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# Blow-Up of Solutions in a Fractionally Damped Plate Equation with Infinite Memory and Logarithmic Nonlinearity

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**Abstract:** In this article, we consider the dynamics of a viscoelastic plate equation with internal fractional damping, a nonlinear logarithmic source, and infinite memory effects. The existence of a local weak solution is shown effectively through the framework of semigroup theory. Furthermore, we show that the blow-up in finite time of the local solution may occur under specific conditions and is demonstrated within the development of a suitable Lyapunov functional. Our result offers an insight into the challenges presented by this class of equations and their relevance to physical systems.

**Keywords:** blow-up phenomena; plate equation; nonlinear logarithmic source term; fractional damping; infinite memory; nonlinear equations; partial differential equations

**MSC:** 93D15 ; 35L70; 35R35; 93D20

## 1. Introduction

In this paper, we investigate the following plate equation:

$$(W)\begin{cases} v_{tt} + \Delta^2 v - \int_0^{+\infty} r(s)\Delta^2 v(t-s)ds + \partial_t^{\eta,\varsigma} v(t) = v|v|^{p-2}\ln|v|^{\ell}, & \text{in } \Omega\times(0,\infty),\\ \frac{\partial v}{\partial \nu} = v = 0, \quad \text{on } \partial\Omega\times(0,\infty),\\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), \quad \text{in } \Omega \end{cases}$$

where $\ell$ is a positive constant, $r$ is a function that will be explained later, $p > 2$, and $\Omega$ is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. The symbol $\partial_t^{\eta,\varsigma}$ denotes the modified Caputo's fractional derivative which is defined by the following (see [1,2]):

$$\partial_t^{\eta,\varsigma} v(t) = \frac{1}{\Gamma(1-\eta)}\int_0^t (t-\tau)^{-\eta} e^{-\varsigma(t-\tau)} v_\tau(\tau)d\tau, \qquad 0 < \eta < 1, \varsigma \geq 0.$$

The viscoelastic equation with fractional damping, infinite memory and logarithmic nonlinear source term results from the need of describing a complicated physical phenomenon in numerous fields of research including engineering, material science and applied mathematics. The emergence of equations of this type signifies an advanced approach to the modeling of real-world phenomena where complex time interactions exist, making it a prominent aspect of study in both theoretical and applied contexts. These kinds of equations are applied to predict the mode of failure in viscoelastic material structures—beams and plates—under dynamic loads. By understanding the behavior of these materials under various loading scenarios, an engineer can design safer and more effective structures. This concept of damping in the model is used to describe systems where the damping force is not

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proportional to the velocity but instead depends on a fractional derivative. Such damping is mainly found in viscoelastic materials. The response of such materials at any moment is governed partly by their present and partly by their past states. Consequently, fractional derivatives will model these types of memory-containing materials, improving the models for their dynamics. The integral term for infinite memory in the constructed model allows the current situation to have influences from all its past. It becomes relevant in materials such as polymers or biological tissues, when past stress or strain is known to influence current behavior. Convolution integrals are frequently involved in the mathematical formulation together with a relaxation function, expressing how past states affect the dynamics of the present state. The logarithmic nonlinearity added to the equation brings with it a new kind of nonlinearity that could be rather difficult to analyze and is interesting enough to deserve consideration in qualitative responses. This nonlinearity is commonly employed to model processes in which the material strength decreases logistically with continued deformation or stress, representing true material behavior under extreme conditions.

Memory effects in viscoelastic wave equations of a plate type significantly influence local existence and blow-up behavior of solutions. Local existence can be established through techniques like the Galerkin method, which shows that weak solutions exist under certain conditions related to initial data and the nature of nonlinearity, ensuring stability in finite time. Blow-up significance arises when initial energy conditions are low, leading to solutions that can become unbounded in finite time. This phenomenon is often linked to the memory effects, where the material retains stress from previous states, potentially resulting in complex dynamics that increase instability. Understanding these interactions is essential for predicting material behavior in engineering applications, particularly in structures subjected to dynamic loads.

Overall, the plate equation is a fundamental component in the analysis of viscoelastic wave equation systems, as it provides a robust mathematical framework for understanding the dynamic behavior of plate-like structures made of viscoelastic materials. Its applications span a wide range of engineering disciplines, from aerospace and civil engineering to materials science and structural dynamics.

In connection with the study of plates, Lagnese [3] investigated a viscoelastic plate equation and demonstrated that, by putting a dissipative mechanism on the system boundary, the energy decays to zero as time approaches infinity. If the memory's kernel decays exponentially, then the first and second order energies related to the solutions of the viscoelastic plate equation similarly decline exponentially, according to Rivera et al. [4]. Komornik [5] examined the feedback function's energy decay in a plate model with weak growth assumptions. Messaoudi [6] investigated the following system:

$$\begin{cases} \sigma_{tt} + \Delta^2\sigma + a\,\sigma_t|\sigma_t|^{m-2} = \sigma|\sigma|^{p-2}, & \text{in } \Omega \times (0,\infty) \\ \sigma = \frac{\partial\sigma}{\partial\nu} = 0, & \text{on } \Omega \times (0,\infty) \\ \sigma(x,0) = \sigma_0(x), \quad \sigma_t(x,0) = \sigma_1(x), & x \in \Omega. \end{cases} \tag{1}$$

and proved that the solution exists globally and blows up in finite time. They also developed an existence result. Chen and Zhou [7] later improved this result.

The following fourth-order viscoelastic plate equation was studied by Messaoudi and Mukiawa [8]:

$$\sigma_{tt} + \Delta^2\sigma - \int_0^t g(t-s)\Delta^2\sigma(s)ds = 0. \tag{2}$$

Cavalcanti et al. [9] addressed the following equation:

$$\sigma_{tt} - \int_0^t g(t-s)\Delta^2\sigma(s)ds + \Delta^2\sigma + \gamma\Delta\sigma_{tt} + a(t)\sigma_t = 0 \tag{3}$$

in $\Omega \times (0, \infty)$ to discuss decay. Rivera et al. [4] demonstrated how the energy decay of the same formula was shown.

In [10], Mukiawa investigated the following equation:

$$\sigma_{tt} - \int_0^t g(t-s)\Delta^2\sigma(s)ds + \Delta^2\sigma + \mu_1\sigma_t + \mu_2\sigma_t(t-s) = 0 \tag{4}$$

with the concept of delay. For this equation, the author obtained the decay of solutions. Mustafa and Kafini studied the following equation in [11]:

$$\sigma_{tt} + \mu_1\sigma_t - \int_0^\infty g(t-s)\Delta^2\sigma(s)ds + \Delta^2\sigma + \mu_2\sigma_t(t-s) = \sigma|\sigma|^\gamma. \tag{5}$$

Furthermore, they demonstrated that solutions decay in general. For a better understanding of the plate equation, we refer to [12–16].

