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Strong k-Skew Commutativity Preserving Maps on Standard Operator Algebras

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Abstract: Let $\mathcal{A}$ be a self-adjoint standard operator algebra on a real or complex Hilbert space of dimension ≥ 2, and let $k \in \{1,2,3\}$. The k-skew commutator for $A, B \in \mathcal{A}$ is defined by ${}_*[A,B]_1 = AB - BA^*$ and ${}_*[A,B]_k = {}_*[A, {}_*[A,B]_{k-1}]_1$. Assume that $\Phi : \mathcal{A} \to \mathcal{A}$ is a map whose range contains all rank-one projections. In this paper, we prove that Φ is strong k-skew-commutativity preserving, that is, ${}_*[\Phi(A),\Phi(B)]_k = {}_*[A,B]_k$ for all $A, B \in \mathcal{A}$ if and only if one of the following statements holds: (i) Φ is either the identity map or the negative identity map whenever $k \in \{1,3\}$; (ii) Φ is the identity map whenever $k = 2$.

Keywords: bounded linear operators; self-standard operator algebras; k-skew commutators; preservers; Hilbert spaces

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1. Introduction

Let X be a real or complex Banach space and denote by $\mathcal{B}(X)$ the algebra of all bounded linear operators on X. For a positive integer k and any elements $A, B \in \mathcal{B}(X)$, the k-commutator or k-Lie product of A and B is defined inductively by $[A,B]_0 = A$, $[A,B]_1 = [A,B] = AB - BA$ and $[A,B]_k = [[A,B]_{k-1}, B]$. The k-commutators appear naturally in the famous Baker–Campbell–Hausdorff formula

$$e^B A e^{-B} = A - \frac{1}{1!}[A,B] + \frac{1}{2!}[[A,B],B] - \frac{1}{3!}[[[A,B],B],B] + \ldots = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}[A,B]_k,$$

which is used in quantum physics frequently and has been investigated extensively.

Let $\mathcal{R}$ be a ring. A map $\Phi : \mathcal{R} \to \mathcal{R}$ is said to be strong k-commutativity preserving if Φ preserves k-commutators invariant, that is, $[\Phi(A),\Phi(B)]_k = [A,B]_k$ for all $A, B \in \mathcal{A}$. The problem of characterizing strong k-commutativity-preserving maps has been studied intensively on various rings and operator algebras. For example, the readers can see [1–6] and the references therein.

If $X = H$ is a real or complex Hilbert space, then $\mathcal{B}(H)$ is a $*$-algebra with $*$ operation the usual adjoint of operators. Thus, the skew Lie product or the skew commutator of A and B may be defined as ${}_*[A,B] = AB - BA^*$ (Ref. [7], where ${}_*[A,B]$ is denoted by $[A,B]_*$). For any positive integer $k \geq 1$, like the concept of the k-commutator, the k-skew commutator of a pair A, B in $\mathcal{B}(H)$ is naturally defined inductively by ${}_*[A,B]_k = {}_*[A, {}_*[A,B]_{k-1}]$ with

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$_*[A,B]_0 = B$, $_*[A,B]_1 = {}_*[A,B] = AB - BA^*$. Note that the k-skew commutators appear in so-called skew Baker–Campbell–Hausdorff formula in a natural way:

$$e^A B e^{-A^*} = \sum_{k=0}^{\infty} \frac{1}{k!} {}_*[A,B]_k.$$

The concept of k-skew Lie product is found playing an important role in some research topics, and their study has attracted many authors' attention (Refs. [8–18]).

Recall that a standard operator algebra $\mathcal{A}$ on H is said to be self-adjoint if $A \in \mathcal{A}$ implies $A^* \in \mathcal{A}$. The concept of the k-skew commutator can be introduced for any ring $\mathcal{R}$ with involution $*$. Similar to the commutator case stated in the previous paragraph, a map $\Phi : \mathcal{R} \to \mathcal{R}$ is said to be strong k-skew-commutativity preserving if Φ keeps the k-skew commutators invariant, i.e., $_*[\Phi(A), \Phi(B)]_k = {}_*[A,B]_k$ for any $A, B \in \mathcal{A}$. For the problem of how to characterize the strong k-skew-commutativity-preserving maps, it was shown in [11] that a nonlinear surjective map Φ on a factor von Neumann algebra $\mathcal{A}$ is strong skew-commutativity preserving if and only if it has the form $A \mapsto \Psi(A) + h(A)I$, where Ψ is a strong skew-commutativity-preserving bijective linear map and $h : \mathcal{A} \to \mathbb{R}$ is a functional with $h(0) = 0$. As usual, $\mathbb{R}$ and $\mathbb{C}$ stand, respectively, for the real number field and the complex number field. In [16], a more precise form of the above surjective map was achieved: $A \mapsto ZA$, where $Z \in \mathcal{Z}_S$, the set of all self-adjoint elements in the center of $\mathcal{A}$. In a pure algebraic frame, it is also shown in [16] that if $\mathcal{R}$ is a prime $*$-ring which contains a nontrivial symmetric idempotent, then every strong skew commutativity preserving surjective map on $\mathcal{R}$ has the form $\Phi(A) = ZA + f(A)$ for all $A \in \mathcal{R}$, where $Z \in \mathcal{Z}_S(\mathcal{R})$ and $f : \mathcal{R} \to \mathcal{Z}_S(\mathcal{R})$. This result later was generalized in [15] and [14] to unital $*$-rings $\mathcal{R}$ which contain a nontrivial symmetric idempotent P satisfying "$A\mathcal{R}P = \{0\} \Rightarrow A = 0$" and "$A\mathcal{R}(I-P) = \{0\} \Rightarrow A = 0$". Recently, ref. [12] has sharpened the results in [14–16] by proving that the symmetric center valued map f is 0 and then gives further a characterization of the strong 2-skew-commutativity-preserving maps; that is, a surjective map $\Phi : \mathcal{R} \to \mathcal{R}$ is strong 2-skew commutativity preserving if and only if $\Phi(A) = \lambda A$ for all $A \in \mathcal{R}$, where $\lambda \in \mathcal{C}_S$ with $\lambda^3 = I$ and $\mathcal{C}_S$ is the symmetric extended centroid of $\mathcal{R}$. This result then is applied to some operator algebras such as prime C*-algebras, factor von Neumann algebras, the indefinite self-adjoint standard operator algebras, and symmetric standard operator algebras.

Suppose that $\mathcal{A}$ is a self-adjoint standard operator algebra on a *complex* Hilbert space of dimension ≥ 2. Without the surjectivity assumption on the maps, it is conjectured in [18] that a map on $\mathcal{A}$ with range containing all rank-one projections is a strong k-skew-commutativity-preserving map if and only if it has the form $A \mapsto \lambda A$, where $\lambda \in \{1, -1\}$ with $\lambda^{k+1} = 1$. However, no approach has been found to provide a proof valid for all cases of dimension and the value of k. In [18], the authors gave a proof of the above result for the case when $\dim H \geq 3$ and $k \geq 4$, which is not valid for the rest of the cases $1 \leq k \leq 3$ or $\dim H = 2$. Thus, the proofs for the cases that $\dim H = 2$ or $k \in \{1, 2, 3\}$ still need to be presented.

One of the purposes of this paper is to prove the main result announced in [18] for the cases $1 \leq k \leq 3$ and $\dim H \geq 2$. But, much more can be said. Though we do not know whether or not the announced result in [18] is still true for the case that the Hilbert spaces are real, we consider here not only the complex but also the real Hilbert spaces H with $\dim H \geq 2$. Under the assumption that the range contains only all rank-one projections, we give a complete characterization of the maps on self-adjoint standard operator algebras that are strong k-skew-commutativity preserving for $1 \leq k \leq 3$. Recall that an operator P on H is called a projection if $P = P^* = P^2$, and the rank of an operator is the dimension of its range.

The following is our main result.

Theorem 1. *Let $k \in \{1,2,3\}$, H be a real or complex Hilbert space with $\dim H \geq 2$ and $\mathcal{A} \subseteq \mathcal{B}(H)$ be a self-adjoint standard operator algebra. Assume that $\Phi : \mathcal{A} \to \mathcal{A}$ is a map whose range contains all rank-one projections. Then, Φ is strong k-skew-commutativity preserving if and only if Φ is the identity map or the negative identity map whenever $k \in \{1,3\}$; Φ is the identity map whenever $k = 2$.*

This paper is organized as follows. Section 2 gives some lemmas that are needed to prove the main result. Some of these lemmas reveal properties of k-skew commutativity and are interesting themselves. Section 3, Section 4, and Section 5 are, respectively, devoted to proving Theorem 1 for the case $k = 1, 2$ and 3.

2. Preliminaries and Lemmas

To prove Theorem 1, we need following lemmas.

Assume that X is a Banach space and $\{A_i,\ B_i\}_{i=1}^m \subset \mathcal{B}(X)$. In [19], the authors proved that if $\sum_{i=1}^m A_i T B_i = 0$ holds for all $T \in \mathcal{B}(X)$ and $A_1, \ldots, A_n$ ($n \leq m$) are linearly independent, then $B_1, \cdots, B_n$ are depend linearly on $B_{n+1}, \ldots, B_m$. By this result, the following lemma is obvious.

