+k} = \gamma_k^{2N-4j-5}\sin X_{N+k},
\end{aligned} \tag{49}$$

for $1 \leq k \leq N$, with

$$\begin{aligned}
X_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{3,\nu}/2 - ie_\nu/2,\\
Y_\nu &= \kappa_\nu x/2 + i\delta_\nu t - ix_{1,\nu}/2 - ie_\nu/2,
\end{aligned}$$

$1 \leq \nu \leq 2N$, with κ_ν, δ_ν, $x_{r,\nu}$, e_ν previously defined.

We define the functions $\psi_{j,k}$, $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ in the same way; we replace only the terms X_k by Y_k.

$$\begin{array}{l} \psi_{4j+1,k} = \gamma_k^{4j-1} \sin Y_k, \quad \psi_{4j+2,k} = \gamma_k^{4j} \cos Y_k, \\ \psi_{4j+3,k} = -\gamma_k^{4j+1} \sin Y_k, \quad \psi_{4j+4,k} = -\gamma_k^{4j+2} \cos Y_k, \end{array} \tag{50}$$

for $1 \leq k \leq N$, and

$$\begin{array}{l} \psi_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos Y_{N+k}, \quad \psi_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin Y_{N+k}, \\ \psi_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos Y_{N+k}, \quad \psi_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin Y_{N+k}, \end{array} \tag{51}$$

for $1 \leq k \leq N$.

We define matrix $(n_{jk})_{j,k\in[1;2N]}$ and $(d_{jk})_{j,k\in[1;2N]}$ by

$$\begin{array}{l} n_{j1} = \varphi_{j,1}(x,t,0),\ 1 \leq j \leq 2N \quad n_{jk} = \dfrac{\partial^{2k-2}\varphi_{j,1}}{\partial\epsilon^{2k-2}}(x,t,0), \\ n_{jN+1} = \varphi_{j,N+1}(x,t,0),\ 1 \leq j \leq 2N \quad n_{jN+k} = \dfrac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\epsilon^{2k-2}}(x,t,0), \\ d_{j1} = \psi_{j,1}(x,t,0),\ 1 \leq j \leq 2N \quad d_{jk} = \dfrac{\partial^{2k-2}\psi_{j,1}}{\partial\epsilon^{2k-2}}(x,t,0), \\ d_{jN+1} = \psi_{j,N+1}(x,t,0),\ 1 \leq j \leq 2N \quad d_{jN+k} = \dfrac{\partial^{2k-2}\psi_{j,N+1}}{\partial\epsilon^{2k-2}}(x,t,0), \\ 2 \leq k \leq N,\ 1 \leq j \leq 2N \end{array}$$

So, we obtain the following [56]:

Theorem 10. *The function v defined by*

$$v(x,t) = \exp(2it - i\varphi) \times \frac{\det((n_{jk})_{j,k\in[1,2N]})}{\det((d_{jk})_{j,k\in[1,2N]})} \tag{52}$$

is a quasi-rational solution of the NLS, quotient of two polynomials $N(x,t)$ and $D(x,t)$ depending on $2N-2$ real parameters a_j and b_j, $1 \leq j \leq N-1$.

N and D are polynomials of degrees $N(N+1)$ in x and t.

We have realized a classification of the solutions to the NLS equation depending on the order N of the solution and the different parameters. All the results are given in a series of papers from order 3 [55] until order 13 [57].

2.3. Study of Solutions of Order $N = 13$

For $N = 13$, we give a complete description of the patterns of the modulus of the solutions to the NLS equation in [57].

We give a construction of the thirteen's Peregrine breather (P_{13} breather). We present also its twenty-four real parameter solutions to the NLS equation. We obtain new families of solutions to the NLS equation in terms of explicit quotients of polynomials of degree 182 in x and t multiplied by an exponential depending on t. Characteristic patterns of the modulus of these solutions are presented in the $(x;t)$ plane, in function of the different parameters.

We recall that, if we take all parameters equal to 0, $a_i = b_i = 0$ for $1 \leq i \leq 10$, we obtain the classical thirteenth Peregrine breather (here, the peak going until 27 has been truncated) (Figure 1).

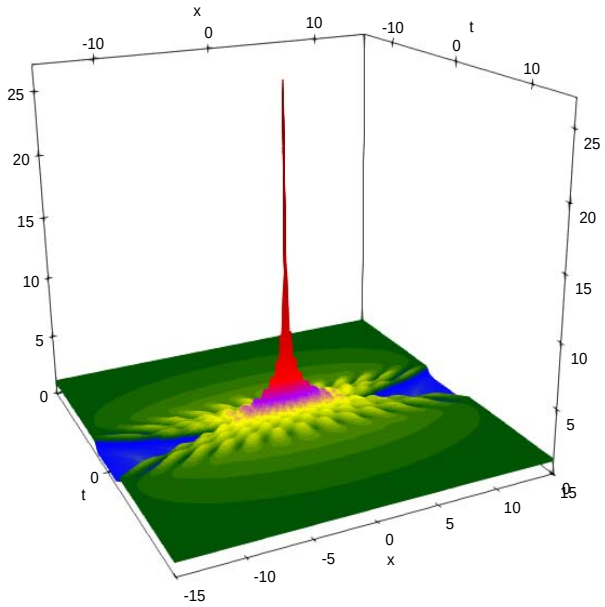


Figure 1. Solution of order $N = 13$ to the NLS equation, the Peregrine breather P_{13}, all parameters equal to 0.

For other choices of parameters, we obtain all types of configurations: triangle and multiple concentric ring configurations with a maximum of 91 peaks.

As the pattern of the solutions to the NLS equation in the $(x; t)$ plane is the same for a_j and b_j, we only present the solutions for the parameters a_j for $1 \leq j \leq 12$. We give figures of deformations with only one parameter non-equal to zero; all other parameters are fixed as equal to zero. For more precision, we present for these figures only the views of the top (Figures 2–13).

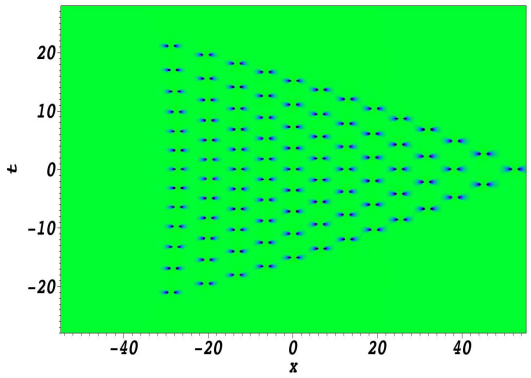


Figure 2. Solution of order $N = 13$ to the NLS equation, $a_1 = 10^3$: triangle with 91 peaks, sight from top.

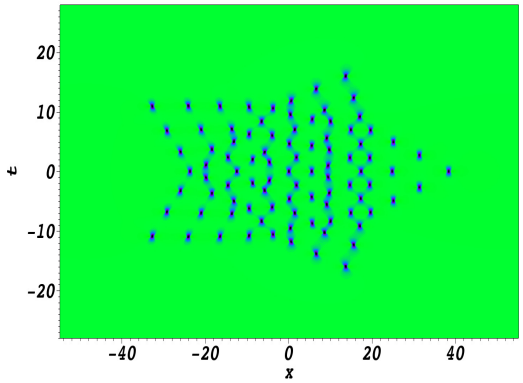


Figure 3. Solution of order $N = 13$ to the NLS equation, $a_2 = 10^4$: 11 rings with 5, 10, 10, 10, 10, 10, 10, 5, 5, 10, 5 peaks and 1 peak in the center, sight from top.