However, it will be unfair if we do not discuss the works involving fractional derivative in wave equations. Numerous technical and scientific fields, such as electro-magnetics, fluid mechanics, electrochemistry, biological population models, optics, signal processing, and viscoelasticity, rely on fractional calculus. They have been used to model technical and physical phenomena that fractional differential equations are best suited to describe. Systems with damping are accurately modeled using fractional derivative models. In recent years, it has been found that classical theory cannot explain phenomena with unusual kinetics using integer-order derivatives. Fractional differential equations have garnered interest due to their convergence to integer-order system responses. Fractional derivatives can accurately explain the dynamics of certain structures. The integer-order differential operator is a local operator. The fractional-order differential operator considers the history of previous states as well as the present state, making it non-local in nature. Fractional-order systems are gaining popularity due to their realistic properties. Fractional-order derivatives are a logical fit for memory-based systems found in many physical and scientific models.

Fractional partial differential equations and other problems requiring special functions of mathematical physics, together with extensions and generalizations in one or more variables, are frequently the results of mathematical modeling of real-world issues. Moreover, most physical processes in fluid dynamics, quantum physics, electricity, ecological systems, and many other models are governed by fractional-order PDEs. Therefore, understanding both traditional and newly developed approaches to solve fractional-order PDEs, as well as how to apply them, is becoming more and more important.

The works [17–19] demonstrate the interest that researchers in the mathematics, biological, and physical sciences have shown in partial differential equations containing fractional derivatives. As demonstrated by the references (see, e.g., [20,21]), these equations have found widespread applications in recent years in fields like electronics, relaxation vibrations, and viscoelasticity.

An asymptotic profile of fractional boundary dissipation was investigated by A. Benaissa and H. Benkheda [22]. Recently, R. Aounallah et al. [23] explored a wave equation having a fractional derivative on the boundary of a system. In [24], S. Boulaaras et al. investigated the same system with finite memory $\int_0^t, g(t-s)\sigma(s)\,ds$. In order to investigate the presence and decay features of desired solutions, the authors in [23,24] employed an improved system to construct the provided problem when a fractional derivative was present at the boundary of a system. In [25], B. Mbodje investigated the decay of wave energy under fractional derivative controls.

N. Doudi and colleagues addressed the global existence, general decay, and blow-up of the logarithmic problem in [26], as follows:

$$\begin{cases} \sigma_{tt} - \Delta\sigma + a\sigma_t = \sigma|\sigma|^{p-1}\ln|\sigma|, & x \in \Omega, t > 0 \\ \frac{\partial\sigma}{\partial v} = -b\partial_t^{\alpha,\eta}\sigma, & x \in \Gamma_0, t > 0 \\ \sigma = 0, & x \in \Gamma_1, t > 0 \\ \sigma(x,0) = \sigma_0(x), \quad \sigma_t(x,0) = \sigma_1(x), & x \in \Omega \end{cases} \tag{6}$$

R. Aounallaha et al. discussed the blow-up and asymptotic behavior of the system with internal fractional time delay in [27].

The blow-up in a system of nonlinear logarithmic wave equations with fractional damping, infinite memory, and high dissipation was studied by M. Fahim Aslam and Jianghao Hao [28]. Moreover, Z. Hajjej [29] recently discussed an asymptotic profile as well as a blow-up result of a suspension bridge with fractional time delay.

The application of fractional derivatives in the modeling of plate equations with viscoelastic effects is an emerging and promising area of research that has not been extensively explored by researchers in the past. Developing a fractional-order plate equation model could provide significant insights and potential benefits compared to traditional integer-order models. Exploring the development and analysis of a fractional-order plate equation model would indeed be a valuable and novel contribution to the field.

Collaboration with experts in these areas would be highly beneficial in advancing this research direction. This study aims to employ semigroup theory to investigate the well-posedness first and the blow-up phenomena of solutions to this equation. This paper is structured as follows. Section 2 provides the assumptions and tools necessary to establish the main results. In Section 3, the existence of a local weak solution is demonstrated using the semigroup theory [30]. Section 4 presents the demonstration of finite-time blow-up for a specific solution employing a carefully constructed Lyapunov functional. Section 5 presents the demonstration of conclusion and future work.

## 2. Preliminaries

This section has some useful results which will be used to prove our findings. Here, the following assertions are needed:

(J1) $r : \mathbb{R}_+ \to \mathbb{R}_+$ is a function $C^1$, such that

$$r(0) > 0, \quad r_0 = \int_0^\infty r(s)ds = 1 - \lambda > 0;$$

(J2) If $\theta > 0$ is constant, then

$$r'(t) \leq -\theta r(t), \quad t \geq 0.$$

**Lemma 1.** *For a function r, the inequality*

$$\int_\Omega \left[\int_0^{+\infty} r(s)\Delta\omega(s)ds\right]^2 dx \leq (1-\lambda)\int_0^{+\infty} r(s)||\Delta\omega(s)||_2^2\, ds.$$

**Proof.** Apply Holder'S Inequality and (J1), we obtain

$$\begin{aligned}\int_\Omega \left[\int_0^{+\infty} r(s)\Delta\omega(s)ds\right]^2 dx &\leq \int_\Omega \left(\int_0^{+\infty} r(s)ds\right)\int_0^{+\infty} r(s)|\Delta\omega(s)|^2 dsdx \\ &\leq (1-\lambda)\int_\Omega \int_0^{+\infty} r(s)|\Delta\omega(s)|^2 dsdx \\ &\leq (1-\lambda)\int_0^{+\infty} r(s)||\Delta\omega(s)||_2^2\, ds. \end{aligned} \tag{7}$$

□

**Lemma 2** (See [25]). *Let $\zeta$ be the following function: $\zeta(\sigma) = |\sigma|^{\frac{(2\eta-1)}{2}}$, $\sigma \in \mathbb{R}, 0 < \eta < 1$ and $b = \frac{\sin(\eta\pi)}{\pi}$; then, the relation between input $U$ and output $O$ of the system is as follows:*

$$\begin{cases} \partial_t\phi(\sigma,t) + (\sigma^2+\varsigma)\phi(\sigma,t) - U(x,t)\zeta(\sigma) = 0, \quad \sigma \in \mathbb{R}, t > 0, \varsigma \geq 0, \\ \phi(\sigma,0) = 0, \\ O(t) := b\int_{-\infty}^{+\infty} \phi(\sigma,t)\zeta(\sigma)d\sigma \end{cases} \tag{8}$$

*is given by*

$$O := I^{1-\eta,\varsigma}U,$$

*where*

$$I^{\eta,\varsigma}u(t) := \frac{1}{\Gamma(\eta)}\int_0^t (t-\tau)^{\eta-1}e^{-\varsigma(t-\tau)}u(\tau)d\tau.$$

