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Review

Abelian Function Fields on Jacobian Varieties

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Abstract: The aim of this paper is an exposition of fields of multiply periodic, or Kleinian, $\wp$-functions. Such a field arises on the Jacobian variety of an algebraic curve, providing natural algebraic models for the Jacobian and Kummer varieties, possessing the addition law, and accommodating dynamical equations with solutions. All of this will be explained in detail for plane algebraic curves in their canonical forms. Examples of hyperelliptic and non-hyperelliptic curves are presented.

Keywords: sigma function; multiply periodic $\wp$-function; addition law

MSC: 33E99; 58C99

1. Introduction

In this review, we consider differential fields of multiply periodic meromorphic functions defined on Jacobian varieties. It is essential that Jacobian varieties arise from algebraic curves.

We work with plane algebraic curves. The related theory of the modular-invariant entire function known as the multi-variable (or multi-dimensional) σ-function, and multiply periodic meromorphic functions, known as Kleinian $\wp$-functions, has been productively developed over the last thirty years, following its abandonment since the beginning of the 20th century. We will frequently recall the results published in the classical monographs of H. F. Baker [1,2]. The former was written as a guide "to analytical developments in Pure Mathematics during the last seventy years" of the 19th century, while the latter serves as "an elementary and self-contained introduction to many of the leading ideas of the theory of multiply periodic functions", illustrated by examples of hyperelliptic functions in genus two, that "reduces the theory in a very practical way".

The results obtained during the last thirty years are widely presented in the reviews [3,4]. A more general algebraic approach can be found in [5]. The present review pursues, first of all, a practical purpose: to demonstrate how to work with abelian functions on the Jacobian variety of a given plane algebraic curve.

We represent abelian functions through $\wp$-functions, which generalize the Weierstrass $\wp$-function to higher genera. The $\wp$-functions associated with a curve of genus g are periodic on the period lattice of rank $2g$ which naturally arises on the curve. Similar to the Weierstrass $\wp$-function, these functions are generated from the σ-function, which is entire and modular-invariant with respect to the period lattice. The concept of the σ-function in higher genera was suggested by Klein [6–8]. Therefore, $\wp$-functions in g variables are called Kleinian in [9], where the results from the 19th century were revisited after nearly a century-long lull, and extended.

Due to the works of K. Weierstrass [10], O. Bolza [11,12], and H. F. Baker [13,14], further progress was made in developing the theory of multi-variable σ-functions asso-

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ciated with hyperelliptic curves, and related multiply periodic $\wp$-functions. In the beginning of the 21st century, V. Enolski, V. Leykin, and V. Buchstaber pushed progress to a new level. The theory of constructing the heat equations which define a multi-variable σ-function as an analytic series was developed in [15–18], as well as other techniques for working with $\wp$-functions [9,19–23]. This gave birth to extensive research on identities for $\wp$-functions associated with hyperelliptic and non-hyperelliptic curves [24–41]. At the same time, other aspects of the theory of multi-variable σ-functions were developed in [5,42–45], and an extension to space curves in [46–48].

List of Notations

n, s	co-prime natural numbers
$\mathbb{N}_0 = \mathbb{N} \cup \{0\}$	non-negative integers
$\equiv$	means 'defined by'
$\mathcal{C}$	an algebraic curve, assumed to be in the canonical form
$\mathcal{C}^n$	the n-th symmetric product of $\mathcal{C}$
g	the genus of $\mathcal{C}$
$\lambda = (\lambda_k)$	a vector of parameters (= coefficients of the equation) of $\mathcal{C}$
$\mathfrak{W} = \{\mathfrak{w}_i \mid i = 1, \dots, g\}$	a Weierstrass gap sequence
$\mathfrak{M} = \{\mathfrak{m}_w \mid w \in \mathbb{N}_0 \setminus \mathfrak{W}\}$	an ordered list of monomials on $\mathcal{C}$
ω, ω'	first kind not normalized $\mathfrak{a}$-, and $\mathfrak{b}$-period matrices
η, η'	second kind not normalized $\mathfrak{a}$-, and $\mathfrak{b}$-period matrices
$\mathrm{Jac}(\mathcal{C}) = \mathbb{C}^g / \{\omega, \omega'\}$	the Jacobian variety of $\mathcal{C}$, w.r.t. not normalized periods
$\mathrm{Kum}(\mathcal{C}) = \mathrm{Jac}(\mathcal{C})/\pm$	the Kummer variety of $\mathcal{C}$
$\mathrm{d}u_{\mathfrak{w}_i} = v_{\mathfrak{w}_i}\mathrm{d}x/\partial_y f$	first kind (or holomorphic) differentials on $\mathcal{C}$
$\mathrm{d}r_{\mathfrak{w}_i} = \rho_{\mathfrak{w}_i}\mathrm{d}x/\partial_y f$	second kind differentials on $\mathcal{C}$
$\mathcal{A}(P), \mathcal{A}(D)$	the Abel image (or first kind integral) of a point P and a divisor D
$\mathcal{B}(P), \mathcal{B}(D)$	the second kind integral at a point P, and a divisor D
Σ	the theta divisor defined by $\{u \in \mathrm{Jac}(\mathcal{C}) \mid \sigma(u) = 0\}$
$\mathfrak{A}(\mathcal{C})$	differential field of $\wp$-functions on $\mathrm{Jac}(\mathcal{C}) \setminus \Sigma$
$\mathcal{R}_w$	a polynomial function of weight w from $\mathbb{C}[x, y]/f(x, y; \lambda)$
$\mathfrak{P}(\mathcal{C})$	the vector space of polynomial functions on $\mathcal{C}$
$v_{\mathfrak{w}_1}, \dots, v_{\mathfrak{w}_g}$	basis monomials in $\mathfrak{P}(\mathcal{C})$

2. Preliminaries

2.1. Canonical Form of Plane Algebraic Curves

As canonical forms of plane curves we use so-called (n, s)-curves, $\gcd(n, s) = 1$, introduced in [15] as follows

$$\mathcal{C} = \{(x, y) \in \mathbb{C}^2 \mid f(x, y; \lambda) \equiv -y^n + x^s + \sum_{j=0}^{n-2} \sum_{i=0}^{s-2} \lambda_{ns-in-js} y^j x^i = 0, \quad \lambda_{k \leqslant 0} = 0, \ \ \lambda_k \in \mathbb{C}\}. \tag{1}$$

The curve equation in (1) arises as a universal unfolding of the Pham singularity $y^n + x^s = 0$, and contains the minimal number of parameters λ_k, which is $\mathfrak{N} = (n-1)(s-1) - M$, where M is the number of $\lambda_{k \leqslant 0}$, called modality. By $\lambda = (\lambda_k)$ we denote a vector of parameters of a curve $\mathcal{C}$.

All curves (1) with $\lambda \in \Lambda \equiv \mathbb{C}^{\mathfrak{N}}$ form a fiber bundle $\mathcal{E}(\Lambda, \pi, \mathcal{C})$, where $\pi(\mathcal{C}) = \lambda$ serves as a projection. As shown in [15], the genus of curves in $\mathcal{E}(\Lambda, \pi, \mathcal{C})$ does not exceed

$$g = \tfrac{1}{2}(n-1)(s-1). \tag{2}$$

Thus, $\mathfrak{N} = 2g - M$. The space Λ of parameters, which serves as the base of $\mathcal{E}(\Lambda, \pi, \mathcal{C})$, is naturally stratified into $g+1$ strata: $\Lambda = \cup_{\tilde{g}=0}^{g} \Lambda_{\tilde{g}}$, such that curves with $\lambda \in \Lambda_{\tilde{g}}$ have the actual genus $\tilde{g}$. In [49] such a stratification in the case of a genus 2 curve is described in detail. The whole theory works for λ from all strata.

An (n, s)-curve possesses the Weierstrass gap sequence generated by n and s, namely,

$$\mathfrak{W} = \mathbb{N}_0 \setminus \{an + bs \mid a, b \in \mathbb{N}_0\}.$$

The length of this sequence is equal to (2). In the presence of double points, the genus of $\mathcal{C}$ decreases, and the gap sequence is truncated.

Remark 1. *In fact, (n, s)-curves are Weierstrass canonical forms (see [1], Chap. V); they generalize the Weierstrass canonical form $-y^2 + 4x^3 - g_2 x - g_3 = 0$ of elliptic curves. Every plane algebraic curve can be bi-rationally transformed into one with a branch point at infinity, where all n sheets wind (see [1], p. 92). The latter is an (n, s)-curve, possibly with double points. (A double point refers to a location where two distinct sheets of a Riemann surface come together. If more than two sheets converge at such a point, the location is described as having two or more double points).*

Infinity on (1) is a Weierstrass point and serves as the base point of the Abel map.

In what follows, we work with $\mathcal{E}(\Lambda_g, \pi, \mathcal{C})$, that is, we assume that the genus of $\mathcal{C}$ is equal to g computed by (2).

Remark 2. *In the presence of one or more double points, the actual genus $\tilde{g}$ is less than g, say $\tilde{g} = g - k$. In this case, we work with λ from Λ_{g-k} or a subspace of Λ_{g-k}, defined by the relations between λ which introduce double points. We call such curves degenerate. The case of a degenerate curve can be derived from the case of the maximal genus g, as shown in [49].*

One can add extra terms in the function f from (1), namely $\lambda_{s-ni} y^{n-1} x^i$, $i = 1, \ldots, [s/n]$, and $\lambda_3 x^{s-1}$, which do not affect (2). For example, a $(3, 4)$-curve with extra terms is worked out in [32]. Although introducing extra terms seems to cover a wider variety of curves, only the parameters λ incorporated into (1) serve as independent arguments of the associated σ-function. Extra terms can be introduced through a proper birational transformation of (1).

Let $\mathcal{P}$ be a polynomial in x of degree s. A curve defined by the equation

$$f(x, y; \lambda) \equiv -y^n + \mathcal{P}(x) = 0 \tag{3}$$

is called *cyclic* (n, s)-curve, or *superelliptic* if $s \geqslant 3$.

2.2. Sato Weight

The Sato weight plays an important role in the theory of (n, s)-curves, and the associated entire and Abelian functions. The *Sato weight* shows the negative exponent of the leading term in expansion near infinity. Let ξ denote a local parameter in the vicinity of infinity, then $\mathcal{C}$ admits the parametrization

$$x = \xi^{-n}, \qquad y = \xi^{-s}(1 + O(\lambda)). \tag{4}$$

Thus, the Sato weights of x and y are $\operatorname{wgt} x = n$, and $\operatorname{wgt} y = s$. Then, $\operatorname{wgt} f(x, y; \lambda) = ns$, and parameters λ of the curve are also assigned with weights: $\operatorname{wgt} \lambda_k = k$. Note that the curve equation in (1) contains only parameters with positive Sato weights.

The Sato weight introduces an order in the space of monomials $\mathfrak{m}_{in+js}(x, y) = x^i y^j$, $i, j \in \mathbb{N}_0$, $j < n$; the weight of a monomial is indicated in the subscript. Let $\mathfrak{M}$ denote the

ordered list of monomials. Evidently, the weights equal to elements of the Weierstrass gap sequence $\mathfrak{W} = \{\mathfrak{w}_i \mid i = 1, \ldots, g\}$ are absent in the list $\mathfrak{M}$.

2.3. Cohomology Basis

Differentials of the first kind $\mathrm{d}u = (\mathrm{d}u_{\mathfrak{w}_1}, \mathrm{d}u_{\mathfrak{w}_2}, \ldots, \mathrm{d}u_{\mathfrak{w}_g})^t$, also known as holomorphic differentials, in the not normalized form are generated by the first g monomials of the list $\mathfrak{M}$, ordered by the Sato weight descendingly. Actually,

$$\mathrm{d}u_{\mathfrak{w}_i} = \frac{\mathfrak{m}_{2g-1-\mathfrak{w}_i}(x,y)\,\mathrm{d}x}{\partial_y f(x,y;\lambda)}, \quad i = 1, 2, \ldots, g. \tag{5}$$

Weights of $\mathrm{d}u$ coincide with the negative Weierstrass gap sequence, namely $\operatorname{wgt} u_{\mathfrak{w}_i} = -\mathfrak{w}_i$, which is clearly seen from expansions near infinity.

Also differentials of the second kind $\mathrm{d}r = (\mathrm{d}r_{\mathfrak{w}_1}, \mathrm{d}r_{\mathfrak{w}_2}, \ldots, \mathrm{d}r_{\mathfrak{w}_g})^t$ associated with the first kind differentials are required, see ([1], Art. 138). These second kind differentials, which are meromorphic differentials with no simple poles, have poles at infinity of orders equal to elements of the Weierstrass gap sequence, and so, $\operatorname{wgt} r_{\mathfrak{w}_i} = \mathfrak{w}_i$. In the vicinity of infinity, $\xi(\infty) = 0$, the following relation holds:

$$\operatorname{res}_{\xi=0}\Big(\int_0^{\xi} \mathrm{d}u(\tilde{\xi})\Big)\mathrm{d}r(\xi)^t = 1_g, \tag{6}$$

where 1_g denotes the identity matrix of order g. This condition completely determines the principle part of $\mathrm{d}r(\xi)$, which is enough for obtaining a solution of the Jacobi inversion problem. At the same time, deriving identities for $\wp$-functions requires a complete specification of $\mathrm{d}r$.

A system of associated first and second kind differentials arises as a part of the process of constructing the fundamental bi-differential of the second kind, see ([23], § 3.2). In the hyperelliptic case, these differentials are given explicitly by ([9], Eq. (1.3)). On an arbitrary (n,s)-curve, the problem of computing the fundamental bi-differential is solved in [50].

In what follows, we use the notation

$$\mathrm{d}u_{\mathfrak{w}_i} = \frac{v_{\mathfrak{w}_i}(x,y)\,\mathrm{d}x}{\partial_y f(x,y;\lambda)}, \qquad \mathrm{d}r_{\mathfrak{w}_i} = \frac{\rho_{\mathfrak{w}_i}(x,y)\,\mathrm{d}x}{\partial_y f(x,y;\lambda)}, \tag{7}$$

and call the first g monomials $v_{\mathfrak{w}_i}(x,y) = \mathfrak{m}_{2g-1-\mathfrak{w}_i}(x,y)$ of $\mathfrak{M}$ basis monomials, since they form a basis in the linear space $\mathfrak{P}(\mathcal{C})$ of polynomial functions on $\mathcal{C}$.

Let $\{\mathfrak{a}_i, \mathfrak{b}_i\}_{i=1}^{g}$ be canonical homology cycles on $\mathcal{C}$. First kind integrals along these cycles give the first kind period matrices (not normalized)

$$\omega = (\omega_{ij}) = \Big(\int_{\mathfrak{a}_j} \mathrm{d}u_i\Big), \qquad \omega' = (\omega'_{ij}) = \Big(\int_{\mathfrak{b}_j} \mathrm{d}u_i\Big). \tag{8}$$

Columns of ω, ω' generate the lattice $\{\omega, \omega'\}$ of periods. Thus, $\operatorname{Jac}(\mathcal{C}) = \mathbb{C}^g / \{\omega, \omega'\}$ is the *Jacobian variety* of $\mathcal{C}$. Second kind period matrices are defined similarly:

$$\eta = (\eta_{ij}) = \Big(\int_{\mathfrak{a}_j} \mathrm{d}r_i\Big), \qquad \eta' = (\eta'_{ij}) = \Big(\int_{\mathfrak{b}_j} \mathrm{d}r_i\Big). \tag{9}$$

Since $\mathrm{d}u$ and $\mathrm{d}r$ form a system of associated differentials, the period matrices ω, ω', η, η' satisfy the Legendre relation, see ([1], Art. 140):

$$\Omega^t \mathrm{J}\,\Omega = 2\pi\imath\mathrm{J}, \tag{10}$$
$$\Omega = \begin{pmatrix} \omega & \omega' \\ \eta & \eta' \end{pmatrix}, \qquad \mathrm{J} = \begin{pmatrix} 0 & -1_g \\ 1_g & 0 \end{pmatrix}.$$

Relation (10) means that $\Omega \in \imath\mathrm{Sp}(2g, \mathbb{C})$, and Ω transforms under the action of the symplectic group of size $2g$.

At the same time, symplectic transformations act on the vector $\mathrm{d}R = \begin{pmatrix} \mathrm{d}u \\ \mathrm{d}r \end{pmatrix}$ composed of the associated $\mathrm{d}u$ and $\mathrm{d}r$. This fact singles out these particular $2g$ differentials, and provides an alternative way of obtaining the required second kind differentials. Vector $\mathrm{d}R$ serves as a basis in the space $\mathcal{H}^1(\mathcal{C}^\circ)$ of holomorphic 1-forms on the curve $\mathcal{C}$ with the puncture at infinity. $\mathrm{d}R$ is obtained along with the symplectic gauge of the Gauss–Manin connection in $\mathcal{H}^1(\mathcal{C}^\circ)$, see ([18], §3.1). In fact, the Gauss–Manin connection is defined in the bundle $\mathcal{E}(\Lambda, \varpi, \mathcal{H}^1(\mathcal{C}^\circ))$ associated with the bundle $\mathcal{E}(\Lambda, \pi, \mathcal{C})$ of curves.

2.4. Examples

$(2, 2g+1)$-Curves. The canonical form of genus g hyperelliptic curves is defined by

$$f(x, y; \lambda) \equiv -y^2 + x^{2g+2} + \sum_{i=1}^{2g} \lambda_{2i+1} x^{2g-i}. \tag{11}$$

The corresponding Weierstrass gap sequence is $\mathfrak{W} = \{2i-1 \mid i = 1, \ldots, g\}$. The associated first and second kind not normalized differentials have the form, see ([1], p. 195 *Ex.* i),

$$\mathrm{d}u_{2i-1} = \frac{x^{g-i}\mathrm{d}x}{\partial_y f(x, y; \lambda)}, \quad i = 1, \ldots, g, \tag{12a}$$
$$\mathrm{d}r_{2i-1} = \frac{\mathrm{d}x}{\partial_y f(x, y; \lambda)} \sum_{k=1}^{2i-1} k\lambda_{4i-2k-2} x^{g-i+k}, \quad i = 1, \ldots, g. \tag{12b}$$

$(2, 7)$-Curve. A $(2, 7)$-curve is defined by

$$f(x, y; \lambda) \equiv -y^2 + x^7 + \lambda_4 x^5 + \lambda_6 x^4 + \lambda_8 x^3 + \lambda_{10} x^2 + \lambda_{12} x + \lambda_{14}, \tag{13}$$

with the gap sequence $\mathfrak{W} = \{1, 3, 5\}$, and the ordered list of monomials

$$\mathfrak{M} = \{1,\ x,\ x^2,\ x^3,\ y,\ x^4,\ yx,\ x^5,\ yx^2,\ x^6,\ \ldots\}. \tag{14}$$

The associated first and second kind differentials are given by

$$\mathrm{d}u = \begin{pmatrix} \mathrm{d}u_1 \\ \mathrm{d}u_3 \\ \mathrm{d}u_5 \end{pmatrix} = \begin{pmatrix} x^2 \\ x \\ 1 \end{pmatrix} \frac{\mathrm{d}x}{\partial_y f(x, y; \lambda)}, \quad \mathrm{d}r = \begin{pmatrix} \mathrm{d}r_1 \\ \mathrm{d}r_3 \\ \mathrm{d}r_5 \end{pmatrix} = \begin{pmatrix} \rho_1(x, y) \\ \rho_3(x, y) \\ \rho_5(x, y) \end{pmatrix} \frac{\mathrm{d}x}{\partial_y f(x, y; \lambda)}, \tag{15}$$
$$\rho_1(x, y) = x^3, \quad \rho_3(x, y) = 3x^4 + \lambda_4 x^2, \quad \rho_5(x, y) = 5x^5 + 3\lambda_4 x^3 + 2\lambda_6 x^2 + \lambda_8 x.$$

Trigonal curves have canonical forms of two types: $(3, 3\mathfrak{m}+1)$, and $(3, 3\mathfrak{m}+2)$, $\mathfrak{m} \in \mathbb{N}$.