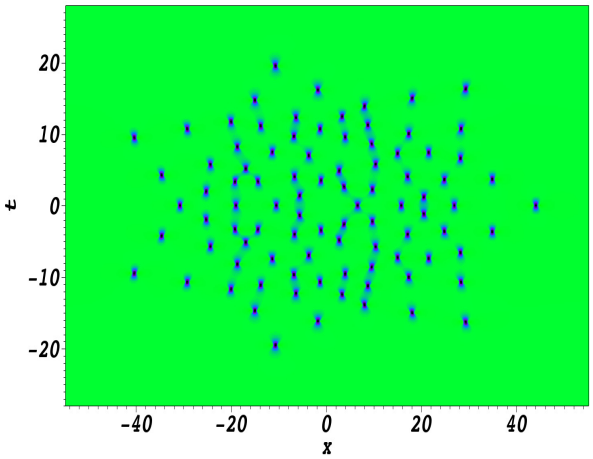


Figure 4. Solution of order $N = 13$ to the NLS equation, $a_3 = 10^6$: 10 rings with 7, 14, 7, 7, 14, 14, 7, 7, 7, 7 peaks, sight from top.

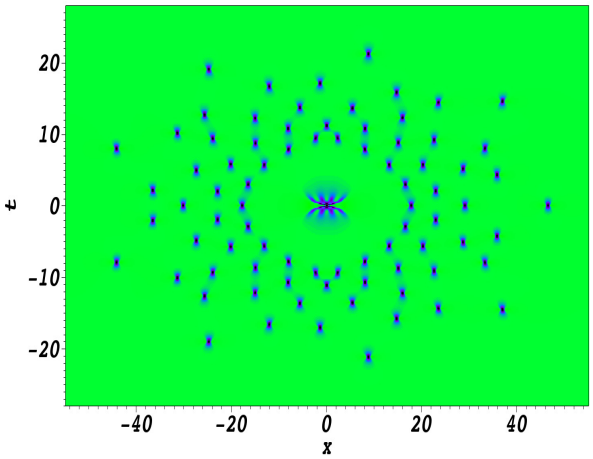


Figure 5. Solution of order $N = 13$ to the NLS equation, $a_4 = 10^8$: 5 rings with 9, 18, 18, 18, 18 peaks, in the center P_4, sight from top.

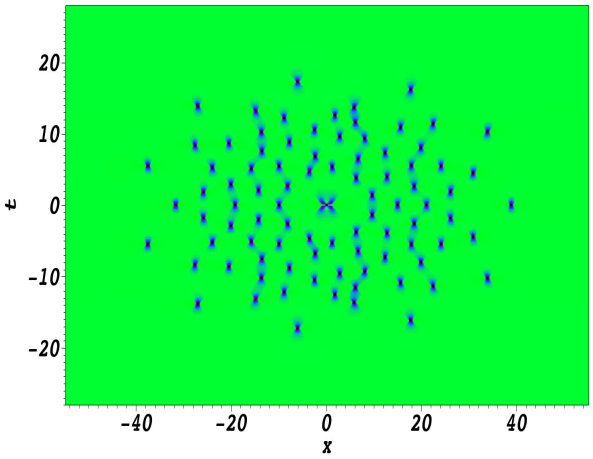


Figure 6. Solution of order $N = 13$ to the NLS equation, $a_5 = 10^9$: 7 rings of 11, 11, 22, 11, 11, 11, 11 peaks and 1 peak in the center P_2, sight from top.

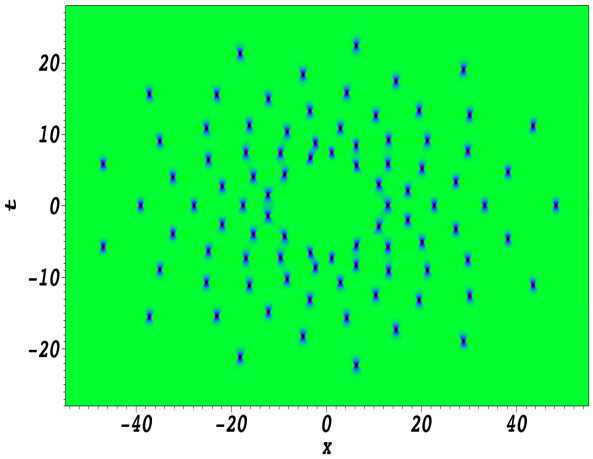


Figure 7. Solution of order $N = 13$ to the NLS equation, $a_6 = 10^{12}$: 7 rings with 13 peaks without peak in the center, sight of top.

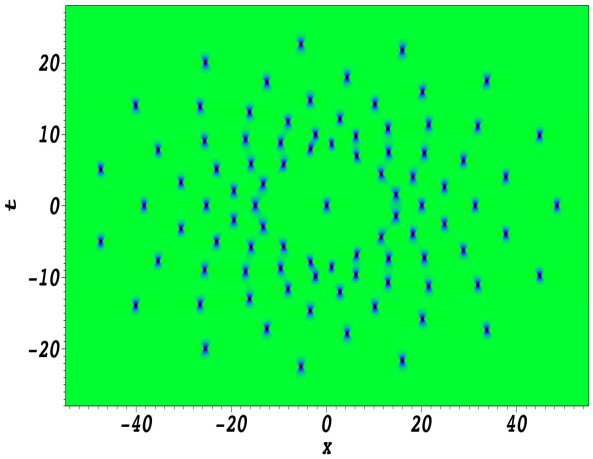


Figure 8. Solution of order $N = 13$ to the NLS equation, $a_7 = 10^{14}$: 6 rings with 15 peaks and, in the center, 1 peak, sight from top.

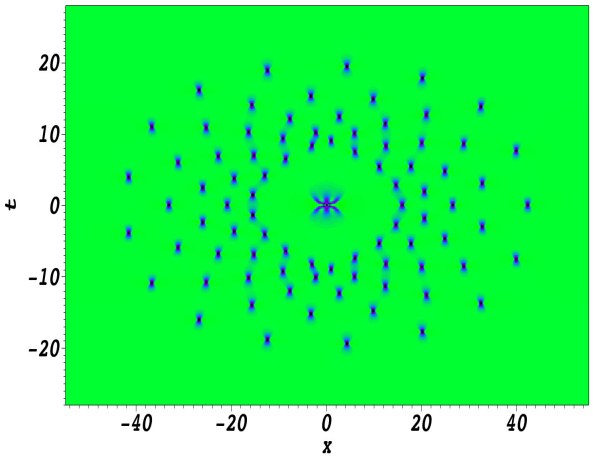


Figure 9. Solution of order $N = 13$ to the NLS equation, $a_8 = 10^{15}$: 5 rings with 17 peaks and, in the center, the Peregrine breather of order 3, sight from top.

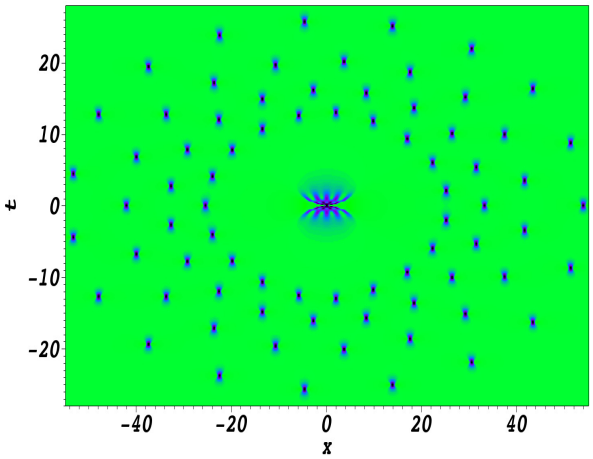


Figure 10. Solution of order $N = 13$ to the NLS equation, $a_9 = 10^{19}$: 4 rings with 19 peaks and, in the center, the Peregrine breather of order 5, sight from top.