Lemma 1. *Let $\mathcal{A}$ be a standard operator algebra on a Banach space, and let $\{A_i,\ B_i\}_{i=1}^n$, $\{C_j,\ D_j\}_{j=1}^m \subset \mathcal{A}$ such that $\sum_{i=1}^n A_i T B_i = \sum_{i=1}^m C_j T D_j$ for all rank-one operators $T \in \mathcal{A}$. If $A_1, \ldots, A_n$ are linearly independent, then each B_i is a linear combination of $D_1, \ldots, D_m$. Similarly, if $B_1, \ldots, B_n$ are linearly independent, then each A_i is a linear combination of $C_1, \ldots, C_m$.*

Bear in mind that

$$ {}_*[A,B]_k = \sum_{i=0}^{k} (-1)^i \mathrm{C}_k^i A^{k-i} B (A^*)^i, $$

where $\mathrm{C}_k^i = \frac{k(k-1)\ldots(k-i+1)}{i!}$, and, particularly,

$$ {}_*[A,B]_1 = AB - BA^*, \quad {}_*[A,B]_2 = A^2B - 2ABA^* + B(A^*)^2 $$

and

$$ {}_*[A,B]_3 = A^3B - 3A^2BA^* + 3AB(A^*)^2 - B(A^*)^3. $$

The following lemma comes from [18]. Notice that the lemma holds automatically if $\dim H = 1$.

Lemma 2 ([18])**.** *Let H be a real or complex Hilbert space, $k \geq 1$ be an integer and $S \in \mathcal{B}(H)$. Then, ${}_*[P,S]_k = 0$ for all rank-one projections P on H if and only if $S = \alpha I$ for some scalar α.*

Let k be a positive integer and S be a bounded linear operator on a complex Hilbert space H. It was shown in [18] that ${}_*[S,P]_k = 0$ for all rank-one projections P if and only if $S = \lambda I + N$ for some $\lambda \in \mathbb{R}$ and $N \in \mathcal{N}_{[\frac{k+1}{2}]}(H)$. Here, $[t]$ stands for the integer part of a real number t and $\mathcal{N}_r(H) = \{A \in \mathcal{B}(H) : A^r = 0\}$. Our next lemma says that this result also holds for real case if $1 \leq k \leq 3$.

Lemma 3. *Let $k \in \{1,2,3\}$, H be a real or complex Hilbert space. Then, for $S \in \mathcal{B}(H)$, the following statements are equivalent.*

(1) ${}_*[S,A]_k = 0$ for all $A \in \mathcal{B}(H)$.

(2) ${}_*[S,P]_k = 0$ for all rank-one projections P on H.

(3) $S = \lambda I + N$ for some real number λ and some nilpotent operator $N \in \mathcal{B}(H)$ with $N^{[\frac{k+1}{2}]} = 0$.

Proof. (3)⇒(1): Note that $[\frac{k+1}{2}] = 1$ if $k = 1, 2$, and $[\frac{k+1}{2}] = 2$ if $k = 3$. So, if S has the form in (3), then $S = \lambda I$ whenever $k = 1$ or 2. In this case it is easily checked that, for any $A \in \mathcal{B}(H)$, ${}_*[S, A]_1 = {}_*[\lambda I, A]_1 = \lambda A - \lambda A = 0$ and ${}_*[S, A]_2 = {}_*[\lambda I, A]_2 = \lambda^2 A - 2\lambda^2 A + \lambda^2 A = 0$. If $k = 3$, then $S = \lambda I + N$ with $N^2 = 0$, and so for any $A \in \mathcal{B}(H)$, we have

$$\begin{aligned} {}_*[S, A]_3 = & \ {}_*[\lambda I + N, {}_*[\lambda I + N, A]_2] = {}_*[N, {}_*[\lambda I + N, A]_2] = \cdots = {}_*[N, A]_3 \\ = & \ N^3 A - 3N^2 A N^* + 3N A (N^*)^2 - A(N^*)^3 = 0. \end{aligned}$$

The implication (1)⇒(2) is obvious.

Let us check (2)⇒(3). Assume that ${}_*[S, P]_k = 0$ holds for every rank-one projection P.

Case 1. $k = 1$. For any unit vector $x \in H$, $P = x \otimes x$ is a rank-one projection. Thus, $Sx \otimes -x \otimes Sx = {}_*[S, x \otimes x]_1 = 0$. It follows that, for any vector x, there exists some scalar α_x depending on x such that $Sx = \alpha_x x$. This implies $S = \lambda I$ for some scalar λ. Moreover, since $0 = {}_*[S, x \otimes x]_1 = (\lambda - \bar{\lambda}) x \otimes x$, λ is real.

Case 2. $k = 2$. If $S = \lambda I$ for some scalar λ, then, picking a rank-one projection P, $(\lambda - \bar{\lambda})^2 P = {}_*[S, P]_2 = 0$ entails that λ is real and thus (3) is true.

So, we may assume that S is linearly independent of I and then induce a contradiction. For any $A \in \mathcal{B}(H)$, define

$$\psi_2(A) = {}_*[S, A]_2 = S^2 A - 2SAS^* + A(S^*)^2.$$

Then, ψ_2 is an elementary operator on $\mathcal{B}(H)$ which is linear and continuous under the weak operator topology (WOT, briefly). Since $\psi_2(P) = 0$ for all rank-one projections P by (2), we see that ψ_2 also vanishes at every finite rank operator. Note that the set of all finite rank operators is dense in $\mathcal{B}(H)$ under WOT. So, $\psi_2 \equiv 0$, that is,

$$S^2 A + A(S^*)^2 = 2SAS^* \tag{1}$$

holds for all $A \in \mathcal{B}(H)$. By Lemma 1, as $S^* \neq 0$, there exist scalars α, β such that $S = \alpha I + \beta S^2$. Thus, the linear independence of S and I implies that $\beta \neq 0$ and hence, $S^2 = \mu I + \gamma S$ for some scalars μ and $\gamma \neq 0$. Substituting this into Equation (1) gives

$$A((\mu + \bar{\mu}) I + \bar{\gamma} S^*) = SA(2S^* - \gamma I)$$

for all $A \in \mathcal{B}(H)$. However, by Lemma 1, this implies that $(\mu + \bar{\mu}) I + \bar{\gamma} S^* = \nu(2S^* - \gamma I)$ for some scalar ν, contradicting to the fact that S^* is linearly independent to I.

Case 3. $k = 3$. For any $A \in \mathcal{B}(H)$, define

$$\psi_3(A) = {}_*[S, A]_3 = \sum_{i=0}^{3} (-1)^i \mathrm{C}_3^i S^{3-i} A (S^*)^i.$$

Then, ψ_3 is a linear transformation from $\mathcal{A}$ into $\mathcal{B}(H)$ and is continuous under WOT. Since $\psi_3(P) = 0$ for any rank-one projection P, $\psi_3(A) = 0$ holds for every $A \in \mathcal{B}(H)$, that is,

$$S^3 A - 3S^2 A S^* = A(S^*)^3 - 3SA(S^*)^2 \ \text{ for all } \ A \in \mathcal{B}(H). \tag{2}$$

If there exists some $\lambda \in \mathbb{F}$ such that $S = \lambda I$, then Equation (2) reduces to

$$\lambda^3 A - 3\lambda^2 \lambda^* A = (\lambda^*)^3 A - 3\lambda(\lambda^*)^2 A$$

for any A, which implies $(\lambda - \lambda^*)^3 A = 0$ for any A. Hence, λ is real and S has the required form in (3).

In the sequel, we assume that S is linearly independent of I. In this case, applying Lemma 1 to Equation (2), there exist $\lambda_1, \lambda_2, \mu_1, \mu_2 \in \mathbb{F}$ such that

$$S^3 = \lambda_1 + \lambda_2 S \text{ and } S^2 = \mu_1 + \mu_2 S.$$

Substituting these into Equation (2), we obtain

$$(\lambda_1 + \lambda_2 S)A - 3(\mu_1 + \mu_2 S)AS^* = A(\bar{\lambda}_1 + \bar{\lambda}_2 S^*) - 3SA(\bar{\mu}_1 + \bar{\mu}_2 S^*),$$

that is,

$$[(\lambda_1 - \bar{\lambda}_1) + (3\bar{\mu}_1 + \lambda_2)S]A + [-(3\mu_1 + \bar{\lambda}_2) - 3(\mu_2 - \bar{\mu}_2)S]AS^* = 0.$$

As S and I are linearly independent, by Lemma 1 again, one obtains

$$(\lambda_1 - \bar{\lambda}_1) + (3\bar{\mu}_1 + \lambda_2)S = 0 \text{ and } (3\mu_1 + \bar{\lambda}_2) - 3(\mu_2 - \bar{\mu}_2)S = 0.$$

Thus, we must have

$$\lambda_1 = \bar{\lambda}_1, \ \ \lambda_2 = -3\bar{\mu}_1 \text{ and } \mu_2 = \bar{\mu}_2. \tag{3}$$

On the other hand, by $\lambda_1 + \lambda_2 S = S^3 = (\mu_1 + \mu_2 S)S = \mu_1\mu_2 + (\mu_1 + \mu_2^2)S$, we obtain

$$\lambda_1 = \mu_1\mu_2 \text{ and } \lambda_2 = \mu_1 + \mu_2^2. \tag{4}$$

Equations (3) and (4) together entail $\mu_1 = \bar{\mu}_1$ and $\mu_2^2 = -4\mu_1$. Let $N = S - \frac{\mu_2}{2} I$. It is easily checked that $N^2 = S^2 - \mu_2 S - \mu_1 = 0$ and $S = \frac{\mu_2}{2} I + N$ has the required form. □

In the following lemmas, we always assume that $\mathcal{A}$ *is a self-adjoint standard operator algebra on a real or complex Hilbert space* H, $k \in \{1, 2, 3\}$ *and* $\Phi : \mathcal{A} \to \mathcal{A}$ *is a strong* k*-commutativity-preserving map whose range contains all rank-one projections.*

Note that, as the surjectivity of Φ is not supposed, one can not assert that Φ is additive as that in [12]. What we can say in the present situation is the following.

Lemma 4. Φ *is almost additive, that is, for any* $A, B \in \mathcal{A}$*, there exists a scalar* $\gamma_{A,B}$ *depending on* A, B *such that* $\Phi(A + B) = \Phi(A) + \Phi(B) + \gamma_{A,B} I$.