$(3, 3\mathfrak{m}+1)$-Curves. The canonical trigonal curve of genus $3\mathfrak{m}$ is defined by

$$f(x, y; \lambda) \equiv -y^3 + x^{3\mathfrak{m}+1} + y \sum_{i=0}^{2\mathfrak{m}} \lambda_{3i+2} x^{2\mathfrak{m}-i} + \sum_{i=1}^{3\mathfrak{m}} \lambda_{3i+3} x^{3\mathfrak{m}-i}. \tag{16}$$

The Weierstrass gap sequence is $\mathfrak{W} = \{3i-2 \mid i = 1, \ldots, \mathfrak{m}\} \cup \{3i-1 \mid i = 1, \ldots, 2\mathfrak{m}\}$, sorted ascendingly. The standard not normalized first kind differentials have the form

$$\begin{aligned} \mathrm{d}u_{3i-2} &= \frac{yx^{\mathfrak{m}-i}\mathrm{d}x}{\partial_y f(x,y;\lambda)}, \quad i=1,\ldots,\mathfrak{m}, \\ \mathrm{d}u_{3i-1} &= \frac{x^{2\mathfrak{m}-i}\mathrm{d}x}{\partial_y f(x,y;\lambda)} \quad i=1,\ldots,2\mathfrak{m}. \end{aligned} \tag{17}$$

$(3,4)$**-Curve.** The simplest curve of this type is $(3,4)$-curve defined by

$$f(x,y;\lambda) \equiv -y^3 + x^4 + \lambda_2 yx^2 + \lambda_5 yx + \lambda_6 x^2 + \lambda_8 y + \lambda_9 x + \lambda_{12}, \tag{18}$$

with the gap sequence $\mathfrak{W} = \{1,2,5\}$, and the ordered list of monomials

$$\mathfrak{M} = \{1,\ x,\ y,\ x^2,\ yx,\ y^2,\ x^3,\ yx^2,\ y^2x,\ \ldots\}. \tag{19}$$

The system of associated first and second kind differentials consists of

$$\mathrm{d}u = \begin{pmatrix} \mathrm{d}u_1 \\ \mathrm{d}u_2 \\ \mathrm{d}u_5 \end{pmatrix} = \begin{pmatrix} y \\ x \\ 1 \end{pmatrix} \frac{\mathrm{d}x}{\partial_y f(x,y;\lambda)}, \quad \mathrm{d}r = \begin{pmatrix} \mathrm{d}r_1 \\ \mathrm{d}r_2 \\ \mathrm{d}r_5 \end{pmatrix} = \begin{pmatrix} \rho_1(x,y) \\ \rho_2(x,y) \\ \rho_5(x,y) \end{pmatrix} \frac{\mathrm{d}x}{\partial_y f(x,y;\lambda)}, \tag{20}$$
$$\rho_1(x,y) = x^2, \quad \rho_2(x,y) = 2xy, \quad \rho_5(x,y) = 5x^2y + \tfrac{2}{3}\lambda_2^2 x^2 + \lambda_6 y + \tfrac{2}{3}\lambda_2\lambda_5 x.$$

$(3,3\mathfrak{m}+2)$**-Curves.** The canonical trigonal curve of genus $3\mathfrak{m}+1$ is defined by

$$f(x,y;\lambda) \equiv -y^3 + x^{3\mathfrak{m}+2} + y\sum_{i=0}^{2\mathfrak{m}+1} \lambda_{3i+1} x^{2\mathfrak{m}+1-i} + \sum_{i=1}^{3\mathfrak{m}+1} \lambda_{3i+3} x^{3\mathfrak{m}+1-i}. \tag{21}$$

The Weierstrass gap sequence is $\mathfrak{W} = \{3i-1 \mid i=1,\ldots,\mathfrak{m}\} \cup \{3i-2 \mid i=1,\ldots,2\mathfrak{m}+1\}$, sorted ascendingly. The standard not normalized first kind differentials have the form

$$\begin{aligned} \mathrm{d}u_{3i-1} &= \frac{yx^{2\mathfrak{m}+1-i}\mathrm{d}x}{\partial_y f(x,y;\lambda)}, \quad i=1,\ldots,2\mathfrak{m}+1, \\ \mathrm{d}u_{3i-2} &= \frac{x^{\mathfrak{m}-i}\mathrm{d}x}{\partial_y f(x,y;\lambda)} \quad i=1,\ldots,\mathfrak{m}. \end{aligned} \tag{22}$$

Remark 3. *In the hyperelliptic case, modality is $M=0$, and on trigonal curves $M=\mathfrak{m}-1$.*

2.5. Abel Map

Let the Abel map $\mathcal{A}: \mathcal{C} \to \mathrm{Jac}(\mathcal{C})$ be defined with respect to not normalized differentials $\mathrm{d}u$:

$$\mathcal{A}(P) = \int_\infty^P \mathrm{d}u, \qquad P=(x,y) \in \mathcal{C}. \tag{23}$$

Recall that infinity serves as the base point. The Abel map is also defined on any symmetric product $\mathcal{C}^n$. Namely, given a positive divisor $D = \sum_{i=1}^n P_i$, we have

$$\mathcal{A}(D) = \sum_{i=1}^n \mathcal{A}(P_i).$$

The Abel map is invertible on $\mathfrak{C}_g \subset \mathcal{C}^g$ which consists of non-special divisors of degree g. The problem of finding D from the equation

$$u(D) = \sum_{i=1}^{g} \int_{\infty}^{P_i} \mathrm{d}u, \qquad D = \sum_{i=1}^{g} P_i \in \mathfrak{C}_g, \tag{24}$$

is known as the *Jacobi inversion problem*, see [51]. A solution of the problem on hyperelliptic curves has been known since the end of the 19th century, see ([1], Chap. IX).

Normalized holomorphic differentials $\mathrm{d}v$, and the normalized period lattice $\{1_g, \tau\}$ are obtained as follows

$$\mathrm{d}v = \omega^{-1}\mathrm{d}u, \qquad \tau = \omega^{-1}\omega'. \tag{25}$$

The Riemann period matrix τ belongs to the Siegel upper half-space $\mathfrak{S}_g$ of degree g, that is, τ is symmetric with a positive imaginary part: $\tau^t = \tau$, $\operatorname{Im}\tau > 0$.

We also define the Abel map $\bar{\mathcal{A}}$ with respect to normalized differentials $\mathrm{d}v$:

$$\bar{\mathcal{A}}(P) = \int_{\infty}^{P} \mathrm{d}v, \qquad P = (x, y) \in \mathcal{C}. \tag{26}$$

In addition to the Abel map $\mathcal{A}$, which produces first kind integrals, we introduce the map $\mathcal{B}$, which produces second kind integrals:

$$\begin{aligned} \mathcal{B}(P) &= \int_{\infty}^{P} \mathrm{d}r, \qquad P = (x, y) \in \mathcal{C}, \\ \mathcal{B}(D) &= \sum_{i=1}^{n} \mathcal{B}(P_i), \qquad D \in \mathcal{C}^n. \end{aligned} \tag{27}$$

Remark 4. *The integral in* (27) *requires regularization since the base point serves as the pole of second kind differentials* $\mathrm{d}r$. *Such a regularization is suggested in [52], in a way similar to the regularization of the Weierstrass* ζ*-function. The problem of regularization involves finding a constant vector, which depends on* λ. *In [52] the regularization constants are obtained for* $(3,4)$, $(3,5)$, $(3,7)$, $(4,5)$*-curves. In the hyperelliptic case, these constants equal zero.*

2.6. Theta Function

In terms of normalized coordinates v, and the Riemann period matrix τ, the *Riemann theta function* $\theta : \mathbb{C}^g \times \mathfrak{S}_g \to \mathbb{C}$ is defined by

$$\theta(v;\tau) = \sum_{n \in \mathbb{Z}^g} \exp\left(\imath\pi n^t \tau n + 2\imath\pi n^t v\right). \tag{28}$$

The definition works for any $\tau \in \mathfrak{S}_g$, though not every τ relates to a Jacobian variety if $g \geqslant 4$. In what follows, we deal only with τ obtained from the period matrices ω and ω' related to a curve $\mathcal{C}$.

Let a *theta function with characteristic* $[\varepsilon] = (\varepsilon', \varepsilon)^t$ be defined by

$$\theta[\varepsilon](v;\tau) = \exp\left(\imath\pi\varepsilon'^t\tau\varepsilon' + 2\imath\pi(v+\varepsilon)^t(\varepsilon')\right)\theta(v+\varepsilon+\tau\varepsilon';\tau), \tag{29}$$

where ε and ε' are g-component vectors with real values within the interval $[0,1)$. Every point u in the fundamental domain of $\mathrm{Jac}(\mathcal{C})$ can be represented by its characteristic $[\varepsilon]$, namely

$$u[\varepsilon] = \omega\varepsilon + \omega'\varepsilon'.$$

2.7. Sigma Function

In the theory presented below, the leading role belongs to the modular-invariant, entire function, known as σ-function.

A generalization of the Weierstrass σ-function is proposed by Klein: an extension to hyperelliptic curves in [6,7], and to an arbitrary curve of genus 3 in [8]. In [2] the σ-function associated with a genus two curve is studied in detail. The fundamental σ-function is defined in ([2], p. 97), and more precisely in ([9], Eq. (2.3)), namely

$$\sigma(u) = C \exp\left(-\tfrac{1}{2} u^t \varkappa u\right) \theta[K](\omega^{-1}u; \omega^{-1}\omega'), \tag{30}$$

where $\varkappa = \eta\omega^{-1}$ is a symmetric matrix constructed from η and ω defined in (9) and (8), constant C does not depend on u, and $[K]$ denotes the characteristic of the vector K of Riemann constants.

On the other hand, a σ-function is an analytic series in coordinates $u \in \mathrm{Jac}(\mathcal{C})$ and parameters λ of $\mathcal{C}$, and arises as a solution of a system of heat equations. This definition generalizes the definition of the Weierstrass σ-function given in [53]. Namely, the entire function $\sigma(u; g_2, g_3)$ associated with the Weierstrass form of elliptic curves is defined by the system of partial differential equations

$$\mathfrak{Q}_0 \sigma = 0, \qquad \mathfrak{Q}_2 \sigma = 0, \tag{31}$$

$$\begin{aligned} \mathfrak{Q}_0 &= 1 - u\partial_u + 4g_2\partial_{g_2} + 6g_3\partial_{g_3}, \\ \mathfrak{Q}_2 &= -\tfrac{1}{24} g_2 u^2 - \tfrac{1}{2}\partial_u^2 + 6g_3\partial_{g_2} + \tfrac{1}{3} g_2^2 \partial_{g_3}, \end{aligned} \tag{32}$$

with the initial condition $\sigma(u; 0, 0) = u$. Here, and in what follows, $\partial_a \equiv \partial/\partial a$. Indices of the differential operators $\mathfrak{Q}_k$ indicate the Sato weight: $\operatorname{wgt} \mathfrak{Q}_k = k$. Note that $\operatorname{wgt} g_2 = 4$, and $\operatorname{wgt} g_3 = 6$. Evidently, $\mathfrak{Q}_0$ is the Euler operator, which shows the weights of all arguments. Operator $\mathfrak{Q}_2$ is of the second order with respect to u, and of the first order with respect to parameters g_2, g_3 of the curve.

After introducing (n, s)-curves, and analyzing the rational case in [15], Buchstaber and Leykin developed the theory of constructing series expansions for σ-functions associated with (n, s)-curves, see [16–18], known as the theory of multi-variable σ-functions. Every (n, s)-curve, more precisely a bundle $\mathcal{E}(\Lambda, \pi, \mathcal{C})$ of curves defined by (1), is equipped with a unique function $\sigma(u; \lambda)$. The operators $\mathfrak{Q}_k$ which annihilate the σ-function are obtained by lifting the Lie algebra of vector fields from the base Λ to the fiber bundle $\mathcal{E}\big(\Lambda, \varpi, \mathrm{Jac}(\mathcal{C})\big)$.

The σ-function associated with $\mathcal{C}$ arises as a unique solution of the system $\{\mathfrak{Q}_k \sigma = 0\}$ of heat equations with the initial condition $\sigma(u; 0) = S_{\mathfrak{p}}(u)$, where $S_{\mathfrak{p}}$ denotes the Schur–Weierstrass polynomial in coordinates u (see [15,42]) generated by the partition

$$\mathfrak{p} = \{\mathfrak{p}_g, \mathfrak{p}_{g-1}, \ldots, \mathfrak{p}_2, \mathfrak{p}_1\}, \qquad \mathfrak{p}_i = \mathfrak{w}_i - (i - 1). \tag{33}$$

The latter is determined by the Weierstrass gap sequence $\mathfrak{W} = \{\mathfrak{w}_1, \mathfrak{w}_2, \ldots, \mathfrak{w}_g\}$ on $\mathcal{C}$. Note, that $\sum_{i=1}^{g} \mathfrak{p}_i = \frac{1}{24}(n^2 - 1)(s^2 - 1)$.

Property 1 ([15]). *The Sato weight of the σ-function associated with an (n, s)-curve is*

$$\operatorname{wgt} \sigma = -\tfrac{1}{24}(n^2 - 1)(s^2 - 1). \tag{34}$$

The system of heat equations generated by $\mathfrak{Q}_k$ produces recurrence relations between coefficients of a series for the σ-function in closed form.

Remark 5. *Note that the series for $\sigma(u; \lambda)$ is analytic with respect to all arguments: u and λ, and so works for all $\lambda \in \Lambda$, including strata with decreased genera.*

In [36] a detailed exposition of the theory of Buchstaber and Leykin on heat equations for multi-variable σ-functions is provided, along with examples for $(2, 3)$, $(2, 5)$, $(2, 7)$,

$(3,4)$-curves, with recurrence relations derived. The power series expansion of the σ-function associated with the most general form of an elliptic curve $y^2 = \lambda_1 yx + \lambda_3 y + x^3 + \lambda_2 x^2 + \lambda_4 x + \lambda_6$ is derived in [54].

Explicit expressions for the annihilating operators $\mathfrak{Q}_k$ in the hyperelliptic case can be found in [55], where $\mathfrak{Q}_k$ are improperly called Schrödinger operators. The heat equations $\mathfrak{Q}_k \sigma = 0$, whether considered as a system or separately, have neither physical interpretation nor the form of the Schrödinger equation.

First terms in the expansions of σ-functions associated with (n,s)-curves of low genera have been computed: for a $(3,4)$-curve with extra terms see [32], for a cyclic (superelliptic) $(3,5)$-curve see [29], for cyclic trigonal curves of genera six and seven see [40], and for a cyclic $(4,5)$-curve see [37].

Relations between σ, θ, and the entire τ-function are explained in detail in [34,44,56]. The theory of σ-functions associated with space curves is developed in [46,48] for $(3,4,5)$-, $(3,7,8)$-, and $(6,13,14,15,16)$-curves; in [57] for telescopic curves; and in [47] for more general cyclic trigonal curves. In [58] a σ-function is defined for a compact Riemann surface which is not directly associated with an algebraic curve. In [45] σ-functions for compact Riemann surfaces are derived from Sato's theory of the universal Grassmannian manifolds.

2.8. Vector of Riemann Constants

A formula for computing the vector of Riemann constants K with respect to a given base point can be found in ([59], Eq. (2.4.14)). In the hyperelliptic case, the vector is computed in ([60], p. 14), and equals the sum of all odd characteristics from the fundamental set of $2g+1$ half-integer characteristics which represent branch points, according to ([1], §§ 200–202).

From ([59], Eq. (2.4.20)) we know that $[K]$ is among half-integer characteristics, if $\mathcal{C}$ is not superelliptic. Property 1 along with (30) leads to the following.

Theorem 1. *$\theta[K]$ as a function of not normalized coordinates u has the maximal weighted order of vanishing at $u = 0$, equal to $\mathfrak{d} = -\operatorname{wgt}\sigma$. That is,*

$$\forall \mathfrak{i} < \mathfrak{d} \quad \partial^{\mathfrak{i}}_{u_1}\theta[K](0;\omega^{-1}\omega') = 0, \quad \partial^{\mathfrak{d}}_{u_1}\theta[K](0;\omega^{-1}\omega') \neq 0. \tag{35}$$

This criterion singles out the characteristic of the vector of Riemann constants on non-hyperelliptic curves, as well as on hyperelliptic curves of even genus. See examples in https://community.wolfram.com/groups/-/m/t/3296279.

2.9. Multiply Periodic $\wp$-Functions

The σ-function associated with $\mathcal{C}$ generates abelian functions on $\mathrm{Jac}(\mathcal{C})$, known as multiply periodic functions after [2], or Kleinian $\wp$-functions after [9], as follows

$$\wp_{i,j}(u) = -\frac{\partial^2 \log\sigma(u)}{\partial u_i \partial u_j}, \qquad \wp_{i,j,k}(u) = -\frac{\partial^3 \log\sigma(u)}{\partial u_i \partial u_j \partial u_k}, \quad \text{etc.} \tag{36}$$

The notation arose in ([1], p. 294). In what follows, we refer to them as $\wp$-functions. All $\wp$-functions are periodic with respect to the period lattice $\{\omega, \omega'\}$. Namely, let m, m' be g-component vectors of integers, then

$$\wp_{i,j}(u + \omega m + \omega' m') = \wp_{i,j}(u).$$

2.10. Divisor Classes

Traditionally, the Jacobian variety $\mathrm{Jac}(\mathcal{C})$ is described as the group Pic^0 of divisors of degree zero factored out by principal divisors. Every class of equivalent divisors on a curve of genus g has a representative $D = \sum_{k=1}^{l} P_k - l\infty$, $0 \leqslant l \leqslant g$, such that the positive part contains no groups of points in involution. Such representatives are called *reduced divisors*. In what follows, we define a reduced divisor by its positive part, namely $D = \sum_{k=1}^{l} P_k$, since the poles are located at infinity, which serves as the base point. We will always assume that a positive divisor D contains no groups of points in involution, that is, no groups of points (a, b_i), $i = 1, \ldots, n$, such that $y = b_i$ solve $f(a, y; \lambda) = 0$. A non-special divisor may contain $n - 1$ points from a group in involution, but not all n, since $\sum_{i=1}^{n}(a, b_i) \sim n\infty$.

Reduced divisors of degree less than g are special, and $\theta[K]$, as well as σ, vanishes on Abel images of such divisors, according to the Riemann vanishing theorem. Reduced divisors of degree g are non-special. Every non-special divisor represents its class uniquely.

Let $\Sigma = \{u \in \mathrm{Jac}(\mathcal{C}) \mid \sigma(u) = 0\}$. Up to the normalization (25), Σ coincides with the theta divisor, see ([59], p. 38).

Remark 6. *As follows from* (36), *$\wp$-functions are defined on* $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$. *Therefore, the abelian function field* $\mathfrak{A}(\mathcal{C})$ *is defined over* $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$. *At the same time, by taking proper limits and canceling singularities, one can obtain fields of functions over strata of* Σ *from* $\mathfrak{A}(\mathcal{C})$.

Let $\mathfrak{C}_g$ be the subspace of $\mathcal{C}^g$ which consists of reduced divisors of degree g. We also introduce $\mathfrak{C}_{g-l} \subset \mathcal{C}^{g-l}$, $l = 1, \ldots, g-1$. Each $\mathfrak{C}_{g-l}$ consists of reduced divisors of the fixed degree equal to $g - l$. The Abel map (24) on $\mathfrak{C}_{g-l}$ is obtained by taking the limit $P_k \to \infty$, $k = g - l + 1, \ldots, g$. That is, $\mathfrak{C}_{g-l} + l\infty \subset \mathcal{C}^g$.

The Riemann vanishing theorem implies $\mathfrak{C}_g = \{D \mid \sigma(\mathcal{A}(D)) \neq 0\}$. Subspaces $\mathfrak{C}_{g-l}$ are also defined by means of the order of vanishing of the σ-function (see [61], Proposition 5.7). Actually,

$$\mathfrak{C}_{g-l} = \{D \mid \sigma(\mathcal{A}(D)) = 0,\ \forall \mathfrak{i} < \mathfrak{r} = \textstyle\sum_{i=1}^{l} \mathfrak{p}_i \quad \partial_{u_1}^{\mathfrak{i}}\sigma(\mathcal{A}(D)) = 0, \quad \partial_{u_1}^{\mathfrak{r}}\sigma(\mathcal{A}(D)) \neq 0\}, \quad l = 1, \ldots, g, \tag{37}$$

where $\mathfrak{p}_i$ are defined in (33). Note that $\mathfrak{C}_0$ consists of the class of points equivalent to $g\infty \in \mathcal{C}^g$, whose Abel image is the origin of $\mathrm{Jac}(\mathcal{C})$, namely, $\mathcal{A}(\mathfrak{C}_0) = 0$.