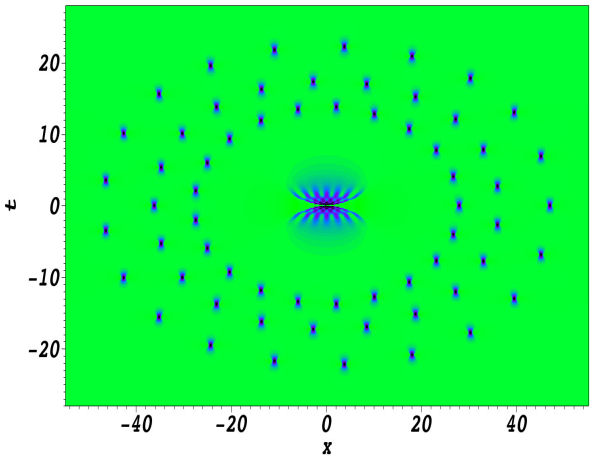


Figure 11. Solution of order $N = 13$ to the NLS equation, $a_{10} = 10^{20}$: three rings with 21 peaks and, in the center, the Peregrine breather of order 7, sight from top.

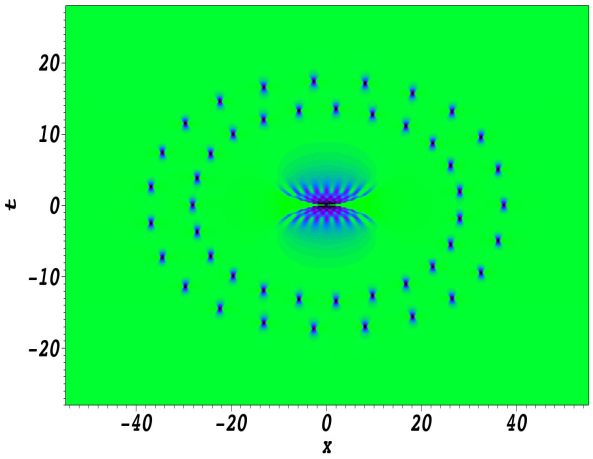


Figure 12. Solution of order $N = 13$ to the NLS equation, $a_{11} = 10^{20}$: two rings with 23 peaks and, in the center, the Peregrine breather of order 9, sight from top.

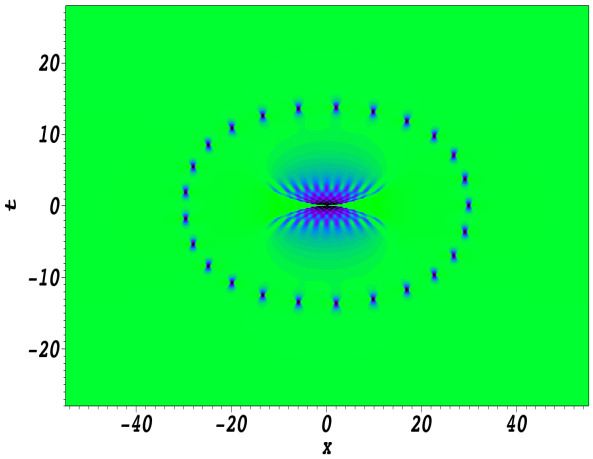


Figure 13. Solution of order $N = 13$ to the NLS equation, $a_{12} = 10^{20}$: one ring with 25 peaks and, in the center, the Peregrine breather of order 11, sight from top.

A classification of the rogue waves at order 13 has been realized.

We obtain two types of configurations: the triangles and the concentric rings for the same index i for a_i or b_i non-equal to 0.

In the cases $a_1 \neq 0$ or $b_1 \neq 0$, we obtain triangles with a maximum of 91 peaks.

For $a_2 \neq 0$ or $b_2 \neq 0$, we obtain rings with 5 or 10 peaks;
For $a_3 \neq 0$ or $b_3 \neq 0$, we obtain rings with 7 or 14 peaks with in the center P_2;
For $a_4 \neq 0$ or $b_4 \neq 0$, we obtain five rings with 9, 18, 18, 19, 18, peaks with, in the center, the Peregrine P_4;
For $a_5 \neq 0$ or $b_5 \neq 0$, we obtain seven rings of 11, 11, 22, 11, 11, 11, 11 peaks with in the center P_3;
For $a_6 \neq 0$ or $b_6 \neq 0$, we have seven rings with 13 peaks on each of them, without a peak in the center;
For $a_7 \neq 0$ or $b_7 \neq 0$, we obtain six rings with 15 peaks on each of them and in the center;
For $a_8 \neq 0$ or $b_8 \neq 0$, we obtain five rings with 17 peaks on each of them and, in the center, the Peregrine breather of order 3;
For $a_9 \neq 0$ or $b_9 \neq 0$, we obtain four rings with 19 peaks and, in the center, the Peregrine breather of order 5. For $a_{10} \neq 0$ or $b_{10} \neq 0$, we have three rings with 21 peaks and, in the center, the Peregrine breather of order 7;
For $a_{11} \neq 0$ or $b_{11} \neq 0$, we obtain only two rings with 23 peaks and, in the center, the Peregrine breather of order 9;
For $a_{12} \neq 0$ or $b_{12} \neq 0$, we obtain only one ring with 25 peaks and, in the center, the Peregrine breather of order 11.

The solutions of the NLS equation were explicitly constructed up to the order 13 by the author when no one had given them before.

The calculations as soon as we exceed the order 10 are very complicated.

The number of terms of the polynomials of the numerator d_3 and denominator d_1 of the solutions is shown in Table 1 when other a_i and b_i are set to 0.

Table 1. Number of terms for the polynomials d_3 and d_1 of the solutions to the NLS equation in the case $N = 13$.

a_i	d_3	d_1
2	1,051,741	531,675
3	559,480	282,841
4	348,403	176,084
5	245,676	124,198
6	183,131	92,581
7	143,053	72,324
8	115,599	58,448
9	94,974	48,014
10	76,243	38,529
11	55,015	27,787
12	30,003	15,154

3. Rogue Waves in the Kadomtsev–Petviashvili (KPI) Equation

3.1. The (KPI) Equation

In this section, the Kadomtsev–Petviashvili (KPI) equation is considered and written in the following normalization:

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0, \tag{53}$$

where the indices x, y and t denote partial derivatives.

Kadomtsev and Petviashvili proposed this equation for the first time in 1970. In 1974, Dryuma proved that the KPI equation could be written in Lax form [58]. Zakharov used the IST method to solve this equation and obtained several types of solutions. Manakov, Zakharov, Bordag and Matveev constructed the first rational solutions in November 1977 [59] and, at the same time as Krichever, published them in January 1978 [60].

To solve this equation, Krichever built for the first time in 1976 the KPI solutions expressed in terms of Riemann theta functions in the algebro-geometric framework in [61]. Almost at the same time, a more general study in this context was carried out by Dubrovin [62]. Much research has been conducted and more general rational solutions of the KPI equation were obtained.

Among all these works, we can mention those of Krichever in 1978 [63], Satsuma and Ablowitz in 1979, Matveev in 1979 [64], Freeman and Nimmo in 1983 [65,66], Matveev in 1987 [67], Pelinovsky and Stepanyants in 1993 [68], Pelinovsky [69], Ablowitz, Villarroel, Chakravarty and Trubatch [70–72] in 1997–2000 and Biondini and Kodama [73–75] in 2003–2007.

3.2. Recent Results

3.2.1. The Darboux Transformation

Xu, Sin, Zhang and Li [76] constructed in 2014 the iterated Darboux transformation for the second- and third-order m-coupled AKNS systems. They found in particular that the KPI equation admits a reduced multi-component Wronskian solution. They obtained the ordinary N-soliton solution to the KPI equation: the bound state at the moment of collision.

3.2.2. The Hirota Bilinear Method

In 2008, Yao, Zhang, Zhu, Meng, Li, Shaw and Tian [77] used the bilinear method for a variable-coefficient KPI equation. They obtained solutions in terms of Wronskians. They proved that this equation could be reduced to a Jacobi identity.