**Lemma 3** (See [22]). *For all $\lambda \in D_\varsigma = \lambda \in \mathbb{C} : \mathrm{Re}(\lambda) + \varsigma > 0 \cup \lambda \in \mathbb{C} : \mathrm{Im}(\lambda) \neq 0$.*

$$A_\lambda := \int_{-\infty}^{+\infty} \frac{\zeta^2(\sigma)}{\lambda+\varsigma+\sigma^2}d\sigma = \frac{\pi}{sin(\eta\pi)}(\lambda+\varsigma)^{\eta-1}$$

Now, similarly to [31,32], we propose new variable:

$$\mu^t(x,s) = v(x,t) - v(x,t-s), \tag{9}$$

The variable $\mu^t$ represents the relative history of $v$ that fulfills the following equation:

$$\mu_t^t(x,s) - v_t(x,t) + \mu_s^t(x,s) = 0, \quad x \in \Omega, \quad t,s > 0. \tag{10}$$

By utilizing Lemma 2 and Equation (9), the system (W) can be expressed in the following manner:

$$(M)\begin{cases} v_{tt} + \lambda\Delta^2 v(t) + \int_0^{+\infty} r(s)\Delta^2\mu^t(x,s)ds & \\ \quad + b\int_{-\infty}^{+\infty}\phi(\sigma,t)\zeta(\sigma)d\sigma = v|v|^{p-2}\ln|v|^\ell, & x \in \Omega, t > 0, \\ \partial_t\phi(\sigma,t) + (\sigma^2+\varsigma)\phi(\sigma,t) - v_t(x,t)\zeta(\sigma) = 0, & \sigma \in \mathbb{R}, t > 0, \varsigma \geq 0, \\ \mu_t^t(x,s) + \mu_s^t(x,s) = v_t(x,t), & x \in \Omega, t,s > 0, \\ v = \mu^t(x,s) = \frac{\partial v}{\partial \nu} = 0, & x \in \partial\Omega, t,s > 0, \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x \in \Omega, \\ \mu^t(x,0) = 0, \quad \mu^0(x,s) = v_0(x) - v_0(x,-s), & x \in \Omega, t,s > 0, \\ \phi(\sigma,0) = 0, & x \in \Omega, \sigma \in \mathbb{R}. \end{cases}$$

**Lemma 4.** *The energy*

$$E(t) := \frac{1}{2}||v_t(t)||_2^2 + \frac{b}{2}\int_\Omega \int_{-\infty}^{+\infty} |\phi(\sigma,t)|^2 d\sigma dx + \frac{\lambda}{2}||\Delta v(t)||_2^2 + \frac{\ell}{p^2}||v(t)||_p^p - \frac{1}{p}\int_\Omega ln|v|^\ell v^p dx + \frac{1}{2}\int_0^{+\infty} r(s)||\Delta\mu^t(s)||_2^2 ds \tag{11}$$

*satisfies*

$$\frac{dE(t)}{dt} = \frac{1}{2}\int_0^{+\infty} r'(s)||\Delta\mu^t(s)||_2^2 ds - b\int_\Omega\int_{-\infty}^{+\infty}(\sigma^2+\varsigma)|\phi(\sigma,t)|^2 d\sigma dx \le 0. \tag{12}$$

**Proof.** Multiply system ($M$) by $v_t$, with its first equation integrating over $\Omega$. Thus, using integration by parts, we obtain

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}\|v_t(t)\|_2^2 + \frac{\lambda}{2}\|\Delta v\|_2^2 - \frac{1}{p}\int_\Omega \ln|v|^\ell |v|^p\, dx + \frac{\ell}{p^2}\|v\|_p^p\right\} \\ &+b\int_\Omega v_t \int_{-\infty}^{+\infty} \phi(\sigma,t)\zeta(\sigma)\, d\sigma + \underbrace{\int_\Omega v_t \int_0^{+\infty} r(s)\Delta^2\mu^t(x,s)\, ds}_{I_1} = 0. \end{aligned} \tag{13}$$

Using (10)

$I_1 = \int_\Omega v_t \int_0^{+\infty} r(s)\Delta^2\mu^t(x,s)ds = \int_0^{+\infty} r(s)\int_\Omega(\mu_t^t(x,s)+\mu_s^t(x,s))\Delta^2\mu^t(x,s)ds$

From integration by parts, we obtain

$I_1 = \frac{d}{dt}\left\{\frac{1}{2}\int_0^{+\infty} r(s)\left\|\Delta\mu^t(x,s)\right\|_2^2 ds\right\} - \frac{1}{2}\int_0^{+\infty} r'(s)\left\|\Delta\mu^t(x,s)\right\|_2^2 ds$

Therefore, taking the value of $I_1$ from the last equation, we obtain the following:

$$\begin{aligned} &\frac{d}{dt}\left\{\frac{1}{2}\|v_t(t)\|_2^2 + \frac{\lambda}{2}\|\Delta v\|_2^2 - \frac{1}{p}\int_\Omega \ln|v|^\ell |v|^p dx + \frac{\ell}{p^2}\|v\|_p^p + \frac{1}{2}\int_0^{+\infty} r(s)\left\|\Delta\mu^t(x,s)\right\|_2^2 ds\right\} \\ &+b\int_\Omega v_t\int_{-\infty}^{+\infty}\phi(\sigma,t)\zeta(\sigma)d\sigma - \frac{1}{2}\int_0^{+\infty} r'(s)\left\|\Delta\mu^t(x,s)\right\|_2^2 ds = 0 \end{aligned} \tag{14}$$

Now, multiply system ($M$) by $b\phi$ with its second equation and integrate over $\Omega\times\mathbb{R}$. Thus, we obtain

$$\begin{aligned} &-b\int_\Omega v_t\int_{-\infty}^{+\infty}\phi(\sigma,t)\zeta(\sigma)d\sigma + \frac{d}{dt}\left(\frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty}|\phi(\sigma,t)|^2\zeta(\sigma)d\sigma dx\right) \\ &+b\int_\Omega\int_{-\infty}^{+\infty}(\sigma^2+\varsigma)|\phi(\sigma,t)|^2 d\sigma dx = 0 \end{aligned} \tag{15}$$

Therefore, from (14) and (15), we obtain

$$\begin{aligned} &\frac{d}{dt}\left\{\begin{array}{c} \frac{1}{2}\|v_t(t)\|_2^2 + \frac{\lambda}{2}\|\Delta v\|_2^2 - \frac{1}{p}\int_\Omega \ln|v|^\ell |v|^p dx + \frac{\ell}{p^2}\|v\|_p^p + \frac{1}{2}\int_0^{+\infty} r(s)\left\|\Delta\mu^t(x,s)\right\|_2^2 ds \\ +\frac{b}{2}\int_\Omega\int_{-\infty}^{+\infty}|\phi(\sigma,t)|^2\zeta(\sigma)d\sigma dx \end{array}\right\} \\ &= \frac{1}{2}\int_0^{+\infty} r'(s)\left\|\Delta\mu^t(x,s)\right\|_2^2 ds - b\int_\Omega\int_{-\infty}^{+\infty}(\sigma^2+\varsigma)|\phi(\sigma,t)|^2 d\sigma dx \end{aligned} \tag{16}$$