Remark 7. *$\mathfrak{C}_{g-l}$, $l = 0, 1, \ldots, g$, are disjoint and define the stratification of* $\mathrm{Jac}(\mathcal{C})$:

$$\mathrm{Jac}(\mathcal{C})\backslash\Sigma = \mathcal{A}(\mathfrak{C}_g), \qquad \Sigma = \cup_{l=1}^{g}\mathcal{A}(\mathfrak{C}_{g-l}). \tag{38}$$

Relations with the Wirtinger varieties W_ℓ (see [33], p. 38) are given by $W_\ell = \cup_{l=0}^{\ell}\mathcal{A}(\mathfrak{C}_l)$.

2.11. Polynomial Functions on a Curve

Since we have fixed the base point at infinity, divisors on $\mathcal{C}$ are described by means of polynomial functions from the ring $\mathbb{C}[x, y]/f(x, y; \lambda)$, according to Abel's theorem. We denote by $\mathcal{R}_w$ a *polynomial function of weight* w; the weight shows the degree of its divisor of zeros $(\mathcal{R}_w)_0$. And $(\mathcal{R}_\mathfrak{w})_0$ is obtained from the system

$$\mathcal{R}_\mathfrak{w}(x, y) = 0, \qquad f(x, y) = 0. \tag{39}$$

Not all points of $(\mathcal{R}_\mathfrak{w})_0$ can be chosen arbitrarily.

Theorem 2. *A polynomial function $\mathcal{R}_w$ of weight $w \geqslant 2g$ is uniquely defined by a positive divisor D of degree $w-g$ such that $D \subset (\mathcal{R}_w)_0$.*

Proof. A polynomial function $\mathcal{R}_w$ is constructed from monomials $\{\mathfrak{m}_{\widetilde{w}} \mid \widetilde{w} \leqslant w\}$, namely

$$\mathcal{R}_w(x,y) = \mathfrak{m}_w + \sum_{\widetilde{w}<w} c_{\widetilde{w}} \mathfrak{m}_{\widetilde{w}}. \tag{40}$$

If $w \geqslant 2g$, there exist $w-g+1$ such monomials. Note that $\mathcal{R}_w$ is monic, that is, the leading coefficient equals 1. Thus, (40) contains $w-g$ unknown coefficients, which are uniquely determined from $w-g$ points of $\mathcal{C}$. Indeed, let $D = \sum_{k=1}^{w-g}(x_k, y_k)$. If all points of D are distinct, then

$$\mathcal{R}_w(x,y;D) = \frac{\begin{vmatrix} \mathfrak{m}_w(x,y) & \mathfrak{m}_{w-1}(x,y) & \dots & \mathfrak{m}_0(x,y) \\ \mathfrak{m}_w(x_1,y_1) & \mathfrak{m}_{w-1}(x_1,y_1) & \dots & \mathfrak{m}_0(x_1,y_1) \\ \vdots & \ddots & \vdots & \\ \mathfrak{m}_w(x_{w-g},y_{w-g}) & \mathfrak{m}_{w-1}(x_{w-g},y_{w-g}) & \dots & \mathfrak{m}_0(x_{w-g},y_{w-g}) \end{vmatrix}}{\begin{vmatrix} \mathfrak{m}_{w-1}(x_1,y_1) & \dots & \mathfrak{m}_0(x_1,y_1) \\ \vdots & \ddots & \vdots \\ \mathfrak{m}_{w-1}(x_{w-g},y_{w-g}) & \dots & \mathfrak{m}_0(x_{w-g},y_{w-g}) \end{vmatrix}}. \tag{41a}$$

If some points coincide, say $P_k = P_1$, $k = 2, \dots m$, then row $k+1$ in the numerator and row k in the denominator of (41a) are replaced with

$$\Big(\frac{\mathrm{d}^{k-1}}{\mathrm{d}x^{k-1}} \mathfrak{m}_{\widetilde{w}}(x, y(x))\Big|_{\substack{x=x_1 \\ y(x_1)=y_1}}\Big). \tag{41b}$$

Formula (41) produces $\mathcal{R}_w$ from $D_{w-g} \subset (\mathcal{R}_w)_0$ uniquely. □

Corollary 1. *A polynomial function $\mathcal{R}_w$ of weight $w \geqslant 2g$ constructed from a positive divisor of degree $w-g$ produces the complement divisor D^* such that $(\mathcal{R}_w)_0 = D + D^*$; D^* is non-special, and* $\deg D^* = g$.

3. Abelian Function Fields Associated with Curves

We denote by $\mathfrak{A}(\mathcal{C})$ the abelian function field associated with a curve $\mathcal{C}$ of genus g. We construct this field from $\wp$-functions generated by the σ-function associated with the curve.

Property 2. *$\mathfrak{A}(\mathcal{C})$ is a differential field (see [18]), which means it has the property: if $\Phi \in \mathfrak{A}(\mathcal{C})$, then $\partial_{u_{\mathfrak{w}_i}} \Phi \in \mathfrak{A}(\mathcal{C})$, $i = 1, \dots, g$. The differentiation satisfies the additive rule and the Leibniz product rule. As a differential field $\mathfrak{A}(\mathcal{C})$ has g generators, see [18], and Theorem 6.*

Property 3. *$\mathfrak{A}(\mathcal{C})$ has the structure of a ring of polynomial functions (in the hyperelliptic case) or rational functions (in the non-hyperelliptic case) in a set of basis $\wp$-functions. The ideal of the ring defines* $\mathrm{Jac}(\mathcal{C}) \backslash \Sigma$.

These properties are derived from the structure of abelian function fields associated with algebraic curves. Such a function field serves to uniformize the curve. There exist relations between $\wp$-functions, which we call identities. In this section we recall how to solve the uniformization problem, known as the Jacobi inversion problem, and describe two techniques of obtaining identities for $\wp$-functions.

In what follows, we focus on hyperelliptic and trigonal curves, which have been intensively studied and broadly illustrated in the literature.

3.1. Jacobi Inversion Problem

Due to fixing the base point at infinity, a solution of the Jacobi inversion problem is guaranteed to be expressed by means of polynomial functions from $\mathbb{C}[x,y]/f(x,y;\lambda)$, according to ([1], § 164). We formulate the problem as follows.

Jacobi inversion problem. *Given a point* $u \in \mathrm{Jac}(\mathcal{C})\backslash\Sigma$, *find a divisor* $D \in \mathfrak{C}_g$ *such that* $\mathcal{A}(D) = u$.

$(2,2g+1)$-Curves. A solution of the Jacobi inversion problem on hyperelliptic curves was given in ([1], § 216), see also ([9], Theorem 2.2). On a curve defined by (11), the divisor D such that $u = \mathcal{A}(D)$ is obtained from the system

$$\begin{aligned} \mathcal{R}_{2g}(x;u) &\equiv x^g - \sum_{i=1}^{g} x^{g-i}\wp_{1,2i-1}(u) = 0, \\ \mathcal{R}_{2g+1}(x,y;u) &\equiv 2y + \sum_{i=1}^{g} x^{g-i}\wp_{1,1,2i-1}(u) = 0. \end{aligned} \tag{42}$$

In other words, D is the common divisor of zeros of the two polynomial functions $\mathcal{R}_{2g}$ and $\mathcal{R}_{2g+1}$ of weights $2g$ and $2g+1$, respectively.

A solution of the Jacobi inversion problem on trigonal curves is derived from the Klein formula in [20]. In the case of superelliptic (n,s)-curves, expressions for $\wp_{1,\mathfrak{w}_i}$, $i = 1,\dots,g$, in terms of coordinates of D are suggested in ([62], Theorem 5.1), which is equivalent to the first equation from the system which solves the Jacobi inversion problem. Although proofs are given for superelliptic curves, the obtained results work for generic (n,s)-curves, as mentioned in ([61], Remark 3.5). The case of special divisors is also addressed in ([62], Theorem 5.1), though part (3) is incorrect for small k, as well as ([62], Lemma 5.6). On the other hand, the stratification of Σ based on the order of vanishing of the associated σ-function is given correctly in ([61], Proposition 5.7). The Jacobi inversion problem on special divisors, in the case of hyperelliptic curves, is addressed in [63].

A complete solution of the Jacobi inversion problem on (n,s)-curves is obtained in [64] by a more elegant technique based on the residue theorem, see (55). Solutions of the Jacobi inversion problem on trigonal, tetragonal, and pentagonal curves are shown as illustrations. Below, the solutions on canonical trigonal curves from [64] are presented.

$(3,3\mathfrak{m}+1)$-Curves. On the canonical $(3,3\mathfrak{m}+1)$-curve defined by (16), the pre-image D of $u = \mathcal{A}(D)$ is given by the system

$$\begin{aligned} \mathcal{R}_{6\mathfrak{m}+1}(x,y;u) &\equiv x^{2\mathfrak{m}} - y\sum_{i=1}^{\mathfrak{m}} \wp_{1,3i-2}(u)x^{\mathfrak{m}-i} - \sum_{i=1}^{2\mathfrak{m}} \wp_{1,3i-1}(u)x^{2\mathfrak{m}-i} = 0, \\ \mathcal{R}_{6\mathfrak{m}+2}(x,y;u) &\equiv 2yx^{\mathfrak{m}} - y\sum_{i=1}^{\mathfrak{m}} \big(\wp_{2,3i-2}(u) - \wp_{1,1,3i-2}(u)\big)x^{\mathfrak{m}-i} \\ &\quad - \sum_{i=1}^{2\mathfrak{m}} \big(\wp_{2,3i-1}(u) - \wp_{1,1,3i-1}(u)\big)x^{2\mathfrak{m}-i} = 0. \end{aligned} \tag{43}$$

$(3,3\mathfrak{m}+2)$-Curves. On the canonical $(3,3\mathfrak{m}+2)$-curve defined by (21), the required D is given by the system

$$\begin{aligned}\mathcal{R}_{6\mathfrak{m}+2}(x,y;u) &\equiv yx^{\mathfrak{m}} - y\sum_{i=1}^{\mathfrak{m}}\wp_{1,3i-1}(u)x^{\mathfrak{m}-i} - \sum_{i=1}^{2\mathfrak{m}+1}\wp_{1,3i-2}(u)x^{2\mathfrak{m}+1-i} = 0,\\ \mathcal{R}_{6\mathfrak{m}+3}(x,y;u) &\equiv 2x^{2\mathfrak{m}+1} - y\sum_{i=1}^{\mathfrak{m}}\big(\wp_{2,3i-1}(u) - \wp_{1,1,3i-1}(u)\big)x^{\mathfrak{m}-i}\\ &\quad - \sum_{i=1}^{2\mathfrak{m}+1}\big(\wp_{2,3i-2}(u) - \wp_{1,1,3i-2}(u)\big)x^{2\mathfrak{m}+1-i} = 0.\end{aligned} \tag{44}$$

$(3,4)$**-Curve.** A solution of the Jacobi inversion problem on $\mathcal{C}$ defined by (18) is given by the system

$$\begin{aligned}\mathcal{R}_6(x,y;u) &= x^2 - y\wp_{1,1}(u) - x\wp_{1,2}(u) - \wp_{1,5}(u),\\ \mathcal{R}_7(x,y;u) &= 2xy - y\big(\wp_{1,2}(u) - \wp_{1,1,1}(u)\big) + x\big(\wp_{2,2}(u) - \wp_{1,1,2}(u)\big)\\ &\quad - \big(\wp_{2,5}(u) - \wp_{1,1,5}(u)\big).\end{aligned} \tag{45}$$

On the other hand, polynomial functions of weights 6 and 7 are constructed from a non-special divisor $D = \sum_{k=1}^{3}(x_k, y_k)$ using Formula (41), with all distinct points, as follows:

$$\mathcal{R}_6(x,y;D) = \frac{\begin{vmatrix} x^2 & y & x & 1\\ x_1^2 & y_1 & x_1 & 1\\ x_2^2 & y_2 & x_2 & 1\\ x_3^2 & y_3 & x_3 & 1\end{vmatrix}}{\begin{vmatrix} y_1 & x_1 & 1\\ y_2 & x_2 & 1\\ y_3 & x_3 & 1\end{vmatrix}}, \qquad \mathcal{R}_7(x,y;D) = 2\frac{\begin{vmatrix} xy & y & x & 1\\ x_1y_1 & y_1 & x_1 & 1\\ x_2y_2 & y_2 & x_2 & 1\\ x_3y_3 & y_3 & x_3 & 1\end{vmatrix}}{\begin{vmatrix} y_1 & x_1 & 1\\ y_2 & x_2 & 1\\ y_3 & x_3 & 1\end{vmatrix}}, \tag{46}$$

Assuming that (45) and (46) define the same functions, expressions for the following basis $\wp$-functions in terms of coordinates of D can be found:

$$\begin{gathered}\wp_{1,1}(u), \quad \wp_{1,2}(u), \quad \wp_{1,5}(u), \quad \wp_{1,1,1}(u) - \wp_{1,2}(u),\\ \wp_{1,1,2}(u) - \wp_{2,2}(u), \quad \wp_{1,1,5}(u) - \wp_{2,5}(u),\end{gathered}$$

According to (37), with $\mathfrak{p} = \{3,1,1\}$, special divisors are stratified as follows

$$\begin{aligned}\mathfrak{C}_2 &= \{D \mid \sigma(\mathcal{A}(D)) = 0,\, \partial_{u_1}\sigma(\mathcal{A}(D)) \neq 0\},\\ \mathfrak{C}_1 &= \{D \mid \sigma(\mathcal{A}(D)) = 0,\, \partial_{u_1}\sigma(\mathcal{A}(D)) = 0, \partial_{u_2}\sigma(\mathcal{A}(D)) \neq 0\},\\ \mathfrak{C}_0 &= \{D \mid \sigma(\mathcal{A}(D)) = 0,\, \forall \mathfrak{i} < 5\ \partial_{u_1}^{\mathfrak{i}}\sigma(\mathcal{A}(D)) = 0, \partial_{u_5}\sigma(\mathcal{A}(u)) \neq 0\}.\end{aligned} \tag{47}$$

Since σ vanishes on Σ, (45) is replaced with

$$\sigma(u)^2\mathcal{R}_6(x,y;u) = 0, \qquad \sigma(u)^3\mathcal{R}_7(x,y;u) = 0, \tag{48}$$

where $\wp$-functions are expressed in terms of derivatives of σ, and all vanishing terms are cancelled. As a result, both $\mathcal{R}_6$ and $\mathcal{R}_7$ reduce to the same equation. Namely,

$$\begin{aligned} D \in \mathfrak{C}_2,\ \ u = \mathcal{A}(D) \qquad & y\sigma_1(u) + x\sigma_2(u) + \sigma_5(u) = 0,\\ D \in \mathfrak{C}_1,\ \ u = \mathcal{A}(D) \qquad & x\sigma_2(u) + \sigma_5(u) = 0,\end{aligned}$$

where $\sigma_{\mathfrak{w}_i}(u) \equiv \partial_{u_{\mathfrak{w}_i}}\sigma(u)$. This particular case agrees with ([62], Theorem 5.1), including part (3).

On the contrary, in the case of $(3,7)$-curve, we have $\mathfrak{W} = \{1,2,4,5,8,11\}$, and $\mathfrak{p} = \{6,4,2,2,1,1\}$. Thus, part (3) fails for $\mathfrak{C}_2$ since $\sigma_5(u) = 0$ occurs in the denominator, and for $\mathfrak{C}_1$ due to the denominator $\sigma_8(u) = 0$.

$(2,7)$**-Curve.** A solution of the Jacobi inversion problem on $\mathcal{C}$ defined by (13) is given by the system

$$\begin{aligned} &\mathcal{R}_6(x;u) = x^3 - x^2\wp_{1,1}(u) - x\wp_{1,3}(u) - \wp_{1,5}(u),\\ &\mathcal{R}_7(x,y;u) = 2y + x^2\wp_{1,1,1}(u) + x\wp_{1,1,2}(u) + \wp_{1,1,5}(u). \end{aligned} \tag{49}$$

According to (37), with $\mathfrak{p} = \{3,2,1\}$, special divisors are stratified as follows

$$\begin{aligned} \mathfrak{C}_2 &= \{D \mid \sigma(\mathcal{A}(D)) = 0,\ \partial_{u_1}\sigma(\mathcal{A}(D)) \neq 0\},\\ \mathfrak{C}_1 &= \{D \mid \sigma(\mathcal{A}(D)) = 0,\ \forall \mathrm{i} < 3\ \partial^{\mathrm{i}}_{u_1}\sigma(\mathcal{A}(D)) = 0, \partial_{u_3}\sigma(\mathcal{A}(D)) \neq 0\},\\ \mathfrak{C}_0 &= \{D \mid \sigma(\mathcal{A}(D)) = 0,\ \forall \mathrm{i} < 6\ \partial^{\mathrm{i}}_{u_1}\sigma(\mathcal{A}(D)) = 0, \partial^{6}_{u_1}\sigma(\mathcal{A}(D)) \neq 0\}. \end{aligned} \tag{50}$$

Since σ vanishes on Σ, we use (48), and obtain, cf. ([62], Theorem 5.1),

$$\begin{aligned} &D \in \mathfrak{C}_2,\ \ u = \mathcal{A}(D) && x^2\sigma_1(u) + x\sigma_3(u) + \sigma_5(u) = 0,\\ &D \in \mathfrak{C}_1,\ \ u = \mathcal{A}(D) && x\sigma_3(u) + \sigma_5(u) = 0. \end{aligned}$$

In the case of $(2,9)$-curve, we have $\mathfrak{W} = \{1,3,5,7\}$, and $\mathfrak{p} = \{4,3,2,1\}$. Then $\forall \mathrm{i} < 6$ $\partial^{\mathrm{i}}_{u_1}\sigma(u) = 0$ on $\mathcal{A}(\mathfrak{C}_2)$, and $\forall \mathrm{i} < 10$ $\partial^{\mathrm{i}}_{u_1}\sigma(u) = 0$ on $\mathcal{A}(\mathfrak{C}_1)$, that agrees with ([61], Proposition 5.7), but contradicts ([62], Theorem 5.1(3)).

3.2. Basis Functions

Coefficients of monomials in the polynomial functions which give a solution of the Jacobi inversion problem serve as a convenient choice of basis $\wp$-functions in $\mathfrak{A}(\mathcal{C})$.

Definition 1. *In the abelian function field $\mathfrak{A}(\mathcal{C})$ associated with a curve $\mathcal{C}$, characterized by the Weierstrass gap sequence $\mathfrak{W} = \{\mathfrak{w}_i \mid i = 1, \ldots, g\}$, we introduce basis $\wp$-functions:*

$$p_{\mathfrak{w}_i+1} = \wp_{1,\mathfrak{w}_i}, \qquad q_{\mathfrak{w}_i+2} = \wp_{1,1,\mathfrak{w}_i} \qquad \text{if } \mathcal{C} \text{ is hyperelliptic;} \tag{51a}$$

$$p_{\mathfrak{w}_i+1} = \wp_{1,\mathfrak{w}_i}, \qquad q_{\mathfrak{w}_i+2} = \wp_{1,1,\mathfrak{w}_i} - \wp_{2,\mathfrak{w}_i} \qquad \text{if } \mathcal{C} \text{ is trigonal.} \tag{51b}$$

In the hyperelliptic case, coefficients of x^k, $k = 0, \ldots, g-1$, in $\mathcal{R}_{2g}$, $\mathcal{R}_{2g+1}$, without representation in terms of $\wp$-functions, were used as coordinates of $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$, see ([65], Chap. IIIa, § 1), and so, named *Mumford coordinates*. Mumford's construction of a hyperelliptic Jacobian variety implies that the affine ring of $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$ is isomorphic to $\mathbb{C}[p_{\mathfrak{w}_i+1}, q_{\mathfrak{w}_i+2}]$, see ([66], Theorem 2.8).

