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Upper *a*-Graphical Topological Spaces with the COVID-19 Form and Its Diffusion

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Abstract: In this work, we use the monophonic eccentric open neighborhood system and the upper approximation neighborhood system to construct a new class of topologies in the theory of undirected simple graphs. We study some fundamental topological properties of this class and characterize the graphs that induce the indiscrete or discrete topology. Next, we present the openness, the connectedness and the continuity properties with isomorphic maps of graphs. Some applications concerning the topological discrete and connectedness properties for some corresponding graphs of the COVID-19 form and its diffusion are introduced.

Keywords: neighbourhood system; simple graph; continuous function; connectedness; topology

MSC: 05C99; 18F60; 05C20

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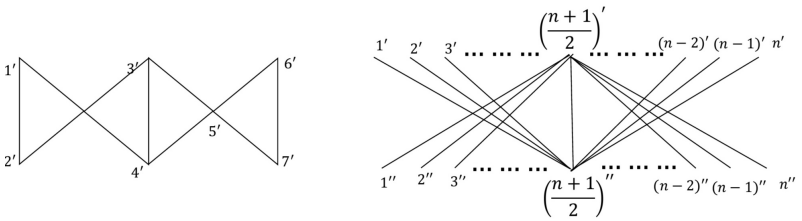
1. Introduction

Graph theory and general topology are important topics in the mathematical field. Several researchers studied the relation between general topology and graph theory by constructing some topologies on the vertex set of graphs and on the edge set for undirected simple graphs or directed graphs. The subject of constructing topologies by using graph theory has recently become a subject of interest to mathematical researchers, which is considered as one of the relationships between graph theory and general topology. This construction is on the vertex set of graphs and on the edge set in directed graphs or undirected graphs, which depends on the neighborhood system used. Many mathematical researchers studied this construction on vertex sets in the theory of undirected simple graphs. The construction of any neighborhood system depends mainly on the type of relations used between the vertices or edges for graphs. Most studies were on the theory of undirected simple graphs and in particular on the vertex sets. Jafarian et al. [1] constructed the graphic topological space of undirected simple graphs $\Omega = (\nu(\Omega), \xi(\Omega))$ as a pair $(\nu(\Omega), T_{\mathcal{A}\nu})$, where $T_{\mathcal{A}\nu}$ is a topology on $\nu(\Omega)$ induced by the class of open neighbourhoods $N(x)$ of vertices in Ω. Nada et al. [2] introduced a relation on graphs to generate new types of topological structures. In 2018, Abdu and Kiliçman [3] introduced a construction of topologies, with an incidence topology on $\nu(\Omega)$ for simple graphs $\Omega = (\nu(\Omega), \xi(\Omega))$ without isolated vertices, which has a sub-basis as the family of end-sets that contain only

end points of every edge. Kiliçman and Abdu [4] used the graphs $\Omega = (\nu(\Omega), \xi(\Omega))$ to introduce two constructions of topologies on the set $\xi(\Omega)$, called compatible edge topology and incompatible edge topology. In 2019, Nianga and Canoy [5] constructed a topology in simple graphs by using the notions of unary and binary operations; in [6], they introduced some topologies on the vertex set in the theory of simple graphs by using the hop neighborhoods of the graphs. For simple graphs without isolated vertices $\Omega = (\nu(\Omega), \xi(\Omega))$, in 2020, Sari and Kopuzlu [7] generated a topology on the vertex set induced by the same basis, which is defined by Amiri et al. [1], and studied the continuity of functions. The discrete property of topologies for special graphs and the minimal neighborhood system of vertices are studied, such as complete graphs K_n, cycle graphs C_n and complete bipartite graphs $K_{n,m}$. Zomam et al. [8] in 2021 used graphic topological spaces, which were introduced by Amiri et al. [1], to satisfy the Alexandroff property by giving some conditions such as a locally finite property. The notion of pathless topological spaces on the vertex set $\nu(\Omega)$ in the theory of directed graphs was introduced by Othman et al. [9] in 2022 and they presented the relation between pathless topological spaces and the relative topologies and E-generated subgraphs and studied the role of pathless topology in the blood circulation of the heart of the human body. The notion of L_2-topological spaces is introduced by Othman et al. [10] via the open neighborhood C-set system. In 2023, Abu-Gdairi et al. [11] explained the role of topological visualization in the medical field through graph analysis and rough sets by using neighbourhood systems. In the theory of approximating a neighborhood system, Yao [12] introduced the notions of upper and lower approximations of any nonempty set as a generalized rough set by using a binary relation. Next, by using graph theory, Atik et al. [13] introduced a new type of rough approximation model using *j*-neighborhood systems. Guler [14] generated different approximations and compared these approximations by using the notion of an ideal collection, for more details see [15–18]. Dammag et al. [19] used monophonic paths in the theory of directed graphs to construct a new topology, called the out mondirected topology. They introduced the important role of out mondirected topological spaces in satisfying the connectedness and discrete properties of the nervous system of the human body. Here, we are going to give some preliminaries on our paper applications concerning COVID-19. Davahli et al. [20] used graph theory to introduce a quantitative assessment by comparing the pandemic spreading of COVID-19 in Kyoto city in Japan and Kentucky state in the USA. Ashokkumar et al. [21] used graph theory to introduce a proposed algorithm for scheduling the timing of the relief funds that helped the Indian government to combat the spread of COVID-19. Bhapkar et al. [22] used graph theory to introduce four representations of COVID-19 types.

In this paper, using the idea of a monophonic eccentric open neighborhood system [23] and an upper approximation neighbourhood system [13] of undirected simple graphs $\Omega = (\nu(\Omega), \xi(\Omega))$, we introduce a new class of topologies on a vertex set $\nu(\Omega)$, called an upper *a*-graphical topology. In Section 2, we first give the operator $\cap^{\Omega} : \nu(\Omega) \to P(\nu(\Omega))$ and next we use the operator $\cap^{\Omega}$ together with a monophonic eccentric open neighborhood system [23] and an upper approximation neighbourhood system [13] to introduce the notion of an upper *a*-neighborhood system. Using this neighborhood system, we give the concept of an upper *a*-graphical topological space and we show the discrete property of path P_n, complete graphs K_n, cycle graphs C_n and complete bipartite graphs $K_{n,m}$. Figure 1a and Figure 1b present some examples for the upper approximation neighbourhood system and upper *a*-Graphical Topological Spaces. In Section 3, we present some isomorphic properties such as connectedness, discreteness, compactness, $\cap$-upper connectedness and $\cap$-upper discreteness. Figure 2 presents example of compactness property. Section 4 presents the discreteness, $\cap$-upper connectedness and $\cap$-upper discreteness for upper *a*-

graphical topological spaces for some corresponding graphs concerning the COVID-19 form and its diffusion, such as Figure 3, which presents a quantitative assessment comparing the COVID-19 pandemic spread of cities Kyoto in Japan and Kentucky in the USA, [20]. Figure 4 presents a proposed algorithm for the timing schedule of the relief funds that helped the Indian government to combat the spread of COVID-19, [21]. Figure 5 presents the four COVID-19 types, [22].



(a) Type of a-graphical 1 (b) Type of a-graphical 2

Figure 1. Upper *a*-graphical topologies.

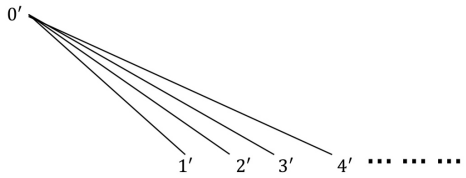


Figure 2. Graphical form with infinite vertex set.

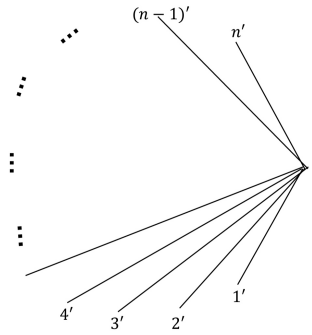
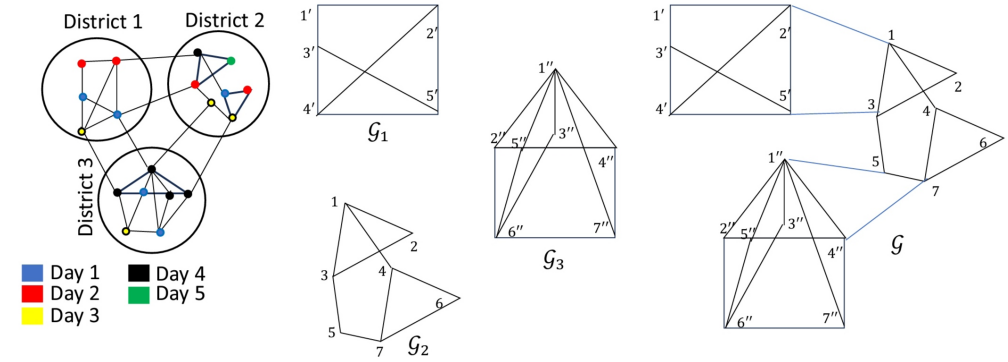
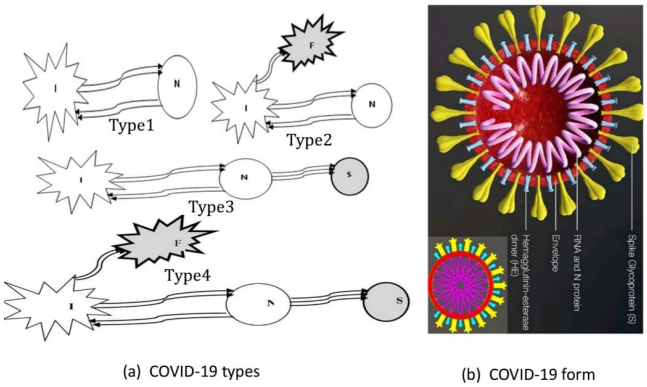


Figure 3. COVID-19 diffusion in Kentucky state and Kyoto city.



(a) Daily graphical diffusion (b) Separable diffusion

Figure 4. Relief scheme and its graphical forms in some cities of India.



(a) COVID-19 types (b) COVID-19 form

Figure 5. COVID-19 form and its types.