This completes the required proof. □

## 3. Well -Posedness

This section outlines the local existence of a system (M). Firstly, the vector function is defined as follows: $$U = (v, v_t, \phi, \mu^t)^T$$

Let

$$u = v_t.$$

Thus, the problem (M) can be written as

$$(\mathrm{M}^1)\begin{cases} U_t(t) + AU(t) = J(U(t)), \\ U(0) = U_0. \end{cases}$$

The operator $A : D(A) \to \mathscr{H}$ is defined by

$$AU = \begin{pmatrix} -u \\ \lambda\Delta^2 v + \int_0^{+\infty} r(s)\Delta^2\mu^t(x,s)ds + b\int_{-\infty}^{+\infty}\phi(x,\sigma,t)\zeta(\sigma)d\sigma \\ (\zeta^2 + \varsigma)\phi - u(x)\zeta(\sigma) \\ \mu_s^t(s) - u \end{pmatrix},$$

$$J(U) = (0, |v|^{p-2}vln|v|^{\ell}, 0, 0)^T, \tag{17}$$

and $\mathscr{H}$ is the energy space given by

$$\mathscr{H} = H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega, \mathbb{R}) \times L_r^2(\mathbb{R}_+, H_0^1(\Omega))$$

such that

$$L_r^2(\mathbb{R}_+, H_0^1(\Omega)) = \left\{\Omega : \mathbb{R}_+ \to H_0^1(\Omega), \int_0^{+\infty} r(s)||\Delta w(s)||_2^2 ds < \infty\right\};$$

the space $L_r^2(\mathbb{R}_+, H_0^1(\Omega))$ is endowed with the following inner product:

$$\langle w_1, w_2\rangle_{L_r^2(\mathbb{R}_+, H_0^1(\Omega))} = \int_0^{+\infty} r(s)\int_\Omega \Delta w_1(s)\Delta w_2(s)dxds.$$

The inner product of any $U$ and $\bar{U}\in \mathscr{H}$ can be defined as

$$\langle U, \bar{U}\rangle_{\mathscr{H}} = \int_\Omega [\lambda\Delta v.\Delta\bar{v} + u\bar{u}]dx + b\int_\Omega\int_{-\infty}^{+\infty}\phi\bar{\phi}d\xi dx$$

$$+ \int_0^{+\infty} r(s)\int_\Omega \Delta\mu^t(s)\Delta\bar{\mu}^t(s)dxds.$$

where $U = (v, u, \phi, \mu^t)^T \in \mathscr{H}$ and $\bar{U} = (\bar{v}, \bar{u}, \bar{\phi}, \bar{\mu}^t)^T \in \mathscr{H}$.

The domain of $A$ is given by

$$D(A) = \left\{\begin{array}{l} U = (v, u, \phi, \mu^t)^T \in \mathscr{H}; v \in H^2(\Omega); u \in H_0^1(\Omega); \\ (\sigma^2 + \varsigma)\phi - u\zeta(\sigma) \in L^2(\Omega, \mathbb{R}); \\ |\sigma|\phi \in L^2(\Omega, \mathbb{R}); \mu_s^t \in L_r^2(\mathbb{R}_+, H_0^1(\Omega)), \end{array}\right\}.$$

To show that the solution of this system exists, we are now ready to reveal this upcoming result.

**Theorem 1.** *Let us assume that*

$$\begin{cases} n = 1, 2, \quad for \quad P > 2. \\ n \geq 3, \quad for \quad 2 < p < \frac{2(n-1)}{n-2}. \end{cases} \tag{18}$$

*suppose further that*

$$U_0 \in \mathscr{H}; \tag{19}$$

*then, the problem* $(M^1)$ *has a unique local solution:*

$$U \in C([0,T), \mathscr{H}). \tag{20}$$

**Proof.** The proof relies on reference [31] as its foundation. To begin, we establish that operator $A$ is monotone on the space $\mathscr{H}$. We commence by demonstrating that, for any $U$ belonging to the domain $A$ utilizing $(M^1)$, we can deduce the following:

$$\begin{aligned}\langle AU, \bar{U}\rangle &= \int_\Omega y\left[\lambda\Delta^2 v + \int_0^\infty r(s)\Delta^2\mu^t(x,s)ds + b\int_{-\infty}^{+\infty}\phi(\sigma,t)\zeta(\sigma)d\sigma\right]dx \\ &\quad - \int_\Omega \lambda\Delta y\Delta v\,dx + b\int_\Omega\left[\int_{-\infty}^{+\infty}(\sigma^2+\varsigma)\phi - y\zeta(\sigma)\right]\phi(x,\xi,t)d\sigma dx \qquad (21)\\ &\quad + \int_0^\infty r(s)\int_\Omega \Delta(\mu_s^t(x,s)-y)\Delta\mu^t(s)ds \\ &= b\int_\Omega\int_\infty^{+\infty}(\sigma^2+\varsigma)|\phi|^2 d\sigma dx \quad - \frac{1}{2}\int_0^{+\infty} r'(s)||\Delta\mu^t(s)||_2^2 ds \geq 0. \qquad (22)\end{aligned}$$

Now, as we know that $A$ is a monotone operator, we proceed to demonstrate the surjectivity of the operator (I + $A$). To do so, let us consider an element $\pounds = (f_1, f_2, f_3, f_4)^T \in \mathscr{H}$. Our aim is to show that there exists a $U$ belonging to the domain of $A$, such that

$$(I + A)U = \pounds;$$

thus,

$$\begin{cases} v - u = f_1 \in H_0^1(\Omega), \\ u + \lambda\Delta^2 v + \int_0^{+\infty} r(s)\Delta^2\mu^t(s)ds + b\int_{-\infty}^{+\infty}\phi(\sigma)\zeta(\sigma)d\sigma = f_2 \in L^2(\Omega), \\ \phi + (\sigma^2+\varsigma)\phi - u\zeta(\sigma) = f_3(\sigma) \in L^2(\Omega,\mathbb{R}), \\ \mu^t + \mu_s^t - u = f_4(s) \in L_r^2(\mathbb{R}_+; H_0^1(\Omega)). \end{cases} \tag{23}$$