Theorem 3. *The abelian function field $\mathfrak{A}(\mathcal{C})$ associated with a hyperelliptic curve $\mathcal{C}$ is a polynomial ring in the basis $\wp$-functions* (51a)*. In other words, every meromorphic function on* $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$ *is represented as a polynomial in these basis $\wp$-functions.*

Conjecture 1. *The abelian function field $\mathfrak{A}(\mathcal{C})$ associated with a non-hyperelliptic curve $\mathcal{C}$ is a ring of rational functions in basis $\wp$-functions, given by* (51b) *for trigonal curves. In other words, every meromorphic function on* $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$ *is represented as a rational function in the basis $\wp$-functions.*

Basis $\wp$-functions in $\mathfrak{A}(\mathcal{C})$ provide uniformization of $\mathcal{C}$: there exists a one-to-one correspondence between the basis functions and coordinates of divisors in $\mathfrak{C}_g$. Examples of

related computations can be found in https://community.wolfram.com/groups/-/m/t/3243472, https://community.wolfram.com/groups/-/m/t/3252458.

Remark 8. *(i) The ring* $\mathbb{C}[x,y]/f(x,y;\lambda)$ *of polynomial functions introduced in Section 2.11 is defined, in fact, over* $\mathfrak{A}(\mathcal{C})$*, since all coefficients of* $\mathcal{R}_w$*,* $w \geqslant 2g$*, are expressed in terms of* $\wp$*-functions on* $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$*.*

(ii) At the same time, if $w \geqslant 3g$ *a divisor* D*,* $\deg D = w - g$*, which defines* $\mathcal{R}_w$ *can be considered as a sum of* $\mathfrak{k} = [w/g] - 1$ *non-special divisors, and* $\mathcal{R}_w$ *is defined not only on* $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$ *but also on* $(\mathrm{Jac}(\mathcal{C})\backslash\Sigma)^{\mathfrak{k}}$*.*

A solution of the Jacobi inversion problem implies

Theorem 4. *Polynomial functions from* $\mathbb{C}[x,y]/f(x,y;\lambda)$ *over* $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$ *form a vector space, which we denote by* $\mathfrak{P}(\mathcal{C})$*. Monomials* $v_{\mathfrak{w}_i}$*,* $i = 1,\dots,g$*, from (7) serve as a basis in* $\mathfrak{P}(\mathcal{C})$*.*

Proof. Indeed, polynomial functions of weights $2g, 2g+1, \dots, 2g+n-1$ which provide solutions of the Jacobi inversion problem (see [64], Theorem 1) serve to reduce monomials of these weights to linear combinations of the basis monomials. Any polynomial function of weight higher than $2g+n-1$ admits a reduction by means of $\mathcal{R}_{2g}, \mathcal{R}_{2g+1}, \dots, \mathcal{R}_{2g+n-1}$. □

Theorem 5. *Polynomial functions* $\mathcal{R}_w$*,* $w \geqslant 2g+n$*, serve as generators of identities for* $\wp$*-functions on* $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$*.*

Proof. Any polynomial function $\mathcal{R}_w$, $w \geqslant 2g+n$, is reduced to a function $\mathcal{R}_w^{\mathrm{red}}$ of weight less than $2g$, according to Theorem 4. Coefficients of $\mathcal{R}_w^{\mathrm{red}}$ are expressed rationally in basis $\wp$-functions. At the same time, the function $\mathcal{R}_w$ with coefficients computed at $u \in \mathrm{Jac}(\mathcal{C})\backslash\Sigma$ has a divisor of zeros of degree $2g$ or greater, since $(\mathcal{R}_w)_0$ contains $\mathcal{A}^{-1}(u) \in \mathfrak{C}_g$ and a degree g complement divisor, as follows from Theorem 2. This contradiction means that $\mathcal{R}_w^{\mathrm{red}}$ vanishes, and coefficients of basis monomials serve as identities for $\wp$-functions. □

Theorem 5 provides a way of obtaining algebraic identities for $\Phi \in \mathfrak{A}(\mathcal{C})$. There exist several techniques of constructing polynomial functions which generate identities. Below, we consider two of them: the one based on the Klein formula, and the other based on the residue theorem.

3.3. The Klein Formula Technique

Definition 2. *[[2], p. 45; [9], § 1.2.4] The fundamental bi-differential of the second-kind* $\mathrm{dB}(x,y;\tilde{x},\tilde{y})$ *is symmetric, that is,* $\mathrm{dB}(x,y;\tilde{x},\tilde{y}) = \mathrm{dB}(\tilde{x},\tilde{y};x,y)$*, and has the only pole of the second order along the diagonal* $x = \tilde{x}$*.*

According to ([9], Eq. (1.8)) (hyperelliptic case), and ([23], Definition 3.4), the fundamental bi-differential dB has the form

$$\mathrm{dB}(x,y;\tilde{x},\tilde{y}) = \frac{\mathcal{F}(x,y;\tilde{x},\tilde{y})\mathrm{d}x\mathrm{d}\tilde{x}}{(x-\tilde{x})^2\big(\partial_y f(x,y;\lambda)\big)\big(\partial_{\tilde{y}} f(\tilde{x},\tilde{y};\lambda)\big)}, \tag{52}$$

where $\mathcal{F}: \mathcal{C}^2 \to \mathbb{C}$ is a polynomial of its variables, symmetric in (x,y) and $(\tilde{x},\tilde{y})$.

In the case of a hyperelliptic curve, see ([1], p. 195 *Ex.* i), ([9], Eq. (1.7)), we have

$$\mathcal{F}(x,y;\tilde{x},\tilde{y}) = 2y\tilde{y} + x^g\tilde{x}^g(x+\tilde{x}) + \sum_{i=1}^{g} x^{g-i}\tilde{x}^{g-i}\Big(\lambda_{4i}(x+\tilde{x}) + 2\lambda_{4i+2}\Big). \tag{53}$$

A formula for $\mathcal{F}$ associated with canonical trigonal curves is given in ([20], Sect. 4). A method of constructing $\mathcal{F}$ in the non-hyperelliptic case is suggested in ([23], § 3.2), ([32], Lemma 2.4). The polynomial $\mathcal{F}$ was computed explicitly for $(3,4)$-curve with extra terms in ([32], Eqs. (A.3), (A.4)), cyclic $(3,5)$-curve in ([30], Theorem 4.1), cyclic trigonal curves of genera six and seven in ([40], Eqs. (10), and § 7.1), and cyclic $(4,5)$-curve in ([37], Eq. (15)).

Let $u = \mathcal{A}(D)$, $D \in \mathfrak{C}_g$, and $(x,y) \in D$. Let $(\tilde{x},\tilde{y})$ be an arbitrary point of $\mathcal{C}$. The Klein formula (see ([1], § 217), ([14], p. 138), ([23], Theorem 3.4)) has the form

$$\sum_{i,j=1}^{g} \wp_{\mathfrak{w}_i,\mathfrak{w}_j}(u-\mathcal{A}(\tilde{x},\tilde{y}))v_{\mathfrak{w}_i}(x,y)v_{\mathfrak{w}_j}(\tilde{x},\tilde{y}) = \frac{\mathcal{F}(x,y;\tilde{x},\tilde{y})}{(x-\tilde{x})^2}. \tag{54}$$

Multiplying (54) by $(x-\tilde{x})^2$, and sending $(\tilde{x},\tilde{y})$ to infinity by introducing the local parameter ξ by (4), we obtain polynomial functions $\mathcal{R}_w$ as coefficients in the expansion. This technique produces polynomial functions of all weights, starting from $2g$ and greater. Functions of the lowest weights give a solution to the Jacobi inversion problem, in particular $\mathcal{R}_{2g}$, and $\mathcal{R}_{2g+1}$ on hyperelliptic and trigonal curves.

3.4. The Residue Theorem Technique

Another technique, proposed in [64], derives identities for $\wp$-functions from the curve equation and a system of associated first and second kind differentials. We start with the formula ([64], Eq. (15)):

$$\sum_{k=1}^{g} r_{\mathfrak{w}_i}(x_k,\, y_k) = -\operatorname{res}_{\xi=0}\left(r_{\mathfrak{w}_i}(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\log\sigma(u-\mathcal{A}(\xi))\right) \equiv \mathcal{B}_{\mathfrak{w}_i}(u), \quad i=1,\dots,g, \tag{55}$$

which is, in fact, an application of the residue theorem. Formula (55) contains series expansions about infinity of antiderivatives $r_{\mathfrak{w}_i}(\xi)$ of the second kind differentials $\mathrm{d}r_{\mathfrak{w}_i}(\xi)$, and antiderivatives $\mathcal{A}(\xi)$ of the first kind differentials $\mathrm{d}u_{\mathfrak{w}_i}(\xi)$. These expansions are derived with the help of (4), obtained from the curve equation (1).

By means of (55) second kind integrals $\mathcal{B}(u)$ corresponding to $\mathrm{d}r$ are computed at $u = \mathcal{A}(D)$, $D \in \mathfrak{C}_g$. Actually,

$$\mathcal{B}_{\mathfrak{w}_i}(u) = -\zeta_{\mathfrak{w}_i}(u) + \text{abelian function}(u) + c_{\mathfrak{w}_i}(\lambda), \quad i=1,\dots,g, \tag{56}$$

where $\zeta_{\mathfrak{w}_i} = \partial\log\sigma(u)/\partial u_{\mathfrak{w}_i}$ serves as a generalization of the Weierstrass ζ-function, and $c_{\mathfrak{w}_i}(\lambda)$ is a component of a regularization constant vector $c(\lambda)$, see Remark 4.

Property 4. *The abelian function in* (56) *can be obtained by differentiating* $\mathcal{B}_{\mathfrak{w}_j}$, $j=1,\dots,i-1$, *and expressed in terms of* $\wp_{1,\mathfrak{w}_j}$, $j=1,\dots,g$, *and their derivatives.*

Polynomial functions $\mathcal{R}_{2g-1+\mathfrak{w}_i}$ with $D = \sum_{k=1}^{g}(x_k,y_k) \subset (\mathcal{R}_{2g-1+\mathfrak{w}_i})_0$ are produced by taking derivatives of (55) with respect to any x_k:

$$\frac{\mathrm{d}r_{\mathfrak{w}_i}(x_k,y_k)}{\mathrm{d}x_k} = \left(\partial_u\mathcal{B}_{\mathfrak{w}_i}(u)\right)^t\frac{\mathrm{d}u(x_k,y_k)}{\mathrm{d}x_k}, \quad i=1,\dots,g.$$

Multiplying these equalities by $\partial_{y_k}f(x_k,y_k;\lambda)$, we obtain g polynomial functions

$$\mathcal{R}_{2g-1+\mathfrak{w}_i}(x,y;u) = \rho_{\mathfrak{w}_i}(x,y) - \sum_{j=1}^{g} v_{\mathfrak{w}_j}(x,y)\partial_{u_{\mathfrak{w}_j}}\mathcal{B}_{\mathfrak{w}_i}(u). \tag{57}$$

We call them *main polynomial functions*. These functions are sufficient to derive all identities for $\wp$-functions in $\mathfrak{A}(\mathcal{C})$.

Remark 9. *Main polynomial functions with $i = 1, \dots, n-1$ give a solution to the Jacobi inversion problem, and so define basis $\wp$-functions in $\mathfrak{A}(\mathcal{C})$. In the hyperelliptic case ($n = 2$), only one function $\mathcal{R}_{2g}$ is obtained by this technique. This reflects the fact that only this function is essential, and $\mathcal{R}_{2g+1}$ is derived from $\mathcal{R}_{2g}$, namely $\mathcal{R}_{2g+1}(x, y; u) = -\partial_{u_1}\mathcal{R}_{2g}(x, y; u)$.*

As indicated in [18], $\mathfrak{A}(\mathcal{C})$ has g generators.

Theorem 6. *The basis functions $p_{\mathfrak{w}_i+1} = \wp_{1,\mathfrak{w}_i}$, $i = 1, \dots, g$, serve as generators in the differential field $\mathfrak{A}(\mathcal{C})$.*

Proof. Suffices it to show that $\wp_{\mathfrak{w}_i,\mathfrak{w}_j}$, $i, j = 2, \dots, g$, are expressible in terms of $\wp_{1,\mathfrak{w}_i}$, $i = 1, \dots, g$, and their derivatives. Indeed, all required relations follow from the main polynomial functions $\mathcal{R}_{2g-1+\mathfrak{w}_i}$, given by (57). As seen from (56), functions $\wp_{\mathfrak{w}_i,\mathfrak{w}_j}$, $i, j = 2, \dots, g$, arise as first terms in $\partial_{u_{\mathfrak{w}_j}}\mathcal{B}_{\mathfrak{w}_i}(u)$. Due to Property 4, the required expressions through $\wp_{1,\mathfrak{w}_i}$, and their derivatives can be obtained. □

$(2,7)$-Curve. On the curve defined by (13), with the first and second kind differentials (15), by means of (55) we obtain the second kind integrals

$$\begin{aligned}
\mathcal{B}_1(u) &= -\zeta_1(u),\\
\mathcal{B}_3(u) &= -\zeta_3(u) + \tfrac{1}{2}\wp_{1,1,1}(u),\\
\mathcal{B}_5(u) &= -\zeta_5(u) + \tfrac{5}{6}\wp_{1,1,3}(u) + \tfrac{1}{24}\wp_{1,1,1,1,1}(u).
\end{aligned} \tag{58}$$

According to Property 4, we have

$$\begin{aligned}
\mathcal{B}_3(u) &= -\zeta_3(u) + \tfrac{1}{2}\partial_{u_1}^2\mathcal{B}_1(u),\\
\mathcal{B}_5(u) &= -\zeta_5(u) + \tfrac{5}{6}\partial_{u_1}^2\mathcal{B}_3(u) - \tfrac{3}{8}\partial_{u_1}^4\mathcal{B}_1(u).
\end{aligned} \tag{59}$$

The lowest weight main polynomial function, obtained from $\mathcal{B}_1$, is $\mathcal{R}_6$, which completely defines a solution of the Jacobi inversion problem, since another function $\mathcal{R}_7$ is derived by $\mathcal{R}_7(x, y; u) = -\partial_{u_1}\mathcal{R}_6(x, y; u)$. Alternatively, $\mathcal{R}_6$ can be obtained by applying (55) to the second kind differential $\mathrm{d}r_2 = 2y$.

From $\mathcal{B}_3$, $\mathcal{B}_5$, we derive two more main polynomial functions:

$$\begin{aligned}
\mathcal{R}_8(x, y; u) &= 3x^4 - x^2\big(\tfrac{1}{2}\wp_{1,1,1,1} + \wp_{1,3} - \lambda_4\big) - x\big(\tfrac{1}{2}\wp_{1,1,1,3} + \wp_{3,3}\big) - \big(\tfrac{1}{2}\wp_{1,1,1,5} + \wp_{3,5}\big),\\
\mathcal{R}_{10}(x, y; u) &= 5x^5 + 3\lambda_4 x^3 - x^2\big(\tfrac{1}{24}\wp_{1,1,1,1,1,1} + \tfrac{5}{6}\wp_{1,1,1,3} + \wp_{1,5} - 2\lambda_6\big)\\
&\quad - x\big(\tfrac{1}{24}\wp_{1,1,1,1,1,3} + \tfrac{5}{6}\wp_{1,1,3,3} + \wp_{3,5} - \lambda_8\big) - \big(\tfrac{1}{24}\wp_{1,1,1,1,1,5} + \tfrac{5}{6}\wp_{1,1,3,5} + \wp_{5,5}\big).
\end{aligned}$$

By reducing these polynomial functions to the basis monomials 1, x, x^2 we find

$$\begin{aligned}
\wp_{3,3} &= -\tfrac{1}{2}\wp_{1,1,1,3} + 3\wp_{1,1}\wp_{1,3} + 3\wp_{1,5},\\
\wp_{3,5} &= -\tfrac{1}{2}\wp_{1,1,1,5} + 3\wp_{1,1}\wp_{1,5},\\
\wp_{5,5} &= -\tfrac{1}{24}\wp_{1,1,1,1,1,5} - \tfrac{5}{6}\wp_{1,1,3,5} + 5\wp_{1,1}^2\wp_{1,5} + 5\wp_{1,3}\wp_{1,5} + 3\lambda_4\wp_{1,5}.
\end{aligned}$$

For brevity, we omit the argument of $\wp$-functions.

$(3,4)$-Curve. On the curve defined by (18), with the first and second kind differentials (20), by means of (55) we obtain the second kind integrals

$$\begin{aligned}
\mathcal{B}_1(u) &= -\zeta_1(u),\\
\mathcal{B}_2(u) &= -\zeta_2(u) - \wp_{1,1}(u),\\
\mathcal{B}_5(u) &= -\zeta_5(u) + \tfrac{5}{12}\lambda_2\wp_{1,2}(u) + \tfrac{5}{8}\wp_{1,2,2}(u) - \tfrac{5}{12}\wp_{1,1,1,2}(u) + \tfrac{1}{24}\wp_{1,1,1,1,1}(u);
\end{aligned} \tag{60}$$

and according to Property 4, we have

$$\begin{aligned}
\mathcal{B}_2(u) &= -\zeta_2(u) - \partial_{u_1}\mathcal{B}_1(u),\\
\mathcal{B}_5(u) &= -\zeta_5(u) + \left(\tfrac{5}{8}\partial_{u_2} + \tfrac{5}{24}\partial_{u_1}^2 + \tfrac{5}{12}\lambda_2\right)\partial_{u_1}\mathcal{B}_2(u) + \left(\tfrac{1}{4}\partial_{u_1}^2 + \tfrac{5}{12}\lambda_2\right)\partial_{u_1}^2\mathcal{B}_1(u).
\end{aligned} \tag{61}$$

The two lowest weight main polynomial functions, obtained from $\mathcal{B}_1$ and $\mathcal{B}_2$, are $\mathcal{R}_6$ and $\mathcal{R}_7$, which give a solution to the Jacobi inversion problem. They produce expressions for $\wp_{2,2}$ and $\wp_{2,5}$. Then, from $\mathcal{B}_5$ we obtain $\mathcal{R}_{10}$, which produces an expression for $\wp_{5,5}$.

3.5. Identities for $\wp$-Functions: Hyperelliptic Case

Identities for $\wp$-functions in the hyperelliptic case are thoroughly elaborated. Expressions for the 4-index $\wp$-functions associated with a genus two curve can be found in ([2], pp. 47–48). In genus three, expressions for 4-index $\wp$-functions are obtained in [14]. In [9] the latter results are extended to arbitrary hyperelliptic curves, and the focus has shifted from 4-index $\wp$-functions to $\wp_{1,1,\mathfrak{w}_i}\wp_{1,1,\mathfrak{w}_j}$, expressions for which are referred to as *fundamental cubic relations*, since they generalize the well-known differential equation for the Weierstrass $\wp$-function, known as the Weierstrass cubic,

$$\wp'(u)^2 = 4\wp(u)^3 - g_2\wp(u) - g_3.$$

Analyzing identities for $\wp$-functions from [9], we conclude the following.

Theorem 7. *Let $\mathcal{C}$ be a hyperelliptic curve in the canonical form (11) with the Weierstrass gap sequence $\mathfrak{W} = \{\mathfrak{w}_i = 2i-1 \mid i = 1, \ldots, g\}$. Then, in $\mathfrak{A}(\mathcal{C})$*

- *every 4-index function $\wp_{i,j,k,l}$ is represented as a polynomial in $\wp_{i,j}$ with coefficients in $\mathbb{Z}[\lambda]$;*
- *every product $\wp_{1,1,\mathfrak{w}_i}\wp_{1,1,\mathfrak{w}_j}$ is represented as a polynomial in $\wp_{i,j}$ with coefficients in $\mathbb{Z}[\lambda]$.*

As we see below, fundamental cubic relations produce an algebraic model of $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$. And so-called quartic relations, induced by

$$\left(\wp_{1,1,\mathfrak{w}_i}\wp_{1,1,\mathfrak{w}_j}\right)\left(\wp_{1,1,\mathfrak{w}_k}\wp_{1,1,\mathfrak{w}_l}\right) - \left(\wp_{1,1,\mathfrak{w}_i}\wp_{1,1,\mathfrak{w}_k}\right)\left(\wp_{1,1,\mathfrak{w}_j}\wp_{1,1,\mathfrak{w}_l}\right) = 0,$$

produce a model of the Kummer variety $\mathrm{Kum}(\mathcal{C}) = \mathrm{Jac}(\mathcal{C})/\pm$.