Fu, Song and Xu studied in 2012 the generalized (n + 1)-dimensional KPI equation with variable coefficients in the paper [78]. They constructed, using the Wronskian technique and the Pfaffian properties, Wronskian and Grammian solutions of this equation.

Recently, in 2021, Rao, Chow, Mihalache and He studied in the paper [79] resonant collisions among localized lumps and line solitons of the KPI equation. For these resonant collisions, they noted that the phase shift could become indefinitely large.

In 2021, Yang and Yang studied in [80] pattern formation in higher-order lumps of the KPI equation for a long time.

They found two types of figures. One type of configuration is given by triangular shapes, and is described by root structures of the Yablonskii–Vorob'ev polynomials. The other type of configuation is given by non-triangular shapes in the outer region described analytically by non-zero root structures of the Wronskian–Hermit polynomials.

3.2.3. The Algebro-Geometric Approach

In 2014, Zhao and Fan proposed in [81] an algebro-geometric method to express the Baker–Ahkiezer function of the KPI hierarchy, and they deduced representations of algebro-geometric solutions for the whole KPI hierarchy.

3.3. The Three Representations of the Solutions to the KPI Equation

This paragraph concerns the results obtained by the author and gives the different representations of the solutions to the equation KPI in terms of Fredholm determinants, Wronskians and degenerate determinants. We obtain what we call lump solutions, which are rational solutions decaying in all directions of space variables.

We construct solutions to the Kadomtsev–Petviashvili equation (KPI) from the solutions of the NLS equation. We build expressions of the solutions of the equation in terms of the Fredholm determinant of order $2N$ depending on $2N-1$ real parameters and in terms of Wronskians of order $2N$ depending on $2N-1$ real parameters. We also build rational solutions of the KPI equation depending on $2N-2$ real parameters. These solutions can be represented as a quotient of two polynomials of degree $2N(N+1)$ in x, y and t. The maximum modulus of these solutions is equal to $2(2N+1)^2$. This is a new approach in order to build explicit solutions for arbitrary orders N depending on $2N-2$ real parameters and try to describe the structure of these rational solutions.

3.3.1. Fredholm Determinant Representation of the Solutions to the KPI Equation

We consider the real numbers λ_j satisfying $-1 < \lambda_\nu < 1$, $\nu = 1, \ldots, 2N$, which depend on a parameter ϵ that will tend toward 0. They can be written as

$$\lambda_j = 1 - 2\epsilon^2 j^2, \quad \lambda_{N+j} = -\lambda_j, \quad 1 \le j \le N, \tag{54}$$

The quantities $\kappa_\nu, \delta_\nu, \gamma_\nu$ and $x_{r,\nu}$ depend on λ_ν, $1 \le \nu \le 2N$ and are given by

$$\begin{aligned}
&\kappa_j = 2\sqrt{1-\lambda_j^2}, \quad \delta_j = \kappa_j \lambda_j, \quad \gamma_j = \sqrt{\frac{1-\lambda_j}{1+\lambda_j}},; \\
&x_{r,j} = (r-1)\ln\frac{\gamma_j - i}{\gamma_j + i}, \quad r = 1, 3, \quad \tau_j = -12i\lambda_j^2\sqrt{1-\lambda_j^2} - 4i(1-\lambda_j^2)\sqrt{1-\lambda_j^2}, \\
&\kappa_{N+j} = \kappa_j, \quad \delta_{N+j} = -\delta_j, \quad \gamma_{N+j} = \gamma_j^{-1}, \\
&x_{r,N+j} = -x_{r,j}, \quad \tau_{N+j} = \tau_j \quad j = 1, \ldots, N.
\end{aligned} \tag{55}$$

e_ν $1 \le \nu \le 2N$ are defined by

$$\begin{array}{l} e_j = i\sum_{k=1}^{N-1} a_k \epsilon^{2k+1} j^{2k+1} - \sum_{k=1}^{N-1} b_k \epsilon^{2k+1} j^{2k+1}, \\ e_{N+j} = i\sum_{k=1}^{N-1} a_k \epsilon^{2k+1} j^{2k+1} + \sum_{k=1}^{N-1} b_k \epsilon^{2k+1} j^{2k+1}, a_i,\ b_i \text{ reals} \quad 1 \le j \le N-1, \end{array} \tag{56}$$

ϵ_ν, $1 \le \nu \le 2N$ are real numbers defined by

$$\epsilon_j = 1, \quad \epsilon_{N+j} = 0 \quad 1 \le j \le N. \tag{57}$$

Let I be the unit matrix and $D_r = (d_{jk})_{1 \le j,k \le 2N}$ be the matrix defined by

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \ne \mu} \left(\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right) \exp(i\kappa_\nu x - 2\delta_\nu y + \tau_\nu t + x_{r,\nu} + e_\nu). \tag{58}$$

Then, we have the following result [82]:

Theorem 11. *The function v defined by*

$$v(x,y,t) = -2\,\frac{|n(x,y,t)|^2}{d(x,y,t)^2} \tag{59}$$

where

$$n(x,y,t) = \det(I + D_3(x,y,t)), \tag{60}$$

$$d(x,y,t) = \det(I + D_1(x,y,t)), \tag{61}$$

and $D_r = (d_{jk})_{1 \le j,k \le 2N}$ *the matrix*

$$d_{\nu\mu} = (-1)^{\epsilon_\nu} \prod_{\eta \ne \mu} \left(\frac{\gamma_\eta + \gamma_\nu}{\gamma_\eta - \gamma_\mu} \right) \exp(i\kappa_\nu x - 2\delta_\nu y + \tau_\nu t + x_{r,\nu} + e_\nu). \tag{62}$$

is a solution to the equation KPI (53), depending on $2N-1$ *real parameters* a_k, b_h, $1 \le k \le N-1$ *and* ϵ.

3.3.2. Wronskian Representation of the Solutions to the KPI Equation

To express the solutions of the KPI equation in terms of Wronskians, we need the following notations:

$$\phi_{r,\nu} = \sin\Theta_{r,\nu}, \quad 1 \le \nu \le N, \quad \phi_{r,\nu} = \cos\Theta_{r,\nu}, \quad N+1 \le \nu \le 2N, \quad r = 1,3, \tag{63}$$

with the arguments

$$\Theta_{r,\nu} = \frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{r,\nu}}{2} - i\frac{\tau_\nu}{2}t + \gamma_\nu w - i\frac{e_\nu}{2}, \quad 1 \le \nu \le 2N. \tag{64}$$

Let $W_r(w)$ be the Wronskian of the functions $\phi_{r,1}, \ldots, \phi_{r,2N}$ defined by

$$W_r(w) = \det[(\partial_w^{\mu-1}\phi_{r,\nu})_{\nu,\,\mu \in [1,\ldots,2N]}]. \tag{65}$$

Let D_r be the matrix $D_r = (d_{\nu\mu})_{\nu,\,\mu \in [1,\ldots,2N]}$ defined by (62). Then, we obtain [82]

Theorem 12.

$$\det(I + D_r) = k_r(0) \times W_r(\phi_{r,1}, \ldots, \phi_{r,2N})(0), \tag{66}$$

where

$$k_r(y) = \frac{2^{2N} \exp(i \sum_{\nu=1}^{2N} \Theta_{r,\nu})}{\prod_{\nu=2}^{2N} \prod_{\mu=1}^{\nu-1} (\gamma_\nu - \gamma_\mu)}.$$

