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Existence and Attractivity of Mild Solutions for Fractional Diffusion Equations Involving the Regularized ψ-Hilfer Fractional Derivatives

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Abstract: The regularized ψ-Hilfer derivative within the sense of Caputo is an improved version of the ψ-Hilfer fractional derivative, primarily because it addresses the issue where the initial conditions of problems involving the ψ-Hilfer fractional derivative lack clear physical significance unless $p = 1$. This article's main contribution is the use of the ψ-Laplace transform, which is the first to provide an explicit expression for mild solutions to the fractional diffusion equations with the regularized ψ-Hilfer derivative. Additionally, we investigate the existence and attractivity of mild solutions for fractional diffusion equations involving the regularized ψ-Hilfer fractional derivatives. Finally, we provide two examples to illustrate our main results.

Keywords: regularized ψ-Hilfer fractional derivative; ψ-Laplace transform; attractivity; existence; mild solution

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1. Introduction

In comparison to integer differential equations, fractional differential equations excel in handling problems involving non-locality and memory effects. Fractional diffusion equations have been extensively researched by numerous scholars in terms of the relevant aspects [1,2].

The ψ-Riemann–Liouville and ψ-Caputo operator are extensions that have generalized the traditional Riemann–Liouville and Caputo operator [3,4]. The distinctive advantage of these fractional operators involving ψ lies in their ability to allow unrestricted choice over both the traditional operator of differentiation and the function denoted by ψ. This enables the unification and derivation of various properties of the earlier mentioned fractional operators.

In 2022, Jajarmi et al. proposed a definition of the regularized version of the ψ-Hilfer fractional derivative within the framework of the Caputo sense [5], thereby addressing the challenge that the initial conditions of problems involving the ψ-Hilfer fractional derivative lack clear physical significance unless $p = 1$ (here, the ψ-Hilfer fractional derivative refers to the definition proposed by Sousa et al. in 2018 [6]). Therefore, this paper will explore an explicit expression for mild solutions to the fractional diffusion equation involving the regularized ψ-Hilfer derivative and further investigate the properties of the solutions to equations involving this derivative.

The existence and attractiveness of solutions to fractional differential equations are issues that many scholars have focused on [7,8]. Zhu [7], employing the Schauder fixed point theorem and the generalized Ascoli–Arzelà theorem, established that the equations

$$\begin{cases} D_{0^+}^{q} x(t) = f(t, x(t)),\ t \in (0, +\infty), \\ \lim\limits_{t \to 0^+} t^{1-q} x(t) = x_0, \end{cases}$$

has at least one globally attractive solution. Where $q \in (0,1)$, $D_{0^+}^{q}$ is the q order Riemann–Liouville derivative and $f : (0, +\infty) \times R \to R$ is continuous and satisfies the following equation:

$$|f(t,x)| \leq l(t)|x|^{\mu} + k(t).$$

Following the research in [7], Jiang et al. [8] researched the global existence, uniqueness and attractivity by using the Leray–Schauder alternative fixed point theorem and more generalized constraints on f than those in [7]. The equations studied in the work in [8] are as follows:

$$\begin{cases} D_{0^+}^{q} x(t) = Ax(t) + f(t, x(t)),\ t \in (0, +\infty), \\ I_{0^+}^{1-\beta} x(t)\big|_{t=0} = x_0. \end{cases}$$

Additionally, we found that due to the regularized ψ-Hilfer fractional derivative being an improved version of the ψ-Hilfer fractional derivative, the initial conditions differ when dealing with fractional equations involving these two operators. In [9], Sousa et al. used the Krasnoselskii fixed point theorem to investigate the attractivity of solutions to the following fractional equations involving the ψ-Hilfer derivative,

$$\begin{cases} {}^{H}D_{t_0^+}^{q,p;\psi} x(t) = f(t, x(t)),\ t \geq t_0, \\ I_{t_0^+}^{1-r;\psi} x(t_0) = x_0. \end{cases}$$

where ${}^{H}D_{t_0^+}^{q,p;\psi}$ and $I_{t_0^+}^{1-r;\psi}$ are ψ-Hilfer fractional derivative and ψ-Riemann–Liouville fractional integral. Then, Jajarmi et al. introduces equations for the initial value problem [5], expressed as

$$\begin{cases} {}^{C}\mathbb{D}_{t}^{q,p;\psi} x(t) = f(t, x(t)),\ t \in (0, +\infty), \\ x(0) = x_{\text{ini}}, \end{cases}$$

where ${}^{C}\mathbb{D}_{t}^{q,p;\psi}$ is the regularized ψ-Hilfer fractional derivative and x_{ini} is the initial condition.