Proof. For any $T \in \mathcal{A}$, we have

$${}_*[\Phi(T), \Phi(A + B) - \Phi(A) - \Phi(B)]_k = {}_*[T, A + B]_k - {}_*[T, A]_k - {}_*[T, B]_k = 0.$$

As the range of Φ contains all rank-one projections, Lemma 2 implies that $\Phi(A + B) - \Phi(A) - \Phi(B) = \gamma_{A,B} I$ for some scalar $\gamma_{A,B}$. □

Similarly, Φ is not symmetric without the surjectivity assumption. But, we have

Lemma 5. *For any* $A \in \mathcal{A}$*, we have* $\Phi(A^*) = \Phi(A)^* + \delta_A I$ *for some scalar* δ_A *depending on* A.

Proof. In fact, for any $A, B \in \mathcal{A}$,

$$\begin{aligned} & {}_*[\Phi(B), \Phi(A^*)]_k = {}_*[B, A^*]_k = (-1)^k {}_*[B, A]_k{}^* \\ = \ & (-1)^k {}_*[\Phi(B), \Phi(A)]_k{}^* = {}_*[\Phi(B), \Phi(A)^*]_k \end{aligned}$$

holds. As the range of Φ contains all rank-one projections, by use of Lemma 2, one obtains $\Phi(A^*) - \Phi(A)^* = \delta_A I$ for some scalar δ_A, a scalar multiple of the identity. □

Lemma 6. *For $A \in \mathcal{A}$, $\Phi(A) \in \mathbb{R}I + \mathcal{N}_{[\frac{k+1}{2}]}(\mathcal{A})$ if and only if $A \in \mathbb{R}I + \mathcal{N}_{[\frac{k+1}{2}]}(\mathcal{A})$. Particularly, for the cases $k = 1, 2$, $\Phi(A) \in \mathbb{R}I$ if and only if $A \in \mathbb{R}I$.*

Proof. For any $A \in \mathbb{R}I + \mathcal{N}_{[\frac{k+1}{2}]}(\mathcal{A})$, by Lemma 3, ${}_*[\Phi(A), \Phi(B)]_k = {}_*[A, B]_k = 0$ holds for all $B \in \mathcal{A}$. As the range of Φ contains all rank-one projections, Lemma 3 also implies that $\Phi(A) \in \mathbb{R}I + \mathcal{N}_{[\frac{k+1}{2}]}(\mathcal{A})$. Conversely, if $S = \Phi(A) \in \mathbb{R}I + \mathcal{N}_{[\frac{k+1}{2}]}(\mathcal{A})$, still by Lemma 3, ${}_*[A, B]_k = {}_*[S, \Phi(B)]_k = 0$ holds for all $B \in \mathcal{A}$, which entails that $A \in \mathbb{R}I + \mathcal{N}_{[\frac{k+1}{2}]}(\mathcal{A})$.

The last assertion holds because $\mathbb{R}I + \mathcal{N}_{[\frac{k+1}{2}]}(\mathcal{A}) = \mathbb{R}I$ if $1 \leq k \leq 2$. □

Lemma 7. *For any projection $P \in \mathcal{A}$, we have*

$$ {}_*[\Phi(P), X]_k \in P\mathcal{A}(I - P) + (I - P)\mathcal{A}P $$

for all $X \in \mathcal{B}(H)$.

Proof. Note that, if P is a projection, we have ${}_*[P, X]_r = {}_*[P, X]_{r+2}$ for any $X \in \mathcal{A}$. Then, for any $A \in \mathcal{A}$,

$$ {}_*[\Phi(P), \Phi(A)]_1 = {}_*[P, A]_1 = PA - AP \in P\mathcal{A}(I - P) + (I - P)\mathcal{A}P, $$

$$ {}_*[\Phi(P), \Phi(A)]_2 = {}_*[P, A]_2 = PA - 2PAP + AP \in P\mathcal{A}(I - P) + (I - P)\mathcal{A}P $$

and

$$ {}_*[\Phi(P), \Phi(A)]_3 = {}_*[P, A]_3 = {}_*[P, A]_1 = PA - AP \in P\mathcal{A}(I - P) + (I - P)\mathcal{A}P. $$

So, ${}_*[\Phi(P), Q]_k \in P\mathcal{A}(I - P) + (I - P)\mathcal{A}P$ for all rank-one projections Q by the assumption on the range of Φ. Now, it is easily checked that ${}_*[\Phi(P), X]_k \in P\mathcal{A}(I - P) + (I - P)\mathcal{A}P$ holds for all operators $X \in \mathcal{B}(H)$. □

The "if" part of Theorem 1 is obviously true. To prove Theorem 1, we need only to check the "only if" part. This will be performed in Sections 3–5 by considering, respectively, $k = 1, 2$, and 3.

3. The Proof of Theorem 1: $k = 1$

In this section, we will check the "only if" part of Theorem 1 for the case $k = 1$.

Let $\mathcal{A}$ be a self-adjoint standard operator algebra on a Hilbert space H of dimension ≥ 2 over the field $\mathbb{F}$ with $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Let $\Phi : \mathcal{A} \to \mathcal{A}$ be a map whose range contains all rank one projections. Assume that Φ is strong skew-commutativity preserving, that is,

$$ \Phi(A)\Phi(B) - \Phi(B)\Phi(A)^* =_* [\Phi(A), \Phi(B)]_1 = {}_*[A, B]_1 = AB - BA^* $$

holds for any $A, B \in \mathcal{A}$. We have to show that $\Phi = \mathrm{id}$ or $\Phi = -\mathrm{id}$, where id stands for the identity map on $\mathcal{A}$. This will be performed in several steps.

Step 3.1. For any nontrivial projection $P \in \mathcal{A}$, we have $\Phi(P) = \lambda_P P + \mu_P I$ for some $\lambda_P, \mu_P \in \mathbb{R}$ with $\lambda_P \neq 0$.

Write $P_1 = P$, $P_2 = I - P$. As P is nontrivial, $P_i \neq 0$, $i = 1, 2$. Write $\Phi(P_1) = S = S_{11} + S_{12} + S_{21} + S_{22}$ with $S_{ij} \in \mathcal{A}_{ij} = P_i\mathcal{A}P_j$, $i, j \in \{1, 2\}$.

Note that, by Lemma 7, $\Phi(P_1)X - X\Phi(P_1)^* \in \mathcal{A}_{12} + \mathcal{A}_{21}$ for any $X \in \mathcal{B}(H)$. Thus, for any $X_{12} \in \mathcal{A}_{12}$, we have

$$S_{21}X_{12} - X_{12}S_{12}^* = P_2(SX_{12} - X_{12}S^*)P_2 = 0.$$

So, $S_{21}X_{12} = 0$ holds for all $X_{12} \in \mathcal{A}_{12}$, which is equivalent to that $S_{21}XP_2 = 0$ holds for all $X \in \mathcal{A}$. By Lemma 1, we must have $S_{21} = 0$. Note that $\Phi(P_1)^* = \Phi(P_1) + \delta I$ for some $\delta \in \mathbb{F}$ by Lemma 5. We see that $S_{12} = S_{21}^* = 0$, and thus $\Phi(P_1) = S_{11} + S_{22}$. It follows that $S_{ii}X_{ii} - X_{ii}S_{ii}^* = P_{i*}[S, X_{ii}]_1P_i = 0$ for any $X_{ii} \in \mathcal{A}_{ii}$, $i = 1, 2$. Since $\mathcal{A}_{ii}$ can be regarded as a self-adjoint standard operator algebra on $P_i(H)$; by Lemma 3, we obtain $S_{ii} = \lambda_i P_i$ for some $\lambda_i \in \mathbb{R}$, $i = 1, 2$. Let $\lambda = \lambda_1 - \lambda_2$ and $\mu = \lambda_2$; then $\Phi(P_1) = \lambda P_1 + \mu I$, as desired. Clearly, $\lambda \neq 0$. Otherwise, $\Phi(P_1) = \mu I$, and Lemma 6 implies that $P_1 = \alpha I$, a contradiction.

Step 3.2. Assume that $\dim H \geq 3$. Then, there exists a nonzero real number λ such that, for any rank-one projection $P \in \mathcal{A}$, $\Phi(P) = \lambda P + \mu_P I$ for some $\mu_P \in \mathbb{R}$.

Let P, Q be two rank-one projections on H so that $P \perp Q$ (i.e., $PQ = 0$). Let $R = P + Q$. Then, R is a nontrivial projection as $\dim H \geq 3$. By Lemma 4 and Step 3.1, we have

$$\lambda_R R + \mu_R I = \Phi(R) = \Phi(P) + \Phi(Q) + \gamma_{P,Q}I = \lambda_P P + \mu_P I + \lambda_Q Q + \mu_Q I + \gamma_{P,Q}I,$$

which implies that $\mu_R = \mu_P + \mu_Q + \gamma_{P,Q}$, and consequently, $\lambda_P = \lambda_Q = \lambda_R$. Now, for any rank-one projections P and Q, the assumption $\dim H \geq 3$ ensures that there exists a rank-one projection Z such that $P \perp Z$ and $Q \perp Z$. Then, by what we just proved above, we have $\lambda_P = \lambda_Z$ and $\lambda_Q = \lambda_Z$. Hence, $\lambda_P = \lambda_Q$ still holds. This means that there exists a nonzero $\lambda \in \mathbb{R}$ so that $\lambda_P = \lambda$ for every rank-one projection P.

Step 3.3. Assume that $\dim H \geq 3$. Then, $\lambda = \pm 1$ and $\Phi(A) = \lambda A$ for every $A \in \mathcal{A}$.