From $(23)_3$,

$$\phi = \frac{f_3 + u\zeta(\sigma)}{\sigma^2+\varsigma+1}. \tag{24}$$

Moreover, from $(23)_4$, we conclude that

$$\mu^t = \left(\int_0^s e^z(f_4(z) + v - f_1)dz\right)e^{-s}. \tag{25}$$

By substituting $u = v - f_1$ from Equations (24) and (25) into Equation $(23)_2$, we obtain the following:

$$\begin{aligned}&(1 + b\int_{-\infty}^{+\infty}\frac{\eta^2(\sigma)}{\sigma^2+\varsigma+1}d\sigma)v + (\lambda + \int_0^{+\infty} r(s)e^{-s}\left(\int_0^s e^z dz\right)ds)\Delta^2 v \\ &= f_2 + \varrho f_1 - b\int_{-\infty}^{+\infty}\frac{\eta(\sigma)f_3(\sigma)}{\sigma^2+\varsigma+1} - \int_0^{+\infty} r(s)e^{-s}\left(\int_0^s e^z\Delta^2(f_4(z)-f_1)dz\right)ds.\end{aligned} \tag{26}$$

Therefore,

$$\varrho v + \bar{\lambda}\Delta^2 v = G, \tag{27}$$

where

$$\varrho = 1 + b\int_{-\infty}^{+\infty} \frac{\eta^2(\sigma)}{\sigma^2+\varsigma+1} d\sigma > 0,$$

$$\bar{\lambda} = \lambda + \int_0^{+\infty} r(s)e^{-s}\left(\int_0^s e^z dz\right) ds$$

$$= 1 - \int_0^{+\infty} r(s)e^{-s} ds > 0,$$

and

$$G = f_2 + \varrho f_1 - b\int_{-\infty}^{+\infty} \frac{\eta(\sigma)f_3(\sigma)}{\sigma^2+\varsigma+1} - \int_0^{+\infty} r(s)e^{-s}\left(\int_0^s e^z \Delta^2(f_4(z) - f_1)dz\right) ds.$$

To solve Equation (27), we approach it by considering the following variational formulation:

$$B(v,\omega) = L(\omega), \quad \forall \omega \in H_0^1(\Omega), \tag{28}$$

where $B$ denotes the bilinear form defined as

$$B(v,\omega) = \varrho \int_\Omega v\omega dx + \bar{\lambda}\int_\Omega \Delta v.\Delta\omega dx, \tag{29}$$

and the linear functional $L$ is provided by

$$L(\omega) = \int_\Omega G\omega dx. \tag{30}$$

The boundedness of $L$ and the coerciveness and boundedness of $B$ are easily verified. Thus, the linear equation which is elliptic (27) has a solution $v \in H_0^1(\Omega)$ for all $\omega \in H_0^1(\Omega)$, which is unique, according to the Lax–Milgram theorem. Here, $u \in H_0^1(\Omega)$ is the result of substituting $v$ into the first equation in Equation (23). When we insert $u$ into Equation (23) and consider its third equation. Thus, we obtain

$$\phi \in L^2(\Omega, \mathbb{R}).$$

Similarly, we possess

$$\mu^t \in L_r^2(\mathbb{R}_+; H_0^1(\Omega)).$$

Using (27), we obtain

$$\varrho \int_\Omega v\omega dx + \bar{\lambda}\int_\Omega \Delta v.\Delta\omega dx = \int_\Omega G\omega dx. \tag{31}$$

By invoking the elliptic regularity theory, we can conclude that $v$ belongs to $H^2(\Omega)$, which implies that (I + A) is onto.

Next, we proceed to establish the local Lipschitz continuity of the operator defined in (17) within the space $\mathscr{H}$.

For any $U$ and $\widetilde{U}$ belonging to $\mathscr{H}$, we have $\mathscr{F}(s) = |s|^{p-2}s\ln|s|^\ell$; thus, $\mathscr{F}'(s) = \ell[1 + (p-1)\ln|s|]|s|^{p-2}$.

Hence,

$$\begin{aligned}\|\mathscr{J}(U)-\mathscr{J}(\bar{U})\|_H^2 &= \left\|\left(0,|v|^{p-2}v\ln|v|^{\ell}-|\bar{v}|^{p-2}\bar{v}\ln|\bar{v}|^{\ell},0,0\right)\right\|_H^2\\ &= \left\||v|^{p-2}v\ln|v|^{\ell}-|\bar{v}|^{p-2}\bar{v}\ln|\bar{v}|^{\ell}\right\|_L^2\\ &= \|\mathscr{F}(U)-\mathscr{F}(\bar{U})\|_L^2.\end{aligned} \tag{32}$$

By the mean value theorem (MVT), we have, for $0 \le \vartheta \le 1$,

$$\begin{aligned}|\mathscr{F}(U)-\mathscr{F}(\bar{U})| &= \left|\mathscr{F}'(v\vartheta+(1-\vartheta)\bar{v})(v-\bar{v})\right|\\ &\le \ell[1+(p-1)\ln|v\vartheta+(1-\vartheta)\bar{v}|]|v\vartheta+(1-\vartheta)\bar{v}|^{p-2}|(v-\bar{v})|\\ &\le \ell|v\vartheta+(1-\vartheta)\bar{v}|^{p-2}|(v-\bar{v})|+\ell(p-1)|j(v\vartheta+(1-\vartheta)\bar{v})||(v-\bar{v})|.\end{aligned} \tag{33}$$

By recalling the lemma above, we have

$$\begin{aligned}|\mathscr{F}(U)-\mathscr{F}(\bar{U})| &= \ell|v\vartheta+(1-\vartheta)\bar{v}|^{p-2}|v-\bar{v}|+\ell(p-1)A|v-\bar{v}|\\ &\quad+\ell(p-1)|v\vartheta+(1-\vartheta)\bar{v}|^{p-2+\varepsilon}|v-\bar{v}|\\ &\le \ell(|v|+|\bar{v}|)^{p-2}|v-\bar{v}|+\ell(p-1)A|v-\bar{v}|\\ &\quad+\ell(p-1)(|v|+|\bar{v}|)^{p-2+\varepsilon}|v-\bar{v}|.\end{aligned} \tag{34}$$

As $v,\bar{v}\in H_0^1(\Omega)$, we then use Holder's inequality and the Sobolev embedding,

$$H_0^1(\Omega)\hookrightarrow L^r(\Omega),\ \forall\ 1\le r\le \tfrac{2n}{n-2},$$

allowing us to obtain

$$\begin{aligned}\int_\Omega\Big[(|v|+|\bar{v}|)^{p-2}|v-\bar{v}|\Big]^2dx &= \int_\Omega\Big[(|v|+|\bar{v}|)^{2(p-2)}|v-\bar{v}|^2\Big]dx\\ &\le C\Big(\int_\Omega(|v|+|\bar{v}|)^{2(p-2)}dx\Big)^{\frac{p-2}{p-1}}\times\Big(\int_\Omega(|v-\bar{v}|)^{2(p-2)}dx\Big)^{\frac{1}{p-1}}\\ &\le C\Big[\|v\|^{2(p-1)}_{L^{2(p-1)}(\Omega)}+\|\bar{v}\|^{2(p-1)}_{L^{2(p-1)}(\Omega)}\Big]^{\frac{p-2}{p-1}}\times\|v-\bar{v}\|^2_{L^{2(p-1)}(\Omega)}\\ &\le C\Big[\|v\|^{2(p-1)}_{H_0^1(\Omega)}+\|\bar{v}\|^{2(p-1)}_{H_0^1(\Omega)}\Big]^{\frac{p-2}{p-1}}\times\|v-\bar{v}\|^2_{H_0^1(\Omega)}.\end{aligned} \tag{35}$$