Following the preceding discussion, this paper will addresses the complexities introduced by the ψ function in the process of obtaining mild solutions. We use the ψ-Laplace transform, known as the generalized Laplace transform [10], which extends the classical Laplace transform and is applicable to generalized fractional integrals and derivatives. Together with the semigroup generated by the operator A and probability density functions, we provide a precise expression for mild solutions. The considered equations is defined as follows:

$$\begin{cases} \partial_{t}^{q,p;\psi} u(x,t) = Au(x,t) + f(x,t,u(x,t)),\ (x,t) \in \Omega \times (0,d), \\ u(x,t) = 0,\ (x,t) \in \partial\Omega \times (0,d), \\ u(x,0) = u_0(x),\ x \in \Omega, \end{cases} \tag{1}$$

where $(0,d)$ is a finite or infinite time interval, $\psi \in C^1(0,d)$ is an increasing and positive function, the partial regularized ψ-Hilfer derivative $\partial_{t}^{q,p;\psi}$ of order $q \in (0,1)$ with type $p \in [0,1]$. $u_0(x) \in L^2(\Omega)$, Ω, a subset of $\mathbb{R}^n$, has a smooth boundary denoted by $\partial\Omega$,

and the nonlinear term f represents a specified function. The definition of the coefficient linear operator A is as follows:

$$Au(x,t) = \sum_{a=1}^{n}\sum_{b=1}^{n}\partial_{x_a}\left[c_{ab}(x)\partial_{x_b}u(x,t)\right] - h(x)u(x,t), \tag{2}$$

where

$$c_{ab}(x) \in C^1(\overline{\Omega}),\ a,b \in [1,n],$$

$$\sum_{a,b=1}^{n} c_{ab}(x)\vartheta_a\vartheta_b \geq \varsigma|\vartheta|^2, \vartheta \in \mathbb{R}^n,\ x \in \overline{\Omega},\ \varsigma > 0,$$

$$h(x) \in C(\overline{\Omega}),\ h(x) \geq h_0 > 0,\ x \in \overline{\Omega}.$$

This article is structured as follows: Section 2 provides fundamental knowledge, including the definition of mild solutions of Equation (1). Section 3 focuses on the existence and uniqueness result on the finite interval $[0,d]$. In Section 4, the attractivity of solutions of Equation (1) is proved on the infinite interval $[0,+\infty)$. We provide two examples that illustrate the results of our study.

2. Preliminaries

This section will provide the foundational knowledge needed for what follows.

Let $q \in (0,1)$, $p \in [0,1]$, $J = [0,d)$, $y,\ \psi \in C^1(J)$, and the function ψ is strictly increasing. The regularized ψ-Hilfer fractional derivative is defined by [5]

$$\begin{aligned} {}^{C}\mathbb{D}_t^{q,p;\psi}y(t) &= I_t^{p-pq;\psi}\ {}^{C}D_t^{q+p-pq;\psi}y(t) \\ &= \frac{1}{\Gamma(p-pq)}\int_0^t {}^{C}D_t^{r;\psi}y(s)(\psi(t)-\psi(s))^{p-pq-1}\psi'(s)ds, \end{aligned} \tag{3}$$

where $I_t^{r-q;\psi}(\cdot)$ and ${}^{C}D_t^{r;\psi}(\cdot)$ denote ψ-Riemann–Liouville fractional integral [4] and ψ-Caputo fractional derivative [3], respectively.

Let $X = L^2(\Omega)$, $Y = \{u|u \in C(J,X), \sup_{t\in J}\|u(t)\| < +\infty\}$, with the norm $\|u\|_\infty = \sup_{t\in J}\|u(t)\|$, and $C_0(J,X)$ be the space of functions in Y with $\lim_{t\to+\infty}\|u(t)\| = 0$. It is easy to find that Y and $C_0(J,X)$ are Banach spaces. We define the operator $A : D(A) \subset X \to X$, where $D(A) = H_0^1(\Omega) \cap H^2(\Omega)$ [11]. The operation of A on u at time t and point x is represented as $(Au)(t)x = Au(x,t)$. Consequently, A generates a uniformly bounded analytic semigroup $\{S(t)\}_{t\geq 0}$ on X. There is a constant $K \geq 1$, such that

$$K := \sup_{t\in J}\|S(t)\| \leq +\infty. \tag{4}$$

We define $0 \in \rho(A)$ as the resolvent set, and $R(\lambda,A) = (\lambda I - A)^{-1}$ as the corresponding resolvent operator for A [12], respectively. According to reference [13], we set

$$\lambda^{q-1}(\lambda^q I - A)^{-1}x = \int_0^{+\infty} e^{-\lambda t}\omega(t)xdt,\ t \geq 0,\ x \in X, \tag{5}$$

and

$$(\lambda^q I - A)^{-1}x = \int_0^{+\infty} e^{-\lambda t}\nu(t)xdt,\ t \geq 0,\ x \in X. \tag{6}$$