Take rank-one projections P, Q so that $PQ - QP \neq 0$. Then

$$0 \neq {}_*[P, Q]_1 = {}_*[\Phi(P), \Phi(Q)]_1 = {}_*[\lambda P + \mu_P I, \lambda Q + \mu_Q]_1 = \lambda^2 {}_*[P, Q]_1$$

gives $\lambda^2 = 1$. So $\lambda = \pm 1$.

For any $A \in \mathcal{A}$, note that ${}_*[\lambda P, \Phi(A)]_1 = {}_*[\Phi(P), \Phi(A)]_1 = {}_*[P, A]_1$ holds for all rank one projections P. By Lemma 2, this implies that $\lambda\Phi(A) - A \in \mathbb{F}I$. As $\lambda^2 = 1$, and one sees that $\Phi(A) = \lambda A + \alpha_A I$ for some scalar α_A.

We claim that $\alpha_A = 0$ for each A. Let Q be a rank one idempotent operator with $Q^* \neq Q$ and consider

$$\begin{aligned} Q - QQ^* = {} & {}_*[Q, Q]_1 = {}_*[\Phi(Q), \Phi(Q)]_1 = {}_*[\lambda Q + \alpha_Q I, \lambda Q + \alpha_Q I]_1 \\ = {} & Q - QQ^* + (2\lambda\alpha_Q - \lambda\bar{\alpha}_Q)Q - \lambda\alpha_Q Q^* + (\alpha_Q^2 - |\alpha_Q|^2)I. \end{aligned}$$

One obtains

$$(2\lambda\alpha_Q - \lambda\bar{\alpha}_Q)Q - \lambda\alpha_Q Q^* + (\alpha_Q^2 - |\alpha_Q|^2)I = 0.$$

Since $\dim H \geq 3$ and $\operatorname{rank}((2\lambda\alpha_Q - \lambda\bar{\alpha}_Q)Q - \lambda\alpha_Q Q^*) \leq 2$, we must have $\alpha_Q \in \mathbb{R}$. This implies that $\alpha_Q = 0$. It follows that

$$\begin{aligned} & QA - AQ^* = {}_*[Q, A]_1 = {}_*[\Phi(Q), \Phi(A)]_1 \\ = {} & {}_*[\lambda Q, \lambda A + \alpha_A I]_1 = QA - AQ^* + \lambda\alpha_A Q - \lambda\alpha_A Q^*, \end{aligned}$$

which gives $\lambda\alpha_A Q - \lambda\alpha_A Q^* = 0$. Hence $\alpha_A = 0$.

Step 3.4. Assume that $\dim H = 2$. Then, $\Phi = \pm\mathrm{id}$, too.

If $\dim H = 2$, then $\mathcal{A} = \mathcal{B}(H)$ can be identified with $\mathcal{M}_2(\mathbb{F})$, the algebra of all 2×2 matrices over $\mathbb{F}$. So we can assume that Φ is a map on $\mathcal{M}_2(\mathbb{F})$. Denote by

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},\ E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

the matrix units for $\mathcal{M}_2(\mathbb{F})$. By Step 3.1, there exist real numbers $\lambda \neq 0, \mu_1$ such that

$$\Phi(E_{11}) = \lambda E_{11} + \mu_1 I. \tag{5}$$

For any $a \in \mathbb{F}$, denote by $\Phi(aE_{12}) = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$. Then, Equation (5) implies that

$$\begin{aligned} aE_{12} &= {}_*[E_{11}, aE_{12}]_1 = {}_*[\Phi(E_{11}), \Phi(aE_{12})]_1 \\ &= {}_*[\lambda E_{11} + \mu_1 I, \textstyle\sum_{i,j=1}^{2} s_{ij}E_{ij}]_1 = \lambda(s_{12}E_{12} - s_{21}E_{21}), \end{aligned}$$

which gives $s_{12} = \lambda^{-1}a$ and $s_{21} = 0$.

Note that

$${}_*[E_{11} + aE_{12}, aE_{12}]_1 = -|a|^2 E_{11} + aE_{12}; \tag{6}$$

and by Lemma 4, we have

$$\begin{aligned} &{}_*[\Phi(E_{11} + aE_{12}), \Phi(aE_{12})]_1 = {}_*[\Phi(E_{11}) + \Phi(aE_{12}) + \gamma_{12}I, \Phi(aE_{12})]_1 \\ =&{}_*[\lambda E_{11} + \mu_1 I + s_{11}E_{11} + a\lambda^{-1}E_{12} + s_{22}E_{22} + \gamma_{12}I, s_{11}E_{11} + a\lambda^{-1}E_{12} + s_{22}E_{22}]_1 \\ =&(s_{11}^2 - |s_{11}|^2 - |a|^2\lambda^{-2} + (\gamma_{12} - \bar{\gamma}_{12})s_{11})E_{11} - \bar{a}\lambda^{-1}s_{22}E_{21} \\ &+ a(1 + \lambda^{-1}s_{11} + \lambda^{-1}(s_{22} - \bar{s}_{22}) + \lambda^{-1}(\gamma_{12} - \bar{\gamma}_{12}))E_{12} \\ &+ (s_{22}^2 - |s_{22}|^2 + (\gamma_{12} - \bar{\gamma}_{12})s_{22})E_{22}. \end{aligned} \tag{7}$$

Comparing Equation (7) with Equation (6), we see that $s_{22} = 0$.

On the other hand, since

$$\begin{aligned} -|a|^2 E_{11} &= {}_*[aE_{12}, aE_{12}]_1 = {}_*[\Phi(aE_{12}), \Phi(aE_{12})]_1 \\ &= {}_*[s_{11}E_{11} + a\lambda^{-1}E_{12}, s_{11}E_{11} + a\lambda^{-1}E_{12}]_1 \\ &= (s_{11}^2 - |s_{11}|^2 - |a|^2\lambda^{-2})E_{11} + a\lambda^{-1}s_{11}E_{12}, \end{aligned}$$

we obtain $s_{11} = 0$ and $\lambda^2 = 1$. Thus, we have proved that

$$\Phi(aE_{12}) = \lambda aE_{12} \text{ holds for each nonzero } a \in \mathbb{F}.$$

Similarly, replacing E_{12} by E_{21} in Equations (6) and (7), one obtains

$$\Phi(aE_{21}) = \lambda aE_{21} \text{ holds for each nonzero } a \in \mathbb{F}.$$

So, by the equation

$$E_{21} - E_{12} = {}_*[E_{21}, E_{11}]_1 = {}_*[\Phi(E_{21}), \Phi(E_{11})]_1 = (1 + \lambda\mu_1)(E_{21} - E_{12}),$$

one obtains $\mu_1 = 0$, and thus $\Phi(E_{11}) = \lambda E_{11}$.

For any $a \in \mathbb{F}$, denote $\Phi(aE_{11}) = t_{11}E_{11} + t_{12}E_{12} + t_{21}E_{21} + t_{22}E_{22}$. Then,

$${}_*[\Phi(E_{11}), \Phi(aE_{11})]_1 = {}_*[E_{11}, aE_{11}]_1 = 0$$

implies that $t_{12} = t_{21} = 0$. Thus, $\Phi(aE_{11}) = t_{11}E_{11} + t_{22}E_{22}$. Write $\Phi(E_{11} + E_{12}) = \Phi(E_{11}) + \Phi(E_{12}) + \gamma I$. As $*[\Phi(E_{11} + E_{12}), \Phi(aE_{11})]_1 = *[E_{11} + E_{12}, aE_{11}]_1 = 0$, one obtains $t_{22} = 0$ and $\gamma \in \mathbb{R}$. It follows that

$$\begin{aligned} & (a - \bar{a})E_{11} - aE_{12} = *[aE_{11}, E_{11} + E_{12}]_1 = *[\Phi(aE_{11}), \Phi(E_{11}) + \Phi(E_{12}) + \gamma I]_1 \\ = \; & *[t_{11}E_{11}, \lambda(E_{11} + E_{12}) + \gamma I]_1 = (\lambda + \gamma)(t_{11} - \bar{t}_{11})E_{11} + \lambda t_{11}E_{12}, \end{aligned}$$

and therefore, $\Phi(aE_{11}) = \lambda aE_{11}$.

Now, for E_{22}, by a similar argument to that of E_{11}, one can show that

$$\Phi(aE_{22}) = \lambda' aE_{22},\ \Phi(aE_{12}) = \lambda' aE_{12} \text{ and } \Phi(aE_{21}) = \lambda' aE_{21}$$

for all $a \in \mathbb{R}$, where $\lambda'^2 = 1$. Note that

$$E_{12} - E_{21} =_* [E_{12}, E_{22}]_1 =_* [\Phi(E_{12}), \Phi(E_{22})]_1 =_* [\lambda E_{12}, \lambda' E_{22}]_1 = \lambda\lambda'(E_{12} - E_{21}).$$

This yields $\lambda\lambda' = 1$, and so $\lambda = \lambda' \in \{-1, 1\}$.

Finally, for any $A \in \mathcal{M}_2$, writing $A = a_{11}E_{11} + a_{12}E_{12} + a_{21}E_{21} + a_{22}E_{22}$, Lemma 4 ensures that

$$\Phi(A) = \sum_{i,j=1}^{2} \Phi(a_{ij}E_{ij}) + \mu_A I = \lambda A + \mu_A I$$

for some scalar μ_A. In addition, by

$$*[E_{11} + E_{12}, E_{11} + E_{12}]_1 =_* [\Phi(E_{11} + E_{12}), \Phi(E_{11} + E_{12})]_1,$$

it is easily checked that $\Phi(E_{11} + E_{12}) = \Phi(E_{11}) + \Phi(E_{12}) + \gamma I$ for some $\gamma \in \mathbb{R}$. Thus, as

$$*[E_{11} + E_{12}, A]_1 = *[\lambda(E_{11} + E_{12}) + \gamma I, \lambda A + \mu_A I]_1 = *[E_{11} + E_{12}, A]_1 + \lambda\mu_A(E_{12} - E_{21}),$$

we obtain $\mu_A = 0$. So $\Phi(A) = \lambda A$ for all A, where $\lambda \in \{-1, 1\}$.