For a any relation $\rtimes$ on any set $S \neq \emptyset$ and for any $A \subseteq S$, [12], the lower $\underline{\rtimes}(A)$ and upper $\overline{\rtimes}(A)$ approximations of a set A are given by $\underline{\rtimes}(A) = \{x \in S : \rtimes_x \subseteq A\}$ and $\overline{\rtimes}(A) = \{x \in S : \rtimes_x \cap A \neq \emptyset\}$, respectively, where $\rtimes_x = \{y \in S : x \rtimes y\}$. By a graph Ω, we mean the pair $(\nu(\Omega), \xi(\Omega))$ of a vertex set $\nu(\Omega) \neq \emptyset$ and edge set $\xi(\Omega)$. For any subgraph $\mathcal{T}$ of Ω, [13] defined rough approximation j-neighborhood systems such that the lower approximations $\underline{N_j}(\nu(\mathcal{T}))$ and upper approximations $\overline{N_j}(\nu(\mathcal{T}))$ of $\mathcal{T}$ are given by $\underline{N_j}(\nu(\mathcal{T})) = \{x \in \nu(\Omega) : N_j(x) \subseteq \nu(\mathcal{T})\}$ and $\overline{N_j}(\nu(\mathcal{T})) = \nu(\mathcal{T}) \cup \{x \in \nu(\Omega) : N_j(x) \cap \nu(\mathcal{T}) \neq \emptyset\}$, respectively, where $j \in \{r, l, r', l', u, i, u', i'\}$, $N_r(x) = \{y \in \nu(\Omega) : x \rtimes y\}$, $N_l(x) = \{y \in \nu(\Omega) : y \rtimes x\}$, $N_{r'}(x) = \cap_{x \in N_r(y)} N_r(y)$, $N_{l'}(x) = \cap_{x \in N_l(y)} N_l(y)$, $N_u(x) = N_r(x) \cup N_l(x)$, $N_i(x) = N_r(x) \cap N_l(x)$, $N_{u'}(x) = N_{r'}(x) \cup N_{l'}(x)$ and $N_{i'}(x) = N_{r'}(x) \cap N_{l'}(x)$. Throughout this paper, all graphs will be assumed to be undirected. If $\gamma \in \xi(\Omega)$ is any an edge in $\xi(\Omega)$ joining x and y in $\nu(\Omega)$, then we write $\nu(\gamma) = \{x, y\}$. For $x \in \nu(\Omega)$, $\mathbb{D}_x$ is called the degree of x, which is defined by the number of vertices that are adjacent to x. The edge γ with $\nu(\gamma) = \{x\}$ is called a loop. For any two edges γ and γ' in $\xi(\Omega)$, if $\nu(\gamma) = \nu(\gamma')$, then they are called multiple edges. A graph Ω is said to be a simple graph if it has no loops and multiple edges. In any simple graph $\Omega = (\nu(\Omega), \xi(\Omega))$, if there is an edge between x and y, it will be denoted by Ω_{xy}. For the vertex $x \in \nu(\Omega)$, the open neighbourhood $\mathbb{N}(x)$ of x is the set of all adjacent vertices with x. A path P is an alternating sequence of distinct edges and distinct vertices. A path that starts and ends at the same vertex is called a closed path. A graph Ω is called connected if we can move along the edges from any vertex to any other vertex in $\nu(\Omega)$. A complete graph K_n with $n > 0$ is a simple graph with n vertices such that $\mathbb{D}_x = n$ for all $x \in \nu(\Omega)$. A complete bipartite graph $K_{n,m}$ with $n, m > 0$ is a simple graph whose vertices can be partitioned into two subsets, V_n with n vertices and V_m with m vertices, where there is no edge that has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph. A cycle graph C_n with $n > 2$ is a simple graph with n vertices and n edges such that $\mathbb{D}_x = 2$ for all $x \in \nu(C_n)$. Let P be any path in a graph Ω and $\gamma \in \xi(\Omega)$. If γ joins two non-adjacent vertices of a path P, then γ is called a chord of P. A path P is called a monophonic path if it has no chord edge. The length of the longest monophonic path that ends with vertices x and y is defined as the monophonic distance $\partial^m_\Omega(x, y)$ between x and y. Note that the function ∂^m_Ω does not satisfy the triangle inequality in the definition of a metric function. The monophonic eccentricity $e^m_\Omega(x)$ is defined as $e^m_\Omega(x) = \max_{y \in \nu(\Omega)} \partial^m_\Omega(x, y)$. A point y in $\nu(\Omega)$ is said to be a monophonic eccentric of x in $\nu(\Omega)$ if $\partial^m_\Omega(x, y) = e^m_\Omega(x)$. In our work, we define the relation $\rtimes$ on $\nu(\Omega)$ by the monophonic eccentric relation, that is, $x \rtimes y$ if x is a monophonic eccentric vertex of y. So the monophonic $eo-$neighborhood $\mathcal{J}^{em}_\Omega(x)$ of a point x in $\nu(\Omega)$ is given by $\mathcal{J}^{em}_\Omega(x) = \{y \in \nu(\Omega) : y \rtimes x\}$. The monophonic $eo-$neighborhood $\mathcal{J}^{em}_\Omega(G)$ of a subset $G \subseteq \nu(\Omega)$ is given by $\mathcal{J}^{em}_\Omega(G) = \cup_{x \in G} \mathcal{J}^{em}_\Omega(x)$. The

collection $\mathcal{J}_{\Omega}^{em} = \{\mathcal{J}_{\Omega}^{em}(x) : x \in \nu(\Omega)\}$ is called a monophonic *eo*−neighborhood system of a graph Ω.

2. The Upper *a*-Neighborhood System

Let $\Omega = (\nu(\Omega), \xi(\Omega))$ be any simple graph. We use the rough approximation *j*-neighborhood systems, which are introduced in [13] to structure a new neighborhood system for the elements of $\nu(\Omega)$. Define two operators $\cap^{\Omega}, \Omega^{u} : \nu(\Omega) \to P(\nu(\Omega))$ by $\cap^{\Omega}(x) = \{y \in \nu(\Omega) : \mathcal{J}_{\Omega}^{em}(x) \cap \mathcal{J}_{\Omega}^{em}(y) \neq \emptyset\}$ and $\Omega^{u}(x) = \cap^{\Omega}(x) \cup \mathcal{J}_{\Omega}^{em}(x)$, respectively, where $P(\nu(\Omega))$ is a power set of $\nu(\Omega)$. The collection $\Omega^{u} = \{\Omega^{u}(x) : x \in \nu(\Omega)\}$ is called an upper approximated neighborhood system (shortly, upper *a*-neighborhood system) of a graph Ω. For a subset $G \subseteq \nu(\Omega)$, the upper *a*-neighborhood $\Omega^{u}(G)$ of G is defined by $\Omega^{u}(G) = \cap^{\Omega}(G) \cup \mathcal{J}_{\Omega}^{em}(G)$, where $\cap^{\Omega}(G) = \cup_{x \in G} \cap^{\Omega}(x)$. That is, $\Omega^{u}(G) = \cup_{x \in G} \Omega^{u}(x)$.

Theorem 1. *Let $\Omega = (\nu(\Omega), \xi(\Omega))$ be any simple graph and $G, K \subseteq \nu(\Omega)$. If $K \subseteq G$, then $\Omega^{u}(K) \subseteq \Omega^{u}(G)$.*

Proof. Let $x \in \Omega^{u}(K)$. Then, $x \in \mathcal{J}_{\Omega}^{em}(K)$ or $x \in \cap^{\Omega}(K)$. If $x \in \mathcal{J}_{\Omega}^{em}(K)$, then $x \in \mathcal{J}_{\Omega}^{em}(y)$ for some $y \in K$. Since $K \subseteq G$, $y \in G$ and hence $x \in \mathcal{J}_{\Omega}^{em}(y) \subseteq \Omega^{u}(G)$. If $x \in \cap^{\Omega}(K)$, then $\mathcal{J}_{\Omega}^{em}(x) \cap \mathcal{J}_{\Omega}^{em}(y) \neq \emptyset$ for some $y \in K$. Since $K \subseteq G$, $y \in G$ and hence $x \in \cap^{\Omega}(G) \subseteq \Omega^{u}(G)$. Hence, $\Omega^{u}(K) \subseteq \Omega^{u}(G)$. □

Theorem 2. *Let $\Omega = (\nu(\Omega), \xi(\Omega))$ be any simple graph and $G, K \subseteq \nu(\Omega)$. Then, $\Omega^{u}(K \cup G) = \Omega^{u}(K) \cup \Omega^{u}(G)$.*

Proof. Since $K, G \subseteq K \cup G$, from Theorem 1, $\Omega^{u}(K) \subseteq \Omega^{u}(K \cup G)$ and $\Omega^{u}(G) \subseteq \Omega^{u}(K \cup G)$, that is, $\Omega^{u}(K) \cup \Omega^{u}(G) \subseteq \Omega^{u}(K \cup G)$. On the other hand, let $x \in \Omega^{u}(K \cup G)$. Then, $x \in \mathcal{J}_{\Omega}^{em}(K \cup G)$ or $x \in \cap^{\Omega}(K \cup G)$. If $x \in \mathcal{J}_{\Omega}^{em}(K \cup G)$, then $x \in \mathcal{J}_{\Omega}^{em}(y)$ for some $y \in K \cup G$. So in this case, if $y \in K$, then $x \in \mathcal{J}_{\Omega}^{em}(y) \subseteq \Omega^{u}(K)$ and similarly if $y \in G$, then $x \in \mathcal{J}_{\Omega}^{em}(y) \subseteq \Omega^{u}(G)$. If $x \in \cap^{\Omega}(K \cup G)$, then $\mathcal{J}_{\Omega}^{em}(x) \cap \mathcal{J}_{\Omega}^{em}(y) \neq \emptyset$ for some $y \in K \cup G$. If $y \in K$, then $x \in \cap^{\Omega}(y) \subseteq \Omega^{u}(K)$ and if $y \in G$, then $x \in \cap^{\Omega}(y) \subseteq \Omega^{u}(G)$. Therefore, $\Omega^{u}(K \cup G) \subseteq \Omega^{u}(K) \cup \Omega^{u}(G)$, that is, $\Omega^{u}(K \cup G) = \Omega^{u}(K) \cup \Omega^{u}(G)$. □

Theorem 3. *Let $\Omega = (\nu(\Omega), \xi(\Omega))$ be any simple graph and $G, K \subseteq \nu(\Omega)$. Then, $\Omega^{u}(K \cap G) \subseteq \Omega^{u}(K) \cap \Omega^{u}(G)$.*

In the theorem above, it is no need for $\Omega^{u}(K \cap G) = \Omega^{u}(K) \cap \Omega^{u}(G)$; for example, if we have the path $\Omega = P : 1' \to 2' \to 3' \to 4' \to 5'$, then

$$\Omega^{u}(1') = \Omega^{u}(2') = \Omega^{u}(3') = \{1', 2', 3', 5'\} \text{ and } \Omega^{u}(4') = \Omega^{u}(4') = \{1', 3', 4', 5'\}.$$

If we take $K = \{1'\}$ and $G = \{4'\}$, then $\Omega^{u}(K \cap G) = \emptyset \neq \{1', 3', 5'\} = \Omega^{u}(K) \cap \Omega^{u}(G)$.