Then,

$$\omega(t) = \int_0^{+\infty} \xi_q(\theta) S(t^q\theta) d\theta,\ \nu(t) = t^{q-1} g(t),\ g(t) = q\int_0^{+\infty} \theta \xi_q(\theta) S(t^q\theta) d\theta, \tag{7}$$

where $\xi_q(\theta) \geq 0$ represents the probability density function established over the interval $(0, +\infty)$ [14]. We note that

$$\int_0^{+\infty} \theta^\alpha \xi_q(\theta) d\theta = \frac{\Gamma(1+\alpha)}{\Gamma(1+q\alpha)},\ \alpha > 0, \tag{8}$$

where $\int_0^{+\infty} e^{z\theta}\xi_q(\theta)d\theta = E_q(-z)$, and $\int_0^{+\infty} q\theta e^{z\theta}\xi_q(\theta)d\theta = E_{q,q}(-z), z \in \mathbb{C}$, where $E_q(\cdot)$, $E_{q,q}(\cdot)$ are the Mittag–Leffler functions [4].

Lemma 1 ([5]). *If $y \in C^1(J)$, $q \in (0,1)$, $p \in [0,1]$, then we have*

$$I_t^{q;\psi}\ {}^C\mathbb{D}_t^{q,p;\psi} y(t) = y(t) - y(0).$$

Lemma 2 ([10]). *The function f is transformed by a generalized Laplace operation, denoted by $\mathcal{L}_\psi\{f(s)\}$, where ψ is strictly increasing and continuous over the interval $[0,\infty)$. This transform is defined by the integral*

$$\mathcal{L}_\psi\{f(s)\}(\lambda) = \int_0^{+\infty} e^{-\lambda(\psi(s)-\psi(0))} f(s)\psi'(s) ds,$$

where the integral is valid for all values of λ.

Lemma 3 ([10]).

$$\mathcal{L}_\psi\{1\}(\lambda) = \frac{1}{\lambda},\ \lambda > 0.$$

Lemma 4 ([10]). *Consider two functions, f and u, each piecewise continuous on $[0,d]$, and exhibiting exponential growth. Thus,*

$$\mathcal{L}_\psi\{f *_\psi u\}(\lambda) = \mathcal{L}_\psi\{f\}\mathcal{L}_\psi\{u\}.$$

Lemma 5 ([10]). *Given $\alpha > 0$ and f being a piecewise continuous function on $[0,T]$, while also possessing ψ exhibiting exponential growth. Thus,*

$$\mathcal{L}_\psi\{I^{\alpha;\psi(t)}u\}(\lambda) = \lambda^{-\alpha}\mathcal{L}_\psi\{u\}.$$

Lemma 6. (i) $\|\omega(\psi(t))\| \leq K$, $\|g(\psi(t))\| \leq \frac{K}{\Gamma(q)}$, for $t \geq 0$.

(ii) The families $\{\omega(\psi(t))\}_{t\geq 0}$ and $\{g(\psi(t))\}_{t\geq 0}$ exhibit strongly continuity, which means that for any $x \in X, t_1, t_2 \in J$, then

$$\|\omega(\psi(t_2))x - \omega(\psi(t_1))x\| \to 0,\ \|g(\psi(t_2))x - g(\psi(t_1))x\| \to 0 \text{ as } t_1 \to t_2.$$

(iii) For $t > 0$, $\omega(\psi(t))$ and $g(\psi(t))$ are compact if $S(t)$ is compact.

(iv) If $\{S(t)\}_{t\geq 0}$ exhibits exponential stability, this means there that are constants $M, \delta > 0$ for which

$$\|S(t)\| \leq Me^{-\delta t},\ \text{for } t \geq 0, \tag{9}$$

then

$$\|\omega(\psi(t))\| \leq ME_q\big(-\delta(\psi(t))^q\big),\ \text{for } t \geq 0, \tag{10}$$

$$\|g(\psi(t))\| \leq ME_{q,q}\big(-\delta(\psi(t))^q\big), \text{ for } t \geq 0. \tag{11}$$

Proof. Due to the fact that $\psi \in C^1(J)$ is an increasing and positive function, (i)–(iii) can be deduced from [13]. We prove (iv). By (7) and (9), we have

$$\begin{aligned}\|\omega(\psi(t))\| &= \left\|\int_0^{+\infty} \xi_q(\theta) S\big((\psi(t))^q\theta\big)d\theta\right\| \leq M\int_0^{+\infty} \xi_q(\theta)e^{-\delta(\psi(t))^q\theta}d\theta \\ &= ME_q\big(-\delta(\psi(t))^q\big),\end{aligned}$$

$$\begin{aligned}\|g(\psi(t))\| &= \left\|q\int_0^{+\infty} \theta\xi_q(\theta) S\big((\psi(t))^q\theta\big)d\theta\right\| \leq qM\int_0^{+\infty} \theta\xi_q(\theta)e^{-\delta(\psi(t))^q\theta}d\theta \\ &= ME_{q,q}\big(-\delta(\psi(t))^q\big).\end{aligned}$$

□

By defining $u(t)(x) = u(x,t)$ and $f(t,u(t))(x) = f(x,t,u(x,t))$, then Equation (1) can be reformulated as a set of abstract equations,

$$\begin{cases} {}^C\mathbb{D}_t^{q,p;\psi}u(t) = Au(t) + f(t,u(t)),\ t \in (0,d), \\ u(0) = u_0, \end{cases} \tag{12}$$

where ${}^C\mathbb{D}_{t_0^+}^{q,p;\psi}$ denotes the q order with type p regularized ψ-Hilfer derivative. Let $f : J \times X \to X$ be a function, where u_0 belongs to X.