A convenient matrix form of the fundamental cubic relations is suggested in ([2], p. 39) for genus two curves. An extension to hyperelliptic curves is given in ([9], § 3), and elaborated in more detail in [26,27]. The matrix form originates in the Klein formula (54). Let $(g+2)\times(g+2)$ symmetric matrices P and L be defined by

$$\begin{aligned}
(x-\tilde{x})^2\sum_{i,j=1}^{g} x^{g-i}\tilde{x}^{g-j}\wp_{2i-1,2j-1} &= \mathbf{x}^t\mathsf{P}\tilde{\mathbf{x}},\\
x^g\tilde{x}^g(x+\tilde{x}) + \sum_{i=1}^{g} x^{g-i}\tilde{x}^{g-i}\Big(\lambda_{4i}(x+\tilde{x}) + 2\lambda_{4i+2}\Big) &= \mathbf{x}^t\mathsf{L}\tilde{\mathbf{x}},
\end{aligned} \tag{62}$$

where $\mathbf{x} = (1, x, x^2, \ldots, x^{g+1})^t$, and $\tilde{\mathbf{x}} = (1, \tilde{x}, \tilde{x}^2, \ldots, \tilde{x}^{g+1})^t$. The Klein formula, cf. ([26], Eq. (4.2)), implies

$$\mathbf{x}^t\mathsf{H}\tilde{\mathbf{x}} + 2y\tilde{y} = 0, \qquad \mathsf{H} = \mathsf{P} - \mathsf{L}. \tag{63}$$

Then, y, x^g, and x^{g+1} are reduced by means of $\mathcal{R}_{2g+1}$ and $\mathcal{R}_{2g}$ as follows

$$x^g = \mathrm{Y}_1^t \mathbf{b}, \qquad y = \mathrm{Y}_2^t \mathbf{b}, \qquad x^{g+1} = \mathrm{Y}_3^t \mathbf{b}, \tag{64}$$

where $\mathbf{b} = (1, x, x^2, \ldots, x^{g-1})^t$, and

$$\mathrm{Y}_1 = \begin{pmatrix} \wp_{1,\mathfrak{w}_g} \\ \vdots \\ \wp_{1,\mathfrak{w}_2} \\ \wp_{1,1} \end{pmatrix}, \quad \mathrm{Y}_1^\circ = \begin{pmatrix} 0 \\ \wp_{1,\mathfrak{w}_g} \\ \vdots \\ \wp_{1,\mathfrak{w}_2} \end{pmatrix}, \quad \mathrm{Y}_2 = -\frac{1}{2} \begin{pmatrix} \wp_{1,1,\mathfrak{w}_g} \\ \vdots \\ \wp_{1,1,\mathfrak{w}_2} \\ \wp_{1,1,1} \end{pmatrix}, \quad \mathrm{Y}_3 = \wp_{1,1}\mathrm{Y}_1 + \mathrm{Y}_1^\circ. \tag{65}$$

Similarly, $\tilde{y}$, $\tilde{x}^g$, and $\tilde{x}^{g+1}$ are reduced to $\widetilde{\mathbf{b}} = (1, \tilde{x}, \tilde{x}^2, \ldots, \tilde{x}^{g-1})^t$. Let $\mathsf{T} = (1_g, \mathrm{Y}_1, \mathrm{Y}_3)^t$, then the reduction is written as $\mathbf{x} = \mathsf{T}\mathbf{b}$; and the fundamental cubic relations are obtained in the matrix form

$$\mathsf{T}^t \mathsf{H} \mathsf{T} + 2\mathrm{Y}_2 \mathrm{Y}_2^t = 0. \tag{66}$$

Note that $\mathrm{Y}_2\mathrm{Y}_2^t$ is a tensor product, and $2\mathrm{Y}_2\mathrm{Y}_2^t = (\frac{1}{2}\wp_{1,1,\mathfrak{w}_i}\wp_{1,1,\mathfrak{w}_j})$.

Theorem 8 ([9], Theorem 3.3)**.** *We have the following:* $\operatorname{rank}\mathsf{H} = 3$ *at points of* $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$ *which are not half-periods;* $\operatorname{rank}\mathsf{H} = 2$ *at half-periods; and* $\operatorname{rank}\sigma(u)^2\mathsf{H} = 3$ *at points of* $\mathcal{A}(\mathfrak{C}_{g-1})$.

A matrix representation of fundamental cubic relations associated with hyperelliptic curves in the form $(2, 2g+2)$ up to genus three is given in [26]. Expressions for 4-index $\wp$-functions and lists of fundamental cubic relations are also obtained.

A covariant approach, based on elementary representation theory, is developed in [24–27]. Families of hyperelliptic curves related by $\mathfrak{sl}(2,\mathbb{C})$ transformations are considered. Let $\mathcal{H}^1(\mathcal{C})$ denote the vector space of holomorphic 1-forms, which is a g-dimensional irreducible $\mathfrak{sl}(2,\mathbb{C})$-module. n-Index $\wp$-functions belong to $\mathrm{Sym}^{\otimes n}\mathcal{H}^1(\mathcal{C})$, which is decomposable into irreducible components, each associated with a highest weight element. Similarly, each identity between $\wp$-functions falls into some irreducible component, and knowledge of the highest weight element is enough to determine all identities in that module. This approach "has a clear calculation advantage, and also reveals the structure of the equations". Further development of the covariant approach is given in [28].

3.6. Identities for $\wp$-Functions: Non-Hyperelliptic Case

Enormous strides have been made in listing identities for non-hyperelliptic curves of genera up to seven. Using the Klein formula technique, expressions for all 4-index $\wp$-functions are derived, along with all identities both linear and quadratic in 3-index $\wp$-functions. $(3,4)$-Curve with extra terms $\lambda_1 xy^2$, $\lambda_4 y^2$, and $\lambda_3 x^3$ is elaborated on in [32]. Identities for $\wp$-functions associated with cyclic (or superelliptic) $(3,5)$-curve are obtained in [30,41]. A comparison of the identities associated with two types of genus three curves: $(2,7)$, and cyclic $(3,4)$ is given in [35]. Identities associated with cyclic $(4,5)$-curve are presented in [37], and associated with cyclic $(3,7)$- and $(3,8)$-curves in [40].

In the next section, we will return to identities for $\wp$-functions, and illustrate the residue theorem technique in obtaining such identities associated with $(3,4)$-curve.

4. Algebraic Models of Jacobian Varieties

4.1. Hyperelliptic Case

In ([65], Chap. IIIa § 1) an algebraic construction of hyperelliptic $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$ is suggested. Although this construction does not involve $\wp$-functions, it proves

Theorem 9 ([66], Theorem 2.9)**.** *The homomorphism*

$$\begin{aligned} \mathbb{C}[p_{2i}, q_{2i+1}]/\langle \mathcal{J}_{4+2g}, \mathcal{J}_{6+2g}, \ldots, \mathcal{J}_{2+4g}\rangle &\to \mathbb{C}[\wp_{1,2i-1}, \wp_{1,1,2i-1}] \\ p_{2i} &\mapsto \wp_{1,2i-1}, \\ q_{2i+1} &\mapsto \wp_{1,1,2i-1} \end{aligned} \tag{67}$$

is well-defined and an isomorphism. In particular, the ideal $\langle \ldots \rangle$ *is formed by the equations*

$$\mathcal{J}_{2+2g+2i}(p_2, p_4, \ldots, p_{2g}, q_3, q_5, \ldots, q_{2g+1}) = 0. \tag{68}$$

Remark 10. *One should take into account that polynomials in* p_{2i}, q_{2i+1}, *denoted by* $\mathbb{C}[p_{2i}, q_{2i+1}]$ *above, are considered over the field* $\mathbb{C}$, *where parameters of the equation of* $\mathcal{C}$ *are supposed to belong. Working with* (n, s)*-curves, we denote parameters by* λ, *and consider the space* $\Lambda_g \subset \mathbb{C}^{\mathfrak{N}}$ *of parameters as the base of* $\mathcal{E}(\Lambda_g, \pi, \mathcal{C})$. *Therefore, the homomorphism is defined on polynomials in* p_{2i}, q_{2i+1} *with coefficients from* $\mathbb{Z}[\lambda]$.

Explicit equations (68), which serve as a model of hyperelliptic $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$, can be obtained from the fundamental cubic relations (66) by eliminating $\wp_{2i-1,2j-1}$ with $i, j = 2, \ldots, g$. Indeed, $\frac{1}{2}g(g+1)$ equations in (66) contain all 2-index $\wp$-functions, of number $\frac{1}{2}g(g+1)$, as seen from the definition of matrix P. Eliminating all but basis functions, we obtain the required g equations (68). This proves

Theorem 10 ([9], Corollary 3.2.1)**.** *The map* $\varphi : \mathrm{Jac}(\mathcal{C})\backslash\Sigma \to \mathbb{C}^{g+g(g+1)/2}$,

$$\varphi(u) = \left(\{\wp_{1,1,2i-1}(u) \mid i = 1, \ldots, g\}, \{\wp_{2i-1,2j-1}(u) \mid i = 1, \ldots, g, j = i, \ldots, g\}\right)$$

is a meromorphic embedding. The range of φ *is the intersection of* $g(g+1)/2$ *cubic surfaces determined by the fundamental cubic relations.*

4.2. Trigonal Case

Theorem 11 ([20], Corollary 3.3)**.** *The variety* $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$ *of a trigonal curve of genus* g *can be realized as an algebraic variety in* $\mathbb{C}^{4g+\delta}$, *where* $\delta = 2(g - 3[g/3])$. *This subvariety is defined as the set of common zeros of a system of* $3g + \delta$ *polynomials generated by the functions* $\mathcal{R}_{2g}$, $\mathcal{R}_{2g+1}$ *which give a solution to the Jacobi inversion problem.*

Now, we explain how to obtain such an algebraic model of $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$ by employing the method from ([20], § 2), which works for both hyperelliptic and trigonal curves.

In the cases of $n = 2$ and $n = 3$, the Jacobi inversion problem is solved by means of two functions $\mathcal{R}_{2g}$ and $\mathcal{R}_{2g+1}$ of weights $2g$ and $2g+1$, respectively. Let the common divisor of zeros of these two functions at $u \in \mathrm{Jac}(\mathcal{C})\backslash\Sigma$ be D such that $u = \mathcal{A}(D)$. We write this fact down as $\left(\mathcal{R}_{2g}(u), \mathcal{R}_{2g+1}(u)\right)_0 = D$. At the same time, $(\mathcal{R}_{2g})_0 = D + D^*$, where D^* is the inverse of D induced by the map $u \mapsto -u$, since $\mathcal{R}_{2g}$ is even with respect to u. That is, $D^* = \left(\mathcal{R}_{2g}(u), \mathcal{R}_{2g+1}(-u)\right)_0$. Let $\left(\mathcal{R}_{2g+1}(u)\right)_0 = D + \widetilde{D}$, and $\left(\mathcal{R}_{2g+1}(-u)\right)_0 = D^* + \widetilde{D}^*$. Then,

$$\mathcal{M}(x, y) = \frac{\mathcal{R}_{2g+1}(x, y; u)\mathcal{R}_{2g+1}(x, y; -u)}{\mathcal{R}_{2g}(x, y; u)}$$

is a polynomial function of weight $2g+2$, and $(\mathcal{M})_0 = \widetilde{D} + \widetilde{D}^*$. Let $\mathcal{M}$ be a linear combination of the first $g+3$ monomials, with weights up to $2g+2$, from the ordered list $\mathfrak{M}$. Then, the polynomial function

$$\mathcal{Q}(x, y; u) \equiv \mathcal{R}_{2g+1}(x, y; u)\mathcal{R}_{2g+1}(x, y; -u) - \mathcal{M}(x, y)\mathcal{R}_{2g}(x, y; u) \tag{69}$$

contains terms with y to the power equal to or greater than n. These terms are eliminated with the help of the curve equation. According to (69), $\mathcal{Q}$ has the weight of $4g+2$. At the same time, after canceling higher weight terms, the actual weight becomes less than $4g+2$, and so we have

$$\mathcal{Q}(x,y;u) - \mathcal{N}(x)f(x,y;\lambda) \equiv 0,$$

where $\operatorname{wgt}\mathcal{N} = (n-2)(s-2)$.

Theorem 12. *On curves with $n = 2$ or 3 the equality*

$$\mathcal{R}_{2g+1}(x,y;u)\mathcal{R}_{2g+1}(x,y;-u) - \mathcal{M}(x,y)\mathcal{R}_{2g}(x,y;u) - \mathcal{N}(x)f(x,y;\lambda) \equiv 0 \tag{70}$$

produces g identities for basis $\wp$-functions $p_{\mathfrak{w}_i+1}$, $q_{\mathfrak{w}_i+2}$, $i = 1, \dots, g$, which we denote by $\mathcal{J}_{\delta+\mathfrak{w}_i}$, $i = 1, \dots, g$, where $\delta + \mathfrak{w}_i$ shows the weight, $\delta = 2g+3$ in the hyperelliptic case, and $\delta = \mathfrak{w}_g + 6 + \mathfrak{w}_i$ in the trigonal case.

Proof. First, we find unknown coefficients of $\mathcal{M}$ and $\mathcal{N}$ expressed in terms of basis $\wp$-functions and parameters λ. Since we know the weights $\operatorname{wgt}\mathcal{M} = 2g+2$ and $\operatorname{wgt}\mathcal{N} = (n-2)(s-2)$, the number of unknowns, namely $\#\mathcal{M}$ in $\mathcal{M}$ and $\#\mathcal{N}$ in $\mathcal{N}$, can be easily counted, as well as the number of equations #Eqs, which arise as coefficients of monomials. By direct computations for each family of curves, and taking into account $\wp_{i,j}(-u) = \wp_{i,j}(u)$, $\wp_{i,j,k}(-u) = -\wp_{i,j,k}(u)$, we find

	$\operatorname{wgt}\mathcal{N}$	$\#\mathcal{N}$	$\#\mathcal{M}$	#Eqs.
$(2, 2g+1)$	0	1	$g+3$	$2g+4$
$(3, 3\mathfrak{m}+1)$	$3\mathfrak{m}-1$	$\mathfrak{m}$	$3\mathfrak{m}+3$	$10\mathfrak{m}+2$
$(3, 3\mathfrak{m}+2)$	$3\mathfrak{m}$	$\mathfrak{m}+1$	$3\mathfrak{m}+4$	$10\mathfrak{m}+5$

Thus, after eliminating all unknown coefficients of $\mathcal{M}$ and $\mathcal{N}$, we obtain

- g equations on a $(2, 2g+1)$-curve;
- $6\mathfrak{m}-1$ equations on a $(3, 3\mathfrak{m}+1)$-curve, which contain $\wp_{1,\mathfrak{w}_i}$, $\wp_{1,1,\mathfrak{w}_i}$ with $i = 1, \dots, g = 3\mathfrak{m}$, and $\wp_{2,\mathfrak{w}_i}$ with $i = 2, \dots, g$;
- $6\mathfrak{m}+2$ equations on a $(3, 3\mathfrak{m}+2)$-curve, which contain $\wp_{1,\mathfrak{w}_i}$, $\wp_{1,1,\mathfrak{w}_i}$ with $i = 1, \dots, g = 3\mathfrak{m}+1$, and $\wp_{2,\mathfrak{w}_i}$ with $i = 2, \dots, g$.

On a hyperelliptic curve $\mathcal{C}$, the obtained g equations give a model of $\operatorname{Jac}(\mathcal{C})\backslash\Sigma$.

On a trigonal curve $\mathcal{C}$, we obtain $2g-1$ equations which provide expressions for $\wp_{2,\mathfrak{w}_i}$, $i = 2, \dots, g$, in terms of the basis $\wp$-functions, and g equations which define a model of $\operatorname{Jac}(\mathcal{C})\backslash\Sigma$. □

Remark 11. *The identities $\mathcal{J}_{\delta+\mathfrak{w}_i} = 0$, $i = 1, \dots, g$, obtained in Theorem 12 give an algebraic model of $\operatorname{Jac}(\mathcal{C})\backslash\Sigma$, if $\mathcal{C}$ is hyperelliptic or trigonal. In the hyperelliptic case, these identities coincide with (68). Thes identities form the ideal of the ring mentioned in Theorem 3, and Conjecture 1.*

In the case of a trigonal curve, we have

Theorem 13. *The homomorphism*

$$\begin{aligned} \mathbb{Z}[\lambda](p_{\mathfrak{w}_i+1}, q_{\mathfrak{w}_i+2}) / \langle \mathcal{J}_{\mathfrak{w}_g+7}, \mathcal{J}_{\mathfrak{w}_g+6+\mathfrak{w}_2}, \dots, \mathcal{J}_{2\mathfrak{w}_g+6} \rangle &\to \mathfrak{A}(\mathcal{C}) \\ p_{\mathfrak{w}_i+1} &\mapsto \wp_{1,\mathfrak{w}_i}, \\ q_{\mathfrak{w}_i+2} &\mapsto \wp_{1,1,\mathfrak{w}_i} - \wp_{2,\mathfrak{w}_i} \end{aligned} \tag{71}$$

is well-defined and an isomorphism. In particular, the ideal $\langle \dots \rangle$ *is formed by the equations*

$$\mathcal{J}_{\mathfrak{w}_g+6+\mathfrak{w}_i}\left(p_2, p_{\mathfrak{w}_2+1}, \dots, p_{\mathfrak{w}_g+1}, q_3, q_{\mathfrak{w}_2+2}, \dots, q_{\mathfrak{w}_g+2}\right) = 0, \tag{72}$$

which are obtained from Theorem 12.