Similarly,

$$\begin{aligned}\int_\Omega\Big[(|v|+|\bar{v}|)^{p-2+\varepsilon}|v-\bar{v}|\Big]^2dx &= \int_\Omega\Big[(|v|+|\bar{v}|)^{2(p-2+\varepsilon)}|v-\bar{v}|^2\Big]dx\\ &\le C\Big(\int_\Omega(|v|+|\bar{v}|)^{\frac{2(p-2+\varepsilon)(p-1)}{(P-2)}}dx\Big)^{\frac{p-2}{p-1}}\times\Big(\int_\Omega(|v-\bar{v}|)^{2(p-2)}dx\Big)^{\frac{1}{p-1}}\\ &\le C\Big[\int_\Omega(|v|+|\bar{v}|)^{2(p-1)+\frac{2\varepsilon(p-1)}{(P-2)}}dx\Big]^{\frac{p-2}{p-1}}\times\|v-\bar{v}\|^2_{L^{2(p-1)}(\Omega)}.\end{aligned} \tag{36}$$

Since $p<(n-1)/(n-2)$, we can choose $\varepsilon>0$, which is so small that

$$p^*=2(p-1)+\tfrac{2\varepsilon(p-1)}{(P-2)}\le\tfrac{2n}{n-2}.$$

Hence, we have

$$\begin{aligned}\int_\Omega\Big[(|v|+|\bar{v}|)^{p-2+\varepsilon}|v-\bar{v}|\Big]^2dx &= C\Big[\|v\|^{p^*}_{L^{p^*(\Omega)}}+\|\bar{v}\|^{p^*}_{L^{p^*(\Omega)}}\Big]^{\frac{p-2}{p-1}}\times\|v-\bar{v}\|^2_{L^{2(p-1)}(\Omega)}\\ &\le C\Big[\|v\|^{p^*}_{H_0^1(\Omega)}+\|\bar{v}\|^{p^*}_{H_0^1(\Omega)}\Big]^{\frac{p-2}{p-1}}\times\|v-\bar{v}\|^2_{H_0^1(\Omega)}.\end{aligned} \tag{37}$$

Therefore, by combining

$$\|\mathscr{J}(v)-\mathscr{J}(\bar{U})\|_H^2 = C\Big[k^2(p-1^2)A^2\Big]\|v-\bar{v}\|_{H_0^1(\Omega)}^2 + C\left[\|v\|_{H_0^1(\Omega)}^{2(p-1)} + \|\bar{v}\|_{H_0^1(\Omega)}^{2(p-1)}\right]^{\frac{p-2}{p-1}}$$
$$+\left[\|v\|_{H_0^1(\Omega)}^{p^*} + \|\bar{v}\|_{H_0^1(\Omega)}^{p^*}\right]^{\frac{p-2}{p-1}}\|v-\bar{v}\|_{H_0^1(\Omega)}^2$$
$$\leq C\Big[\|v\|_{H_0^1(\Omega)} + \|\bar{v}\|_{H_0^1(\Omega)}\Big]^{\frac{p-2}{p-1}} \times \|v-\bar{v}\|_{H_0^1(\Omega)}^2. \tag{38}$$

Therefore, $\mathscr{J}$ is locally Lipschitz and the proof is completed. □

## 4. Blow-Up Result

In this part, we demonstrate that some solutions can undergo blow-up in a finite amount of time via a rigorous application of the Lyapunov functional. We need the following lemmas in order to accomplish our goal:

**Lemma 5** (See [33]). *For every $2 \leq s \leq p$ and $v \in L^P(\Omega)$, the following inequality is true for a constant $C > 0$ that relies on $\Omega$:*

$$||v||_p^s \leq C\Big(||v||_p^p + ||\Delta v||_2^2\Big), \tag{39}$$

**Lemma 6** (See [33]). *There exists a positive constant $C > 0$ depending on $\Omega$, such that*

$$\Big(\int_\Omega |v|^p ln|v|^\ell dx\Big)^{\frac{s}{p}} \leq C\Big(\int_\Omega |v|^p ln|v|^\ell dx + ||\Delta v||_2^2\Big). \tag{40}$$

*for any $v \in L^{p+1}(\Omega)$ and $2 \leq s \leq p$, provided that $\int_\Omega |v|^p ln|v|^\ell dx \geq 0$.*

**Lemma 7** (See [33]). *Depending on $\Omega$, there is a positive constant $C > 0$, such that*

$$||v||_p^p \leq C\Big(\int_\Omega |v|^p ln|v|^\ell dx + ||\Delta v||_2^2\Big), \tag{41}$$

*for any $v \in L^P(\Omega)$ and $2 \leq s \leq p$, provided that $\int_\Omega |v|^p ln|v|^\ell dx \geq 0$.*

**Corollary 1** (See [33]). *There exists a positive constant $C > 0$ depending on $\Omega$, such that*

$$||v||_2^2 \leq C\Big[\Big(\int_\Omega |v|^p ln|v|^\ell dx\Big)^{\frac{2}{p}} + ||\Delta v||_2^{\frac{4}{p}}\Big], \tag{42}$$

*provided that $\int_\Omega |v|^p ln|v|^\ell dx \geq 0$.*

Let

$$H(t) = -E(t). \tag{43}$$

**Theorem 2.** *Assume that (18) is satisfied and $E(0) < 0$ holds. Thus, the solution of the system (M) blows up in a finite time.*

**Proof.** From (11), we have

$$E(t) \leq E(0) < 0. \tag{44}$$

Thus, we obtain

$$\begin{aligned} H'(t) = -E'(t) = -\frac{1}{2}\int_0^{+\infty} r'(s)||\Delta\mu^t(s)||_2^2 ds \\ + b\int_\Omega \int_{-\infty}^{+\infty} (\sigma^2+\varsigma)|\phi(\sigma,t)|^2 d\sigma dx \geq 0. \end{aligned} \tag{45}$$