This completes the proof of Theorem 1 for $k = 1$.

4. The Proof of Theorem 1: $k = 2$

In this section, we will check the "only if" part of Theorem 1 for case $k = 2$.

Assume that $\Phi : \mathcal{A} \to \mathcal{A}$ is strong 2-skew-commutativity preserving, that is,

$$*[\Phi(A), \Phi(B)]_2 = *[A, B]_2$$

for any $A, B \in \mathcal{A}$, of which the range contains all rank-one projections. We have to show that Φ is the identity map.

Step 4.1. For any nontrivial projection $P \in \mathcal{A}$, there exists some $\mu_P \in \mathbb{R}$ such that $\Phi(P) = P + \mu_P I$.

Let $P \in \mathcal{A}$ be any nontrivial projection. Write $P_1 = P$, $P_2 = I - P$ and $\mathcal{A}_{ij} = P_i\mathcal{A}P_j$, $i, j \in \{1, 2\}$. Then, there are $S_{ij} \in \mathcal{A}_{ij}$ so that $\Phi(P_1) = S_{11} + S_{12} + S_{21} + S_{22}$.

Taking $X = X_{11} \in \mathcal{A}_{11}$ and applying Lemma 7, we have

$$0 = P_{2*}[\Phi(P_1), X_{11}]_2 P_2 = -2P_2\Phi(P_1)X_{11}\Phi(P_1)^* P_2 = -2S_{21}X_{11}S_{21}^*,$$

which implies that $S_{21} = 0$ by Lemma 1. Then, $S_{12} = 0$ by Lemma 5. Hence, $\Phi(P) = S_{11} + S_{22}$. Applying Lemma 7 again, one obtains that

$$*[\Phi(P_1), X_{11}]_2 = S_{11}^2 X_{11} - 2S_{11}X_{11}S_{11}^* + X_{11}(S_{11}^*)^2 = 0$$

holds for all $X_{11} \in \mathcal{A}_{11}$. That is, ${}_*[S_{11}, X_{11}]_2 = 0$ for all $X_{11} \in \mathcal{A}_{11}$. By Lemma 3, $S_{11} = \lambda_1 P_1$ for some real number λ_1.

Analogously, there exists some real number λ_2 such that $S_{22} = \lambda_2 P_2$. It follows that

$$\Phi(P) = \Phi(P_1) = \lambda_1 P_1 + \lambda_2 P_2 = \lambda_P P_1 + \mu_P I,$$

where $\lambda_P = \lambda_1 - \lambda_2,\ \mu_P = \lambda_2 \in \mathbb{R}$.

We claim that $\lambda_P \neq 0$. If not, then $\Phi(P) = \mu_P I$, which implies that $P \in \mathbb{R}I$ by Lemma 6, a contradiction as $\dim H \geq 2$.

Now, take any nontrivial projection Q so that $PQ \neq QP$. Then,

$$\begin{aligned} 0 \neq \ & {}_*[P, Q]_2 = {}_*[\Phi(P), \Phi(Q)]_2 = {}_*[\lambda_P P + \mu_P I, \lambda_Q Q + \mu_Q I]_2 \\ = \ & {}_*[\lambda_P P, \lambda_Q Q]_2 = \lambda_P^2 \lambda_Q {}_*[P, Q]_2, \end{aligned}$$

which gives $\lambda_P^2 \lambda_Q = 1$. Similarly, $\lambda_P \lambda_Q^2 = 1$. So $\lambda_P = \lambda_P^4$, and consequently, $\lambda_P = 1$. The proof of the step is completed.

Step 4.2. $\Phi(A) = A$ holds for any $A \in \mathcal{A}$.

Let $A \in \mathcal{A}$ be arbitrary. For any rank-one projection $P \in \mathcal{A}$; by Step 4.1, we have

$${}_*[P, A]_2 = {}_*[\Phi(P), \Phi(A)]_2 = {}_*[P + \mu_P I, \Phi(A)]_2 = {}_*[P, \Phi(A)]_2.$$

By Lemma 2, we see that $\Phi(A) = A + \mu_A I$ for some scalar μ_A.

Thus, we have shown that there exists a functional $\mu : \mathcal{A} \to \mathbb{F}$ such that

$$\Phi(A) = A + \mu(A) I \ \text{ for all } \ A \in \mathcal{A}. \tag{8}$$

In the following, let us prove $\mu \equiv 0$ by considering two cases.

Case 4.2.1. $\dim H \geq 3$.

We first claim that $\mu(Q) = 0$ for any rank-one idempotent Q with $Q^* \neq Q$. In fact, by Equation (8),

$$\begin{aligned} Q - QQ^* = \ & {}_*[Q, Q]_2 = {}_*[Q + \mu(Q) I, Q + \mu(Q) I]_2 \\ = \ & aQ + bQQ^* + cQ^* + \mu(Q)(\mu(Q) - \mu(Q)^*)^2 I \end{aligned}$$

for some scalars a, b, c, that is,

$$\mu(Q)(\mu(Q) - \mu(Q)^*)^2 I = (1 - a)Q - (1 + b)QQ^* - cQ^*.$$

Note that the rank of $(1 - a)Q - (1 + b)QQ^* - cQ^*$ is not greater than 2 as Q is of rank one. Since $\dim H \geq 3$, we must have $\mu(Q)(\mu(Q) - \mu(Q)^*)^2 = 0$, which implies $\mu(Q) \in \mathbb{R}$. Thus,

$${}_*[Q, Q]_2 = {}_*[Q + \mu(Q) I, Q + \mu(Q) I]_2 = {}_*[Q, Q]_2 + \mu(Q)(Q - 2QQ^* + Q^*),$$

which entails that $\mu(Q) = 0$ as $Q - 2QQ^* + Q^* \neq 0$.

Now, for any $A \in \mathcal{A}$, taking a non self-adjoint rank one idempotent Q and by using of Equation (8), we have

$${}_*[Q, A]_2 = {}_*[\Phi(Q), \Phi(A)]_2 = {}_*[Q, A + \mu(A) I]_2 = {}_*[Q, A]_2 + \mu(A)(Q - 2QQ^* + Q^*).$$

So $\mu(A) = 0$, as desired.

Case 4.2.2. $\dim H = 2$.

In this case we can identify $\mathcal{A}$ with $\mathcal{M}_2(\mathbb{F})$. Let $Q = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and denote $\mu = \mu(Q)$. Then,

$$\begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} = {}_*[Q,Q]_2 = {}_*[Q+\mu I, Q+\mu I]_2 = \begin{pmatrix} a_{11} & a_{12} \\ -2\mu^2 + 2|\mu|^2 + \mu & \mu(\mu-\bar{\mu})^2 \end{pmatrix}.$$

So, $\mu(\mu-\bar{\mu})^2 = 0$ and $-2\mu^2 + 2|\mu|^2 + \mu = 0$, which entail $\mu = 0$. Now, by a similar argument to that of the corresponding part in Case 4.2.1, one can show that $\mu(A) = 0$ for each $A \in \mathcal{A}$.

The proof of Theorem 1 for $k = 2$ is completed.

5. The Proof of Theorem 1: $k = 3$

In this section, we assume that H is a real or complex Hilbert space with $\dim H \geq 2$, $\mathcal{A} \subseteq \mathcal{B}(H)$ is a self-adjoint standard operator algebra, and $\Phi : \mathcal{A} \to \mathcal{A}$ is a map of which the range contains all rank-one projections. To complete the proof of Theorem 1 for the case $k = 3$, one only needs to show that if Φ is strong 3-skew-commutativity preserving, that is, if Φ satisfies

$$\sum_{i=0}^{3} (-1)^i \mathrm{C}_3^i \Phi(A)^{3-i} \Phi(B) (\Phi(A)^*)^i = \sum_{i=0}^{3} (-1)^i \mathrm{C}_3^i A^{3-i} B (A^*)^i$$

for any $A, B \in \mathcal{A}$, then Φ is the identity map or the negative identity map. We will perform this in several steps.

Step 5.1. For any rank-one projection $P \in \mathcal{A}$, there exist real scalars λ_P, μ_P with $\lambda_P \neq 0$ such that $\Phi(P) = \lambda_P P + \mu_P I$.

Let P be any nontrivial projection in $\mathcal{A}$. Still, denote $P_1 = P$, $P_2 = I - P$ and $\mathcal{A}_{ij} = P_i \mathcal{A} P_j$, $i, j = 1, 2$. Then, $\mathcal{A}_{ii}$ is a self-adjoint standard operator algebra on $H_i = P_i(H)$, $i = 1, 2$.

Thus, there are $S_{ij} \in \mathcal{A}_{ij}$ so that

$$\Phi(P_1) = S_{11} + S_{12} + S_{21} + S_{22}. \tag{9}$$

As $P_1 = P_1^*$, by Lemma 5, we have $\Phi(P_1)^* = \Phi(P_1) + \delta I$ for some scalar δ, which implies that $S_{ij} = S_{ji}^*$ for $i \neq j \in \{1, 2\}$.

We have to consider the real case and the complex case separately.

Case 5.1.1. $\mathbb{F} = \mathbb{R}$.

In this case, we in fact show that Step 5.1 holds for any nontrivial projection P. Write $\Phi(P)$ as in Equation (9).

Assume that $S_{21} \neq 0$ in Equation (9). Obviously, $S_{12} \neq 0$ as $S_{12} = S_{21}^*$.