Theorem 4. *Let $\Omega = (\nu(\Omega), \xi(\Omega))$ be any simple graph without isolated vertices and Ω_{x}^{u} denote the intersection set of all upper a-neighbourhoods containing x for all $x \in \nu(\Omega)$. Then, the collection $\mathcal{B}_{\Omega}^{u} = \{\Omega_{x}^{u} : x \in \nu(\Omega)\}$ forms a basis of a topology on $\nu(\Omega)$.*

Proof. Since $\nu(\Omega)$ is a total set, it is clear that $\Omega_{x}^{u} \subseteq \nu(\Omega)$ for all $x \in \nu(\Omega)$. That is, $\cup_{x \in \nu(\Omega)} \Omega_{x}^{u} \subseteq \nu(\Omega)$. Since Ω is without isolated vertices, $\mathcal{J}_{\Omega}^{em}(x) \neq \emptyset$ for all $x \in \nu(\Omega)$ and hence $\mathcal{J}_{\Omega}^{em}(x) \cap \mathcal{J}_{\Omega}^{em}(x) = \mathcal{J}_{\Omega}^{em}(x) \neq \emptyset$ for all $x \in \nu(\Omega)$. Then, $x \in \cap^{\Omega}(x) \subseteq \Omega^{u}(x)$ for all $x \in \nu(\Omega)$. That is, $\nu(\Omega) \subseteq \cup_{x \in \nu(\Omega)} \Omega_{x}^{u}$. So we obtain that $\nu(\Omega) = \cup_{x \in \nu(\Omega)} \Omega_{x}^{u}$. Next, we prove that for any two elements $\Omega_{x}^{u}, \Omega_{y}^{u} \in \mathcal{B}_{\Omega}^{u}$, there is $M \subseteq \nu(\Omega)$ such that $\Omega_{x}^{u} \cap \Omega_{y}^{u} = \cup_{w \in M} \Omega_{w}^{u}$. Let Ω_{x}^{u} and Ω_{y}^{u} be any two elements in $\mathcal{B}_{\Omega}^{u}$. If $\Omega_{x}^{u} \cap \Omega_{y}^{u} = \emptyset$,

then take $M = \emptyset$ to indicate that the proof is complete. Let $\Omega^u_x \cap \Omega^u_y \neq \emptyset$. Then, there exists $w \in \nu(\Omega)$ such that $w \in \Omega^u_x$ and $w \in \Omega^u_y$. By $w \in \Omega^u_x$, we obtain $w \in \Omega^u(x')$ for all $x' \in \nu(\Omega)$ with $x \in \Omega^u(x')$ and by $w \in \Omega^u_y$ we obtain that $w \in \Omega^u(y')$ for all $y' \in \nu(\Omega)$ with $x \in \Omega^u(y')$. For any of the previous cases, we obtain that $w \in \cup_{w \in \Omega^u(x') \cap \Omega^u(y')} \Omega^u_w$. Take $M = \Omega^u(x') \cap \Omega^u(y') \subseteq \nu(\Omega)$ to obtain $\Omega^u_x \cap \Omega^u_y \subseteq \cup_{w \in M} \Omega^u_w$. On the other hand, let $\theta \in \cup_{w \in M} \Omega^u_w$, that is, $\theta \in \cup_{w \in \Omega^u(x') \cap \Omega^u(y')} \Omega^u_w$. Hence, $\theta \in \Omega^u_w$ for some $\theta \in \Omega^u(x') \cap \Omega^u(y')$. Then, $\theta \in \cap_{w \in \Omega^u(v)} \Omega^u(v)$ and this implies $\theta \in \Omega^u(v)$ for all $w \in \Omega^u(v)$. So we have $\theta \in \Omega^u(x') \cap \Omega^u(y')$. Hence, $\theta \in \cap_{x \in \Omega^u(x')} \Omega^u(x')$ and $\theta \in \cap_{y \in \Omega^u(y')} \Omega^u(y')$, that is, $\theta \in \Omega^u_x$ and $\theta \in \Omega^u_y$. Hence, $\cup_{w \in M} \Omega^u_w \subseteq \Omega^u_x \cap \Omega^u_y$.Therefore, $\Omega^u_x \cap \Omega^u_y = \cup_{w \in M} \Omega^u_w$. Hence, $\mathcal{B}^u_\Omega$ forms a basis of a topology on $\nu(\Omega)$. □

Remark 1. *In the class of a simple graph $\Omega = (\nu(\Omega), \xi(\Omega))$ without isolated vertices, the topology that has the basis $\mathcal{B}^u_\Omega$ in Theorem 4 is called an upper a-graphical topology of a graph Ω and denoted by T_{Ω^u}.*

Remark 2. *In the class of a simple graph $\Omega = (\nu(\Omega), \xi(\Omega))$ without isolated vertices, from the definitions of an upper a−neighbourhood system Ω^u and the basis $\mathcal{B}^u_\Omega$, we obtain that the family Ω^u forms a sub-basis for an upper a-graphical topological space $(\nu(\Omega), T_{\Omega^u})$.*

Example 1. *A graph $\Omega = (\nu(\Omega), \xi(\Omega))$, shown in Figure 1a, has an upper a-neighborhood system Ω^u, which is given by*

$$\Omega^u(1') = \Omega^u(2') = \nu(\Omega) \setminus \{5'\} \text{ and } \Omega^u(k') = \nu(\Omega)$$

for all $k = 3, 4, 5, 6, 7$. So the basis

$$\mathcal{B}^u_\Omega = \left\{ \Omega^u_{k'} = \nu(\Omega) \setminus \{5'\}, \Omega^u_{5'} = \nu(\Omega) : k = 1, 2, 3, 4, 6, 7 \right\}$$

induces the upper a-graphical topology

$$T_{\Omega^u} = \left\{ \emptyset, \nu(\Omega), \nu(\Omega) \setminus \{5'\} \right\}.$$

In Figure 1b, if n is odd number greater than 3, then the graph $\Omega = (\nu(\Omega), \xi(\Omega))$ has an upper a-neighborhood system Ω^u, which is given by

$$\Omega^u(k') = \Omega^u((\frac{n+1}{2})'') = \nu(\Omega) \setminus \{(\frac{n+1}{2})'\}$$

and $\Omega^u(k'') = \Omega^u((\frac{n+1}{2})') = \nu(\Omega) \setminus \{(\frac{n+1}{2})''\}$ for all $k = 1, 2, \cdots, \frac{n}{2} - \frac{1}{2}, \frac{n}{2} + \frac{1}{2}, \cdots, n$. So the basis

$$\mathcal{B}^u_\Omega = \Big\{ \Omega^u(k') = \nu(\Omega) \setminus \{(\frac{n+1}{2})'\}, \Omega^u((\frac{n+1}{2})'') = \nu(\Omega) \setminus \{(\frac{n+1}{2})'\}, \Omega^u(k'')$$

$$= \nu(\Omega) \setminus \{(\frac{n+1}{2})''\}, \Omega^u((\frac{n+1}{2})') = \nu(\Omega) \setminus \{(\frac{n+1}{2})''\} :$$

$$k = 1, 2, \cdots, \frac{n}{2} - \frac{1}{2}', \frac{n}{2} + \frac{1}{2}, \cdots, n \Big\}$$

induces the upper a-graphical topology

$$T_{\Omega^u} = \left\{ \emptyset, \nu(\Omega), \nu(\Omega) \setminus \{(\frac{n+1}{2})'\}, \nu(\Omega) \setminus \{(\frac{n+1}{2})''\}, \nu(\Omega) \setminus \{(\frac{n+1}{2})', (\frac{n+1}{2})''\} \right\}.$$

Let $\Omega = (\nu(\Omega), \xi(\Omega))$ be any simple graph. If $\Omega_{xy} \in \xi(\Omega)$ is an isolated edge, then we have $\Omega^u(x) = \Omega^u(y) = \{x, y\}$ in the basis of the upper *a*-graphical topological space $(\nu(\Omega), T_{\Omega^u})$. If we have a graph as a path P_3 of the form $P_3 = \{\Omega_{1'2'}, \Omega_{2'3'}\}$, then P_3 has an upper *a*-neighborhood system P_3^u, which is given by $P_3^u(1') = P_3^u(2') = P_3^u(3') = \{1', 2', 3'\}$ in the upper *a*-graphical topological space $(\nu(P_3), T_{P_3^u})$.

Theorem 5. *Let P_n be a path of the form*

$$P_n = \{P_{1'2'}, P_{2'3'}, \cdots, P_{(n-2)'(n-1)'}, P_{(n-1)'n'}\},$$

where $n > 2$. If n is even, then the basis of $(\nu(P_n), T_{P_n^u})$ is given by

$$\mathcal{B}_{P_n}^u = \Big\{\{1', n'\}, \{n', k' : k \leq \frac{n}{2}\}, \{1', k' : k > \frac{n}{2}\}\Big\}.$$

If n is odd, then the basis of $(\nu(P_n), T_{P_n^u})$ is given by

$$\mathcal{B}_{P_n}^u = \Big\{\{1', (\frac{n+1}{2})', n'\}, \{n', k' : k \leq \frac{n+1}{2}\}, \{1', k' : k \geq \frac{n+1}{2}\}\Big\}.$$

Proof. In this case, n is even. Note that for all $k \leq \frac{n}{2}$, $\mathcal{J}_{P_n}^{em}(k') = \{n'\}$ and $\cap^{P_n}(k') = \{j' : j \leq \frac{n}{2}\}$. This is similar for all $k > \frac{n}{2}$, where $\mathcal{J}_{P_n}^{em}(k') = \{1'\}$ and $\cap^{P_n}(k') = \{j' : j > \frac{n}{2}\}$. So the upper *a*-neighborhood system P_n^u is given by $P_n^u(k') = \{n', j' : j \leq \frac{n}{2}\}$ for all $k \leq \frac{n}{2}$ and $P_n^u(k') = \{1', j' : j > \frac{n}{2}\}$ for all $k > \frac{n}{2}$. If $k = 1, n$, then

$$(P_n)_{k'}^u = \Big\{n', j' : j \leq \frac{n}{2}\} \cap \{1', j' : j > \frac{n}{2}\Big\} = \{1', n'\}.$$