According to Lemma 2, we can deduce that Equation (12) is equivalent to the integral equation,

$$u(t) = u_0 + I_t^{q;\psi}[Au(t) + f(t,u(t))]. \tag{13}$$

We apply the ψ-Laplace transform in Lemma 2

$$U(\lambda) = \mathcal{L}_\psi\{u(s)\}(\lambda) = \int_0^{+\infty} e^{-\lambda(\psi(s)-\psi(0))}\psi'(s)u(s)ds,$$

and

$$F(\lambda) = \mathcal{L}_\psi\{f(s,u(s))\}(\lambda) = \int_0^{+\infty} e^{-\lambda(\psi(s)-\psi(0))}\psi'(s)f(s,u(s))ds,\ \lambda > 0,$$

to (13) and, by using the properties of the ψ-Laplace transform Lemmas 3–5, we have

$$\begin{aligned}U(\lambda) &= \frac{1}{\lambda}u_0 + \frac{1}{\lambda^q}AU(\lambda) + \frac{1}{\lambda^q}F(\lambda) \\ &= \lambda^{q-1}(\lambda^q I - A)^{-1}u_0 + (\lambda^q I - A)^{-1}F(\lambda).\end{aligned} \tag{14}$$

Using (5), (6), $(P_\psi f)(x) = f(\psi(x))$, $\mathcal{L}_\psi^{-1} = P_\psi \circ \mathcal{L}^{-1}$, and Theorem 3.4 from [15], we can obtain the solution using the inverse ψ-Laplace transform,

$$u(t) = \omega(\psi(t)-\psi(0))u_0 + \int_0^t (\psi(t)-\psi(s))^{q-1}g(\psi(t)-\psi(s))\psi'(s)f(s,u(s))ds. \tag{15}$$

Definition 1. *If u satisfies (15), the function u in the space $C(J,X)$ is called a mild solution of Equation (12).*

Definition 2. *The mild solution $u(t)$ of Equation (12) is said to be attractive if $u(t) \to 0$ as $t \to +\infty$.*

To estimate the solution operator reasonably, we also need the following inequality.

Lemma 7 ([11]). *If $\alpha < 2$, $\beta \in \mathbb{R}$, and $\pi\alpha/2 < \mu < min\{\pi, \pi\alpha\}$, then there exists a constant N, such that*

$$|E_{\alpha,\beta}(z)| \leq \frac{N}{1+|z|},$$

$$\mu \leq |\arg(z)| \leq \pi, \ z \in \mathbb{C}.$$

Lemma 8 ([16]). *$G \subset C_0(J, X)$ is relatively compact, and is equivalent to the following conditions:*
(i)The function in G is equicontinuous on $[0, b]$, for $b > 0$.
(ii) $G(t) = \{x(t) : x \in G\}$ is relatively compact in X, for $t \in [0, +\infty)$.
(iii) $\lim_{t \to +\infty} |x(t)| = 0$ uniformly for $x \in G$.

Lemma 9 ([17,18]). *For a completely continuous mapping $F : X \to X$, where X is characterized as a convex subset within a Banach space and containing 0, one of the following must hold: either F has a fixed point, or the set $\{z \in X | z = \varsigma F(z)\}$ is unbounded, where $0 < \varsigma < 1$.*

3. Existence and Uniqueness of Mild Solutions

To study the existence and uniqueness of mild solutions of Equation (12) on the finite interval $[0, d]$, the subsequent assumptions are given.

(H_1) Let L_1, f be continuous functions, and $f : J \times X \to X$ satisfies the following condition:

$$\|f(t, u_1(t)) - f(t, u_2(t))\| \leq L_1(t)\|u_1(t) - u_2(t)\|, \text{ for } t \in J, \ u_1, \ u_2 \in Y.$$

(H_1') Assume that L_1, f are continuous functions, and $f : J \times X \to X$ satisfies the following condition:

$$\|f(t, u(t))\| \leq L_1(t)\|u(t)\|, \text{ for } t \in J, \ u \in Y.$$

(H_2) Let $0 < L < 1$ be a constant satisfies the condition

$$\sup_{t \in J} K I_t^{q;\psi} L_1(t) \leq L,$$

where K is given by (4).

(H_3) $\{S(t)\}_{t \geq 0}$ is compact.