$(3,4)$**-Curve.** Equations obtained from (70) have the form

$$\begin{aligned}
\mathcal{G}_6 &\equiv -\wp_{1,1,1}^2 + 4\wp_{1,1}^3 - 4\wp_{1,1}\wp_{2,2} + 4\wp_{1,5} + \wp_{1,2}^2 + 4\lambda_2\wp_{1,1}^2 = 0,\\
\mathcal{G}_7 &\equiv -2\wp_{1,1,1}\wp_{1,1,2} + 8\wp_{1,1}^2\wp_{1,2} - 2\wp_{1,2}\wp_{2,2} - 4\wp_{2,5} + 4\lambda_2\wp_{1,1}\wp_{1,2} + 4\lambda_5\wp_{1,1} = 0,\\
\mathcal{G}_{10} &\equiv -2\wp_{1,1,1}\wp_{1,1,5} + 8\wp_{1,1}^2\wp_{1,5} + 2\wp_{1,2}\wp_{2,5} - 4\wp_{1,5}\wp_{2,2} + 4\lambda_2\wp_{1,1}\wp_{1,5} + 4\lambda_8\wp_{1,1}\\
&\quad + \wp_{1,1}\left(-\wp_{1,1,2}^2 + 4\wp_{1,1}(\wp_{1,2}^2 + \lambda_6) + \wp_{2,2}^2\right) = 0,\\
\mathcal{G}_{11} &\equiv -2\wp_{1,1,2}\wp_{1,1,5} + 8\wp_{1,1}\wp_{1,2}\wp_{1,5} + 2\wp_{2,2}\wp_{2,5} + 4\lambda_9\wp_{1,1}\\
&\quad + \wp_{1,2}\left(-\wp_{1,1,2}^2 + 4\wp_{1,1}(\wp_{1,2}^2 + \lambda_6) + \wp_{2,2}^2\right) = 0,\\
\mathcal{G}_{14} &\equiv -\wp_{1,1,5}^2 + 4\wp_{1,1}\wp_{1,5}^2 + \wp_{2,5}^2 + 4\lambda_{12}\wp_{1,1}\\
&\quad + \wp_{1,5}\left(-\wp_{1,1,2}^2 + 4\wp_{1,1}(\wp_{1,2}^2 + \lambda_6) + \wp_{2,2}^2\right) = 0.
\end{aligned} \tag{73}$$

From the first two equations, which are linear in $\wp_{2,2}$, $\wp_{2,5}$, we find

$$\begin{aligned}
\wp_{2,2} &= -p_2^{-1}\left(\tfrac{1}{4}q_3(q_3+2p_3) - p_6\right) + p_2(p_2+\lambda_2),\\
\wp_{2,5} &= -\tfrac{1}{2}(q_3+p_3)q_4 - \tfrac{1}{2}p_2(p_2+\lambda_2)q_3 + p_2(p_2p_3+\lambda_5)\\
&\quad + \tfrac{1}{2}p_2^{-1}\left(\tfrac{1}{4}q_3(q_3+2p_3) - p_6\right)(q_3+2p_3).
\end{aligned} \tag{74}$$

By eliminating (74) from the remaining three equations of (73), we obtain an algebraic model of $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$ $(3,4)$-curve $\mathcal{C}$, namely

$$\begin{aligned}
\mathcal{J}_{12} &\equiv -2p_2(q_3+p_3)(q_7 - q_3q_4) - p_2^2q_4^2 - q_3^2\left(\tfrac{1}{2}p_2q_4 + (\tfrac{1}{2}q_3+p_3)^2 - p_6\right)\\
&\quad - (2p_2q_4 - q_3^2)(p_6 + p_2^2(p_2+\lambda_2)) + 2q_3\left(2p_3p_6 - p_2^2(p_2p_3+\lambda_5)\right)\\
&\quad - 4p_6^2 + 4p_2^3(p_6 + p_3^2 + \lambda_6) + \lambda_8p_2^2 = 0,\\
\mathcal{J}_{13} &\equiv -(2q_7 - q_3q_4)\left(p_2q_4 - \tfrac{1}{4}q_3(q_3+2p_3) + p_6 + p_2^2(p_2+\lambda_2)\right)\\
&\quad - 2p_2^2q_4\left(p_3(p_2+\lambda_2) + p_2p_3 + \lambda_5\right) + 4p_2^2\left(p_3(p_6+p_3^2+\lambda_6) + p_3p_6 + \lambda_9\right)) = 0,\\
\mathcal{J}_{16} &\equiv -p_2(q_7^2 + p_6q_4^2 - (q_3+p_3)q_7q_4) - (q_7q_3 - 2p_6q_4)\left(\tfrac{1}{4}q_3^2 - p_6 - p_2^2(p_2+\lambda_2)\right)\\
&\quad - p_3q_3(q_7q_3 - p_6q_4) - q_7\left((p_3q_3 - 2p_6)p_3 + 2p_2^2(p_2p_3+\lambda_5)\right)\\
&\quad + 4p_2^2\left(p_6(p_6+p_3^2+\lambda_6) + \lambda_{12}\right) = 0.
\end{aligned} \tag{75}$$

Note that (75) are written in terms of the basis $\wp$-functions in $\mathfrak{A}(\mathcal{C})$:

$$\begin{aligned}
&p_2 = \wp_{1,1}, && p_3 = \wp_{1,2}, && p_6 = \wp_{1,5},\\
&q_3 = \wp_{1,1,1} - \wp_{1,2}, && q_4 = \wp_{1,1,2} - \wp_{2,2}, && q_7 = \wp_{1,1,5} - \wp_{2,5}.
\end{aligned} \tag{76}$$

$(2,7)$**-Curve.** From (70) we obtain the following equations, which serve as the algebraic model of $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$ of $(2,7)$-curve $\mathcal{C}$:

$$\begin{aligned}
\mathcal{J}_{10} &\equiv -2q_3q_7 - q_5^2 - 2p_2q_3q_5 - (p_4+p_2^2)q_3^2 + 12p_2^2p_6 + 12p_2p_4^2 + 16p_2^3p_4 \\
&\quad + 4p_2^5 + 8p_4p_6 + 4\lambda_4(p_6 + 2p_2p_4 + p_2^3) + 4\lambda_6(p_4+p_2^2) + 4\lambda_8p_2 + \lambda_{10} = 0, \\
\mathcal{J}_{12} &\equiv -2q_5q_7 - 2p_4q_3q_5 - (p_6+p_2p_4)q_3^2 + 16p_2p_4p_6 + 12p_2^2p_4^2 + 4p_6^2 + 4p_2^3p_6 \\
&\quad + 4p_2^4p_4 + 4p_4^3 + 4\lambda_4(p_2p_6 + p_4^2 + p_2^2p_4) + 4\lambda_6(p_6 + p_2p_4) + 4\lambda_8p_4 + 4\lambda_{12} = 0, \\
\mathcal{J}_{14} &\equiv -q_7^2 - 2p_6q_3q_5 - p_2p_6q_3^2 + 8p_2p_6^2 + 4p_4^2p_6 + 12p_2^2p_4p_6 + 4p_2^4p_6 \\
&\quad + 4\lambda_4p_6(p_4+p_2^2) + 4\lambda_6p_2p_6 + 4\lambda_8p_6 + 4\lambda_{14} = 0,
\end{aligned} \tag{77}$$

written in terms of the basis $\wp$-functions in $\mathfrak{A}(\mathcal{C})$:

$$p_2 = \wp_{1,1}, \quad p_4 = \wp_{1,3}, \quad p_6 = \wp_{1,5}, \quad q_3 = \wp_{1,1,1}, \quad q_5 = \wp_{1,1,3}, \quad q_7 = \wp_{1,1,5}. \tag{78}$$

The same equations are derived from the fundamental cubic relations in the form (66) with

$$\mathsf{H} = \mathsf{P} - \mathsf{L}, \qquad \mathsf{P} = \begin{pmatrix} 0 & 0 & \wp_{5,5} & \wp_{3,5} & \wp_{1,5} \\ 0 & -2\wp_{5,5} & -\wp_{3,5} & \wp_{3,3} - 2\wp_{1,5} & \wp_{1,3} \\ \wp_{5,5} & -\wp_{3,5} & 2\wp_{1,5} - 2\wp_{3,3} & -\wp_{1,3} & \wp_{1,1} \\ \wp_{3,5} & \wp_{3,3} - 2\wp_{1,5} & -\wp_{1,3} & -2\wp_{1,1} & 0 \\ \wp_{1,5} & \wp_{1,3} & \wp_{1,1} & 0 & 0 \end{pmatrix},$$

$$\mathsf{L} = \begin{pmatrix} 2\lambda_{14} & \lambda_{12} & 0 & 0 & 0 \\ \lambda_{12} & 2\lambda_{10} & \lambda_8 & 0 & 0 \\ 0 & \lambda_8 & 2\lambda_6 & \lambda_4 & \\ 0 & 0 & \lambda_4 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathsf{T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \wp_{1,5} & \wp_{1,3} & \wp_{1,1} \\ \wp_{1,1}\wp_{1,5} & \wp_{1,1}\wp_{1,3} + \wp_{1,5} & \wp_{1,1}^2 + \wp_{1,3} \end{pmatrix},$$

by eliminating $\wp_{3,3}$, $\wp_{3,5}$, $\wp_{5,5}$.

4.3. Identities for $\wp$-Functions on $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$

Keeping in mind Theorem 3 and Conjecture 1, we aim at finding rational (or polynomial) expressions for all $\Phi \in \mathfrak{A}(\mathcal{C})$ through basis $\wp$-functions. We will use the residue theorem technique and Theorem 12.

The main polynomial functions $\mathcal{R}_{2g-1+\mathfrak{w}_i}$, obtained from (55), and reduced to basis monomials in $\mathfrak{P}(\mathcal{C})$, produce identities for $\wp$-functions on $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$. Actually,

$$\begin{aligned}
&\mathcal{R}_{2g-1+\mathfrak{w}_i}(x,y;u) - \mathcal{M}(x,y)\mathcal{R}_{2g}(x,y;u) - \mathcal{N}(x,y)\mathcal{R}_{2g+1}(x,y;u) \\
&\qquad = \sum_{j=1}^{g} v_{\mathfrak{w}_j}(x,y)\mathrm{F}_{\mathfrak{w}_i+\mathfrak{w}_j}(u), \quad i \geqslant n.
\end{aligned} \tag{79}$$

Then, the identities have the form

$$\mathrm{F}_{\mathfrak{w}_i+\mathfrak{w}_j}(u) = 0, \quad i \geqslant n, \ \ j = 1, \dots, g. \tag{80}$$

Explicit expressions $\mathrm{F}_{\mathfrak{w}_i+\mathfrak{w}_j}$ are obtained after finding unknown coefficients of $\mathcal{M}$, and $\mathcal{N}$ of weights $\mathrm{wgt}\,\mathcal{M} = \mathfrak{w}_i - 1$, $\mathrm{wgt}\,\mathcal{N} = \mathfrak{w}_i - 2$.

In the hyperelliptic case, each F_w has a certain parity. In the trigonal case, F_w have no certain parity, and are split into even and odd parts:

$$\mathrm{F}_w^{\mathrm{o}} = \tfrac{1}{2}\big(\mathrm{F}_w(u) - \mathrm{F}_w(-u)\big), \qquad \mathrm{F}_w^{\mathrm{e}} = \tfrac{1}{2}\big(\mathrm{F}_w(u) + \mathrm{F}_w(-u)\big). \tag{81}$$

$(3,4)$-Curve. Expressions for $\wp_{2,2}$, and $\wp_{2,5}$ in terms of the basis $\wp$-functions (76) are given by (74). Further, we use these two $\wp$-functions as an extension of the basis $\wp$-functions,

in order to simplify expressions. Except for $\mathcal{R}_6$, $\mathcal{R}_7$, which give a solution to the Jacobi inversion problem, and produce the identities (73), we have the main polynomial function

$$\begin{aligned}\mathcal{R}_{10}(x,y;u) - \Big(\tfrac{5}{2}(\wp_{2,2}(u) - \wp_{1,1,2}(u)) + \tfrac{2}{3}\lambda_2^2\Big)\mathcal{R}_6(x,y;u)\\ - \Big(\tfrac{5}{2}x + \tfrac{5}{4}(\wp_{1,2}(u) - \wp_{1,1,1}(u))\Big)\mathcal{R}_7(x,y;u) = y\mathrm{F}_6(u) + x\mathrm{F}_7(u) + \mathrm{F}_{10}(u).\end{aligned} \tag{82}$$

Then, we find

$$\begin{aligned}\mathrm{F}^{\mathrm{o}}_{5+\mathfrak{w}_j} &= -\tfrac{5}{4}\wp_{1,2}\wp_{1,1,\mathfrak{w}_j} - \tfrac{5}{2}\wp_{1,\mathfrak{w}_j}\wp_{1,1,2} - \tfrac{5}{4}\wp_{2,\mathfrak{w}_j}\wp_{1,1,1} - \tfrac{5}{12}\lambda_2\wp_{1,2,\mathfrak{w}_j}\\ &\quad + \tfrac{5}{12}\wp_{1,1,1,2,\mathfrak{w}_j} - \tfrac{5}{2}\wp_{1,1,5}\,\delta_{j,2},\\ \mathrm{F}^{\mathrm{e}}_{5+\mathfrak{w}_j} &= -\wp_{5,\mathfrak{w}_j} + \tfrac{5}{4}\wp_{1,2}\wp_{2,\mathfrak{w}_j} + \big(\tfrac{5}{2}\wp_{2,2} + \tfrac{2}{3}\lambda_2^2\big)\wp_{1,\mathfrak{w}_j} + \tfrac{5}{4}\wp_{1,1,1}\wp_{1,1,\mathfrak{w}_j}\\ &\quad - \tfrac{5}{8}\wp_{1,2,2,\mathfrak{w}_j} - \tfrac{1}{24}\wp_{1,1,1,1,1,\mathfrak{w}_j} + \lambda_6\delta_{j,1} + \big(\tfrac{5}{2}\wp_{2,5} + \tfrac{2}{3}\lambda_2\lambda_5\big)\delta_{j,2}.\end{aligned} \tag{83}$$

From F_w and $\mathcal{G}_w$ we obtain

$$\begin{aligned}&\wp_{1,1,1,1} = \wp_{1,1}(6\wp_{1,1} + 4\lambda_2) - 3\wp_{2,2},\\ &\wp_{1,1,1,2} = \wp_{1,2}(6\wp_{1,1} + \lambda_2) + \lambda_5,\\ &\wp_{1,1,1,5} = \wp_{1,5}(6\wp_{1,1} + \lambda_2) + \lambda_8 + \tfrac{3}{4}\big(-\wp_{1,1,2}^2 + 4\wp_{1,1}(\wp_{1,2}^2 + \lambda_6) + \wp_{2,2}^2\big),\\ &\wp_{5,5} = \wp_{1,1,2}^2\big(\tfrac{1}{2}(\wp_{1,1} + \lambda_2) - \tfrac{3}{8}\wp_{1,1}^{-1}\wp_{2,2}\big) + \wp_{2,5}\big(2\wp_{1,2} + \tfrac{1}{2}\wp_{1,1}^{-1}(\lambda_2\wp_{1,2} + \lambda_5)\big)\\ &\quad - \tfrac{1}{8}\wp_{1,1}^{-1}\wp_{2,2}^3 + + \tfrac{1}{2}\wp_{2,2}\big(\wp_{1,2}^2 - \lambda_6 + \wp_{1,1}^{-1}(\lambda_2(\wp_{1,2}^2 + \wp_{1,5}) + \lambda_5\wp_{1,2} + \lambda_8)\big)\\ &\quad - 2(\wp_{1,1} + \lambda_2)\wp_{1,1}\wp_{1,2}^2 - 2\lambda_5\wp_{1,1}\wp_{1,2} - \lambda_2\lambda_5\wp_{1,2} - \tfrac{2}{3}\lambda_2\lambda_8 - \tfrac{1}{2}\lambda_5^2\\ &\quad + \wp_{1,1}^{-1}\big(\tfrac{1}{2}\wp_{1,2}^4 + \wp_{1,5}^2 + 2\wp_{1,2}^2\wp_{1,5} + \lambda_6(\tfrac{1}{2}\wp_{1,2}^2 + \wp_{1,5}) + \tfrac{1}{2}\lambda_9\wp_{1,2} + \lambda_{12}\big).\end{aligned} \tag{84}$$

Using (84) and (74), any $\wp$-function can be represented as a rational function of the basis $\wp$-functions (76). Indeed, every derivative of expressions in (84) is a rational function, expressible through the basis $\wp$-functions by means of (74) and (84).

5. Algebraic Models of Kummer Variety

In ([9], § 4) we find an explicit realization of $\mathrm{Kum}(\mathcal{C})$ of a hyperelliptic curve $\mathcal{C}$. Let H be as defined in (63), and $\mathsf{K}(\mathsf{H})$ be a $g \times g$ matrix with entries

$$\mathsf{k}_{i,j} = \det\big(\mathsf{H}\big[{}^{i,g+1,g+2}_{j,g+1,g+2}\big]\big), \tag{85}$$

where $\mathsf{H}\big[{}^{i,k,l}_{j,m,n}\big]$ denotes a minor of order 3 of H, composed as follows

$$\mathsf{H}\big[{}^{i,k,l}_{j,m,n}\big] = \begin{vmatrix}\mathsf{h}_{i,j} & \mathsf{h}_{i,m} & \mathsf{h}_{i,n}\\ \mathsf{h}_{k,j} & \mathsf{h}_{k,m} & \mathsf{h}_{k,n}\\ \mathsf{h}_{l,j} & \mathsf{h}_{l,m} & \mathsf{h}_{l,n}\end{vmatrix}.$$

The fundamental cubic relations associated with $\mathcal{C}$ (see [9], Eq. (3.15)) acquire the form

$$\tfrac{1}{2}\wp_{1,1,\mathfrak{w}_i}\wp_{1,1,\mathfrak{w}_j} = \mathsf{k}_{i,j}. \tag{86}$$

Evidently, rank $\mathsf{K}(\mathsf{H})$ does not exceed 1.

Theorem 14 (Hyperelliptic case)**.** *Minors of order* 2 *of* $\mathsf{K}(\mathsf{H})$ *define* $\mathrm{Kum}(\mathcal{C})$.

Proof. There are $\binom{g}{2} = \frac{1}{2}g(g-1)$ independent minors of order 2 of $\mathsf{K}(\mathsf{H})$ on $\mathrm{Jac}(\mathcal{C})\backslash\Sigma = \mathcal{A}(\mathfrak{C}_g)$, since rank $\mathsf{K}(\mathsf{H}) = 1$. The minors are expressed in terms of 2-index

$\wp$-functions, which are even. All $\frac{1}{2}g(g+1)$ functions $\wp_{w,j}$, $i,j=1,\ldots,g$, are involved, which follows from the construction of matrix P. Thus, minors of order 2 of $\mathsf{K}(\mathsf{H})$ define a g-dimensional subspace in $\mathcal{A}(\mathfrak{C}_g)/\pm$. $\square$

Remark 12. *The matrix form of the fundamental cubic relations gives* $\mathsf{K}(\mathsf{H})=-\mathsf{T}^t\mathsf{H}\mathsf{T}$. *In the hyperelliptic case, a proof can be found in ([9], Section 3). A similar matrix construction in the case of trigonal curves is suggested in ([20], Section 2).*

(2,7)-Curve. Equations which define $\mathrm{Kum}(\mathcal{C})$ can be constructed as follows

$$\mathsf{k}_{1,1}\mathsf{k}_{3,3}-\mathsf{k}_{1,3}^2=0,\qquad \mathsf{k}_{1,1}\mathsf{k}_{5,5}-\mathsf{k}_{1,5}^2=0,\qquad \mathsf{k}_{3,3}\mathsf{k}_{5,5}-\mathsf{k}_{3,5}^2=0. \tag{87}$$

(3,4)-Curve. The expression for $\wp_{1,1,1,5}$ from (84) enables to single out the cubic with $\wp_{1,1,2}^2$. Then, fundamental cubic relations associated with $(3,4)$-curve are obtained from (73):

$$\begin{aligned}
&\mathcal{G}_6=0,\qquad \mathcal{G}_7=0,\\
&\widetilde{\mathcal{G}}_8\equiv-\wp_{1,1,2}^2+4\wp_{1,1}\wp_{1,2}^2+\wp_{2,2}^2+4\lambda_6\wp_{1,1}+8\wp_{1,1}\wp_{1,5}-\tfrac{4}{3}\big(\wp_{1,1,1,5}-\lambda_2\wp_{1,5}-\lambda_8\big)=0,\\
&\widetilde{\mathcal{G}}_{10}\equiv-2\wp_{1,1,1}\wp_{1,1,5}+2\wp_{1,2}\wp_{2,5}-4\wp_{1,5}\wp_{2,2}+\tfrac{4}{3}\wp_{1,1}\big(\wp_{1,1,1,5}+2\lambda_2\wp_{1,5}+2\lambda_8\big)=0,\\
&\widetilde{\mathcal{G}}_{11}\equiv-2\wp_{1,1,2}\wp_{1,1,5}+2\wp_{2,2}\wp_{2,5}+4\lambda_9\wp_{1,1}+\tfrac{4}{3}\wp_{1,2}\big(\wp_{1,1,1,5}-\lambda_2\wp_{1,5}-\lambda_8\big)=0,\\
&\widetilde{\mathcal{G}}_{14}\equiv-\wp_{1,1,5}^2-4\wp_{1,1}\wp_{1,5}^2+\wp_{2,5}^2+4\lambda_{12}\wp_{1,1}+\tfrac{4}{3}\wp_{1,5}\big(\wp_{1,1,1,5}-\lambda_2\wp_{1,5}-\lambda_8\big)=0.
\end{aligned} \tag{88}$$

Using (88), the matrix $\mathsf{K}=(\mathsf{k}_{i,j})$ is defined by (86). All entries of K are expressed through 6 even functions: $\wp_{1,1}$, $\wp_{1,2}$, $\wp_{1,5}$, $\wp_{2,2}$, $\wp_{2,5}$, and $\wp_{1,1,1,5}$. Three independent equations which define $\mathrm{Kum}(\mathcal{C})$ can be constructed as follows

$$\mathsf{k}_{1,1}\mathsf{k}_{2,2}-\mathsf{k}_{1,2}^2=0,\qquad \mathsf{k}_{1,1}\mathsf{k}_{5,5}-\mathsf{k}_{1,5}^2=0,\qquad \mathsf{k}_{2,2}\mathsf{k}_{5,5}-\mathsf{k}_{2,5}^2=0. \tag{89}$$

6. Addition Laws on Jacobian Varieties

Every abelian function field $\mathfrak{A}(\mathcal{C})$ possesses the addition law, which means that every $\wp$-function at $u+\tilde{u}$ can be expressed in terms of basis $\wp$-functions at u and $\tilde{u}$.