For $1 < k \leq \frac{n}{2}$, $(P_n)_{k'}^u = \{n', j' : j \leq \frac{n}{2}\}$. For $\frac{n}{2} < k < n$, $(P_n)_{k'}^u = \{1', j' : j > \frac{n}{2}\}$. Hence, the basis $\mathcal{B}_{P_n}^u$ is given by

$$\mathcal{B}_{P_n}^u = \Big\{\{1', n'\}, \{n', k' : k \leq \frac{n}{2}\}, \{1', k' : k > \frac{n}{2}\}\Big\}.$$

In this case, n is odd. Note that for all $k < \frac{n+1}{2}$, $\mathcal{J}_{P_n}^{em}(k') = \{n'\}$ and $\cap^{P_n}(k') = \{j' : j \leq \frac{n+1}{2}\}$. For all $k > \frac{n+1}{2}$, $\mathcal{J}_{P_n}^{em}(k') = \{1'\}$ and $\cap^{P_n}(k') = \{j' : j > \frac{n+1}{2}\}$. At $k = \frac{n+1}{2}$, we obtain that $\mathcal{J}_{P_n}^{em}((\frac{n+1}{2})') = \{1', n'\}$ and $\cap^{P_n}((\frac{n+1}{2})') = \nu(P_n)$. So the upper *a*-neighborhood system P_n^u is given by $P_n^u(k') = \{n', j' : j \leq \frac{n+1}{2}\}$ for all $k < \frac{n+1}{2}$, $P_n^u(k') = \{1', j' : j \geq \frac{n+1}{2}\}$ for all $k > \frac{n}{2}$ and $P_n^u((\frac{n+1}{2})') = \nu(P_n)$. If $k = 1, \frac{n+1}{2}, n$, then

$$(P_n)_{k'}^u = \{n', j' : j \leq \frac{n+1}{2}\} \cap \{1', j' : j \geq \frac{n+1}{2}\} \cap \{1', (\frac{n+1}{2})', n'\} = \{1', (\frac{n+1}{2})', n'\}.$$

For $1 < k \leq \frac{n+1}{2}$, $(P_n)_{k'}^u = \{n', j' : j \leq \frac{n+1}{2}\}$. For $\frac{n+1}{2} < k < n$, $(P_n)_{k'}^u = \{1', j' : j \geq \frac{n+1}{2}\}$. Then the basis $\mathcal{B}_{P_n}^u$ is given by

$$\mathcal{B}_{P_n}^u = \Big\{\{1', (\frac{n+1}{2})', n'\}, \{n', k' : k \leq \frac{n+1}{2}\}, \{1', k' : k \geq \frac{n+1}{2}\}\Big\}.$$

□

For a cycle graph $C_3 : 1' \to 2' \to 3' \to 1'$, the upper *a*-graphical topological space $(\nu(C_3), T_{C_3^u})$ is indiscrete, with an upper *a*-neighborhood system $C_3^u(1') = C_3^u(2') = C_3^u(3') = \{1', 2', 3'\}$. For a cycle graph $C_4 : 1' \to 2' \to 3' \to 4' \to 1'$, the upper *a*-graphical topological space $(\nu(C_4), T_{C_4^u})$ is quasi-discrete and has an upper *a*-neighborhood system $C_4^u(1') = C_4^u(3') = \{1', 3'\}$ and $C_4^u(2') = C_4^u(4') = \{2', 4'\}$. For any $x \in \nu(C_n)$ in the cycle graph C_n, it is clear to see that $e_\Omega^m(x) = n - 2$.

Lemma 1. *The upper a-neighborhood system C_n^u of a cycle graph $C_n : 1' \to 2' \to \cdots \to n' \to 1'$ with $n \geq 5$ is given by*

$$C_n^u(k') = \begin{cases} \{(n-4+k)', (n-2+k)', k', (k+2)', (k+4)'\}, & for \quad k = 1, 2; \\ \{(n-4+k)', (k-2)', k', (k+2)', (k+4)'\}, & for \quad k = 3, 4; \\ \{(k-4)', (k-2)', k', (k+2)', (k-n+4)'\}, & for \quad k = n-3, n-2; \\ \{(k-4)', (k-2)', k', (k-n+2)', (k-n+4)'\}, & for \quad k = n-1, n; \\ \{(k-4)', (k-2)', k', (k+2)', (k+4)'\}, & for \quad 5 \leq k \leq n-4. \end{cases}$$

Proof. From Table 1, we can obtain the upper *a*-neighborhood system C_n^u, which is given in Table 2. □

Table 1. Monophonic *a*-neighborhood and operator $\cap^{C_n}$ of C_n.

x	$\mathcal{J}_{C_n}^{em}(x)$	$\cap^{C_n}(x)$
$1'$	$\{(n-1)', 3'\}$	$\{(n-3)', 1', 5'\}$
$2'$	$\{n', 4'\}$	$\{(n-2)', 2', 6'\}$
$3'$	$\{1', 5'\}$	$\{(n-1)', 3', 7'\}$
$4'$	$\{2', 6'\}$	$\{n', 4', 8'\}$
$5' \leq k' \leq (n-4)'$	$\{(k-2)', (k+2)'\}$	$\{(k-4)', k', (k+4)'\}$
$(n-3)'$	$\{(n-5)', (n-1)'\}$	$\{(n-7)', (n-3)', 1'\}$
$(n-2)'$	$\{(n-4)', n'\}$	$\{(n-6)', (n-2)', 2'\}$
$(n-1)'$	$\{(n-3)', 1'\}$	$\{(n-5)', (n-1)', 3'\}$
n'	$\{(n-2)', 2'\}$	$\{(n-4)', n', 4'\}$

Table 2. An upper *a*-neighborhood system of C_n.

x	$C_n^u(x)$
$1'$	$\{(n-3)', (n-1)', 1', 3', 5'\}$
$2'$	$\{(n-2)', n', 2', 4', 6'\}$
$3'$	$\{(n-1)', 1', 3', 5', 7'\}$
$4'$	$\{n', 2', 4', 6', 8'\}$
$5' \leq k' \leq (n-4)'$	$\{(k-4)', (k-2)', k', (k+2)', (k+4)'\}$
$(n-3)'$	$\{(n-7)', (n-5)', (n-3)', (n-1)', 1'\}$
$(n-2)'$	$\{(n-6)', (n-4)', (n-2)', n', 2'\}$
$(n-1)'$	$\{(n-5)', (n-3)', (n-1)', 1', 3'\}$
n'	$\{(n-4)', (n-2)', n', 2', 4'\}$

Theorem 6. *The upper a-graphical topological space $(\nu(C_n), T_{C_n^u})$ of a cycle graph C_n is a discrete space for all $n \geq 5$.*

Proof. From Lemma 1 and Table 3, we obtain that for all $x \in \nu(C_n)$, $(C_n)_x^u = \{x\}$. That is, $(\nu(C_n), T_{C_n^u})$ is a discrete space. □

Theorem 7. *The upper a-graphical topological space $(\nu(K_n), T_{K_n^u})$ of a complete graph K_n is an indiscrete space for all $n > 0$.*

Proof. For any $x \in \nu(K_n)$, $\mathcal{J}_{K_n}^{em}(x) = \nu(K_n) \setminus \{x\}$ and $\cap^{K_n}(x) = \nu(K_n)$. Hence, the upper *a*-neighborhood system K_n^u is given by

$$K_n^u(x) = \mathcal{J}_{K_n}^{em}(x) \cup \cap^{K_n}(x) = \nu(K_n) \setminus \{x\} \cup \nu(K_n) = \nu(K_n)$$

for all $x \in \nu(K_n)$. That is, $(\nu(K_n), T_{K_n^u})$ is an indiscrete space. □

If $n = 1$ and $m \geq 1$, then the upper a-neighborhood system $K^u_{n,m}$ is given by $K^u_{n,m}(x) = \nu(K_{n,m})$ for all $x \in \nu(K_{n,m})$. That is, the upper a-graphical topological space $(\nu(K_{n,m}), T_{K^u_{n,m}})$ is indiscrete.

Table 3. The basis $\mathcal{B}^u_{C_n}$ of C_n.

x	$(C_n)^u_x$
$1'$	$C^u_n((n-3)') \cap C^u_n((n-1)') \cap C^u_n(1') \cap C^u_n(3') \cap C^u_n(5') = \{1'\}$
$2'$	$C^u_n((n-2)') \cap C^u_n(n') \cap C^u_n(2') \cap C^u_n(4') \cap C^u_n(6') = \{2'\}$
$3'$	$C^u_n((n-1)') \cap C^u_n(1') \cap C^u_n(3') \cap C^u_n(5') \cap C^u_n(7') = \{3'\}$
$4'$	$C^u_n(n') \cap C^u_n(2') \cap C^u_n(4') \cap C^u_n(6') \cap C^u_n(8') = \{4'\}$
$\vdots$	$\vdots$
k'	$C^u_n((k-4)') \cap C^u_n((k-2)') \cap C^u_n(k') \cap C^u_n((k+2)') \cap C^u_n((k+4)') = \{k'\}$
$\vdots$	$\vdots$
$(n-3)'$	$C^u_n((n-7)' \cap C^u_n((n-5)') \cap C^u_n((n-3)') \cap C^u_n((n-1)') \cap C^u_n(1') = \{(n-3)'\}$
$(n-2)'$	$C^u_n((n-6)') \cap C^u_n((n-4)') \cap C^u_n((n-2)') \cap C^u_n(n') \cap C^u_n(2') = \{(n-2)'\}$
$(n-1)'$	$C^u_n((n-5)') \cap C^u_n((n-3)') \cap C^u_n((n-1)') \cap C^u_n(1') \cap C^u_n(3') = \{(n-1)'\}$
n'	$C^u_n((n-4)') \cap C^u_n((n-2)') \cap C^u_n(n') \cap C^u_n(2') \cap C^u_n(4') = \{n'\}$

Theorem 8. *The upper a-graphical topological space $(\nu(K_{n,m}), T_{K^u_{n,m}})$ of a complete bipartite graph $K_{n,m}$ is a quasi-discrete space for all $n, m > 1$.*