(H_4) For every $t \in J$, the function $f(t, u(t))$ is continuous with respect to $u(t) \in X$. Moreover, for any $u(t) \in X$, the function $f(t, u(t))$ is strongly measurable in $t \in J$.

Theorem 1. *Equation (12) has a unique mild solution, provided that Assumptions (H_1) and (H_2) are satisfied.*

Proof. One can consider the operator defined as

$$Qu(t) = \omega(\psi(t) - \psi(0))u_0 + \int_0^t (\psi(t) - \psi(s))^{q-1} g(\psi(t) - \psi(s))\psi'(s) f(s, u(s)) ds. \quad (16)$$

Owing to Lemma 6, (H_1), and (H_2), the existence of a mild solution to Equation (12) is equivalent to the presence of a fixed point for the operator $Q : Y \to Y$. We have

$$\begin{aligned}
&\|Qu_1(t) - Qu_2(t)\| \\
&\leq \left\| \int_0^t (\psi(t) - \psi(s))^{q-1} g(\psi(t) - \psi(s)) \psi'(s) \big(f(s, u_1(s)) - f(s, u_2(s))\big) ds \right\| \\
&\leq \frac{K\|u_1 - u_2\|_\infty}{\Gamma(q)} \int_0^t (\psi(t) - \psi(s))^{q-1} \psi'(s) L_1(s) ds \\
&\leq L\|u_1 - u_2\|_\infty. \qquad (17)
\end{aligned}$$

Therefore, Banach contraction mapping principle asserts that Equation (12) has a unique mild solution within the space Y. □

Theorem 2. *Given that hypotheses (H_1') and (H_2)–(H_4) are fulfilled, it can be deduced that Equation (12) has a mild solution.*

Proof. Regarding K_0 as a positive constant, $B_{K_0} = \{u \in C(J, X) : \|u\| \leq K_0\}$. It is apparent that B_{K_0} forms a bounded, closed, and convex subset within the function space $C(J, X)$.

According to Lemma 6, we can deduce that

$$\|\omega(\psi(t))u_0\| \leq K\|u_0\|.$$

In accordance with Assumption (H_4), the function f is measurable over the interval J. By Lemma 6, (H_1'), and (H_2), we obtain

$$\begin{aligned}
&\int_0^t \|(\psi(t) - \psi(s))^{q-1} g(\psi(t) - \psi(s)) \psi'(s) f(s, u(s))\| ds \\
&\leq \frac{KK_0}{\Gamma(q)} \int_0^t (\psi(t) - \psi(s))^{q-1} \psi'(s) L_1(s) ds \\
&\leq K_0 L.
\end{aligned}$$

Therefore, the norm of $(\psi(t) - \psi(s))^{q-1} g(\psi(t) - \psi(s)) \psi'(s) f(s, u(s))$ is Lebesgue integrable over $s, t \in J$; then, according to Bochner's theorem, it can be deduced that it is also Bochner integrable over $s, t \in J$.

It is imperative to establish the existence of a fixed point for Q on B_{K_0}, where Q is defined by (16). Our proof will be structured into three sequential steps.

Step 1. Operator Q is continuous on B_{K_0}.

For any $u_n, u \in B_{K_0}$, we have $\lim_{n \to +\infty} \|u_n - u\| = 0$. Hence, by the properties of f stated in (H_4), we obtain $\lim_{n \to +\infty} f(t, u_n) = f(t, u)$. Thus,

$$\begin{aligned}
&\|(Qu_n)(t) - (Qu)(t)\| \\
&\leq \int_0^t \|(\psi(t) - \psi(s))^{q-1} g(\psi(t) - \psi(s)) \psi'(s) [f(s, u_n(s)) - f(s, u(s))]\| ds.
\end{aligned}$$

By the Lebesgue dominated convergence theorem, we obtain $\|(Qu_n)(t) - (Qu)(t)\| \to 0$, as $n \to +\infty$, indicating that Q is continuous.

Step 2. $\{Qu | u \in B_{K_0}\}$ is relatively compact.

We need to demonstrate that $\{Qu | u \in B_{K_0}\}$ is uniformly bounded and equicontinuous. Additionally, for all $t \in J$, $\{Qu | u \in B_{K_0}\}$ is relatively compact in X.

Firstly, a positive constant K_0 is present, satisfying

$$\frac{K\|u_0\|}{1 - L} \leq K_0.$$

This implies that $\|(Qu)(t)\| \leq K_0$, which means Q is uniformly bounded.