Taking any $X = X_{11} \in \mathcal{A}_{11}$ and applying Lemma 7, we have

$$\begin{aligned} 0 &= P_{2*}[\Phi(P_1), X_{11}]_3 P_2 \\ &= -3(S_{21}S_{11} + S_{22}S_{21}) X_{11} S_{12} + 3 S_{21} X_{11} (S_{11}^* S_{12} + S_{12} S_{22}^*). \end{aligned}$$

It follows that

$$(S_{21}S_{11} + S_{22}S_{21}) P_1 X P_1 S_{12} = S_{21} P_1 X P_1 (S_{11}^* S_{12} + S_{12} S_{22}^*) \quad \text{for all } X \in \mathcal{A}.$$

Then, by Lemma 1, there exists some $\lambda_1 \in \mathbb{R}$ such that

$$S_{21}S_{11} + S_{22}S_{21} = \lambda_1 S_{21}, \tag{10}$$

and consequently, $\lambda_1 S_{12} = S_{11}^* S_{12} + S_{12}S_{22}^*$.

Similarly, it follows from the relation ${}_*[\Phi(P_1), X_{22}]_3 P_1 = 0$ that there exists a real number $\gamma \in \mathbb{R}$ such that

$$S_{11}S_{12} + S_{12}S_{22} = \gamma S_{12} \text{ and } S_{21}S_{11}^* + S_{22}^* S_{21} = \gamma S_{21}. \tag{11}$$

Next, taking $X = X_{12} \in \mathcal{A}_{12}$ and applying Lemma 7, we have $P_{1*}[\Phi(P_1), X_{12}]_3 P_1 = 0$, which, together with Equations (10) and (11), implies that

$$P_1 X(\gamma S_{21}S_{11}^* + S_{21}S_{12}S_{21} + (S_{22}^*)^2 S_{21}) = 3(\lambda_1 S_{11} - S_{11}^2 - S_{12}S_{21})P_1 X P_2 S_{21}$$

holds for all $X \in \mathcal{A}$. Since $S_{21} \neq 0$, by Lemma 1 again, there exists some $\lambda_2 \in \mathbb{R}$ such that

$$3(\lambda_1 S_{11} - S_{11}^2 - S_{12}S_{21}) = \lambda_2 P_1 \tag{12}$$

and

$$S_{21}(S_{11}^*)^2 + S_{22}^* S_{21} S_{11}^* + S_{21}S_{12}S_{21} + (S_{22}^*)^2 S_{21} = \lambda_2 S_{21}. \tag{13}$$

Similarly, for any $X_{21} \in \mathcal{A}_{21}$, by the relation $P_{2*}[\Phi(P_1), X_{12}]_3 P_2 = 0$, there exists some $\lambda_3 \in \mathbb{R}$ such that

$$3(\lambda_1 S_{22}^* - (S_{22}^*)^2 - S_{21}S_{12}) = \lambda_3 P_2 \tag{14}$$

and

$$S_{21}S_{11}^2 + S_{22}S_{21}S_{11} + S_{21}S_{12}S_{21} + S_{22}^2 S_{21} = \lambda_3 S_{21}. \tag{15}$$

Note that $\Phi(P_1)^* = \Phi(P_1) + \delta I$. It follows from Equation (12) that

$$\lambda_2 P_1 = 3(\lambda_1 S_{11}^* - (S_{11}^*)^2 - S_{12}S_{21}) = \lambda_2 P_1 + 3\delta(\lambda_1 - \delta)P_1 - 6\delta S_{11}.$$

Hence,

$$\delta S_{11} = \frac{1}{2}\delta(\lambda_1 - \delta)P_1.$$

Similarly, it follows from Equation (14) that

$$\delta S_{22} = \frac{1}{2}\delta(\lambda_1 - \delta)P_2.$$

If $\delta \neq 0$, then $S_{ii} = \frac{1}{2}(\lambda_1 - \delta)P_i$ $(i = 1, 2)$. So, by Equation (10), $(\lambda_1 - \delta)S_{21} = \lambda_1 S_{21}$, which implies $S_{21} = 0$, contradicting to the assumption. Therefore, we must have $\delta = 0$, and thus $\Phi(P_1)^* = \Phi(P_1)$. This yields $S_{ii}^* = S_{ii}$, $i = 1, 2$. Then, Equation (13) becomes to

$$S_{21}S_{11}^2 + S_{22}S_{21}S_{11} + S_{21}S_{12}S_{21} + S_{22}^2 S_{21} = \lambda_2 S_{21}. \tag{16}$$

Particularly, we have $\lambda_3 = \lambda_2$ by Equations (15) and (16). Multiplying by S_{21} from the left of Equation (12) gives

$$3(\lambda_1 S_{21}S_{11} - S_{21}S_{11}^2 - S_{21}S_{12}S_{21}) = \lambda_2 S_{21}.$$

Comparing this with Equation (16), one obtains

$$3\lambda_1 S_{21}S_{11} + 3S_{22}S_{21}S_{11} + 3S_{22}^2 S_{21} = 4\lambda_2 S_{21}.$$

By using Equation (10) twice for the above equation, we obtain $(3\lambda_1^2 - 4\lambda_2)S_{21} = 0$. As $S_{21} \neq 0$, we have $3\lambda_1^2 = 4\lambda_2$. Let $N = \Phi(P_1) - \frac{\lambda_1}{2}I$; then $\Phi(P_1) = \frac{\lambda_1}{2}I + N$. Combining Equations (10)–(14), one obtains

$$\begin{aligned} N^2 = & (\Phi(P_1) - \tfrac{\lambda_1}{2}I)^2 = (S_{11} + S_{12} + S_{21} + S_{22} - \tfrac{\lambda_1}{2}I)^2 \\ = & (S_{11}^2 + S_{12}S_{21}) + (S_{11}S_{12} + S_{12}S_{22}) + (S_{21}S_{11} + S_{22}S_{21}) \\ & + (S_{22}^2 + S_{21}S_{12}) - \lambda_1 S_{11} - \lambda_1 S_{12} - \lambda_1 S_{21} - \lambda_1 S_{22} + \tfrac{\lambda_1^2}{4}I \\ = & (\lambda_1 S_{11} - \tfrac{\lambda_2}{3}P_1) + \lambda_1 S_{12} + \lambda_1 S_{21} + (\lambda_1 S_{22} - \tfrac{\lambda_2}{3}P_2) \\ & - \lambda_1 S_{11} - \lambda_1 S_{12} - \lambda_1 S_{21} - \lambda_1 S_{22} + \tfrac{\lambda_1^2}{4}I \\ = & (\lambda_1 S_{11} - \tfrac{\lambda_1^2}{4}P_1) + (\lambda_1 S_{22} - \tfrac{\lambda_1^2}{4}P_2) - \lambda_1 S_{11} - \lambda_1 S_{22} + \tfrac{\lambda_1^2}{4}I = 0. \end{aligned}$$

So $\Phi(P_1) \in (\mathbb{R}I + \mathcal{N}_2(\mathcal{A}))$. However, this and Lemma 6 imply $P_1 \in (\mathbb{R}I + \mathcal{N}_2(\mathcal{A}))$, a contradiction. Therefore, we must have $S_{12} = S_{21} = 0$, and thus $\Phi(P_1) = S_{11} + S_{22}$.

Now, for any $X_{11} \in \mathcal{R}_{11}$, by Lemma 7, we have

$$0 = P_{1*}[\Phi(P_1), X_{11}]_3 P_1 = P_{1*}[S_{11} + S_{22}, X_{11}]_3 P_1 = {}_*[S_{11}, X_{11}]_3.$$

As $\mathcal{A}_{11}$ is also a self-adjoint standard operator algebra, it follows from Lemma 3 that

$$S_{11} = \lambda_1 P_1 + N_1 \text{ for some } \lambda_1 \in \mathbb{R} \text{ and } N_1 \in \mathcal{N}_2(\mathcal{A}_{11}).$$

Replacing X_{11} by X_{22}, a similar argument ensures that

$$S_{22} = \mu P_2 + N_2 \text{ for some } \mu \in \mathbb{R} \text{ and } N_2 \in \mathcal{N}_2(\mathcal{A}_{22}).$$

Hence,

$$\Phi(P_1) = \lambda_1 P_1 + N_1 + \mu P_2 + N_2 = \lambda P_1 + \mu I + N,$$

where $\lambda = \lambda_1 - \mu \in \mathbb{R}$, $N = N_1 + N_2 \in \mathcal{N}_2(\mathcal{A})$ as $N_1 N_2 = N_2 N_1 = 0$. Since $\Phi(P_1)^* = \Phi(P_1) + \delta I$, one sees that $N^* = \delta I + N$, which entails $\delta = 0$ and $N = 0$ as $NN^* = N^2 = 0$. Consequently, $\Phi(P_1) = \lambda P_1 + \mu I$. Finally, we claim $\lambda \neq 0$. Otherwise, $\Phi(P_1) = \mu I \in \mathbb{R}I + \mathcal{N}_2(\mathcal{A})$, which, together with Lemma 6, entails $P_1 \in \mathbb{R}I + \mathcal{N}_2(\mathcal{A})$, a contradiction.

This completes the proof of Step 5.1 for the real case.

Case 5.1.2. $\mathbb{F} = \mathbb{C}$.

Assume that P_1 is of rank one; then, $\mathcal{A}_{11} = \mathbb{C}P_1$ and every $X \in \mathcal{A}_{12} + \mathcal{A}_{21}$ has rank ≤ 2. Thus, by Lemma 7, for any $A \in \mathcal{B}(H)$, $\text{rank}({}_*[\Phi(P_1), A]_3) \leq 2$. Particularly, for each unit vector $x \in H$, applying the above fact to the rank-one projection $x \otimes x$, we see that the rank of

$${}_*[\Phi(P_1), x \otimes x]_3 = \Phi(P_1)^3 x \otimes x - 3\Phi(P_1)^2 x \otimes \Phi(P_1)x + 3\Phi(P_1)x \otimes \Phi(P_1)^2 x - x \otimes \Phi(P_1)^3 x$$

is not greater than 2. Denote $S = \Phi(P_1) = S_{11} + S_{12} + S_{21} + S_{22}$ with S_{ij} as in Equation (9). Above equation implies that $\{x, Sx, S^2x, S^3x\}$ is a linearly dependent set for each $x \in H$. By the Kaplansky theorem, S is an algebraic operator of degree ≤ 3; that is, there exists a polynomial $p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$ such that $p(S) = a_0 + a_1 S + a_2 S^2 + a_3 S^3 = 0$. So the spectra $\sigma(S)$ of S consists of at most three eigenvalues.