Proof. Let $\nu(K_{n,m}) = \nu_n \cup \nu_m$ and $x \in \nu(K_{n,m})$ be any vertex. Since $\nu_n \cap \nu_m = \emptyset$, $x \in \nu_n$ or $x \in \nu_m$. Let $x \in \nu_n$. Here, $n, m > 1$ and by the notion of a complete bipartite graph $K_{n,m}$, $\mathcal{J}^{em}_{K_{n,m}}(x) = \nu_n \setminus \{x\}$ and $\cap^{K_{n,m}}(x) = \nu_n$. Similarly, if $x \in \nu_m$, then $\mathcal{J}^{em}_{K_{n,m}}(x) = \nu_m \setminus \{x\}$ and $\cap^{K_{n,m}}(x) = \nu_m$. Hence, the upper a-neighborhood system $K^u_{n,m}$ is given by

$$K^u_{n,m}(x) = \mathcal{J}^{em}_{K_{n,m}}(x) \cup \cap^{K_{n,m}}(x) = \nu_n \setminus \{x\} \cup \nu_n = \nu_n$$

for all $x \in \nu_n$ and

$$K^u_{n,m}(x) = \mathcal{J}^{em}_{K_{n,m}}(x) \cup \cap^{K_{n,m}}(x) = \nu_m \setminus \{x\} \cup \nu_m = \nu_m$$

for all $x \in \nu_m$. Hence, the upper a-graphical topological space $(\nu(K_{n,m}), T_{K^u_{n,m}})$ is given by $T_{K^u_{n,m}} = \{\emptyset, \nu(K_{n,m}), \nu_n, \nu_m\}$. That is, $(\nu(K_{n,m}), T_{K^u_{n,m}})$ is a quasi-discrete space. □

3. On Topological Properties

For compactness, note that the upper a-graphical topological space $(\nu(\Omega), T_{\Omega^u})$ of any simple graph $\Omega = (\nu(\Omega), \xi(\Omega))$ is a compact space if $\nu(\Omega)$ is finite, while for the infinite graph there is no need to induce a compact upper a-graphical topological space. For example, in Figure 2, the simple graph $\Omega = (\nu(\Omega), \xi(\Omega))$ has the infinite vertex set $\nu(\Omega) = \{0', 1', 2', 3', \cdots\}$. For $x \in \nu(\Omega)$, $\mathcal{J}^{em}_{\Omega}(k') = \nu(\Omega) \setminus \{k'\}$ and $\cap^{\Omega}(k') = \nu(\Omega)\}$ for all $k' \in \nu(\Omega)$. So the upper a-neighborhood system Ω^u is given by $\Omega^u(k') = \nu(\Omega)$ for all $k' \in \nu(\Omega)$. That is, the upper a-graphical topological space $(\nu(\Omega), T_{\Omega^u})$ is indiscrete

and hence is a compact space. Let $\Omega_1 = (\nu(\Omega_1), \xi(\Omega_1))$ and $\Omega_2 = (\nu(\Omega_2), \xi(\Omega_2))$ be two simple graphs without isolated vertices. These two graphs Ω_1 and Ω_2 are called isomorphic and we write $\Omega_1 \cong \Omega_2$ if there is a bijective function $\zeta : \nu(\Omega_1) \to \nu(\Omega_2)$ such that $(\Omega_1)_{xy} \in \xi(\Omega_1)$ if and only if $(\Omega_2)_{\zeta(x)\zeta(y)} \in \xi(\Omega_2)$ for all $x, y \in \nu(\Omega_1)$. A function $h : (Y_1, \tau_1) \to (Y_2, \tau_2)$ of a topological space (Y_1, τ_1) into a topological space (Y_2, τ_2) is called continuous if $h^{-1}(O)$ is an open set in (Y_1, τ_1) for every open set O in (Y_2, τ_2). A function $h : (Y_1, \tau_1) \to (Y_2, \tau_2)$ is called an open function if $h(O)$ is an open set in Y_2 for every open set $O \subseteq Y_1$. Recall from [24] that a function $h : (Y_1, \tau_1) \to (Y_2, \tau_2)$ is a homeomorphism if it is a bijective, open and continuous function.

Remark 3. *From Theorem 7, since the upper a-graphical topological space $(\nu(K_n), T_{K_n^u})$ of a complete graph K_n is an indiscrete space for all $n > 0$, any function $\zeta : (\nu(\Omega), T_{\Omega^u}) \to (\nu(K_n), T_{K_n^u})$ is continuous for any simple graph $\Omega = (\nu(\Omega), \xi(\Omega))$ without isolated vertices. By Theorem 6, since the upper a-graphical topological space $(\nu(C_n), T_{C_n^u})$ of a cycle graph C_n is a discrete space for all $n \geq 5$, any function $\zeta : (\nu(C_n), T_{C_n^u}) \to (\nu(\Omega), T_{\Omega^u})$ is continuous for any simple graph $\Omega = (\nu(\Omega), \xi(\Omega))$ without isolated vertices.*

Remark 4. *By Theorem 4, we note that for any simple graph $\Omega = (\nu(\Omega), \xi(\Omega))$ without isolated vertices and for $a \in \nu(\Omega)$, Ω_a^u is the smallest open neighborhood of a in $(\nu(\Omega), T_{\Omega^u})$.*

Lemma 2. *Let $\Omega_1 = (\nu(\Omega_1), \xi(\Omega_1))$ and $\Omega_2 = (\nu(\Omega_2), \xi(\Omega_2))$ be two simple graphs without isolated vertices. A function $\zeta : (\nu(\Omega_1), T_{\Omega_1^u}) \to (\nu(\Omega_2), T_{\Omega_2^u})$ is an open function if and only if $(\Omega_2)^u_{\zeta(x)} \subseteq \zeta[(\Omega_1)^u_x]$ for all $x \in \nu(\Omega_1)$.*

Proof. Let ζ be an open function and $x \in \nu(\Omega_1)$ be any vertex in $\nu(\Omega_1)$. Since $(\Omega_1)^u_x$ is an open set in $(\nu(\Omega_1), T_{\Omega_1^u})$ containing x and ζ is open, $\zeta[(\Omega_1)^u_x]$ is an open set in $(\nu(\Omega_2), T_{\Omega_2^u})$ containing $\zeta(x)$. By the definition of $(\Omega_2)^u_{\zeta(x)}$, we obtain $(\Omega_2)^u_{\zeta(x)} \subseteq \zeta[(\Omega_1)^u_x]$. Now, suppose that $(\Omega_2)^u_{\zeta(x)} \subseteq \zeta[(\Omega_1)^u_x]$ for all $x \in \nu(\Omega_1)$. Let O be any open set in $(\nu(\Omega_1), T_{\Omega_1^u})$ and $y \in \zeta(O)$. Then, $\zeta^{-1}(y) \subseteq O$. Then, there is $B \in \mathcal{B}^u_{\Omega_1}$ such that $\zeta^{-1}(y) \subseteq B \subseteq O$. Since B is an open set containing $\zeta^{-1}(y)$ and by the definition of $(\Omega_1)^u_{\zeta^{-1}(y)}$, $\zeta^{-1}(y) \subseteq (\Omega_1)^u_{\zeta^{-1}(y)} \subseteq B \subseteq O$. By this hypothesis, we obtain that $(\Omega_2)^u_{\zeta(\zeta^{-1}(y))} \subseteq \zeta[(\Omega_1)^u_{\zeta^{-1}(y)}] \subseteq \zeta(B) \subseteq \zeta(O)$, that is, $(\Omega_2)^u_y \subseteq \zeta^{-1}[(\Omega_2)^u_{\zeta^{-1}(y)}] \subseteq \zeta(O)$. Since $(\Omega_2)^u_y$ is an open set and y is arbitrary, $\zeta(O)$ is an open set. Hence, ζ is an open function. □

Lemma 3. *Let $\Omega_1 = (\nu(\Omega_1), \xi(\Omega_1))$ and $\Omega_2 = (\nu(\Omega_2), \xi(\Omega_2))$ be two simple graphs without isolated vertices. A function $\zeta : (\nu(\Omega_1), T_{\Omega_1^u}) \to (\nu(\Omega_2), T_{\Omega_2^u})$ is continuous if and only if $\zeta[(\Omega_1)^u_x] \subseteq (\Omega_2)^u_{\zeta(x)}$ for all $x \in \nu(\Omega_1)$.*

Proof. Let ζ be continuous and $x \in \nu(\Omega_1)$ be any vertex in $\nu(\Omega_1)$. Since $(\Omega_2)^u_{\zeta(x)}$ is an open set in $(\nu(\Omega_2), T_{\Omega_2^u})$ containing $\zeta(x)$ and ζ is continuous, $\zeta^{-1}[(\Omega_2)^u_{\zeta(x)}]$ is an open set in $(\nu(\Omega_1), T_{\Omega_1^u})$ containing x. By the definition of $(\Omega_1)^u_x$, we obtain $(\Omega_1)^u_x \subseteq \zeta^{-1}[(\Omega_2)^u_{\zeta(x)}]$, that is, $\zeta[(\Omega_1)^u_x] \subseteq (\Omega_2)^u_{\zeta(x)}$. Conversely, suppose that $\zeta[(\Omega_1)^u_x] \subseteq (\Omega_2)^u_{\zeta(x)}$ for all $x \in \nu(\Omega_1)$. We prove that ζ is continuous. Let O be any open set in $(\nu(\Omega_2), T_{\Omega_2^u})$ and $x \in \zeta^{-1}(O)$. Then, $\zeta(x) \in O$. Then, there is $B \in \mathcal{B}^u_{\Omega_2}$ such that $\zeta(x) \in B \subseteq O$. Since B is an open set containing $\zeta(x)$ and by the definition of $(\Omega_2)^u_{\zeta(x)}$, $\zeta(x) \in (\Omega_2)^u_{\zeta(x)} \subseteq B \subseteq O$. By this hypothesis, we obtain that $\zeta[(\Omega_1)^u_x] \subseteq (\Omega_2)^u_{\zeta(x)} \subseteq B \subseteq O$, that is, $(\Omega_1)^u_x \subseteq \zeta^{-1}[(\Omega_2)^u_{\zeta(x)}] \subseteq \zeta^{-1}(O)$. Since $(\Omega_1)^u_x$ is an open set, $\zeta^{-1}(O)$ is an open set. Hence, ζ is continuous. □

The proof of the following theorem is clear from Lemmas 4 and 2.