Then, for any $t_1 < t_2$, $t_1, t_2 \in J$ and $u \in B_{K_0}$, we obtain $\|(Qu)(t_2) - (Qu)(t_1)\| \leq I_1 + I_2 = I_3$. The details are as follows:

$$\begin{aligned}
I_1 &= \|\omega(\psi(t_2) - \psi(0))u_0 - \omega(\psi(t_1) - \psi(0))u_0\|, \\
I_2 &= \int_{t_1}^{t_2} \|(\psi(t_2) - \psi(s))^{q-1} g(\psi(t_2) - \psi(s))\psi'(s) f(s, u(s))\| ds, \\
I_3 &= \int_0^{t_1} \Big\| \big[(\psi(t_2) - \psi(s))^{q-1} g(\psi(t_2) - \psi(s)) - (\psi(t_1) - \psi(s))^{q-1} \\
&\quad g(\psi(t_1) - \psi(s))\big]\psi'(s) f(s, u(s))\Big\| ds.
\end{aligned}$$

By Lemma 6, we can readily deduce $I_1 \to 0$ as $t_1 \to t_2$. In addition to (H_1'), we obtain

$$\begin{aligned}
I_2 &\leq \frac{KK_0}{\Gamma(q)} \int_{t_1}^{t_2} (\psi(t_2) - \psi(s))^{q-1} \psi'(s) ||L_1(s)|| ds \\
&\leq \frac{KK_0 \|L_1\|_\infty}{q\Gamma(q)} (\psi(t_2) - \psi(t_1))^q.
\end{aligned}$$

Due to the continuity of ψ, we can deduce $I_2 \to 0$ as $t_1 \to t_2$.

$$\begin{aligned}
I_3 &\leq \int_0^{t_1} \left[(\psi(t_2) - \psi(s))^{q-1} - (\psi(t_1) - \psi(s))^{q-1}\right] \big\| g(\psi(t_2) - \psi(s))\psi'(s) f(s, u(s))\big\| ds \\
&\quad + \int_0^{t_1} (\psi(t_1) - \psi(s))^{q-1} \psi'(s) \big\| g(\psi(t_2) - \psi(s)) - g(\psi(t_1) - \psi(s))\big\| \|f(s, u(s))\| ds \\
&\leq \frac{KK_0 \|L_1\|_\infty}{q\Gamma(q)} \left[(\psi(t_2) - \psi(0))^q - (\psi(t_1) - \psi(0))^q - (\psi(t_2) - \psi(t_1))^q\right] \\
&\quad + K_0 \|L_1\|_\infty \int_{t_1 - \varepsilon}^{t_1} (\psi(t_1) - \psi(s))^{q-1} \psi'(s) ds \\
&\quad \sup_{s \in [0, t_1 - \varepsilon]} \big\| g(\psi(t_2) - \psi(s)) - g(\psi(t_1) - \psi(s))\big\| \\
&\quad + \frac{2KK_0 \|L_1\|_\infty}{q\Gamma(q)} (\psi(t_1) - \psi(t_1 - \varepsilon))^q \to 0 \text{ as } t_1 \to t_2 \text{ and } \varepsilon \to 0.
\end{aligned}$$

Therefore, we obtain that $\{Qu | u \in B_{K_0}\}$ is equicontinuous.
Obviously, $\{(Qu)(0) | u \in B_{K_0}\}$ is relatively compact in X. We set

$$\begin{aligned}
&(Q_{\varepsilon,\delta} u)(t) \\
&\quad = \omega(\psi(t) - \psi(0))u_0 + S(\varepsilon^q \delta) \int_0^{t-\varepsilon} \int_\delta^{+\infty} q\theta(\psi(t) - \psi(s))^{q-1} \xi_q(\theta) \\
&\quad S((\psi(t) - \psi(s))^q \theta - \varepsilon^q \delta)\psi'(s) f(s, u(s)) d\theta ds.
\end{aligned}$$

For $\varepsilon \in (0, t)$ and $\delta > 0$, the compactness of $S(\varepsilon^q \delta)$ implies that $\{(Q_{\varepsilon,\delta} u)(t) | u \in B_{K_0}\}$ is relatively compact in X. Furthermore, for $u \in B_{K_0}$, by Lemma 6, (H_1'), and (H_2), we obtain

$$\begin{aligned}
&\|(Qu)(t) - (Q_{\varepsilon,\delta} u)(t)\| \\
&\leq \int_0^t \int_0^\delta \big\| q\theta(\psi(t) - \psi(s))^{q-1} \xi_q(\theta) S((\psi(t) - \psi(s))^q \theta)\psi'(s) f(s, u(s)) d\theta\big\| ds \\
&\quad + \int_{t-\varepsilon}^t \int_\delta^{+\infty} \big\| q\theta(\psi(t) - \psi(s))^{q-1} \xi_q(\theta) S((\psi(t) - \psi(s))^q \theta)\psi'(s) f(s, u(s)) d\theta\big\| ds \\
&\leq qK_0 L\Gamma(q) \int_0^\delta \theta \xi_q(\theta) d\theta + qKK_0 \|L_1\|_\infty (\psi(\varepsilon) - \psi(0))^q.
\end{aligned}$$

Hence, given the proximity of relatively compact sets to $\{(Q_{\varepsilon,\delta} u)(t) | u \in B_{K_0}\}$ for $t > 0$, for any $t \in J$, by Lemma 8, we know that Q is relatively compact.