But, the solution v of the KPI equation can be written as

$$v(x,y,t) = -2 \frac{|\det(I + D_3(x,y,t))|^2}{(\det(I + D_1(x,y,t)))^2}.$$

By using (66), we obtain the link between Fredholm determinants and Wronskians:

$$\det(I + D_3) = k_3(0) \times W_3(\phi_{r,1}, \ldots, \phi_{r,2N})(0)$$

and

$$\det(I + D_1) = k_1(0) \times W_1(\phi_{r,1}, \ldots, \phi_{r,2N})(0).$$

As $k_3(0) = k_1(0)$, we obtain the following result [82]:

Theorem 13. *The function v defined by*

$$v(x,y,t) = -2 \frac{|W_3(\phi_{3,1}, \ldots, \phi_{3,2N})(0)|^2}{(W_1(\phi_{1,1}, \ldots, \phi_{1,2N})(0))^2}$$

is a solution of the KPI equation depending on $2N-1$ real parameters and a_k, b_k and ϵ, with ϕ_ν^r defined by

$$\begin{aligned}
\phi_{r,\nu} &= \sin\left(\frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{r,\nu}}{2} - i\frac{\tau_\nu}{2} t + \gamma_\nu w - i\frac{e_\nu}{2}\right), \quad 1 \le \nu \le N, \\
\phi_{r,\nu} &= \cos\left(\frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{r,\nu}}{2} - i\frac{\tau_\nu}{2} t + \gamma_\nu w - i\frac{e_\nu}{2}\right), \quad N+1 \le \nu \le 2N, \quad r = 1,3,
\end{aligned}$$

where κ_ν, δ_ν, $x_{r,\nu}$, γ_ν, e_ν are defined in (54)–(56).

3.3.3. Rational Solutions of Order N to the KPI Equation Depending on $2N-2$ Parameters

From the two previous results, we can build rational solutions of the KPI equation as a ratio of two determinants. We use the following notations:

$$X_\nu = \frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{3,\nu}}{2} - i\frac{\tau_\nu}{2} t - i\frac{e_\nu}{2},$$

$$Y_\nu = \frac{\kappa_\nu x}{2} + i\delta_\nu y - i\frac{x_{1,\nu}}{2} - i\frac{\tau_\nu}{2} t - i\frac{e_\nu}{2},$$

for $1 \le \nu \le 2N$, with κ_ν, δ_ν, $x_{r,\nu}$ and the parameters e_ν defined previously.

We consider the following functions:

$$\begin{aligned}
\varphi_{4j+1,k} &= \gamma_k^{4j-1} \sin X_k, \quad \varphi_{4j+2,k} = \gamma_k^{4j} \cos X_k, \\
\varphi_{4j+3,k} &= -\gamma_k^{4j+1} \sin X_k, \quad \varphi_{4j+4,k} = -\gamma_k^{4j+2} \cos X_k,
\end{aligned} \tag{67}$$

for $1 \le k \le N$, and

$$\begin{aligned}
\varphi_{4j+1,N+k} &= \gamma_k^{2N-4j-2} \cos X_{N+k}, \quad \varphi_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin X_{N+k}, \\
\varphi_{4j+3,N+k} &= -\gamma_k^{2N-4j-4} \cos X_{N+k}, \quad \varphi_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin X_{N+k},
\end{aligned} \tag{68}$$

for $1 \leq k \leq N$.

We define the functions $\psi_{j,k}$ for $1 \leq j \leq 2N$, $1 \leq k \leq 2N$ in the same way; the term X_k is only replaced by Y_k.

$$\begin{array}{ll} \psi_{4j+1,k} = \gamma_k^{4j-1} \sin Y_k, & \psi_{4j+2,k} = \gamma_k^{4j} \cos Y_k, \\ \psi_{4j+3,k} = -\gamma_k^{4j+1} \sin Y_k, & \psi_{4j+4,k} = -\gamma_k^{4j+2} \cos Y_k, \end{array} \tag{69}$$

for $1 \leq k \leq N$, and

$$\begin{array}{ll} \psi_{4j+1,N+k} = \gamma_k^{2N-4j-2} \cos Y_{N+k}, & \psi_{4j+2,N+k} = -\gamma_k^{2N-4j-3} \sin Y_{N+k}, \\ \psi_{4j+3,N+k} = -\gamma_k^{2N-4j-4} \cos Y_{N+k}, & \psi_{4j+4,N+k} = \gamma_k^{2N-4j-5} \sin Y_{N+k}, \end{array} \tag{70}$$

for $1 \leq k \leq N$.

The following quotient

$$q(x,t) := \frac{W_3(0)}{W_1(0)}$$

can be rewritten as

$$q(x,t) = \frac{\Delta_3}{\Delta_1} = \frac{\det(\varphi_{j,k})_{j,k\in[1,2N]}}{\det(\psi_{j,k})_{j,k\in[1,2N]}}. \tag{71}$$

Then, we use the previous result to obtain [83].

Theorem 14. *The function v defined by*

$$v(x,y,t) = -2\frac{|\det((n_{jk})_{j,k\in[1,2N]})|^2}{\det((d_{jk})_{j,k\in[1,2N]})^2} \tag{72}$$

is a rational solution of the KPI Equation (53).

$$(4u_t - 6uu_x + u_{xxx})_x - 3u_{yy} = 0,$$

with

$$\begin{array}{ll} n_{j1} = \varphi_{j,1}(x,y,t,0),\ 1 \leq j \leq 2N & n_{jk} = \dfrac{\partial^{2k-2}\varphi_{j,1}}{\partial\epsilon^{2k-2}}(x,y,t,0), \\ n_{jN+1} = \varphi_{j,N+1}(x,y,t,0),\ 1 \leq j \leq 2N & n_{jN+k} = \dfrac{\partial^{2k-2}\varphi_{j,N+1}}{\partial\epsilon^{2k-2}}(x,y,t,0), \\ d_{j1} = \psi_{j,1}(x,y,t,0),\ 1 \leq j \leq 2N & d_{jk} = \dfrac{\partial^{2k-2}\psi_{j,1}}{\partial\epsilon^{2k-2}}(x,y,t,0), \\ d_{jN+1} = \psi_{j,N+1}(x,y,t,0),\ 1 \leq j \leq 2N & d_{jN+k} = \dfrac{\partial^{2k-2}\psi_{j,N+1}}{\partial\epsilon^{2k-2}}(x,y,t,0), \\ 2 \leq k \leq N,\ 1 \leq j \leq 2N & \end{array} \tag{73}$$

The functions φ and ψ are defined in (67)–(70).

3.4. Another Representation of the Solutions to the KPI Equation

By using the previous results, we can construct other types of solutions to the KPI equation depending only on the terms γ_ν, $1 \leq \nu \leq 2N$. This is carried out by expressing κ_ν, δ_ν, τ_ν and $x_{r,\nu}$ in function of γ_ν, for $1 \leq \nu \leq 2N$, and we obtain

$$\kappa_j = \frac{4\gamma_j}{(1+\gamma_j^2)}, \quad \delta_j = \frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}, \quad x_{r,j} = (r-1)\ln\frac{\gamma_j - i}{\gamma_j + i}, \quad \tau_j = \frac{-8i\gamma_j(3-2\gamma_j^2+3\gamma_j^4)}{(1+\gamma_j^2)^3},$$
$$1 \le j \le N,$$
$$\kappa_j = \frac{4\gamma_j}{(1+\gamma_j^2)}, \quad \delta_j = -\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}, \quad x_{r,j} = (r-1)\ln\frac{\gamma_j + i}{\gamma_j - i}, \quad \tau_j = \frac{-8i\gamma_j(3-2\gamma_j^2+3\gamma_j^4)}{(1+\gamma_j^2)^3},$$
$$N+1 \le j \le 2N.$$

We obtain the following representation [83]:

Theorem 15. *The function v defined by*

$$v(x,y,t) = -2\frac{\left|\det[(\partial_w^{\mu-1}\tilde{\phi}_{3,\nu}(0))_{\nu,\,\mu\in[1,\ldots,2N]}]\right|^2}{(\det[(\partial_w^{\mu-1}\tilde{\phi}_{1,\nu}(0))_{\nu,\,\mu\in[1,\ldots,2N]}])^2} \tag{74}$$

is a solution of the KPI Equation (53).