Theorem 9. *Let $\Omega_1 = (\nu(\Omega_1), \xi(\Omega_1))$ and $\Omega_2 = (\nu(\Omega_2), \xi(\Omega_2))$ be two simple graphs without isolated vertices. A bijective function $\zeta : (\nu(\Omega_1), T_{\Omega_1^u}) \to (\nu(\Omega_2), T_{\Omega_2^u})$ is a homeomorphism if and only if $\zeta[(\Omega_1)_x^u] = (\Omega_2)_{\zeta(x)}^u$ for all $x \in \nu(\Omega_1)$.*

Theorem 10. *Let $\Omega_1 = (\nu(\Omega_1), \xi(\Omega_1))$ and $\Omega_2 = (\nu(\Omega_2), \xi(\Omega_2))$ be two simple graphs without isolated vertices. If Ω_1 and Ω_2 are isomorphic, then $(\nu(\Omega_1), T_{\Omega_1^u})$ and $(\nu(\Omega_2), T_{\Omega_2^u})$ are homeomorphic.*

Proof. Since Ω_1 and Ω_2 are isomorphic, there is a bijective function $\zeta : \nu(\Omega_1) \to \nu(\Omega_2)$ such that $(\Omega_1)_{xy} \in \xi(\Omega_1)$ if and only if $(\Omega_2)_{\zeta(x)\zeta(y)} \in \xi(\Omega_2)$ for all $x, y \in \nu(\Omega_1)$. By Theorem 9, it is enough to prove that $\zeta[(\Omega_1)_x^u] = (\Omega_2)_{\zeta(x)}^u$ for all $x \in \nu(\Omega_1)$. Let $x \in \nu(\Omega_1)$ be any vertex and $y \in \zeta[(\Omega_1)_x^u]$. Since ζ is a injective, there is only one vertex $z \in (\Omega_1)_x^u$ such that $y = \zeta(z)$. Hence, $z \in (\Omega_1)^u(\alpha)$ for all $x \in (\Omega_1)^u(\alpha)$, that is, $z \in \mathcal{J}_{\Omega_1}^{em}(\alpha)$ or $\mathcal{J}_{\Omega_1}^{em}(z) \cap \mathcal{J}_{\Omega_1}^{em}(\alpha) \neq \emptyset$ for all $x \in (\Omega_1)^u(\alpha)$. By the condition of Ω_1 and Ω_2 being isomorphic, we obtain that $y = \zeta(z) \in \mathcal{J}_{\Omega_2}^{em}(\zeta(\alpha))$ or $\mathcal{J}_{\Omega_2}^{em}(\zeta(z)) \cap \mathcal{J}_{\Omega_2}^{em}(\zeta(\alpha)) \neq \emptyset$ for all $\zeta(x) \in (\Omega_2)^u(\zeta(\alpha))$. That is, $y \in (\Omega_2)_{\zeta(x)}^u$ and hence $\zeta[(\Omega_1)_x^u] \subseteq (\Omega_2)_{\zeta(x)}^u$. For the other side, let $y \in (\Omega_2)_{\zeta(x)}^u$. Since ζ is a injective, there is only one vertex $z \in (\Omega_1)_x^u$ such that $y = \zeta(z)$. Hence, $y \in (\Omega_2)^u(\beta)$ for all $\zeta(x) \in (\Omega_2)^u(\beta)$, that is, $y \in \mathcal{J}_{\Omega_2}^{em}(\beta)$ or $\mathcal{J}_{\Omega_2}^{em}(y) \cap \mathcal{J}_{\Omega_2}^{em}(\beta) \neq \emptyset$ for all $\zeta(x) \in (\Omega_2)^u(\beta)$. Since ζ is a injective, there is only one vertex $\theta \in (\Omega_1)_x^u$ such that $\beta = \zeta(\theta)$. By the condition of Ω_1 and Ω_2 being isomorphic, we obtain that $z \in \mathcal{J}_{\Omega_1}^{em}(\theta)$ or $\mathcal{J}_{\Omega_1}^{em}(z) \cap \mathcal{J}_{\Omega_1}^{em}(\theta) \neq \emptyset$ for all $x \in (\Omega_1)^u(\theta)$. That is, $z \in (\Omega_1)_x^u$ and hence $y = \zeta(z) \in \zeta[(\Omega_1)_x^u]$. So we obtain $(\Omega_2)_{\zeta(x)}^u \subseteq \zeta[(\Omega_1)_x^u]$. □

If we have the homeomorphic property between the upper *a*-graphical topological spaces, then we have no need to have the isomorphic property between their corresponding graphs. For example, in Figure 2, if we take $\nu(\Omega) = \{0', 1', 2', \cdots, n'\}$, then the upper *a*-graphical topological space $(\nu(\Omega), T_{\Omega^u})$ is still indiscrete and $|\nu(\Omega)| = n + 1$. From Theorem 7, the upper *a*-graphical topological space $(\nu(K_{n+1}), T_{K_{n+1}^u})$ is discrete. So $(\nu(\Omega), T_{\Omega^u})$ and $(\nu(K_{n+1}), T_{K_{n+1}^u})$ are homeomorphic while Ω and K_{n+1} are not isomorphic.

For the connectedness properties of graphs and the upper *a*-graphical topological spaces, the following theorem shows the relationship between those of simple graphs.

Theorem 11. *Let $\Omega = (\nu(\Omega), \xi(\Omega))$ be any simple graph that has no isolated vertices. If $(\nu(\Omega), T_{\Omega^u})$ is a connected space, then Ω is a connected graph.*

Proof. Let $\Omega = (\nu(\Omega), \xi(\Omega))$ be a disconnected simple graph. Hence, take $\mathcal{I} := \{\mathcal{I}_m : m \in M\}$ as the family of all components in Ω, where $\mathcal{I}_m = (\nu(\mathcal{I}_m), \xi(\mathcal{I}_m))$ for all $m \in M$. Now, for all $m \in M$, $\nu(\mathcal{I}_m) = \cup_{x \in \nu(\mathcal{I}_m)} \Omega^u(x)$. Then, $M := \nu(\mathcal{I}_{m_o}) \neq \emptyset$ is a proper open subset of $\nu(\Omega)$ where $m_o \in M$. Then, $[\nu(\mathcal{I}_m)]^c = \cup_{m \in M \setminus \{m_o\}} \nu(\mathcal{I}_m) \neq \emptyset$ is also a proper open subset of $\nu(\Omega)$. That is, $(\nu(\Omega), T_{\Omega^u})$ is a disconnected space and this is a contradiction with the connectedness of $(\nu(\Omega), T_{\Omega^u})$. Hence, Ω is a connected graph. □

The converse of the theorem above does not need to be true; for example, by Theorem 6, the upper *a*-graphical topological space $(\nu(C_n), T_{C_n^u})$ is discrete and so disconnected, while the cycle graph C_n is a connected graph.

Let $\Omega = (\nu(\Omega), \xi(\Omega))$ be any simple graph. Define a subgraph $\mathcal{I}_\Omega$ of Ω by the subgraph of $\mathcal{I}_\Omega$ with vertex set $\nu(\mathcal{I}_\Omega)$, which is given as the subset of $\nu(\Omega)$ containing all vertices x with $|\cap^\Omega(x)| \leq 2$. A simple graph $\Omega = (\nu(\Omega), \xi(\Omega))$ is called an $\cap$-upper connected graph if the subgraph $\mathcal{I}_\Omega$ of Ω is connected. If the relative topology $T_{\Omega^u}|_{\nu(\mathcal{I}_\Omega)}$ is discrete on a set $\nu(\mathcal{I}_\Omega)$, then the upper *a*-graphical topological space $(\nu(\Omega), T_{\Omega^u})$ is called $\cap$-upper discrete. If $\nu(\mathcal{I}_\Omega) = \emptyset$, then we assume Ω is $\cap$-upper connected and $(\nu(\Omega), T_{\Omega^u})$ is $\cap$-upper discrete. Recall from Theorem 5 that, in the path P_n, if $2 \leq n \leq 4$,

then $\nu(\mathcal{I}_{P_n}) = \nu(P_n)$. Hence, P_n is an $\cap$-upper connected graph and the upper a-graphical topological space $(\nu(P_n), T_{P_n^u})$ is not $\cap$-upper discrete. If $n \geq 5$, then $\nu(\mathcal{I}_{P_n}) = \emptyset$. Hence, P_n is an $\cap$-upper connected graph and the upper a-graphical topological space $(\nu(P_n), T_{P_n^u})$ is also $\cap$-upper discrete. Recall from Theorem 6 that, in the cycle graph C_n, if $n = 3$ or $n \geq 5$, then $\nu(\mathcal{I}_{C_n}) = \emptyset$. Hence, C_n is an $\cap$-upper connected graph and the upper a-graphical topological space $(\nu(C_n), T_{C_n^u})$ is $\cap$-upper discrete. If $n = 4$, then $\nu(\mathcal{I}_{C_n}) = \nu(C_n)$. Hence, C_n is an $\cap$-upper connected graph and the upper a-graphical topological space $(\nu(C_n), T_{C_n^u})$ is not $\cap$-upper discrete. Recall from Theorem 6 that, in the complete graph K_n, if $n \leq 2$, then $\nu(\mathcal{I}_{K_n}) = \nu(K_n)$. Hence, K_n is an $\cap$-upper connected graph and the upper a-graphical topological space $(\nu(K_n), T_{K_n^u})$ is not $\cap$-upper discrete. If $n \geq 3$, then $\nu(\mathcal{I}_{K_n}) = \emptyset$. Hence, K_n is an $\cap$-upper connected graph and the upper a-graphical topological space $(\nu(C_n), T_{K_n^u})$ is $\cap$-upper discrete. Recall from Theorem 8 that, in the complete bipartite graph $K_{n,m}$ with $\nu(K_{n,m}) = \nu_n \cap \nu_m$, if $n = 1$ and $m \leq 1$, then $\nu(\mathcal{I}_{K_{n,m}}) = \nu(K_{n,m})$. Hence, $K_{n,m}$ is an $\cap$-upper connected graph and the upper a-graphical topological space $(\nu(K_{n,m}), T_{K_{n,m}^u})$ is not $\cap$-upper discrete. If $n = 2$ and $m = 2, 3$, then $\nu(\mathcal{I}_{K_{n,m}}) = \nu(K_{n,m})$. Hence, $K_{n,m}$ is an $\cap$-upper connected graph and the upper a-graphical topological space $(\nu(K_{n,m}), T_{K_{n,m}^u})$ is not $\cap$-upper discrete. If $n = 2, 3$ and $m > 3$, then $\nu(\mathcal{I}_{K_{n,m}}) = \nu_2$. Hence, $K_{n,m}$ is not an $\cap$-upper connected graph and the upper a-graphical topological space $(\nu(K_{n,m}), T_{K_{n,m}^u})$ is not $\cap$-upper discrete. If $n, m > 3$, then $\nu(\mathcal{I}_{K_{n,m}}) = \emptyset$. Hence, $K_{n,m}$ is an $\cap$-upper connected graph and the upper a-graphical topological space $(\nu(K_{n,m}), T_{K_{n,m}^u})$ is $\cap$-upper discrete.