Furthermore,

$$0 < H(0) \leq H(t) \leq \frac{1}{p}\int_\Omega ln|v|^\ell |v|^p dx. \tag{46}$$

Let

$$A(t) = H^{1-\gamma}(t) + \epsilon\int_\Omega v v_t dx, \tag{47}$$

where $\epsilon > 0$, which will have to be specified afterwards. Moreover,

$$\frac{2(p-2)}{p^2} < \gamma < \frac{p-2}{2p} < 1. \tag{48}$$

By differentiating (47) and using (M), we obtain

$$\begin{aligned} A'(t) = \epsilon||v_t||_2^2 - \epsilon\lambda||\Delta v||_2^2 \\ + (1-\gamma)H^{-\gamma}(t)H'(t) - b\epsilon\int_\Omega v\int_{-\infty}^{+\infty}\zeta(\sigma)\phi(x,\sigma,t)d\sigma dx \\ - \epsilon\int_\Omega v\int_0^\infty r(s)\Delta^2\mu^t(s)ds dx + \epsilon\int_\Omega |v|^p ln|v|^\ell dx. \end{aligned} \tag{49}$$

After applying the Green formula on the last term of Equation (49), we obtain

$$\begin{aligned} A'(t) = \epsilon||v_t||_2^2 - \epsilon\lambda||\Delta v||_2^2 \\ + (1-\gamma)H^{-\gamma}(t)H'(t) - b\epsilon\int_\Omega v\int_{-\infty}^{+\infty}\zeta(\sigma)\phi(x,\sigma,t)d\sigma dx + \epsilon\int_\Omega |v|^p ln|v|^\ell dx \\ - \epsilon\int_\Omega v\int_0^\infty r(s)\Delta\mu^t(s)ds dx. \end{aligned} \tag{50}$$

Lemma 1 and Young's inequality allow us to determine

$$\begin{aligned} \int_\Omega v(t)\int_0^{+\infty} r(s)\Delta\mu^t(s)ds dx. \\ \leq (1-\lambda)||\Delta v(t)||_2^2 + \frac{1}{4}\int_0^{+\infty} r(s)||\Delta\mu^t(s)||_2^2 ds \end{aligned} \tag{51}$$

By substituting (51) in (50), we obtain

$$\begin{aligned} A'(t) \geq (1-\gamma)H^{-\gamma}(t)H'(t) + \epsilon||v_t||_2^2 - \epsilon||\Delta v||_2^2 \\ - b\epsilon\int_\Omega v\int_{-\infty}^{+\infty}\zeta(\sigma)\phi(x,\sigma,t)d\sigma dx + \epsilon\int_\Omega |v|^p ln|v|^\ell dx \\ - \frac{\epsilon}{4}\int_0^{+\infty} r(s)||\Delta\mu^t(s)||_2^2 ds. \end{aligned} \tag{52}$$

By utilizing (45) and Young's inequality, we arrive at

$$\begin{aligned} b\int_\Omega v\int_{-\infty}^{+\infty}\zeta(\sigma)\phi(x,\sigma,t)d\sigma dx \\ \leq \delta C_1||v||_2^2 + \frac{b}{4\delta}\int_\Omega\int_{-\infty}^{+\infty}(\sigma^2+\beta)|\phi(x,\sigma,t)|^2 d\sigma dx \\ \leq \delta C_1||v||_2^2 + \frac{1}{4\delta}H'(t), \end{aligned} \tag{53}$$

for $C_1 := b\int_{-\infty}^{+\infty}\frac{\zeta^2(\sigma)}{\zeta^2+\varsigma}d\sigma$ and $\delta > 0$, which could vary according to $t$. By substituting (53) in (52), we have

$$\begin{aligned} A'(t) \geq & \left((1-\gamma)H^{-\gamma}(t) - \frac{\epsilon}{4\delta}\right)H'(t) \\ & + \epsilon||v_t||_2^2 - \epsilon||\Delta v||_2^2 - \epsilon\delta C_1||v||_2^2 \\ & - \frac{\epsilon}{4}\int_0^{+\infty} r(s)||\Delta\mu^t(s)||_2^2 ds + \epsilon\int_\Omega |v|^p ln|v|^\ell dx. \end{aligned} \tag{54}$$

Next, we select a suitable $\delta$ as follows:

$$\frac{1}{4\delta} = kH^{-\gamma}(t), \tag{55}$$

where $k$ is a positive constant to be determined later. Substituting (55) into (54) yields

$$\begin{aligned} A'(t) \geq & \epsilon||v_t||_2^2 \\ & + \left[(1-\gamma) - \epsilon k\right]H^{-\gamma}(t)H'(t) - \epsilon||\Delta v||_2^2 - \frac{\epsilon C_1}{4k}H^{\gamma}(t)||v||_2^2 \\ & + \epsilon\int_\Omega |v|^p ln|v|^\ell dx - \frac{\epsilon}{4}\int_0^{+\infty} r(s)||\Delta\mu^t(s)||_2^2 ds. \end{aligned} \tag{56}$$

For $0 < d < 1$, and by inserting value of $\epsilon\int_\Omega |v|^p ln|v|^\ell dx$ by using $H(t) = -E(t)$, (56) becomes

$$\begin{aligned} A'(t) \geq & \left[(1-\gamma) - \epsilon k\right]H^{-\gamma}(t)H'(t) + \epsilon\Big(\frac{p(1-d)}{2} + 1\Big)||v_t||_2^2 \\ & + \frac{\epsilon\ell(1-d)}{p}||v||_p^p + \epsilon\Big[\frac{\lambda p(1-d)}{2} - 1\Big]||\Delta v||_2^2 \\ & + \frac{\epsilon bp(1-d)}{2}\int_\Omega\int_{-\infty}^{+\infty}|\phi(x,\sigma,t)|^2 d\sigma dx \\ & + \epsilon p(1-d)H(t) - \epsilon\frac{C_1}{4k}H^{-\gamma}(t)||v||_2^2 + \epsilon d\int_\Omega |v|^p ln|v|^\ell dx \\ & + \epsilon\Big(\frac{p(1-d)}{2} - \frac{1}{4}\Big)\int_0^{+\infty} r(s)||\Delta\mu^t(s)||_2^2 ds. \end{aligned} \tag{57}$$

By applying Corollary 1 and Young's inequality, we have

$$\begin{aligned} H^{\gamma}(t)||v||_2^2 \leq & \int_\Omega (|v|^p \ln|v|^\ell dx)^\gamma ||v||_2^2 \\ & \leq c\left(\int_\Omega (|v|^p \ln|v|^\ell dx)^{\gamma+\frac{2}{p}} + (\int_\Omega |v|^p \ln|v|^\ell dx)^\gamma\right)||\Delta v||_2^{\frac{4}{p}} \\ & \leq c\left(\int_\Omega (|v|^p \ln|v|^\ell dx)^{\frac{p\gamma+2}{p}} + (\int_\Omega |v|^p \ln|v|^\ell dx)^{\frac{p\gamma}{(p-2)}} + ||\Delta v||_2^2\right). \end{aligned}$$