The functions $\tilde{\phi}_{r,\nu}$ are defined by

$$\tilde{\phi}_{r,j}(w) = \sin\left(\frac{2\gamma_j}{(1+\gamma_j^2)}x + i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t - 4\frac{\gamma_j(3-2\gamma_j^2+3\gamma_j^4)}{(1+\gamma_j^2)^3}y - i\frac{(r-1)}{2}\ln\frac{\gamma_j - i}{\gamma_j + i} + \gamma_j w - i\frac{e_j}{2}\right),$$
$$\tilde{\phi}_{r,N+j}(w) = \cos\left(\frac{2\gamma_j}{(1+\gamma_j^2)}x - i\frac{4\gamma_j(1-\gamma_j^2)}{(1+\gamma_j^2)^2}t - 4\frac{\gamma_j(3-2\gamma_j^2+3\gamma_j^4)}{(1+\gamma_j^2)^3} + i\frac{(r-1)}{2}\ln\frac{\gamma_j - i}{\gamma_j + i} + \frac{1}{\gamma_j}w - i\frac{e_{N+j}}{2}\right), \tag{75}$$
$$1 \le j, \le N,$$
with
$$e_j = i\sum_{k=1}^{N-1} a_k\epsilon^{2k+1}j^{2k+1} - \sum_{k=1}^{N-1} b_k\epsilon^{2k+1}j^{2k+1},$$
$$e_{N+j} = i\sum_{k=1}^{N-1} a_k\epsilon^{2k+1}j^{2k+1} + \sum_{k=1}^{N-1} b_k\epsilon^{2k+1}j^{2k+1}, a_i,\ b_i \text{ reals} \quad 1 \le j \le N-1.$$

Remark 2. *In the relation (74), the determinants $\det[(\partial_w^{\mu-1}f_\nu(0))_{\nu,\,\mu\in[1,\ldots,2N]}]$ are the Wronskians of the functions $f_1,\ldots,f_{2N}$ evaluated in $w=0$. In particular, $\partial_w^0 f_\nu$ means f_ν.*

3.4.1. Structure of Rational Solutions to the KPI Equation

The structure of the rational solutions to the KPI equation is given by the following theorem [83]:

Theorem 16. *The function v defined by*

$$v(x,y,t) = -2\frac{\left|\det((n_{jk})_{j,k\in[1,2N]})\right|^2}{(\det((d_{jk})_{j,k\in[1,2N]}))^2} \tag{76}$$

is a rational solution of the KPI equation quotient of two polynomials $n(x,y,t)$ and $d(x,y,t)$ depending on $2N-2$ real parameters a_j and b_j, $1 \le j \le N-1$.

n and d are polynomials of degrees $2N(N+1)$ in x, y and t.

The terms n_{jk} and d_{jk} are defined by (73) and the functions φ and ψ are defined by (67)–(70).

3.4.2. The Maximum Modulus of the Solution of Order N to the KPI Equation

We are entirely free to choose γ_j in such a way that the conditions on λ_j are verified. In order to obtain the simplest expressions in the preceding families, particular solutions

are chosen. Here, we choose $\gamma_j = j\epsilon$ for $1 \leq j \leq N$ so that the conditions on λ_j are verified. Then, we obtain the following statement [83]:

Theorem 17. *The function v_0 defined by*

$$v_0(x,y,t) = -2\left(\frac{|\det((n_{jk})_{j,k\in[1,2N]})|^2}{\left(\det((d_{jk})_{j,k\in[1,2N]})\right)^2}\right)_{(a_j=b_j=0,1\leq j\leq N-1)} \tag{77}$$

is the solution of order N of the KPI Equation (53) whose maximum modulus is equal to $2(2N+1)^2$.

Remark 3. *In (77), matrices $(n_{jk})_{j,k\in[1,2N]}$ and $(d_{jk})_{j,k\in[1,2N]}$ are defined by (73).*

By studying the solutions in the (x,y) plane of coordinates for different values of time t, we can see that different structures appear.

In the case $N=1$, we obtain a peak whose height decreases very quickly as t grows.

For $N=2$, there is the formation of three peaks when the parameters a_1 or b_1 are not equal to 0.

In the case $N=3$, for $a_1 \neq 0$ or $b_1 \neq 0$ and other parameters equal to zero, we obtain a triangle with six peaks; for $a_2 neq 0$ or $b_2 \neq 0$ and other parameters equal to zero, we obtain a ring of five peaks with a peak in the center.

In the case $N=4$, for $a_1 \neq 0$ or $b_1 \neq 0$ and other parameters equal to zero, we obtain a triangle with 10 peaks; for $a_2 \neq 0$ or $b_2 \neq 0$, and others parameters equal to zero, two concentric rings of 5 peaks on each of them are obtained; in the last case, when $a_3 \neq 0$ or $b_3 \neq 0$, and the other parameters equal to zero, we obtain a ring with 7 peaks.

For $N=5$, for a given t value, when a parameter grows and the other parameters are equal to zero, we obtain a triangle or concentric rings. There are four types of configurations. For $a_1 \neq 0$ or $b_1 \neq 0$ and other parameters equal to zero, we obtain a triangle with 15 peaks. For $a_2 \neq 0$ or $b_2 \neq 0$ and other parameters equal to zero, we obtain three concentric rings with five peaks on each of them. For $a_3 \neq 0$ or $b_3 \neq 0$ and other parameters equal to zero, we obtain two concentric rings with seven peaks on each of them with a central peak; in the latter case, when $a_4 \neq 0$ or $b_4 \neq 0$ and the other parameters are equal to zero, we obtain a ring with nine peaks with the "lump" L_3 in the center.

For the last case studied, $N=6$, for a given t, when a parameter grows and other parameters equal zero, one obtains triangles, rings or concentric rings. There are five types of configurations. For $a_1 \neq 0$ or $b_1 \neq 0$ and other parameters equal to zero, we obtain a triangle with 21 peaks. For $a_2 \neq 0$ or $b_2 \neq 0$ and other parameters equal to zero, we obtain three concentric rings of 5, 10 and 5 peaks, respectively. For $a_3 \neq 0$ or $b_3 \neq 0$ and other parameters equal to zero, we obtain three concentric rings with seven peaks on each of them. In case $a_4 \neq 0$ or $b_4 \neq 0$ and other parameters equal to zero, we obtain two concentric rings with nine peaks, with the "lump" L_4 in the middle. In the last case, where $a_5 \neq 0$ or $b_5 \neq 0$ and other parameters equal zero, we obtain a rings of 11 peaks with the "lump" L_4 in the center.

3.4.3. Another Approach to Obtaining Solution to the KPI Equation

We consider the matrix M defined by

$$m_{ij} = \sum_{k=0}^{i} c_{i-k}\left(\frac{\sqrt{p^2-4}}{3}\partial_p\right)^k \sum_{l=0}^{j} c_{j-l}\left(\frac{\sqrt{q^2-4}}{3}\partial_q\right)^l \times\left(\frac{1}{p+q}\exp\left(\frac{1}{2}(p+q)(-x+\frac{3}{4}t)-\frac{1}{4}(p^2-q^2)y\right)\right)_{p=q=-1}. \tag{78}$$

The coefficients c_j are defined by

$$c_{2j} = 0, \quad c_{2j+1} = a_j + ib_j \quad 1 \leq j \leq N-1, \tag{79}$$

where a_j and b_j are arbitrary real numbers.