Lemma 4. *If two simple graphs without isolated vertices* $\Omega_1 = (\nu(\Omega_1), \xi(\Omega_1))$ *and* $\Omega_2 = (\nu(\Omega_2), \xi(\Omega_2))$ *are isomorphic by* $\zeta : \nu(\Omega_1) \to \nu(\Omega_2)$*, then* $\zeta[\cap^{\Omega_1}(x)] = \cap^{\Omega_2}(\zeta(x))$ *for all* $x \in \nu(\Omega_1)$*.*

Proof. Let $x \in \nu(\Omega_1)$ be any vertex and $\alpha \in \zeta[\cap^{\Omega_1}(x)]$. Since ζ is a injective, there is only one vertex $\alpha' \in \cap^{\Omega_1}(x)$ such that $\alpha = \zeta(\alpha')$. Since $\alpha' \in \cap^{\Omega_1}(x)$, $\mathcal{J}^{em}_{\Omega_1}(\alpha') \cap \mathcal{J}^{em}_{\Omega_1}(x) \neq \emptyset$, that is, there is at least $\theta \in \nu(\Omega_1)$ such that $\theta \in \mathcal{J}^{em}_{\Omega_1}(\alpha')$ and $\theta \in \mathcal{J}^{em}_{\Omega_1}(x)$. Hence, $\partial^m_{\Omega_1}(x, \theta) = e^m_{\Omega_1}(x)$ and $\partial^m_{\Omega_1}(\alpha', \theta) = e^m_{\Omega_1}(\alpha')$. Since the image monophonic path in Ω_1 under the isomorphism ζ is also a monophonic path in Ω_2, $\partial^m_{\Omega_2}(\zeta(x), \zeta(\theta)) = e^m_{\Omega_2}(\zeta(x))$ and $\partial^m_{\Omega_2}(\alpha, \zeta(\theta)) = e^m_{\Omega_2}(\alpha)$. Hence, $\zeta(\theta) \in \mathcal{J}^{em}_{\Omega_2}(\alpha)$ and $\zeta(\theta) \in \mathcal{J}^{em}_{\Omega_2}(\zeta(x))$. So we obtain $\mathcal{J}^{em}_{\Omega_2}(\alpha) \cap \mathcal{J}^{em}_{\Omega_2}(\zeta(x)) \neq \emptyset$, that is, $\alpha \in \cap^{\Omega_2}(\zeta(x))$. Hence, $\zeta[\cap^{\Omega_1}(x)] \subseteq \cap^{\Omega_2}(\zeta(x))$. For the other side, let $\beta \in \cap^{\Omega_2}(\zeta(x))$. Then, $\mathcal{J}^{em}_{\Omega_2}(\beta) \cap \mathcal{J}^{em}_{\Omega_2}(\zeta(x)) \neq \emptyset$, that is, there is at least $\lambda \in \nu(\Omega_2)$ such that $\lambda \in \mathcal{J}^{em}_{\Omega_2}(\beta)$ and $\lambda \in \mathcal{J}^{em}_{\Omega_1}(\zeta(x))$. Hence, $\partial^m_{\Omega_2}(\zeta(x), \lambda) = e^m_{\Omega_2}(\zeta(x))$ and $\partial^m_{\Omega_2}(\beta, \lambda) = e^m_{\Omega_2}(\beta)$. Since ζ is a bijective, there are two vertices $\beta', \lambda' \in \nu(\Omega_1)$ such that $\beta = \zeta(\beta')$ and $\lambda = \zeta(\lambda')$. Since the inverse image monophonic path in Ω_2 under the isomorphism ζ is also a monophonic path in Ω_1, $\partial^m_{\Omega_1}(x, \lambda')) = e^m_{\Omega_1}(x)$ and $\partial^m_{\Omega_1}(\beta', \lambda') = e^m_{\Omega_1}(\beta')$. Hence, $\lambda' \in \mathcal{J}^{em}_{\Omega_1}(\beta')$ and $\lambda' \in \mathcal{J}^{em}_{\Omega_1}(x)$. So we obtain $\mathcal{J}^{em}_{\Omega_1}(\beta') \cap \mathcal{J}^{em}_{\Omega_1}(x) \neq \emptyset$ and so $\beta' \in \cap^{\Omega_1}(x)$. Hence, $\beta = \zeta(\beta') \in \zeta[\cap^{\Omega_1}(x)]$, that is, $\cap^{\Omega_2}(\zeta(x)) \subseteq \zeta[\cap^{\Omega_1}(x)]$. □

Theorem 12. *In a class of simple graphs without isolated vertices,* $\cap$*-upper connectedness is an isomorphic property.*

Proof. Let $\Omega_1 = (\nu(\Omega_1), \xi(\Omega_1))$ and $\Omega_2 = (\nu(\Omega_2), \xi(\Omega_2))$ be any two simple graphs without isolated vertices, $\Omega_1 \cong \Omega_2$ and Ω_1 is $\cap$-upper connected. We prove that Ω_2 is $\cap$-upper connected. Since $\Omega_1 \cong \Omega_2$, there is a bijective function $\zeta : \nu(\Omega_1) \to \nu(\Omega_2)$ such that $(\Omega_1)_{xy} \in \xi(\Omega_1)$ if and only if $(\Omega_2)_{\zeta(x)\zeta(y)} \in \xi(\Omega_2)$ for all $x, y \in \nu(\Omega_1)$. Suppose that Ω_2 is not $\cap$-upper connected. Then, the subgraph $\mathcal{I}_{\Omega_2}$ is a disconnected graph; that is, there are at least two vertices $\alpha, \beta \in \nu(\mathcal{I}_{\Omega_2})$ such that there is no path between them in Ω_2. Since ζ is bijective, there are $\alpha', \beta' \in \nu(\Omega_1)$ such that $\alpha = \zeta(\alpha')$ and $\beta = \zeta(\beta')$. Since $\alpha, \beta \in \nu(\mathcal{I}_{\Omega_2})$, by Lemma 4, $\alpha', \beta' \in \nu(\mathcal{I}_{\Omega_1})$. Since α and β are not joined by a path in Ω_2, α'

and β' are also not joined by a path in Ω_1. Hence, $\mathcal{I}_{\Omega_1}$ is a disconnected graph; that is, Ω_1 is not $\cap$-upper connected and this is a contradiction. Therefore, Ω_1 is $\cap$-upper connected. □

Theorem 13. *In a class of simple graphs without isolated vertices, $\cap$-upper discreteness is a topological property.*

Proof. Let $\Omega_1 = (\nu(\Omega_1), \xi(\Omega_1))$ and $\Omega_2 = (\nu(\Omega_2), \xi(\Omega_2))$ be two simple graphs without isolated vertices and $(\nu(\Omega_1), T_{\Omega_1^u})$ be $\cap$-upper discrete. Let $\zeta : \nu(\Omega_1) \to \nu(\Omega_2)$ be a homeomorphism. We will prove that $(\nu(\Omega_2), T_{\Omega_2^u})$ is $\cap$-upper discrete. Let $\alpha \in \mathbb{D}_{\pm}(\Omega_2)$ be an arbitrary vertex. Since ζ is bijective, there is $\alpha' \in \mathbb{D}_{\pm}(\Omega_1)$ such that $\alpha = \zeta(\alpha')$. Since $(\nu(\Omega_1), T_{\Omega_1^u})$ is $\cap$-upper discrete, $\{\alpha'\}$ is an open set in a relative topological space $(\nu(\mathcal{I}_{\Omega_1}), T_{\Omega_1^u}|_{\nu(\mathcal{I}_{\Omega_1})})$. Since ζ is an open function and bijective, $\zeta(\{\alpha'\}) = \{\zeta(\alpha')\} = \{\alpha\}$ is an open set in a relative topological space $(\nu(\mathcal{I}_{\Omega_2}), T_{\Omega_2^u}|_{\nu(\mathcal{I}_{\Omega_2})})$. That is, $(\nu(\Omega_2), T_{\Omega_2^u})$ is $\cap$-upper discrete. □

4. On Graphs of COVID-19 Form and Its Diffusion

The novel coronavirus 2019 (COVID-19) appeared in December 2019 in Wuhan—the capital of Hubei, China. See in [20,21] that Figures 3 and 4a present the transformation of the virus in several countries in a short time. Many researchers studied the diffusion networks of COVID-19 and presented this diffusion through representation graphs and adopting the types of pandemic diffusion network dynamics, consisting of path length, local or global efficiency and clustering coefficient. In this section, we investigate and present the topological connectedness and $\cap$-upper discrete properties of the corresponding graphs of some diagrams concern COVID-19 diffusion networks, which are introduced in [20–22].

Recall [20] that some diagrams are a representation of pandemic diffusion graphs of COVID-19 in Kentucky, USA, and Kyoto, Japan. The graph $\Omega = (\nu(\Omega), \xi(\Omega))$ in Figure 3 with the vertex set $\nu(\Omega) = \{0', 1', 2', \cdots, n'\}$ corresponds to these diagrams. The upper a-neighborhood system Ω^u is given by

$$\mathcal{J}_{\Omega}^{em}(0') = \nu(\Omega) \setminus \{0'\},\ \mathcal{J}_{\Omega}^{em}(k') = \nu(\Omega) \setminus \{0', k'\},\ \cap^{\Omega}(j') = \nu(\Omega),$$

and $\Omega_{j'}^u = \nu(\Omega)$ for all $k = 1, 2, \cdots n$, $j = 0, 1, 2, \cdots n$. So the upper a-graphical topological space $(\nu(\Omega), T_{\Omega^u})$ is indiscrete and so connected. Since $n > 3$ and $\nu(\mathcal{I}_{\Omega}) = \emptyset$, Ω is $\cap$-upper connected and $(\nu(\Omega), T_{\Omega^u})$ is $\cap$-upper discrete.