Using (48), the result is

$$2 < p\gamma + 2 \leq p, and, 2 < \frac{\gamma p^2}{p-2} \leq p. \tag{58}$$

By Lemma 6, we have

$$H^{\gamma}(t)||v||_2^2 \leq c\Big(\int_\Omega |v|^p ln|v|^\ell dx + ||\Delta v||_2^2\Big). \tag{59}$$

for some $cC_1 = C_2 > 0$. By merging Equations (59) and (57), we obtain

$$\begin{aligned} A'(t) \geq & \big[(1-\gamma) - \epsilon k\big] H^{-\gamma}(t) H'(t) + \epsilon\Big(\frac{p(1-d)}{2} + 1\Big)||v_t||_2^2 \\ & + \frac{\epsilon\ell(1-d)}{p}||v||_p^p + \epsilon\Big[(\frac{\lambda p(1-d)}{2} - 1) - \frac{C_2}{4k}\Big]||\Delta v||_2^2 \\ & + \frac{\epsilon b p(1-d)}{2}\int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\sigma,t)|^2 d\sigma dx \\ & + \epsilon p(1-d)H(t) + \epsilon\Big(d - \frac{C_2}{4k}\Big)\int_\Omega |v|^p ln|v|^\ell dx \\ & + \epsilon\Big(\frac{p(1-d)}{2} - \frac{1}{4}\Big)\int_0^{+\infty} r(s)||\Delta\mu^t(s)||_2^2 ds. \end{aligned} \tag{60}$$

Now, we select $d$ to be small enough so that

$$\frac{\lambda p(1-d)}{2} - 1) > 0.$$

Next, we select $k$ to be large enough so that

$$(\frac{\lambda p(1-d)}{2} - 1) - \frac{C_2}{4k} > 0, \quad (d - \frac{C_2}{4k}\Big) > 0.$$

We choose a $\epsilon$ that is small enough while $k$ is fixed so that

$$(1-\gamma) - \epsilon k > 0.$$

Thus, for some $C_3 > 0$, the estimate of (60) is

$$\begin{aligned} \mathcal{A}'(t) \geq & C_3\Big\{H(t) + \|v_t\|_2^2 + \|\Delta v\|_2^2 + \int_\Omega |v|^p \ln|v|^\ell dx + \|v\|_p^p \\ & + b\int_\Omega \int_{-\infty}^{+\infty} |\phi(x,\sigma,t)|^2 d\sigma dx + \int_0^{+\infty} r(s)\|\Delta\mu^t(s)\|_2^2 ds\Big\}, \end{aligned} \tag{61}$$

and

$$\mathcal{A}(t) \geq \mathcal{A}(0) > 0, \quad t > 0. \tag{62}$$

On the other hand, we have

$$\mathcal{A}^{\frac{1}{1-\gamma}}(t) = \Big[H^{1-\gamma} + \varepsilon\int_\Omega v v_t dx\Big]^{\frac{1}{1-\gamma}} \tag{63}$$

$$\leq c\Big[H(t) + \Big|\int_\Omega v v_t dx\Big|^{\frac{1}{1-\gamma}}\Big]. \tag{64}$$

Next, employing Holder's inequality along with the embedding theorem, we obtain

$$\Big|\int_\Omega v v_t dx\Big| \leq \|v\|_2 \cdot \|v_t\|_2 \leq C_4\|v\|_p \cdot \|v_t\|_2. \tag{65}$$

Therefore, by Young's inequality, we have

$$\begin{aligned} \Big|\int_\Omega v v_t dx\Big|^{\frac{1}{1-\gamma}} & \leq c\|v\|_p^{\frac{1}{1-\gamma}} \cdot \|v_t\|_2^{\frac{1}{1-\gamma}} \\ & \leq C_5\Big[\|v\|_p^{\frac{\mu}{1-\gamma}} + \|v_t\|_2^{\frac{\theta}{1-\gamma}}\Big]. \end{aligned} \tag{66}$$

where $\frac{1}{\theta} + \frac{1}{\theta_1} = 1$. We take $\theta_1 = 2(1-\gamma)$, so we obtain

$$\frac{\theta}{1-\gamma} = \frac{2}{1-2\gamma} \leq p.$$

Therefore, if $s = 2/(1-2\gamma)$, we then obtain

$$\left| \int_\Omega v v_t dx \right|^{\frac{1}{1-\gamma}} \leq c \Big[ \|v\|_p^s + \|v_t\|_2^2 \Big].$$

Hence, Lemma 5 provides

$$\left| \int_\Omega v v_t dx \right|^{\frac{1}{1-\gamma}} \leq C_5 \Big[ \|\Delta v\|_2^2 + \|v_t\|_2^2 + \|v\|_p^p \Big]. \tag{67}$$

By putting (67) in (64), we have

$$\begin{aligned} \mathcal{A}^{\frac{1}{1-\gamma}}(t) &\leq C_7 \left[ H(t) + \left| \int_\Omega v v_t dx \right|^{\frac{1}{1-\gamma}} \right] \\ &\leq C_8 \Big[ H(t) + \|\Delta v\|_2^2 + \|v_t\|_2^2 + \|v\|_p^p \Big]. \end{aligned} \tag{68}$$

From (61) and (68), we obtain

$$\mathcal{A}'(t) \geq D \mathcal{A}^{\frac{1}{1-\gamma}}(t). \tag{69}$$

where $C_3, C_4, C_5, C_6, C_7, C_8, D > 0$ are constants Via a simple integration of (69), we obtain

$$\mathcal{A}^{\frac{\gamma}{1-\gamma}}(t) \geq \frac{1}{\mathcal{K}^{\frac{-\gamma}{1-\gamma}}(0) - D\frac{\gamma}{(1-\gamma)}t}.$$

Therefore, $\mathcal{A}(t)$ blows up in time as follows:

$$T \leq T^* = \frac{1-\gamma}{D\gamma \mathcal{A}^{\gamma/(1-\gamma)}(0)}.$$

This completes the proof. □

## 5. Conclusions and Future Work

In this article, an important theoretical analysis of blow-up phenomena in a class of nonlinear logarithmic plate wave equations with fractional damping and infinite memory was presented through the rigorous application of the semigroup theory. The authors established the existence of a local weak solution. The most significant finding, however, is the demonstration that under certain conditions, this local solution can blow up in finite time by constructing an appropriate Lyapunov functional. This suggests an important design principle for controlling the dynamics of nonlinear wave propagation processes governed by similar mathematical models. The global existence and asymptotic behavior of this wave equation will remain to be investigated for future research. Investigating the estimated decay rate of the energy in this system is a crucial next step, as it could provide important insights into the global dynamics and stability of solutions.

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