Step 3. This, coupled with the continuity of Q, will prove that the mapping $Q : B_{K_0} \to B_{K_0}$ is completely continuous.

We can define

$$N_0 = \{u \in B_{K_0} | u = \alpha Q u,\ 0 < \alpha < 1\}.$$

Certainly, $0 \in N_0$. For $u \in N_0$, we obtain

$$\|u(t)\| \leq \alpha(K\|u_0\| + K_0 L),$$

for $t \in J$. Using Lemma 9 as a basis, we can derive that Q has a fixed point in B_{K_0}, which means Equation (12) has a mild solution. □

4. Attractivity of Mild Solutions

To study the attractivity of Equation (12) on the infinite interval $[0, \infty)$, we also need the following assumptions.

(H_5) $f : [0, +\infty) \times X \to X$ is continuous, and for all $t \geq 0$, we have $\|f(t,u)\| < L_2(\psi(t) - \psi(0))^{-\beta}$ and $u \in \big(C_0([0, +\infty), X)\big)$, where $L_2 > 0$ is a constant, $q < \beta < 1$.

(H_6) $\psi(t) : [0, +\infty) \to \mathbb{R}^+$ is compulsory. That is, for sequences $\{t_n\} \subset [0, +\infty)$, and $\lim\limits_{n\to\infty} t_n = +\infty$, we have $\lim\limits_{n\to+\infty} \psi(t_n) = +\infty$.

Theorem 3. *Assume that $\{S(t)\}_{t\geq 0}$ is a compact and exponentially stable semigroup, and satisfies (9). If (H_5) and (H_6) hold, then Equation (12) has an attractive mild solution.*

Proof. For a constant $\gamma < q$, we select a constant T, such that

$$\frac{NM}{\delta}(\psi(t) - \psi(0))^{\gamma - q} + L_2 MNB(1 - \beta, q)(\psi(t) - \psi(0))^{\gamma + q - \beta} \leq 1, \tag{18}$$

for $t \geq T$. We set $\Omega = \{u | u \in C([0, +\infty), X),\ u(t) \leq (\psi(t) - \psi(0))^{-\gamma}, \}$. Then, Ω forms a nonempty, closed, convex and bounded set in $C_0([0, +\infty), X)$, given that condition (H_5) is satisfied. As a result of (H_5), there exists $T_1 > 0$, satisfying $L_2 MNB(1 - \beta, q)(\psi(t) - \psi(0))^{q-\beta} < \frac{\epsilon}{4}$, $\frac{NM\|u_0\|}{1+\delta(\psi(t)-\psi(0))^q} < \frac{\epsilon}{4}$ for $t > T_1$. We need to prove that Q has a fixed point in Ω, where Q is given by (16).

Step 1. We show that $\{Qu | u \in \Omega\}$is equicontinuous and uniformly bounded for $u \in \Omega$.

For $u \in \Omega$, $t_1, t_2 \geq T_1$, by Lemma 6 and Lemma 7, we have

$$\begin{aligned}
&\|Qu(t_2) - Qu(t_1)\| \\
&\quad \leq \|\omega(\psi(t_2) - \psi(0))u_0\| + \|\omega(\psi(t_1) - \psi(0))u_0\| \\
&\qquad + \int_0^{t_2} (\psi(t_2) - \psi(s))^{q-1} \|g(\psi(t_2) - \psi(s))\psi'(s) f(s, u(s))\| ds \\
&\qquad + \int_0^{t_1} (\psi(t_1) - \psi(s))^{q-1} \|g(\psi(t_1) - \psi(s))\psi'(s) f(s, u(s))\| ds \\
&\quad \leq \frac{\epsilon}{2} + L_2 MN \int_0^{\psi(t_2)-\psi(0)} \tau^{q-1} (\psi(t_2) - \psi(0) - \tau)^{-\beta} d\tau \\
&\qquad + L_2 MN \int_0^{\psi(t_1)-\psi(0)} \tau^{q-1} (\psi(t_1) - \psi(0) - \tau)^{-\beta} d\tau \\
&\quad \leq \frac{\epsilon}{2} + L_2 MNB(1 - \beta, q)[(\psi(t_2) - \psi(0))^{q-\beta} + (\psi(t_1) - \psi(0))^{q-\beta}] \\
&\quad \leq \epsilon,
\end{aligned} \tag{19}$$

where $B(\cdot, \cdot)$ is the Beta function.