Addition theorems in $\mathfrak{A}(\mathcal{C})$ associated with elliptic curves are known in several forms:

- the addition formula for the σ-function

$$\frac{\sigma(u+\tilde{u})\sigma(u-\tilde{u})}{\sigma(u)^2\sigma(\tilde{u})^2}=\wp(\tilde{u})-\wp(u); \tag{90}$$

- the addition formula for the ζ-function, derived from (90),

$$\big(\zeta(u)+\zeta(\tilde{u})+\zeta(\hat{u})\big)^2=\wp(u)+\wp(\tilde{u})+\wp(\hat{u}),\qquad u+\tilde{u}+\hat{u}=0; \tag{91}$$

- the addition formula for the $\wp$-function, which can be obtained by differentiating (91),

$$\wp(u+\tilde{u})=-\wp(u)-\wp(\tilde{u})+\frac{1}{4}\left(\frac{\wp'(u)-\wp'(\tilde{u})}{\wp(u)-\wp(\tilde{u})}\right)^2. \tag{92}$$

Below, we consider how these formulas are extended to curves of higher genera.

6.1. Groupoid Structure of Jacobian Varieties

In [21,22] the addition law is derived from the groupoid structure of $\mathrm{Jac}(\mathcal{C})$.

Definition 3. *A space X together with an anchor mapping $p_X : X \to Y$ is called a groupoid over Y, if the two structure mappings over Y:*

$$\mathsf{add} : X \times_Y X \to X, \qquad \mathsf{inv}\, X \to X, \tag{93}$$

are defined and satisfy the axioms

1. $\mathsf{add}(\mathsf{add}(x_1, x_2), x_3) = \mathsf{add}(x_1, \mathsf{add}(x_2, x_3))$, *provided* $p_X(x_1) = p_X(x_2) = p_X(x_3)$,
2. $\mathsf{add}(\mathsf{add}(x_1, x_2), \mathsf{inv}(x_2)) = x_1$, *provided* $p_X(x_1) = p_X(x_2)$.

A groupoid structure is commutative if $\mathsf{add}(\mathsf{x}_1, \mathsf{x}_2) = \mathsf{add}(\mathsf{x}_2, \mathsf{x}_1)$, and algebraic if p_X as well as add and inv are algebraic.

Let Y be Λ_g, the space of parameters of a not degenerate algebraic curve $\mathcal{C}$, and X be $\mathcal{E}(\Lambda_g, \pi, \mathcal{C}^g)$, the bundle of the g-th symmetric power of $\mathcal{C}$. Let x be $D \in \mathfrak{C}_g$, and $\mathsf{p}_X(D) = \lambda$.

Theorem 15 ([21], Lemmas 2.2, 2.3)**.** *The polynomial function $\mathcal{R}_{2g}$ of order $2g$ on $\mathcal{C}$ defines the inverse mapping* inv*, and the polynomial function $\mathcal{R}_{3g}$ of order $3g$ defines the addition mapping* add*.*

Remark 13. *Theorem 15 reflects the fact that for any $w \geqslant 2g$ a polynomial function $\mathcal{R}_w$ is uniquely defined by a degree $w - g$ divisor D and produces an inverse divisor $D^* \in \mathfrak{C}_g$, such that $(\mathcal{R}_w)_0 = D + D^*$, according to Theorem 2 and Corollary 1.*

The function $\mathcal{R}_{2g}$, which is uniquely defined by $D \in \mathfrak{C}_g$ such that $\mathcal{A}(D) = u$, produces the inverse divisor $D^* \in \mathfrak{C}_g$ such that $\mathcal{A}(D^*) = -u$.

The function $\mathcal{R}_{3g}$, which is uniquely defined by $D \in \mathfrak{C}_g^2$ such that $\mathcal{A}(D) = u + \tilde{u}$, see Remark 8(ii), produces the inverse divisor $D^* \in \mathfrak{C}_g$ such that $\mathcal{A}(D^*) = -(u + \tilde{u})$.

In [21,22] this approach is illustrated by obtaining hyperelliptic addition laws explicitly, in a concise form for an arbitrary genus. The addition law contains expressions for all basis $\wp$-functions at $u + \tilde{u}$ in terms of basis $\wp$-functions at u and $\tilde{u}$. Below, we show the process of obtaining such expressions in genus 3.

(2,7)-Curve. Let $D, \tilde{D} \in \mathfrak{C}_g$, and $u = \mathcal{A}(D)$, $\tilde{u} = \mathcal{A}(\tilde{D})$. Each divisor is defined by the two functions $\mathcal{R}_6$, $\mathcal{R}_7$ of the form (49), computed at u, and $\tilde{u}$, or by the two collections of basis $\wp$-functions:

$$\begin{aligned} D: &\quad \wp_{1,1}(u), \wp_{1,3}(u), \wp_{1,5}(u), \wp_{1,1,1}(u), \wp_{1,1,3}(u), \wp_{1,1,5}(u); \\ \tilde{D}: &\quad \wp_{1,1}(\tilde{u}), \wp_{1,3}(\tilde{u}), \wp_{1,5}(\tilde{u}), \wp_{1,1,1}(\tilde{u}), \wp_{1,1,3}(\tilde{u}), \wp_{1,1,5}(\tilde{u}). \end{aligned} \tag{94}$$

Let a polynomial function of weight $3g = 9$ have the form

$$\mathcal{R}_9(x, y; \gamma) = yx + \gamma_1 x^4 + \gamma_2 y + \gamma_3 x^3 + \gamma_5 x^2 + \gamma_7 x + \gamma_9, \tag{95}$$

and $D + \tilde{D} \subset (\mathcal{R}_9)_0$. According to Theorem 2, this divisor defines $\mathcal{R}_9$ uniquely, which means $\gamma = (\gamma_k)$ are expressible in terms of the basis $\wp$-functions (94). First, we reduce $\mathcal{R}_9$ with the help of $\mathcal{R}_6$, and $\mathcal{R}_7$, namely

$$\begin{aligned} \mathcal{R}_9(x, y; \gamma) - \big(\gamma_1 x + \gamma_3 + \gamma_1 \wp_{1,1}(u) - \tfrac{1}{2}\wp_{1,1,1}(u)\big)\mathcal{R}_6(x; u) - \tfrac{1}{2}\big(x + \gamma_2\big)\mathcal{R}_7(x, y; u) \\ = x^2 \mathrm{F}_5(u; \gamma) + x\mathrm{F}_7(u; \gamma) + \mathrm{F}_9(u; \gamma). \end{aligned} \tag{96}$$

This implies

$$\mathrm{F}_w(u; \gamma) = 0, \quad \mathrm{F}_w(\tilde{u}; \gamma) = 0, \quad w = 5, 7, 9, \tag{97}$$

or in the matrix form

$$\begin{pmatrix} 1_3 & \mathsf{A}(u) \\ 1_3 & \mathsf{A}(\tilde{u}) \end{pmatrix} \begin{pmatrix} \check{\gamma} \\ \bar{\gamma} \end{pmatrix} + \begin{pmatrix} \mathsf{b}(u) \\ \mathsf{b}(\tilde{u}) \end{pmatrix} = 0, \tag{98a}$$

where 1_3 denotes the identity matrix of order 3, and

$$\begin{aligned} \mathsf{A}(u) &= \begin{pmatrix} \wp_{1,5}(u) & -\frac{1}{2}\wp_{1,1,5}(u) & \wp_{1,1}(u)\wp_{1,5}(u) \\ \wp_{1,3}(u) & -\frac{1}{2}\wp_{1,1,3}(u) & \wp_{1,1}(u)\wp_{1,3}(u) + \wp_{1,5}(u) \\ \wp_{1,1}(u) & -\frac{1}{2}\wp_{1,1,1}(u) & \wp_{1,1}(u)^2 + \wp_{1,3}(u) \end{pmatrix}, \\ \mathsf{b}(u) &= \begin{pmatrix} -\frac{1}{2}\wp_{1,1,1}(u)\wp_{1,5}(u) \\ -\frac{1}{2}\wp_{1,1,1}(u)\wp_{1,3}(u) - \frac{1}{2}\wp_{1,1,5}(u) \\ -\frac{1}{2}\wp_{1,1,1}(u)\wp_{1,1}(u) - \frac{1}{2}\wp_{1,1,3}(u) \end{pmatrix}, \quad \check{\gamma} = \begin{pmatrix} \gamma_9 \\ \gamma_7 \\ \gamma_5 \end{pmatrix}, \quad \bar{\gamma} = \begin{pmatrix} \gamma_3 \\ \gamma_2 \\ \gamma_1 \end{pmatrix}. \end{aligned} \tag{98b}$$

Note that $\mathsf{A}(u) = (\mathrm{Y}_1, \mathrm{Y}_2, \mathrm{Y}_3)$, cf. (64), and $\mathsf{b}(u) = -\frac{1}{2}\wp_{1,1,1}\mathrm{Y}_1 + \mathrm{Y}_2^\circ$, where $\mathrm{Y}_2^\circ = (0, -\frac{1}{2}\wp_{1,1,5}, -\frac{1}{2}\wp_{1,1,3})^t$. Then,

$$\begin{aligned} \bar{\gamma} &= -\big(\mathsf{A}(u) - \mathsf{A}(\tilde{u})\big)^{-1}\big(\mathsf{b}(u) - \mathsf{b}(\tilde{u})\big), \\ \check{\gamma} &= -\mathsf{b}(u) + \mathsf{A}(u)\big(\mathsf{A}(u) - \mathsf{A}(\tilde{u})\big)^{-1}\big(\mathsf{b}(u) - \mathsf{b}(\tilde{u})\big). \end{aligned} \tag{99}$$

On the other hand, the polynomial function $\mathcal{R}_9$ defined by (95) and (99) produces a divisor $\hat{D} \in \mathfrak{C}_3$, such that $(\mathcal{R}_9)_0 = D + \tilde{D} + \hat{D}$. Let $\hat{u} = \mathcal{A}(\hat{D})$, and we also have

$$\check{\gamma} + \mathsf{A}(\hat{u})\bar{\gamma} + \mathsf{b}(\hat{u}) = 0. \tag{100}$$

Let $D^* = -D$, $\tilde{D}^* = -\tilde{D}$, $\hat{D}^* = -\hat{D}$, which implies $(\mathcal{R}_6(u))_0 = D + D^*$, $(\mathcal{R}_6(\tilde{u}))_0 = \tilde{D} + \tilde{D}^*$, $(\mathcal{R}_6(\hat{u}))_0 = \hat{D} + \hat{D}^*$. Now, we construct $\mathcal{R}_9^-$ by the map $u \mapsto -u$, and so $(\mathcal{R}_9^-)_0 = D^* + \tilde{D}^* + \hat{D}^*$. As seen from (98), γ_k with odd indices are odd functions in u, and γ_2 is an even function. Thus,

$$\mathcal{R}_9^-(x, y; \gamma) = yx - \gamma_1 x^4 + \gamma_2 y - \gamma_3 x^3 - \gamma_5 x^2 - \gamma_7 x - \gamma_9.$$

Then, the equality

$$\mathcal{R}_9(x, y; \gamma)\mathcal{R}_9^-(x, y; \gamma) - \mathcal{R}_6(x; u)\mathcal{R}_6(x; \tilde{u})\mathcal{R}_6(x; \hat{u}) = -(x + \gamma_2)^2 f(x, y; \lambda) \tag{101}$$

enables to determine the required divisor $\hat{D} = -(D + \tilde{D})$. In fact, $\mathcal{R}_9(x, y; \gamma)\mathcal{R}_9^-(x, y; \gamma) + (x + \gamma_2)^2 f(x, y; \lambda)$ is a polynomial in x of degree 9, and coefficients of x^8, x^7, and x^6 produce

$$\begin{aligned} \wp_{1,1}(\hat{u}) &= -\wp_{1,1}(u) - \wp_{1,1}(\tilde{u}) - 2\gamma_2 + \gamma_1^2, \\ \wp_{1,3}(\hat{u}) &= -\wp_{1,3}(u) - \wp_{1,3}(\tilde{u}) + \wp_{1,1}(u)\wp_{1,1}(\tilde{u}) + \big(\wp_{1,1}(u) + \wp_{1,1}(\tilde{u})\big)\wp_{1,1}(\hat{u}) \\ &\quad + 2\gamma_1\gamma_3 - \gamma_2^2 - \lambda_4, \\ \wp_{1,5}(\hat{u}) &= -\wp_{1,5}(u) - \wp_{1,5}(\tilde{u}) + \wp_{1,3}(u)\wp_{1,1}(\tilde{u}) + \wp_{1,3}(\tilde{u})\wp_{1,1}(u) \\ &\quad + \big(\wp_{1,1}(u) + \wp_{1,1}(\tilde{u})\big)\wp_{1,3}(\hat{u}) \\ &\quad + \big(\wp_{1,3}(u) + \wp_{1,3}(\tilde{u}) - \wp_{1,1}(u)\wp_{1,1}(\tilde{u})\big)\wp_{1,1}(\hat{u}) \\ &\quad + \gamma_3^2 + 2\gamma_1\gamma_5 - 2\gamma_2\lambda_4 - \lambda_6. \end{aligned} \tag{102}$$

Finally, $\wp_{1,1,1}(\hat{u})$, $\wp_{1,1,3}(\hat{u})$, $\wp_{1,1,5}(\hat{u})$ are obtained from (100).

6.2. Polylinear Relations

Definition 4. *We introduce the primitive function*

$$\psi(P) = \exp\Big(-\int_{\infty}^{P} \mathcal{B}(\tilde{P})\mathrm{d}u(\tilde{P})\Big), \quad P = (x,y) \in \mathcal{C}. \tag{103}$$

Let $\psi(\xi) = \psi\big(x(\xi), y(\xi)\big)$ denote the expansion of ψ in the vicinity of infinity in the local parameter ξ, introduced by (4). Then, (see [22], Eq. 2.2)

$$\psi(\xi) = \xi^{g} \exp\Big(\mathcal{T}(\xi,\lambda)\Big), \tag{104}$$

where $\mathcal{T}$ is a holomorphic function of its arguments ξ and λ, $\mathcal{T}(\xi,0) = 0$. This implies that ψ is an entire function, and has the weight $-g$.

Theorem 16 ([22], Theorem 2.14)**.** *Let* $\mathfrak{n} = -\operatorname{wgt}\sigma - g$*, then*

$$\partial_{u_1}^{\mathfrak{n}}\sigma(u)\big|_{u\to\mathcal{A}(x,y)} = c\psi(x,y), \tag{105}$$

where c is constant.

Generalizing ([22], Theorem 3.11), we have

Theorem 17. *Let* $u^{[k]} \in \mathrm{Jac}(\mathcal{C})\backslash\Sigma$, $k = 1,\dots,\ell \geqslant 2$*, and* $\mathcal{R}_{\ell g}$ *be a monic polynomial function of weight* ℓg *from* $\mathfrak{P}(\mathcal{C})$ *with* $(\mathcal{R}_{\ell g})_0 = \sum_{k=1}^{\ell}\mathcal{A}^{-1}(u^{[k]})$*. Then,*

$$\mathcal{R}_{\ell g}(x,y) = \prod_{k=1}^{\ell}\frac{\sigma(u^{[k]} - \mathcal{A}(x,y))}{\psi(x,y)\sigma(u^{[k]})}, \qquad \sum_{k=1}^{\ell} u^{[k]} = 0. \tag{106}$$

With a fixed ℓ, equality (106) generates *polylinear relations*, which serve as identities for $\wp$-functions of $u^{[k]}$, $k = 1,\dots,\ell$. Indeed, by sending (x,y) to infinity, and applying (4), the expansion is obtained

$$\mathcal{R}_{\ell g}(x,y) = \xi^{-\ell g}\Big(1 + \sum_{s\geqslant 1}\xi^{s}T_{s}^{[\ell]}\big(u^{[1]}, u^{[2]}, \dots, u^{[\ell]}\big)\Big). \tag{107}$$

Thus, a sequence of identities for $\wp$-functions at $u^{[1]}, u^{[2]}, \dots, u^{[\ell]}$ is generated.

If $\ell = 2$, equality (106) acquires the form

$$\mathcal{R}_{2g}(x,y) = \frac{\sigma\big(u - \mathcal{A}(x,y)\big)\sigma\big(\tilde{u} - \mathcal{A}(x,y)\big)}{\psi(x,y)^{2}\sigma(u)\sigma(\tilde{u})}, \qquad u + \tilde{u} = 0, \tag{108}$$

where $\mathcal{R}_{2g}$ is the polynomial function of weight $2g$, which generates the inversion mapping on $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$, and arises in a solution of the Jacobi inversion problem. Equality (108), after expanding in ξ as $(x,y) \to \infty$, gives rise to a *hierarchy of bilinear relations*, which are identities for $\wp$-functions of u, since $\tilde{u} = -u$.

If $\ell = 3$, equality (106) acquires the form

$$\mathcal{R}_{3g}(x,y) = \frac{\sigma\big(u - \mathcal{A}(x,y)\big)\sigma\big(\tilde{u} - \mathcal{A}(x,y)\big)\sigma\big(\hat{u} - \mathcal{A}(x,y)\big)}{\psi(x,y)^{3}\sigma(u)\sigma(\tilde{u})\sigma(\hat{u})}, \quad u + \tilde{u} + \hat{u} = 0, \tag{109}$$

where $\mathcal{R}_{3g}$ is the polynomial function of weight $3g$, which generates the addition mapping on $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$. Equality (109) gives rise to *trilinear relations*, which produce the addition law for $\wp$-functions, since $\hat{u} = -(\tilde{u} + u)$.

Remark 14. *The second kind integral $\mathcal{B}$ on $\mathcal{C}$ in Definition 4 requires specifying a regularization constant vector $c(\lambda)$, see Remark 4. Namely*

$$\mathcal{B}(\xi) = c(\lambda) + \int_0^{\xi} \mathrm{d}r(\xi).$$

These constants $c(\lambda)$ can be found from Theorem 16 if a series for the σ-function associated with on $\mathcal{C}$ is known, or from comparing the identities for $\wp$-functions obtained from (108) with the same identities derived by the techniques described in Section 3, for more details see [52].

Further development in this direction allows us to represent polylinear relations in terms of operators. Let $\mathcal{D} = (\mathcal{D}_{\mathfrak{w}_1}, \mathcal{D}_{\mathfrak{w}_2}, \dots, \mathcal{D}_{\mathfrak{w}_g})^t$ be symbols which stand for linear operators

$$\mathfrak{D} = \left(\mathfrak{D}_{\mathfrak{w}_1}, \mathfrak{D}_{\mathfrak{w}_2}, \dots, \mathfrak{D}_{\mathfrak{w}_g}\right)^t, \qquad \mathfrak{D}_{\mathfrak{w}_i} = \sum_{k=1}^{\ell} \partial_{u^{[k]}_{\mathfrak{w}_i}}, \tag{110}$$

defined as follows

$$\mathcal{D}_{\mathfrak{w}_i} = \frac{\mathfrak{D}_{\mathfrak{w}_i} \prod_{k=1}^{\ell} \sigma(u^{[k]})}{\prod_{k=1}^{\ell} \sigma(u^{[k]})}, \qquad \sum_{k=1}^{\ell} u^{[k]} = 0. \tag{111}$$

Then, $T_s^{[\ell]}$ in (107) can be expressed in terms of $\mathcal{D}$. We call $T_s^{[\ell]}$ *polylinear operators*.