Ashokkumar et al. [21] proposed an algorithm to table the timing of relief funds by using graph theory, Figure 4a, so that the Indian government had the ability to implement its relief scheme whilst taking into account social distancing. The graph $\Omega_1 = (\nu(\Omega_1), \xi(\Omega_1))$ in Figure 4b with the vertex set $\nu(\Omega_1) = \{1', 2', 3', 4', 5'\}$ is the corresponding graph of District(1) in the graph G_1 of Figure 4b. The upper a-neighborhood system Ω_1^u is given by

$$(\Omega_1)_{1'}^u = \{1'\},\ (\Omega_1)_{2'}^u = (\Omega_1)_{3'}^u = \{2', 3'\},\ (\Omega_1)_{4'}^u = \{1', 4'\},\ (\Omega_1)_{5'}^u = \{1', 5'\}.$$

Hence the upper a-graphical topological space $(\nu(\Omega_1), T_{\Omega_1^u})$ is disconnected by an open–closed set $\{2, 3'\}$. Since $\nu(\mathcal{I}_{\Omega_1}) = \nu(\Omega_1)$, Ω_1 is $\cap$-upper connected and $(\nu(\Omega), T_{\Omega^u})$ is not $\cap$-upper discrete. The graph $\Omega_2 = (\nu(\Omega_2), \xi(\Omega_2))$ in the graph G_2 of Figure 4c with the vertex set $\nu(\Omega_2) = \{1, 2, 3, 4, 5, 6, 7\}$ is the corresponding graph of District(2) in Figure 4a. The upper a-neighborhood system Ω_2^u is given by

$$(\Omega_2)_1^u = \{1, 2, 5, 6\},\ (\Omega_2)_2^u = \{2, 5, 6\},\ (\Omega_2)_3^u = \{1, 2, 3, 5, 6\},\ (\Omega_2)_4^u = \{2, 4, 5, 6\},$$

$$(\Omega_2)_5^u = (\Omega_2)_6^u = \{2, 5, 6\},\ (\Omega_2)_7^u = \{2, 4, 5, 6, 7\}.$$

That is, the upper a-graphical topological space $(\nu(\Omega_2), T_{\Omega_2^u})$ is connected. Since $\nu(\mathcal{I}_{\Omega_2}) = \emptyset$, Ω_2 is $\cap$-upper connected and $(\nu(\Omega), T_{\Omega^u})$ is $\cap$-upper discrete. The graph $\Omega_3 = (\nu(\Omega_3), \xi(\Omega_3))$ in the graph G_3 of Figure 4b with the vertex set $\nu(\Omega_3) = \{1'', 2'', 3'', 4'', 5'', 6'', 7''\}$ is the corresponding graph of District(3) in Figure 4a. The upper a-neighborhood system Ω_3^u is given by

$$(\Omega_3)^u_{1''} = (\Omega_3)^u_{7''} = \{1'', 2'', 7''\},\ (\Omega_3)^u_{2''} = \{2''\},\ (\Omega_3)^u_{3''} = (\Omega_3)^u_{6''} = \{3'', 6''\}$$

$$(\Omega_3)^u_{4''} = \{4''\},\ (\Omega_3)^u_{5''} = \{2'', 5''\}.$$

That is, the upper a-graphical topological space $(\nu(\Omega_3), T_{\Omega_3^u})$ is disconnected by an closed–open set $\{3'', 6''\}$. Since $\nu(\mathcal{I}_{\Omega_3}) = \{3'', 4'', 7''\}$, Ω_3 is $\cap$-upper connected and $(\nu(\Omega), T_{\Omega^u})$ is $\cap$-upper discrete. The total graph $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3 = (\nu(\Omega), \xi(\Omega))$ in the graph G of Figure 4 is the corresponding graph of all districts in Figure 4a. The upper a-neighborhood system Ω^u is given by Table 4, and we obtain that the upper a-graphical topological space $(\nu(\Omega), T_{\Omega^u})$ is disconnected by a closed–open set $\{1'\}$. Since $\nu(\mathcal{I}_\Omega) = \{5'', 6''\}$, Ω is $\cap$-upper connected and $(\nu(\Omega), T_{\Omega^u})$ is $\cap$-upper discrete.

Table 4. The basis $\mathcal{B}^u_\Omega$ of $\Omega = \Omega_1 \cup \Omega_2 \cup \Omega_3$.

x	$\cap_{x \in \Omega^u_k} \Omega^u_k$	Ω^u_x
1	$k = 1, 2, 3, 6, 7, 1', 4''$	$\{1\}$
2	$k \in \nu(\Omega) \setminus \{6, 7, 4'', 6'', 7''\}$	$\{2\}$
3	$k = 1, 2, 3, 4, 1', 4''$	$\{2, 3\}$
4	$k = 2, 3, 4, 5, 1', 2', 3', 4', 5'$	$\{2, 4\}$
5	$k = 2, 4, 5, 1', 2', 3', 4', 5'$	$\{1, 2, 4, 5\}$
6	$k = 1, 2, 3$	$\{1, 6, 4''\}$
7	$k = 6, 7, 4''$	$\{1, 6, 7, 4'\}$
$1'$	$k = 2, 3, 4, 6, 7, 1', 2', 3', 4', 5'$	$\{'1\}$
$2'$	$k = 2, 4, 5, 1', 2', 3', 4', 5'$	$\{2, 4, 5, 2', 3', 5'\}$
$3'$	$k = 2, 4, 5, 1', 2', 3', 4', 5'$	$\{3'\}$
$4'$	$k = 4, 5, 1', 2', 3', 4', 5'$	$\{2, 4, 5, 2', 3', 4', 5'\}$
$5'$	$k = 2, 4, 5, 1', 2', 3', 4', 5'$	$\{2, 4, 5, 2', 3', 4', 5'\}$
$1''$	$k = 2, 1'', 2''$	$\{2, 3', 1'', 2''\}$
$2''$	$k = 2, 1'', 2'',$	$\{2, 3', 1'', 2''\}$
$3''$	$k = 3'', 4''$	$\{3'', 5'', 6''\}$
$4''$	$k = 2, 3, 6, 7, 3'', 4'', 7''$	$\{4''\}$
$5''$	$k = 3'', 4'', 5'', 6'', 7''$	$\{5'', 6''\}$
$6''$	$k = 1'', 2'', 3'', 4'', 5'', 6''$	$\{5'', 6''\}$
$7''$	$k = 3'', 4'', 7''$	$\{4'', 5'', 7''\}$

According to the type of COVID-19, there are four types of virus graphs (see Figure 5a) and by depending on these cases of virus graphs, Bhapkar et al. [22] defined the variable graphs, the variable set and their types of variable edge sets and variable vertex sets. Figure 5b represents the COVID-19 form, which is taken from [25]. Here, in Figure 6, we give the general graphical representation Ω for COVID-19, which is in Figure 5b. Since $n \geq 5$ in Figure 6, by Lemma 1, the upper a-neighborhood system Ω^u of the upper a-graphical topological space $(\nu(\Omega), T_{\Omega^u})$ is given in Table 5, where $A_k = \{k^1, k^2, k^3, \cdots, k^{mk}\}$, $\overline{A}_k = \{k'\} \cup A_k$ for all $k = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, mk$. The basis $\mathcal{B}^u_\Omega$ is given by $\Omega^u_{k'} = \Omega^u_{kj} = \overline{A}_k$ for all $k = 1, 2, 3, \cdots, n$ and $j = 1, 2, 3, \cdots, mk$. That is, the upper a-graphical topological space $(\nu(\Omega), T_{\Omega^u})$ is quasi-discrete and so disconnected. Since $\nu(\mathcal{I}_\Omega) = \emptyset$, Ω is an $\cap$-upper connected graph and the upper a-graphical topological space $(\nu(\Omega), T_{\Omega^u})$ is an $\cap$-upper discrete space.

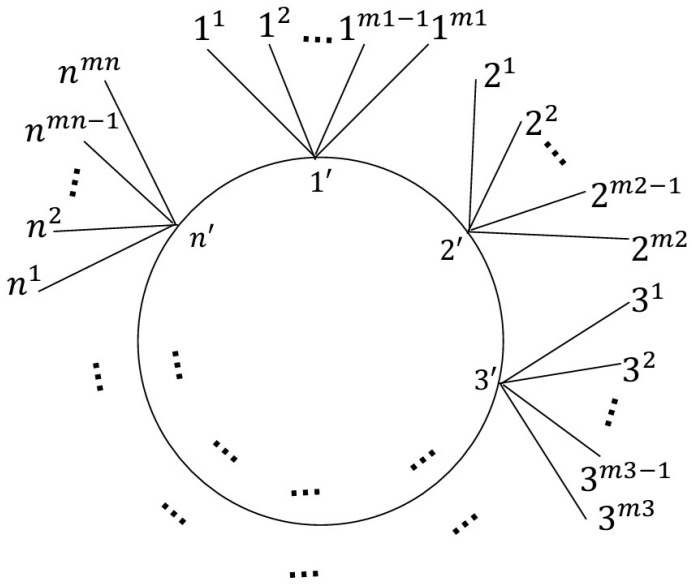


Figure 6. General form of COVID-19.

Table 5. An upper *a*-neighborhood system for Figure 6.

k	$\Omega^u(k') = \Omega^u(k^j)$
$1, 2$	$\overline{A}_{n-4+k} \cup A_{n-2+k} \cup \overline{A}_k \cup A_{k+2} \cup \overline{A}_{k+4}$
$3, 4$	$\overline{A}_{n-4+k} \cup A_{k-2} \cup \overline{A}_k \cup A_{k+2} \cup \overline{A}_{k+4}$
$\vdots$	$\vdots$
$5 \leq k \leq n-4$	$\overline{A}_{k-4} \cup A_{k-2} \cup \overline{A}_k \cup A_{k+2} \cup \overline{A}_{k+4}$
$\vdots$	$\vdots$
$n-3, n-2$	$\overline{A}_{k-4} \cup A_{k-2} \cup \overline{A}_k \cup A_{k+2} \cup \overline{A}_{k-n+4}$
$n-1, n$	$\overline{A}_{k-4} \cup A_{k-2} \cup \overline{A}_k \cup A_{k-n+2} \cup \overline{A}_{k-n+4}$

5. Conclusions

Note that the class of monophonic paths in the theory of undirected graphs is a subclass of the collection of paths and the concept of a rough approximation neighbourhood system in [13] of undirected simple graphs is an important concept in the theory of rough sets. Here, we used this subclass and a rough approximation neighbourhood system in the theory of simple graphs to define the operator $\cap^{\Omega} : \nu(\Omega) \to P(\nu(\Omega))$. By using this operator, monophonic eccentric open neighborhood systems in [23] and an upper approximation neighbourhood system [13], we introduced a new neighborhood system, an upper *a*-neighborhood system. This neighborhood system gave us the concept of an upper *a*-graphical topological space and we proved the discrete property of path P_n, complete graphs K_n, cycle graphs C_n and complete bipartite graphs $K_{n,m}$. By this class of *a*-graphical topological spaces, we investigated and presented discrete, $\cap$-upper connectedness and $\cap$-upper discrete properties for some corresponding graphs concerning the COVID-19 form and its diffusion depending on some studies, such as the study of the COVID-19 pandemic spread of Kyoto city in Japan and Kentucky state in the USA, which were introduced in [20], the spread of COVID-19 in [21] and the four COVID-19 types, Ref. [22]. For future work, we suggest two cases, firstly, to study the connectedness and discrete properties for the form and diffusion of COVID-19 in the above studies by using out mondirected topological spaces, which are introduced in [19]. Secondly, we suggest studying the discrete, $\cap$-upper

connectedness and ∩-upper discrete properties for the networks of the nervous system of the human body, which are introduced in [19].

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