If there exists a $\lambda \in \sigma(S)$ with $\lambda \notin \mathbb{R}$, then $Sx = \lambda x$ for some unit vector x and

$${}_*[S, x \otimes x]_3 = (\lambda - \bar{\lambda})^3 x \otimes x \in \mathcal{A}_{12} + \mathcal{A}_{21},$$

which induces a contradiction $x \otimes x = (x \otimes x)^2 = 0$. So, we must have $\sigma(S) \subset \mathbb{R}$. This, together with the fact $S^* = S + \delta I$ by Lemma 5, gives $\delta = 0$ and $S^* = S$. Note that $\sigma(S)$

has at least two points. Otherwise, as S is self-adjoint, we have $\Phi(P_1) = S = \gamma I$, which is impossible by Lemma 6.

We next claim that the one dimensional subspace $\mathrm{ran}(P_1)$ is invariant under S. To see this, take $\lambda_1, \lambda_2 \in \sigma(S)$ and unit vectors $x_1, x_2 \in H$ so that $Sx_i = \lambda_i x_i$, $i = 1, 2$. Then,

$$ {}_*[S, x_1 \otimes x_2]_3 = (\lambda_1 - \lambda_2)^3 x_1 \otimes x_2 \in \mathcal{A}_{12} + \mathcal{A}_{21}.$$

So, $P_1x_1 \otimes P_1x_2 = 0$ and $P_2x_1 \otimes P_2x_2 = 0$. As P_1x_j and P_2x_j cannot be 0 at the same time, we see that $P_1x_1 \neq 0 \Rightarrow P_2x_2 = x_2$ and $P_1x_1 = x_1$; $P_1x_1 = 0 \Rightarrow P_2x_1 = x_1$ and $P_1x_2 = x_2$. Without loss of generality, say $P_1x_1 = x_1$ and $P_2x_2 = x_2$. Then, $\mathrm{ran}(P_1) = \mathbb{C}x_1$ is invariant under S. Consequently, $S = \lambda_1 P_1 + S_{22}$ and $\lambda_2 \in \sigma(S_{22})$. In fact, $\sigma(S_{22}) = \{\lambda_2\}$. Assume, on the contrary, that there exists $\lambda_3 \in \sigma(S_{22}) \setminus \{\lambda_2\}$. Then, $S_{22}x_3 = \lambda_3 x_3$ for some unit vector $x_3 \in P_2(H)$. However,

$$ {}_*[S, x_3 \otimes x_2]_3 = (\lambda_3 - \lambda_2)^3 x_3 \otimes x_2 \in \mathcal{A}_{12} + \mathcal{A}_{21}$$

implies $P_1x_3 = x_3$, a contradiction. Hence, $S_{22} = \lambda_2 P_2$, and thus $\Phi(P_1) = \lambda_1 P_1 + \lambda_2 P_2 = \lambda P_1 + \mu I$ with $\lambda = \lambda_1 - \lambda_2$ and $\mu = \lambda_2$.

So far we have proved that, for any rank-one projection P, there are real scalars λ_P, μ_P such that

$$\Phi(P) = \lambda_P P + \mu_P I,$$

that is, Step 5.1 is also true for complex case.

Step 5.2. If $\dim H \geq 3$, then there exists a $\lambda \in \{-1, 1\}$ such that $\Phi(P) = \lambda P + \mu_P I$ holds for every rank-one projection $P \in \mathcal{A}$.

Let $P \in \mathcal{A}$ be any rank-two projection. Fix a unit vector $x \in P(H)$ and take a unit vector $y \in P(H)$ so that $x \perp y$. Then, for any scalars α, β with $|\alpha|^2 + |\beta|^2 = 1$, we have

$$(\alpha x + \beta y) \otimes (\alpha x + \beta y) + (\bar{\beta} x - \bar{\alpha} y) \otimes (\bar{\beta} x - \bar{\alpha} y) = P.$$

Denote by $P_1 = x \otimes x$ and $P_2 = y \otimes y$. By Lemma 4 and Step 5.1, one has

$$\begin{aligned} & \lambda_1 P_1 + \lambda_2 P_2 + \mu I = \Phi(P) \\ = \; & \lambda_{\alpha,\beta}(\alpha x + \beta y) \otimes (\alpha x + \beta y) + \lambda_{\bar{\beta},-\bar{\alpha}}(\bar{\beta} x - \bar{\alpha} y) \otimes (\bar{\beta} x - \bar{\alpha} y) + \mu_{\alpha,\beta} I \end{aligned}$$

for some scalars $\mu, \mu_{\alpha,\beta}$. Obviously, this implies $\lambda_{\alpha,\beta} = \lambda_{\bar{\beta},-\bar{\alpha}}$ and $\lambda_1 = \lambda_2$.

Thus, we in fact have shown that, for any rank-one projections P, Q, $\lambda_P = \lambda_Q$ whenever $PQ = 0$.

Since $\dim H \geq 3$, for any rank-one projections P, Q, there exists a rank-one projection P' so that $PP' = QP' = 0$. By what we just proved above, one sees that $\lambda_P = \lambda_{P'} = \lambda_Q$. So, there exists a real number λ such that

$$\Phi(P) = \lambda P + \mu_P I \;\text{ holds for all rank one projections }\; P \in \mathcal{A}.$$

Now, pick two rank-one projections P, Q so that $PQ \neq QP$. Then, by

$$\begin{aligned} 0 \neq \; & {}_*[P, Q]_3 = {}_*[\Phi(P), \Phi(Q)]_3 = {}_*[\lambda P + \mu_P I, \lambda Q + \mu_Q I]_3 \\ = \; & {}_*[\lambda P + \mu_P I, \lambda Q]_3 = {}_*[\lambda P + \mu_P I, {}_*[\lambda P + \mu_P I, \lambda Q]_2] \\ = \; & {}_*[\lambda P, {}_*[\lambda P + \mu_P I, \lambda Q]_2] = {}_*[\lambda P, \lambda Q]_3 = \lambda^4 {}_*[P, Q]_3, \end{aligned}$$

one obtains $\lambda^4 = 1$. It follows that $\lambda = \pm 1$ as λ is real.

Step 5.3. If $\dim H \geq 3$, then $\Phi(A) = \lambda A$ for all $A \in \mathcal{A}$.
Let $A \in \mathcal{A}$. For any rank-one projection $P \in \mathcal{A}$; by Step 5.2, we have

$$*[P, A]_3 = *[\Phi(P), \Phi(A)]_3 = *[\lambda P + \mu_P I, \Phi(A)]_3 = *[P, \lambda^3 \Phi(A)]_3.$$

It follows from Lemma 2 that $\lambda^3 \Phi(A) = A + \nu_A I$ for some scalar ν_A, and so,

$$\Phi(A) = \lambda^{-3} A + \lambda^{-3} \nu_A I = \lambda A + \lambda \nu_A I. \tag{17}$$

What remains is to check $\mu_A = \lambda \nu_A = 0$. Assume $\mu_A \neq 0$ for some A. By Step 5.1, $\mu_P \in \mathbb{R}$ for any rank-one projection P. We claim further that $\mu_Q \in \mathbb{R}$ for any rank one idempotent operator Q. Note that, if Q is an idempotent, then $*[Q, Q]_3 = Q - QQ^*$. Thus, by Equation (17), we have

$$\begin{aligned} & Q - QQ^* =_* [\Phi(Q), \Phi(Q)]_3 = *[\lambda Q + \mu_Q I, \lambda Q + \mu_Q I]_3 \\ = \; & (\lambda Q + \mu_Q I)^3 (\lambda Q + \mu_Q I) - 3(\lambda Q + \mu_Q I)^2 (\lambda Q + \mu_Q I)(\lambda Q^* + \bar{\mu}_Q I) \\ & + 3(\lambda Q + \mu_Q I)(\lambda Q + \mu_Q I)(\lambda Q^* + \bar{\mu}_Q I)^2 - (\lambda Q + \mu_Q I)(\lambda Q^* + \bar{\mu}_Q I)^3 \\ = \; & aQ + bQ^* + cQQ^* + \mu_Q (\mu_Q - \bar{\mu}_Q)^3 I \end{aligned}$$

for some scalars a, b, c, that is,

$$(a-1)Q + bQ^* + (c+1)QQ^* = \mu_Q (\bar{\mu}_Q - \mu_Q)^3 I.$$

Since $\dim H \geq 3$ and $\operatorname{rank}((a-1)Q + bQ^* + (c+1)QQ^*) \leq 2$, we must have $\bar{\mu}_Q = \mu_Q$. So

$$Q - QQ^* = *[\lambda Q + \mu_Q I, \lambda Q + \mu_Q I]_3 = Q - QQ^* + \lambda^3 \mu_Q (Q - Q^*),$$

which entails that $\mu_Q = 0$ whenever $Q^* \neq Q$. Hence we have shown that

$$\Phi(Q) = \lambda Q \text{ holds for any rank one idempotent } Q \text{ with } Q \neq Q^*.$$

Now, for any $A \in \mathcal{A}$, taking a rank one idempotent Q with $Q^* \neq Q$ and applying Equation (17), we have

$$*[Q, A]_3 = *[\lambda Q, \lambda A + \mu_A I]_3 =_* [Q, A]_3 + \lambda^3 \mu_A (Q - Q^*).$$

It follows that $\mu_A = 0$ holds for any $A \in \mathcal{A}$.
This completes the proof of Theorem 1 for the case $\dim H \geq 3$ and $k = 3$.
In the following Steps 5.4-5.6, we assume that $\dim H = 2$. In this situation, we have $\mathcal{A} = \mathcal{B}(H)$, and so we can identify $\mathcal{A}$ as the two by two-matrix algebra $\mathcal{M}_2(\mathbb{F})$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. Denote by

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

the matrix units for $\mathcal{M}_2(\mathbb{F})$. By Step 5.1, there exist real numbers $\lambda \neq 0$ and μ_1 such that

$$\Phi(E_{11}) = \lambda E_{11} + \mu_1 I. \tag{18}$$

Step 5.4. If $i \neq j \in \{1, 2\}$, then, for any $a_{ij} \in \mathbb{F}$, there exists $\mu_{a_{ij}} \in \mathbb{R}$ such that

$$\Phi(a_{ij} E_{ij}) = \lambda^{-3} a_{ij} E_{ij} + \mu_{a_{ij}} I.$$

Here, we only give the details of the proof for the case $(i,j)=(1,2)$. The other case $(i,j)=(2,1)$ is dealt with similarly.