Theorem 18 ([22], Lemma 3.13)**.** *The generating function of polylinear operators has the form*

$$1 + \sum_{s \geqslant 1} \xi^s T_s^{[\ell]} = \exp\Big((\mathcal{A}(\xi), \mathcal{D}) - \ell \mathcal{T}(\xi, \lambda)\Big), \quad \ell \geqslant 2, \tag{112}$$

where $\exp \mathcal{T}(\xi, \lambda) = \xi^{-g} \psi(\xi)$, *cf.* (104).

6.3. Bilinear Relations and Baker–Hirota Operators

In ([2], p. 49) the following operators are introduced:

$$\Delta_{\mathfrak{w}_i} = \frac{\partial}{\partial u_{\mathfrak{w}_i}} - \frac{\partial}{\partial \bar{u}_{\mathfrak{w}_i}}, \quad \bar{u} = u. \tag{113}$$

Later, these operators were used by Hirota, and named after him. In [39] they are referred to as Baker–Hirota operators, and used to produce identities for $\wp$-functions; the latter are defined by

$$\begin{aligned} \wp_{i,j}(u) &= -\tfrac{1}{2}\sigma(u)^{-2} \Delta_i \Delta_j \sigma(u)\sigma(\bar{u})\big|_{\bar{u}=u}, \\ Q_{i,j,k,l}(u) &= -\tfrac{1}{2}\sigma(u)^{-2} \Delta_i \Delta_j \Delta_k \Delta_l \sigma(u)\sigma(\bar{u})\big|_{\bar{u}=u}. \end{aligned} \tag{114}$$

Remark 15. *The linear operators $\mathfrak{D}$ used in constructing bilinear operators $T_s^{[2]}$ are, in fact, the Baker–Hirota operators, cf. (110). Therefore, bilinear relations produce dynamical equations.*

Below, we illustrate how to use bilinear operators in finding identities for $\wp$-functions.

$(3,4)$-Curve. On $(3,4)$-curve we have $\psi(\xi) = \partial_{u_1}^2 \sigma(u)|_{u=u(\xi)}$, and $c(\lambda) = (0, -\frac{1}{3}\lambda_2, -\frac{1}{6}\lambda_5)$, see ([52], Lemma 4.2, Theorem 4.1). Defining ψ by (103), we find

$$\mathcal{T}(\xi, \lambda) = -\tfrac{1}{12}\lambda_5 \xi^5 - \tfrac{7}{60}\lambda_6 \xi^6 - \tfrac{1}{90}\lambda_2\lambda_5 \xi^7 - \left(\tfrac{13}{168}\lambda_8 - \tfrac{3}{112}\lambda_2\lambda_6\right)\xi^8 + O(\xi^9). \tag{115}$$

Using (112), we generate $T_s^{[2]}$, and split each operator into odd $T_s^{[2,\mathrm{o}]}$ and even $T_s^{[2,\mathrm{e}]}$ parts. Note that $T_s^{[2,\mathrm{o}]}=0$ due to $\tilde{u}=-u$. Assuming that $\mathcal{R}_6(x,y)=x^2+\alpha_2 y+\alpha_3 x+\alpha_6$, we find bilinear relations in terms of bilinear operators, namely

$$\begin{aligned}
\alpha_2 &= T_2^{[2,\mathrm{e}]} \equiv \tfrac{1}{2}\mathcal{D}_1^2,\\
\alpha_3 &= T_3^{[2,\mathrm{e}]} \equiv \tfrac{1}{2}\mathcal{D}_1\mathcal{D}_2,\\
\tfrac{1}{3}\lambda_2\alpha_2 &= T_4^{[2,\mathrm{e}]} \equiv \tfrac{1}{8}\mathcal{D}_2^2+\tfrac{1}{4!}\mathcal{D}_1^4,\\
0 &= T_5^{[2,\mathrm{e}]} \equiv \tfrac{1}{12}\big(\mathcal{D}_1^2-\lambda_2\big)\mathcal{D}_1\mathcal{D}_2+\tfrac{1}{6}\lambda_5,\\
\alpha_6 &= T_6^{[2,\mathrm{e}]} \equiv \tfrac{1}{5}\mathcal{D}_1\mathcal{D}_5+\big(\tfrac{1}{16}\mathcal{D}_1^2-\tfrac{1}{24}\lambda_2\big)\mathcal{D}_2^2+\tfrac{1}{6!}\mathcal{D}_1^6-\tfrac{1}{45}\lambda_2^2\mathcal{D}_1^2+\tfrac{7}{30}\lambda_6,\quad \ldots
\end{aligned} \tag{116}$$

By expanding the right-hand side of (108) in ζ, we find the corresponding relations in terms of $\wp$-functions:

$$\alpha_2 = -\wp_{1,1}(u), \tag{117a}$$

$$\alpha_3 = -\wp_{1,2}(u), \tag{117b}$$

$$\tfrac{1}{3}\lambda_2\alpha_2 = \tfrac{1}{12}\big(6\wp_{1,1}(u)^2-3\wp_{2,2}(u)-\wp_{1,1,1,1}(u)\big), \tag{117c}$$

$$0 = \tfrac{1}{6}\big(\lambda_5+\lambda_2\wp_{1,2}(u)+6\wp_{1,1}(u)\wp_{1,2}(u)-\wp_{1,1,1,2}(u)\big), \tag{117d}$$

$$\begin{aligned}
\alpha_6 = -\wp_{1,5}(u)+\tfrac{7}{60}\big(&2\lambda_6+\lambda_2\wp_{2,2}(u)+4\wp_{1,2}(u)^2+4\wp_{1,5}(u)\\
&+2\wp_{1,1}(u)\wp_{2,2}(u)-\wp_{1,1,2,2}(u)\big),\quad \ldots
\end{aligned} \tag{117e}$$

From (117c) and (117d) the first two identities in the list (84) are obtained in a simpler way. Then, (117e) is simplified with the help of these two identities and the cubic relation $\mathcal{G}_6$ from (73). To proceed with producing identities, we need a solution of the Jacobi inversion problem, according to which $\alpha_6=-\wp_{1,5}(u)$.

On the other hand, all relations in (117) come directly from expressions (116) written in terms of bilinear operators, which are significantly easier to construct.

Remark 16. *In the hierarchy of bilinear equations associated with a hyperelliptic curve, the identity* $\wp_{1,1,1,1}(u)-6\wp_{1,1}(u)^2-4\wp_{1,3}(u)=2\lambda_4$ *represents the first integral of the Korteweg–de Vries (KdV) equation, which has Hirota's bilinear form, cf. ([44], p. 5),*

$$-\tfrac{1}{2}\mathcal{D}_1^4+2\mathcal{D}_1\mathcal{D}_3=2\lambda_4.$$

Such an equation exists in the hierarchy in any genus greater than or equal to 2.

6.4. Trilinear Relations and Addition Laws

The structure of the hierarchy of trilinear operators associated with the family of hyperelliptic curves is described in ([22], Theorem 3.15). Below, we give an example of constructing addition formulas on the simplest trigonal curve.

$(3,4)$-Curve. Assuming that $\mathcal{R}_9$ has the form (95), we generate trilinear relations in terms of operator symbols $\mathcal{D}$ by means of (112), and split $T_s^{[3]}$ into odd $T_s^{[3,\mathrm{e}]}$ and even $T_s^{[3,\mathrm{o}]}$ parts:

$$\begin{aligned}
\gamma_1^{\mathrm{o}} &= T_1^{[3,\mathrm{o}]} \equiv \mathcal{D}_1, & \gamma_1^{\mathrm{e}} &= T_1^{[3,\mathrm{e}]} \equiv 0,\\
\gamma_2^{\mathrm{o}} &= T_2^{[3,\mathrm{o}]} \equiv \tfrac{1}{2}\mathcal{D}_2 & \gamma_2^{\mathrm{e}} &= T_2^{[3,\mathrm{e}]} \equiv \tfrac{1}{2}\mathcal{D}_1^2,\\
\gamma_3^{\mathrm{o}} + \tfrac{2}{3}\lambda_2\gamma_1^{\mathrm{o}} &= T_3^{[3,\mathrm{o}]} \equiv \tfrac{1}{3!}\mathcal{D}_1^3, & \gamma_3^{\mathrm{e}} + \tfrac{2}{3}\lambda_2\gamma_1^{\mathrm{e}} &= T_3^{[3,\mathrm{e}]} \equiv \tfrac{1}{2}\mathcal{D}_1\mathcal{D}_2\\
\tfrac{1}{3}\lambda_2\gamma_2^{\mathrm{o}} &= T_4^{[3,\mathrm{o}]} \equiv \tfrac{1}{4}\mathcal{D}_1^2\mathcal{D}_2 - \tfrac{1}{12}\lambda_2\mathcal{D}_2, & \tfrac{1}{3}\lambda_2\gamma_2^{\mathrm{e}} &= T_4^{[3,\mathrm{e}]} \equiv \tfrac{1}{8}\mathcal{D}_2^2 + \tfrac{1}{4!}\mathcal{D}_1^4,\\
\gamma_5^{\mathrm{o}} + \tfrac{1}{9}\lambda_2^2\gamma_1^{\mathrm{o}} &= T_5^{[3,\mathrm{o}]} \equiv \tfrac{1}{5}\mathcal{D}_5 + \tfrac{1}{8}\mathcal{D}_1\mathcal{D}_2^2 + \tfrac{1}{5!}\mathcal{D}_1^5 - \tfrac{1}{45}\lambda_2^2\mathcal{D}_1,\\
\gamma_5^{\mathrm{e}} + \tfrac{1}{9}\lambda_2^2\gamma_1^{\mathrm{e}} &= T_5^{[3,\mathrm{e}]} \equiv \tfrac{1}{12}(\mathcal{D}_1^2 - \lambda_2)\mathcal{D}_1\mathcal{D}_2 + \tfrac{1}{4}\lambda_5, \quad \dots
\end{aligned}$$

Let $\mathfrak{s}_{i,\dots}$ denote the sum $-(\wp_{i,\dots}(u) + \wp_{i,\dots}(\tilde{u}) + \wp_{i,\dots}(\hat{u}))$. By expanding the right-hand side of (109), we find the trilinear relations in terms of $\mathfrak{s}_{i,\dots}$. Solving them, we obtain the addition formulas for ζ-functions, cf. ([52], Theorem 5.5):

$$\begin{aligned}
&\mathfrak{s}_1 = \gamma_1^{\mathrm{o}}, \qquad \mathfrak{s}_2 = 2\gamma_2^{\mathrm{o}},\\
&\mathfrak{s}_5 = 5\gamma_5^{\mathrm{o}} - 5\gamma_3^{\mathrm{o}}\gamma_2^{\mathrm{e}} - 5\gamma_3^{\mathrm{e}}\gamma_2^{\mathrm{o}} + 5\gamma_3^{\mathrm{o}}(\gamma_1^{\mathrm{o}})^2 + 5\big((\gamma_2^{\mathrm{e}})^2 + (\gamma_2^{\mathrm{o}})^2\big)\gamma_1^{\mathrm{o}} - 5\gamma_2^{\mathrm{e}}\gamma_1^{\mathrm{o}}\big((\gamma_1^{\mathrm{o}})^2 + \lambda_2\big) + (\gamma_1^{\mathrm{o}})^5\\
&\quad + \tfrac{2}{3}\lambda_2\gamma_1^{\mathrm{o}}\big(5(\gamma_1^{\mathrm{o}})^2 + \lambda_2\big) - \tfrac{5}{8}\mathfrak{s}_{1,2,2} - \tfrac{1}{24}\mathfrak{s}_{1,1,1,1,1},
\end{aligned}$$

and the addition law for basis $\wp$-functions. Coefficients γ are expressed in terms of basis $\wp$-functions at u and $\tilde{u}$ by (99).

6.5. Addition Formulas for σ-Function

The first generalization of (90) to the hyperelliptic case for genera two and three was given by Baker in [13]. For example, in genus two we have

$$\frac{\sigma(u+\tilde{u})\sigma(u-\tilde{u})}{\sigma(u)^2\sigma(\tilde{u})^2} = \wp_{1,1}(u)\wp_{1,3}(\tilde{u}) + \wp_{3,3}(\tilde{u}) - \big(\wp_{1,1}(\tilde{u})\wp_{1,3}(u) + \wp_{3,3}(u)\big). \tag{118}$$

A generalization to higher genera is suggested in [19] in the concise form

$$\frac{\sigma(u+\tilde{u})\sigma(u-\tilde{u})}{\sigma(u)^2\sigma(\tilde{u})^2} = \mathcal{S}(\tilde{u},u) - \mathcal{S}(u,\tilde{u}), \tag{119}$$

and $\mathcal{S}$ is computed for a hyperelliptic curve of any genus. Addition formulas for ζ-functions can be obtained by applying operator $\mathfrak{D}$.

Further, $\mathcal{S}$ for non-hyperelliptic and superelliptic curves have been computed:

- for $(3,4)$-curve with extra terms, see ([32], Theorem 9.1);
- for cyclic $(3,5)$-curve, see ([30], Theorem 8.1);
- for cyclic $(3,7)$-curve, see ([40], Theorem 4, and Appendix B);
- for cyclic $(4,5)$-curve, see ([37], Appendix D).

A more general than (90) addition formula for the Weierstrass σ-function

$$\begin{aligned}
(-1)^{(l-1)(l-2)/2}1!2!\cdots(l-1)!&\frac{\sigma\big(\sum_{k=1}^{l}u^{[k]}\big)\prod_{i<j}\sigma(u^{[i]}-u^{[j]})}{\prod_{k=1}^{l}\sigma(u^{[k]})^l}\\
&= \begin{vmatrix}
1 & \wp(u^{[1]}) & \wp'(u^{[1]}) & \dots & \wp^{(l-2)}(u^{[1]})\\
1 & \wp(u^{[2]}) & \wp'(u^{[2]}) & \dots & \wp^{(l-2)}(u^{[2]})\\
\vdots & \vdots & \vdots & \cdots & \vdots\\
1 & \wp(u^{[l]}) & \wp'(u^{[l]}) & \dots & \wp^{(l-2)}(u^{[l]})
\end{vmatrix}
\end{aligned} \tag{120}$$

and can be found in [67]. The formula for all equal $u^{[k]}$ is known as the Kiepert formula, see [68]. The function $\sigma(lu)/\sigma(u)^{l^2}$ is derived through a limiting process from (120), and defines division polynomials, see [66].

A generalization of (120) written in terms of monomials $(x^i y^j)(u^{[k]})$ parameterized by $u^{[k]} \in \mathcal{A}(\mathfrak{C}_1)$ is given in [69–71] for hyperelliptic curves of genera two, three, and an arbitrary genus, respectively. An explicit expression in terms of $\wp$-functions on $\mathrm{Jac}(\mathcal{C})\backslash\Sigma$ in the case of genus two and $l = 3$ is obtained in [31].

7. Dynamical Equations and Other Applications

It is well-known that completely integrable dynamical equations, such as the Korteweg–de Vries equation (KdV), the sine-Gordon equation (SG), the non-linear Schrödinger equation (NLS), etc., are solved in terms of abelian functions. According to the definition, a completely integrable equation possesses infinitely many integrals of motion. As a result, such an equation is associated with an infinite hierarchy of integrable systems. Each system provides a finite gap solution, which is expressed in terms of a particular abelian function on the Jacobian variety of the corresponding spectral curve. Solutions of the mentioned equations, expressed in terms of theta functions, are well known, see [72].

These solutions can be expressed in terms of $\wp$-functions. For example, $\wp_{1,1}$ on hyperelliptic Jacobian varieties serves as the finite-gap solution of the KdV equation, and $\wp_{1,2g-1}$ as the solution of the SG equation, as stated in ([9], Theorems 4.12 and 4.13). To be precise, the indicated complex-valued $\wp$-functions solve the KdV and SG equations with both complex time and complex coordinate. In fact, a completely integrable dynamical equation is represented by a bilinear relation, which is an identity for $\wp$-functions, see Remark 16. In order to obtain physically meaningful solutions, reality conditions should be specified. That is, a real-valued and bounded solution as a function of real time and real coordinate is required. Reality conditions for the KdV equation are completely described in [73].

Solutions of completely integrable equations expressed in terms of $\wp$-functions remain largely unexplored. To the best of the author's knowledge, the results mentioned above encompass all that has been presented in the literature regarding solutions in terms of $\wp$-funcitons.

Some progress has been achieved in computing $\wp$-functions. Direct computations of $\wp$-functions are presented in [74], and quasi-periodic solutions of the KdV equation are computed directly in [73]. In [75] the $\wp_{1,1}$-solution of the KdV equation in genus two is computed by means of Euler's numerical quadrature method. In [76] a solution of the MKdV equation in genus two is computed similarly, using a reduction to the $\wp_{1,1}$-solution of KdV by means of the Miura transformation [77].

Another direction, where $\wp$-functions recently found an application, is the algebraic theory of the KdV hierarchy, see ([78], § 5–6). $\wp$-Functions are introduced explicitly in ([78], Theorem 53). On the other hand, the KdV hierarchy can be constructed directly by methods of the Lie group theory, see [73]. The hamiltonian system associated with a genus g hyperelliptic curve is defined by g hamiltonians, equal to the integral of motions $\lambda_{2g+2+2k}$, $k = 1, \ldots, g$, see ([73], § 3.2). The corresponding expressions in terms of $\wp$-functions can be obtained from equations of the Jacobian variety of the spectral curve. Thus, integrable hierarchies arise from algebraic models of Jacobian varieties.

The problem of differentiating $\wp$-functions with respect to parameters of a curve in the canonical form is stated in [18]. The $\wp$-functions are considered over the bundle $\mathcal{E}(\Lambda_g, \omega, \mathrm{Jac}(\mathcal{C}))$ such that the restrictions to each fiber $\mathrm{Jac}(\mathcal{C})$ are abelian functions. Let the field of such extended $\wp$-functions be denoted by $\mathfrak{F}$. The tangent vector fields on the bundle form the $\mathfrak{F}$-module $\mathrm{Der}\,\mathfrak{F}$ of derivations of $\mathfrak{F}$. The problem consists of several parts: (i) find generators of $\mathrm{Der}\,\mathfrak{F}$; (ii) describe the Lie algebra structure of $\mathrm{Der}\,\mathfrak{F}$ over $\mathfrak{F}$ by presenting commutation relations for the generators of $\mathrm{Der}\,\mathfrak{F}$; (iii) describe the action of $\mathrm{Der}\,\mathfrak{F}$ on $\mathfrak{F}$. The problem was solved in [18], and illustrated by the case of $(2, 5)$-curve. A modification

of this problem for hyperelliptic curves is considered in [79], with illustrations in genera 2, 3, and 4.

8. Discussion

Abelian function fields appear to be a perfect tool for studying Jacobian varieties. Such a field provides (i) uniformization of a curve, which is now well understood and computed, (ii) algebraic models of the Jacobian and Kummer varieties, (iii) a connection between differentiation rules and an algebraic structure of the ring of functions, (iv) addition laws.

The presented construction of abelian function fields was developed on (n, s)-curves, where it arises naturally due to an exceptional role of the point at infinity, which is also a good choice of the base point. (n, s)-Curves remain not fully understood, and so underestimated. They are often considered as a special class of curves. On the contrary, they are Weierstrass canonical forms of plane algebraic curves, as clearly seen from ([1], §§60–63). The wrong impression is caused by the absence in the literature of examples of degenerate (n, s)-curves, which fill the gaps in establishing connections between Weierstrass gap sequences, including truncated ones, and plane algebraic curves.

A solution of the Jacobi inversion problem obtained in [64] shows that the number of equations defining the Jacobian variety increases linearly with the genus of a curve. This is a great improvement in comparison with the construction based on theta identities (see [80], §10), where the number of equations defining the Jacobian variety increases exponentially.

Curves of gonality greater than 3 deserve more attention, especially since solutions of the Jacobi inversion problem for such curves became known. Open problems include discovering the ring structure of abelian function fields, and clarifying uniformization of these curves.

The approach involving polylinear operators is virgin territory, abundant in open problems and new horizons. Hierarchies of polylinear relations definitely have a finite number of generating operators and carry a distinct similarity within families of canonical curves.

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