Then, we have the following result [84]:

Theorem 18. *The function v defined by*

$$v(x,y,t) = 2\,\partial_x^2\left(\ln\det(m_{2i-1,2j-1})_{1\leq i,j\leq N}\right) \tag{80}$$

is a solution to the KPI Equation (53) depending on $2N-2$ *parameters* a_k, b_k, $1 \leq k \leq N-1$.

In this result, the order of the solution is the order of the determinant that defines it.

The solutions at order N depend on $2N-2$ parameters for $N \geq 2$. For $N = 1$, the solution does not depend on parameters. We formulate the first example of solutions of this hierarchy giving rogue waves.

Example 1. ***Rational solutions of order*** 1

The function v defined by

$$v(x,y,t) = -2\,\frac{n(x,y,t)}{d(x,y,t)^2} \tag{81}$$

is a rational solution to the KPI Equation (53), with

$n(x,y,t) = -512\,x^2 + (768\,t + 1024)x + 512\,y^2 - 768\,t - 288\,t^2$,

and

$d(x,y,t) = 16\,x^2 + (-24\,t - 32)x + 9\,t^2 + 16\,y^2 + 24\,t + 32$

4. The Lakshmanan–Porsezian–Daniel Equation and Rogue Waves

4.1. The Lakshmanan–Porsezian–Daniel Equation

The Lakshmanan–Porsezian–Daniel (LPD) equation is considered in the following normalization:

$$iu_t + u_{xxxx} + 8|u|^2u_{xx} + 2u^2\overline{u}_{xx} + 6u_x^2\overline{u} + 4u|u_x|^2 + 6|u|^4u = 0, \tag{82}$$

where the subscripts mean the partial derivatives.

One of the most important models for describing the dynamics of soliton propagation is the nonlinear Lakshmanan–Porsezian–Daniel (LPD) equation. This model was introduced in the context of the Heisenberg spin chain [85]. The LPD equation has been studied in the polarization-preserving fibers context [86] in the study of the propagation of periodic ultrashort pulses in optical fibers [87].

Many mathematical methods have been employed to study this equation. We can quote the method of undetermined coefficients in [88], the extended trial equation method used to construct bright solitons, dark solitons, periodic solitary waves and rational and elliptic solutions in [89], the semi-inverse variational principle in [90] used to construct bright solitons with Kerr and power laws of nonlinearity, the extended Jacobi elliptic function approach [91] used to obtain dark and singular optical solitons, the $\exp(-\phi(\epsilon))$-expansion method in [92], the Lie symmetry analysis in [93], the Riccati equation method [94], the sine–Gordon expansion method in [95] and the modified extended direct algebraic method [96,97].

Recently, in [98], new wave solutions to the LPD equation were built with Kerr nonlinearity using the Bäcklund transformation method based on the Riccati equation, Kudryashov method and a new auxiliary ordinary differential equation (ODE). Akram et al. [99] constructed bright, dark, singular, kink and periodic optical solitons solutions of the LPD equation by an improved $\tan \psi(\eta)/2$ method. In [100], the new bright, periodic wave and singular optical soliton solutions were constructed via the $m + \frac{G'}{G}$ expansion method. Yepez constructed [101] new analytical solutions to the LPD equation using Jacobi elliptic functions.

4.2. Quasi-Rational Solutions to the Lakshmanan–Porsezian–Daniel Equation

We give some partial results obtained by the author and given in [102], where quasi-rational solutions to the Lakshmanan–Porsezian–Daniel equation are presented. Explicit expressions of these solutions for the first orders depending on real parameters were constructed.

Configurations of the modulus of the solutions in the (x, t) plane in function of the different parameters have been studied.

In the case of order 2, three rogue waves have been observed as moving according to the values of parameters.

In the case of order 3, six rogue waves have been observed with special configurations moving according to the four parameters (Figure 14).

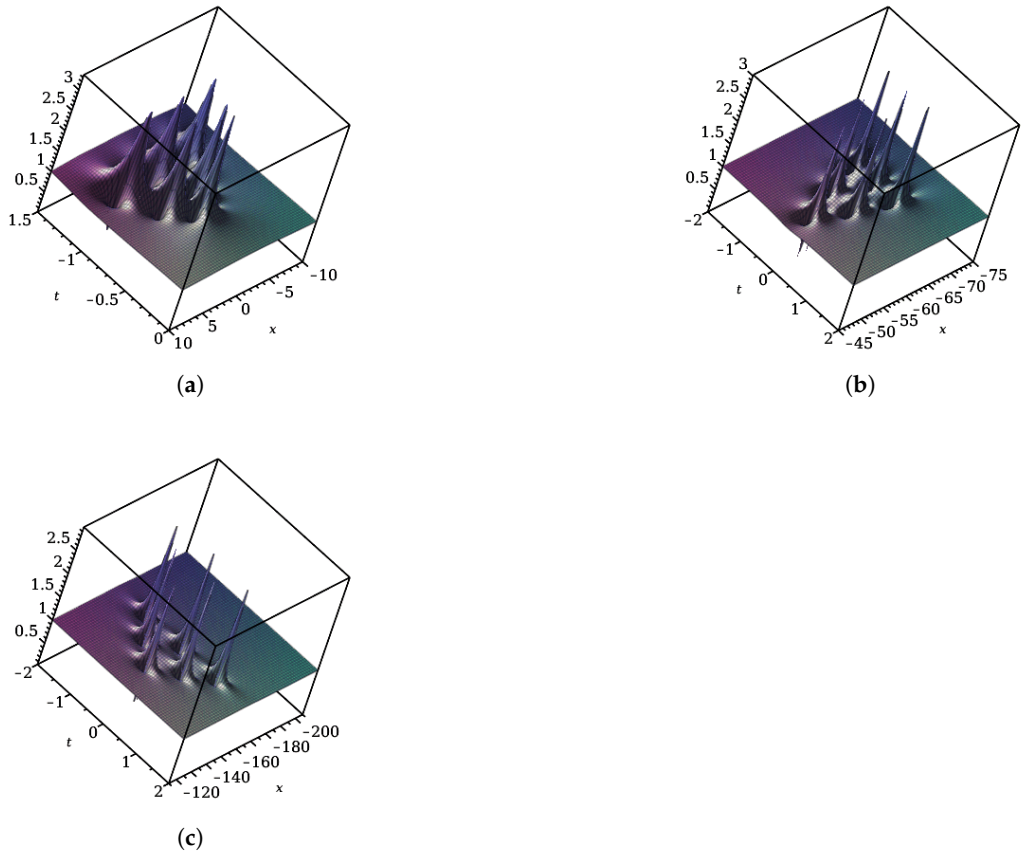


Figure 14. Solution of order 3 to the LPD Equation (82): (**a**) $a_1 = 5$, $b_1 = 0$, $a_2 = 0$, $b_2 = 0$; (**b**) $a_1 = 0$, $b_1 = 10$, $a_2 = 0$, $b_2 = 0$; (**c**) $a_1 = 0$, $b_1 = 0$, $a_2 = 5$, $b_2 = 0$; with $\alpha = \beta = 1$.

For example, we obtain the following solutions of order 1.

Example 2. *The function $v(x,t)$ defined by*

$$v(x,t) = \left(1 - 4\,\frac{1 + 24\,it}{1 + 4\,x^2 + 576\,t^2}\right) e^{6\,it} \tag{83}$$

is a solution to the Lakshmanan–Porsezian–Daniel Equation (82)

If interested, the reader will find in the article the other expressions of the solutions and studies of the obtained configurations. For example, we give some patterns in the case of order 3.

For the case of order 2, the structure of the solutions evolves toward three peaks when parameters a_1 or b_1 grow.

For the case of order 3, the evolution of the structure of the solutions toward six peaks is observed when parameters a_1, a_2, a_1, a_2 grow.