Write $\Phi(a_{12}E_{12})=\begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}$. By Equation (18), a simple computation reveals that

$$\begin{aligned} & a_{12}E_{12} = {}_*[E_{11}, a_{12}E_{12}]_3 = {}_*[\Phi(E_{11}), \Phi(a_{12})E_{12})]_3 \\ = \; & {}_*[\lambda E_{11}+\mu_1 I, \textstyle\sum_{i,j=1}^2 s_{ij}E_{ij}]_3 = \lambda^3 {}_*[E_{11}, \sum_{i,j=1}^2 s_{ij}E_{ij}]_3 = \lambda^3(s_{12}E_{12}-s_{21}E_{21}). \end{aligned}$$

So $s_{21}=0$ and $s_{12}=\lambda^{-3}a_{12}$; that is,

$$\Phi(a_{12}E_{12}) = s_{11}E_{11}+\lambda^{-3}a_{12}E_{12}+s_{22}E_{22}. \tag{19}$$

On the other hand, by Lemma 6, $\Phi(a_{12}E_{12})=\mu_{a_{12}}I+N$ for some real scalar $\mu_{a_{12}}$ and N with $N^2=0$. Comparing this with Equation (19) gives $s_{11}=s_{22}=\mu_{a_{12}}$ and $\Phi(a_{12}E_{12})=\lambda^{-3}a_{12}E_{12}+\mu_{a_{12}}I$, as required.

Step 5.5. $\lambda=\pm 1$, and, for any $a_{ij}\in\mathbb{F}$, we have $\Phi(a_{ij}E_{ij})=\lambda a_{ij}E_{ij}$, $i,j\in\{1,2\}$.

Let $\Phi(a_{11}E_{11})=t_{11}E_{11}+t_{12}E_{12}+t_{21}E_{21}+t_{22}E_{22}$. Since

$$\begin{aligned} 0 = \; & {}_*[E_{11}, a_{11}E_{11}]_3 = {}_*[\Phi(E_{11}), \Phi(a_{11}E_{11})]_3 \\ = \; & {}_*[\lambda E_{11}+\mu_1 I, \textstyle\sum_{i,j=1}^2 t_{ij}E_{ij}]_3 = \lambda^3(t_{12}E_{12}-t_{21}E_{21}), \end{aligned}$$

we see that $t_{12}=t_{21}=0$.

Now let us check that, for $(i,j)=(1,2)$ or $(2,1)$, it is true that

$$\Phi(E_{11}+E_{ij}) = \Phi(E_{11})+\Phi(E_{ij})+\gamma_{ij}I \quad \text{for some } \gamma_{ij}\in\mathbb{R}. \tag{20}$$

In fact, if $(i,j)=(1,2)$, then, by Lemma 4, $\Phi(E_{11}+E_{12})=\Phi(E_{11})+\Phi(E_{12})+\gamma_{12}I$ for some $\gamma_{12}\in\mathbb{F}$. Note that, by Equation (18) and Step 5.4,

$$\begin{aligned} E_{12}-E_{21} = \; & {}_*[E_{11}+E_{12}, E_{11}]_3 = {}_*[\Phi(E_{11}+E_{12}), \Phi(E_{11})]_3 \\ = \; & {}_*[\lambda E_{11}+\mu_1 I+\lambda^{-3}E_{12}+\mu_{12}I+\gamma_{12}I, \lambda E_{11}+\mu_1 I]_3 \\ = \; & {}_*[\lambda E_{11}+\lambda^{-3}E_{12}+\gamma_{12}I, \lambda E_{11}+\mu_1 I]_3 \\ = \; & (\lambda+\mu_1)(\gamma_{12}-\bar{\gamma}_{12})^3 E_{11}+bE_{12}+cE_{21}+\mu_1(\gamma_{12}-\bar{\gamma}_{12})^3 I. \end{aligned}$$

We see that $(\lambda+\mu_1)(\gamma_{12}-\bar{\gamma}_{12})^3=0$ and $\mu_1(\gamma_{12}-\bar{\gamma}_{12})^3=0$. If γ_{12} is not real, then $\lambda+\mu_1=0$ and $\mu_1=0$, which leads to a contradiction $\lambda=0$. The case $(i,j)=(2,1)$ can be dealt with similarly. So Equation (20) holds.

Now, by Equation (18), Equation (20), Lemma 4 and Step 5.4, we obtain

$$\begin{aligned} 0 = \; & {}_*[E_{11}+E_{12}, a_{11}E_{11}]_3 = {}_*[\Phi(E_{11}+E_{12}), \Phi(a_{11}E_{11})]_3 \\ = \; & {}_*[\lambda E_{11}+\mu_1 I+\lambda^{-3}E_{12}+\mu_{12}I+\gamma_{12}I, t_{11}E_{11}+t_{22}E_{22}]_3 \\ = \; & {}_*[\lambda E_{11}+\lambda^{-3}E_{12}, t_{11}E_{11}+t_{22}E_{22}]_3 = \lambda^{-1}t_{22}(E_{12}-E_{21}) \end{aligned}$$

and

$$\begin{aligned} a_{11}(E_{21}-E_{12}) = \; & {}_*[E_{11}+E_{21}, a_{11}E_{11}]_3 = {}_*[\Phi(E_{11}+E_{21}), \Phi(a_{11}E_{11})]_3 \\ = \; & {}_*[\lambda E_{11}+\mu_1 I+\lambda^{-3}E_{21}+\mu_{21}I+\gamma_{21}I, t_{11}E_{11}+t_{22}E_{22}]_3 \\ = \; & {}_*[\lambda E_{11}+\lambda^{-3}E_{21}, t_{11}E_{11}+t_{22}E_{22}]_3 = \lambda^{-1}t_{11}(E_{21}-E_{12}). \end{aligned}$$

These imply that $t_{22}=0$ and $t_{11}=\lambda a_{11}$. So $\Phi(a_{11}E_{11})=\lambda a_{11}E_{11}$ holds for any $a_{11}\in\mathbb{F}$.

Similarly, one can prove that $\Phi(a_{22}E_{22})=\lambda a_{22}E_{22}$ holds for any $a_{22}\in\mathbb{F}$.

Finally, since

$$\begin{aligned} a_{21}(E_{11}-E_{21}) = & {}_*[E_{11}+E_{12}, a_{21}E_{21}]_3 = {}_*[\Phi(E_{11}+E_{12}), \Phi(a_{21}E_{21})]_3 \\ = & {}_*[\lambda E_{11}+\lambda^{-3}E_{12}, \lambda^{-3}a_{21}E_{21}+\mu_{a_{21}}I]_3 \\ = & \lambda^{-4}a_{21}E_{11}+\lambda^{-1}\mu_{a_{21}}E_{12}-(a_{21}+\lambda^{-1}\mu_{a_{21}})E_{21}, \end{aligned}$$

we obtain $\lambda^4 = 1$ and $\mu_{a_{21}} = 0$. Therefore, $\lambda = \pm 1$ and $\Phi(a_{21}E_{21}) = \lambda^{-3}a_{21}E_{21} = \lambda a_{21}E_{21}$. The fact $\Phi(a_{12}E_{12}) = \lambda a_{12}E_{12}$ can be checked similarly.

Step 5.6. $\Phi(A) = \lambda A$ for every $A \in \mathcal{M}_2(\mathbb{F})$.

For any $A \in \mathcal{M}_2(\mathbb{F})$, denote by $A = a_{11}E_{11} + a_{12}E_{12} + a_{21}E_{21} + a_{22}E_{22}$. By Lemma 4 and Step 5.5, there is a scalar μ_A so that

$$\Phi(A) = \Phi(a_{11}E_{11}) + \Phi(a_{12}E_{12}) + \Phi(a_{21}E_{21}) + \Phi(a_{22}E_{22}) + \mu_A I = \lambda A + \mu_A I.$$

Furthermore, by Equation (20) and Step 5.5, we have

$$\begin{aligned} {}_*[E_{11}+E_{12}, A]_3 = & {}_*[\Phi(E_{11}+E_{12}), A]_3 = {}_*[\lambda(E_{11}+E_{12}), \lambda A+\mu_A I]_3 \\ = & {}_*[E_{11}+E_{12}, A]_3 + \lambda^3\mu_A(E_{12}-E_{21}). \end{aligned}$$

Hence, $\mu_A = 0$, and $\Phi(A) = \lambda A$ for every $A \in \mathcal{M}_2(\mathbb{F})$. This completes the proof of the "only if" part for the case $k = 3$ and $\dim H = 2$.

The proof of Theorem 1 for $k = 3$ is finished.

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