5. The Hirota Equation and Rogue Waves

5.1. The Hirota Equation

In this section, the Hirota (H) equation is considered.

$$iu_t + \alpha(u_{xx} + 2|u|^2 u) - i\beta(u_{xxx} + 6|u|^2 u_x) = 0, \tag{84}$$

where the subscript represents partial derivatives and α, β represent real numbers.

This equation was first presented in a different formulation by Hirota in 1973:

$$iu_t + \rho_1 u_{xx} + \delta_1 |u|^2 u + i\sigma_1 u_{xxx} + 3i\alpha_1 |u|^2 u_x = 0,$$

where α_1, ρ_1, σ_1, δ_1 are real numbers satisfying $\alpha_1\rho_1 = \sigma_1\delta_1$. Hirota constructed a kind of soliton in [50].

This last equation can be rewritten as (84) by choosing $\alpha_1 = -2\beta$, $\delta_1 = 2\alpha$, $\rho_1 = \alpha$, $\sigma_1 = -\beta$.

In 1998, a two-dimensional extension of this equation was introduced by Maccari [103]. This equation is used to describe the evolution of the slowly varying amplitude of a nonlinear train in weakly nonlinear systems. The Hirota equation is used in different branches of physics, such as the propagation of optical pulses [104] or in plasma physics [105].

We can quote some methods used to solve this equation, such as the general projective Riccati equation method [106], the Darboux transformation [107,108] or the trace method [109].

Recently, Demontis, Ortenzi and Van der Mee [110], by using the inverse scattering transform, constructed an explicit soliton solution formula for the Hirota equation. This formula allows one to obtain the N-soliton solutions.

By using the Hirota direct method, lump-soliton solutions for the Hirota equation were given by Zhou, Manukure and Ma [111].

Liu and Yu [112] constructed mixed localized wave solutions for the Hirota equation through the modified Darboux transformation.

Here, we construct quasi-rational solutions for the first orders.

5.2. Quasi-Rational Solutions to the Hirota Equation

We present in the following some results obtained by the author in [113], where quasi-rational solutions to the Hirota equation are constructed. Quasi-rational solutions to the focusing NLS equation and also rational solutions to the mKdV equation are obtained.

Configurations of these configurations in the (x,t) plane are studied.

For example, in the case of order 1, we obtain the following solution:

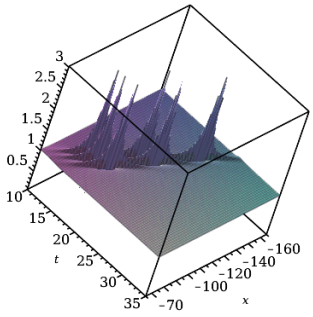
Example 3. *The function* $v(x,t)$ *defined by*

$$v(x,t) = \left(1 - 4\,\frac{1 + 4\,i\alpha t}{1 + (2\,x + 12\,\beta t)^2 + 16\,\alpha^2 t^2}\right) e^{2\,i\alpha t} \tag{85}$$

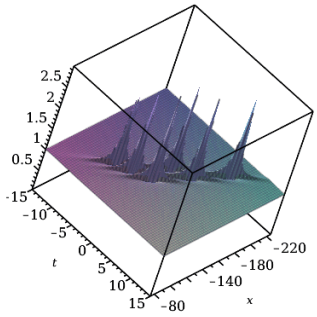
is a solution to the Hirota Equation (84)

$$iu_t + \alpha(u_{xx} + 2|u|^2 u) - i\beta(u_{xxx} + 6|u|^2 u_x) = 0.$$

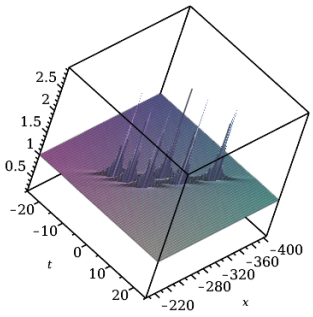
In the case of order 3, we give some patterns of the modulus of the solutions in function of parameters (Figure 15).



(**a**)



(**b**)



(**c**)

Figure 15. Solution to the Hirota equation of order 3 to (84): (**a**) $a_1 = 3$, $b_1 = 0$, $a_2 = 8$, $b_2 = 0$; (**b**) $a_1 = 0$, $b_1 = 5$, $a_2 = 0$, $b_2 = 0$; (**c**) $a_1 = 0$, $b_1 = 10$, $a_2 = 10$, $b_2 = 0$; with $\alpha = \beta = 1$.

For the case of order 2, the structure of the solutions evolves toward three peaks when parameters a_1 or b_1 grow.

For the case of order 3, we observed also the evolution of the structure of the solutions toward six peaks when parameters a_1, a_2, a_1, a_2 grow.

6. Conclusions

Most of this study concerns the NLS equation and the KPI equation. The Darboux method used by Akhmediev and his collaborators led to the construction of solutions up to order 9. The generalized Darboux method proposed by Guo, Ling and Liu provided solutions until the fourth order. The method of Ohta and Yang based on Hirota's method provided solutions up to order 5. V.B. Matveev managed to construct solutions up to order 5 with difficulties in isolating the famous Peregrine breathers.

With the formulation in terms of the degeneracy of Riemann theta functions [53], we succeed in constructing explicitly quasi-rational solutions to the NLS equation depending on $2N - 2$ real parameters until order 13.

This is a significant advance in the study of rogue waves. NLS solutions can be written as a quotient of two $2N$-order determinants depending on $2N-2$ parameters.

It is noteworthy that, in this type of representation of solutions to the NLS equation, the Peregrine breathers are obtained when all parameters are equal to 0. This method is the most efficient of the different methods. While the other methods have difficulty in building explicitly quasi-rational solutions beyond the fifth order, this one allows one to build the quasi-rational solutions of order N depending on $2N-2$ parameters up to the order $N=13$ and the associated Peregrine breathers.

It was found that the values of these parameters very strongly influence the structure of the solutions. At order N, there is the formation of at most $\frac{N(N+1)}{2}$ peaks (results given in Dubard's thesis [114] in 2011) for the modulus of solutions represented in the $(x;t)$ plane of variables, up to the order 13.

At order N, the solutions to the NLS equation depend on $2N-2$ real parameters a_j and b_j. The ranking of indices is crucial in the configurations of solutions.

There is the formation of triangular structures and concentric ring structures according to the different values of parameters a_j and b_j. We obtain the following results:

For $j=1$, there is the formation of a triangle with $\frac{N(N+1)}{2}$ peaks, $(N\geq 4)$ (results already appearing in the works of several authors, in particular, those of Matveev and Akhmediev);

For $j=N-3$, there is the formation of three concentric rings of $2N-5$ peaks with, in the center, the Peregrine breather P_{N-6}, $N\geq 7$;

For $j=N-2$, there is the formation of two concentric rings of $2N-3$ peaks with, in the center, the Peregrine breather P_{N-4}, $(N\geq 5)$;

For $j=N-1$, there is the formation of a ring with $2N-1$ peaks with, in its center, the Peregrine breather P_{N-2};

Generally, for $j=N-k$, in the case of $N>0$ and $2k<N-1$, there is the formation of k concentric rings of $2N-2k+1$ peaks with, in the center, the Peregrine breather P_{N-2k}.

From different studies realized by the author, it has been highlighted that the maximums of the modulus of the solutions to the NLS equation were in connection with the zeros of the Yablonski–Vorobev polynomials [115,116]. This last property was made by the author in 2019 [51], before the study realized in 2021 by Y. Yang [117].

For the LPD and Hirota equations, it will be relevant to find a general formulation of the solutions at order N. Some partial results in this direction are found, as well as for some equations of the AKNS hierarchy. We hope to give new results in a forthcoming paper.

Funding: This research received no external funding.

Data Availability Statement: The raw data supporting the conclusions of this article will be made available by the authors on request.

Conflicts of Interest: The author declares no conflicts of interest.

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