Applying Lemma 6 and Lebesgue dominated convergence theorem, following $0 \le t_1 < t_2 \le T_1$, we have

$$\begin{aligned}
&\|(Qu)(t_2) - (Qu)(t_1)\| \\
&\quad \le \|\omega(\psi(t_2) - \psi(0))u_0 - \omega(\psi(t_1) - \psi(0))u_0\| \\
&\qquad + \left\| \int_0^{t_1} \left((\psi(t_2) - \psi(s))^{q-1} - (\psi(t_1) - \psi(s))^{q-1}\right) g(\psi(t_2) - \psi(s))\psi'(s) f(s, u(s)) ds \right\| \\
&\qquad + \left\| \int_{t_1}^{t_2} (\psi(t_2) - \psi(s))^{q-1} g(\psi(t_2) - \psi(s))\psi'(s) f(s, u(s)) ds \right\| \\
&\qquad + \left\| \int_0^{t_1} (\psi(t_1) - \psi(s))^{q-1} \big(g(\psi(t_2) - \psi(s)) - g(\psi(t_1) - \psi(s))\big)\psi'(s) f(s, u(s)) ds \right\| \\
&\quad \le \|\omega(\psi(t_2) - \psi(0))u_0 - \omega(\psi(t_1) - \psi(0))u_0\| \\
&\qquad + \frac{MN}{q} \sup_{t\in[0,T_1]} \|f(t, u(t))\| [(\psi(t_2) - \psi(0))^q - (\psi(t_1) - \psi(0))^q + (\psi(t_2) - \psi(t_1))^q] \\
&\qquad + \frac{MN}{q} \sup_{t\in[0,T_1]} \|f(t, u(t))\| (\psi(t_2) - \psi(t_1))^q \\
&\qquad + \int_0^{t_1} (\psi(t_1) - \psi(s))^{q-1} \big\| \big(g(\psi(t_2) - \psi(s)) - g(\psi(t_1) - \psi(s))\big) f(s, u(s)) ds \big\| \\
&\quad \to 0 \text{ as } t_2 \to t_1.
\end{aligned} \tag{20}$$

Due to (19) and (20), $\{Qu|\ u \in \Omega\}$ is equicontinuous. Next, we prove that $\lim_{n\to+\infty} \|Qu(t)\| = 0$ uniformly for $u \in \Omega$. In fact, similarly to (19), for $u \in \Omega$, (H_6) implies that

$$\|(Qu)(t)\| \le \frac{MN\|u_0\|}{1 + \delta(\psi(t) - \psi(0))^q} + L_2 MNB(1-\beta, q)(\psi(t) - \psi(0))^{q-\beta} \to 0 \text{ as } t \to +\infty.$$

Step 2. $Q : \Omega \to \Omega$ is continuous.

For $u \in \Omega$, by (19) and (20), we obtain $Qu \in C_0([0, +\infty), X)$. Due to ($H_5$), Lemma 6, and (18), we have

$$\begin{aligned}
&\|(Qu)(t)\| \\
&\quad \le \left[\frac{NM}{\delta}(\psi(t) - \psi(0))^{\gamma-q} + L_2 MNB(1-\beta, q)(\psi(t) - \psi(0))^{\gamma+q-\beta}\right][\psi(t) - \psi(0)]^{-\gamma} \\
&\quad \le (\psi(t) - \psi(0))^{-\gamma},
\end{aligned}$$

for $t \ge T$. That is, $Q\Omega \subset \Omega$.

Next, we show that Q is continuous in Ω. For $u_n, u \in \Omega$ with $u_n \to u$ as $n \to +\infty$, one obtains

$$\begin{aligned}
&\|(Qu_n)(t) - (Qu)(t)\| \\
&\quad \le \int_0^t (\psi(t) - \psi(s))^{q-1} \|g(\psi(t) - \psi(s))\psi'(s)\big(f(s, u_n(s)) - f(s, u(s))\big)\| ds \\
&\quad \le 2L_2 MN \int_0^t (\psi(t) - \psi(s))^{q-1} (\psi(s) - \psi(0))^{-\beta} ds \\
&\quad \le 2L_2 MNB(1-\beta, q)(\psi(t) - \psi(0))^{q-\beta}.
\end{aligned}$$

From (H_4), there exists $T_2 > 0$, such that $(Qu_n)(t) \to (Qu)(t)$ as $n \to \infty$ for $t > T_2$. If $0 \le t \le T_2$, it follows that

$$\|(Qu_n)(t) - (Qu)(t)\| \le MN \int_0^t (\psi(t) - \psi(s))^{q-1} \psi'(s) \|f(s, u_n(s)) - f(s, u(s))\| ds.$$

In view of the continuity of f, the property of ψ, and Lebesgue dominated convergence theorem, we have $\|(Qu_n)(t) - (Qu)(t)\| \to 0$, as $n \to +\infty$. Then, $\|Qu_n - Qu\| \to 0$, as $n \to +\infty$, which implies that Q is continuous.

Step 3. By Step 2, it is obtained that the mapping $Q : \Omega \to \Omega$ is bounded and continuous; we shall show that Q is relatively compact. Based on the results of Step 1, we conclude that the set $\{Qu | u \in n\}$ is equicontinuous, and $\lim_{n \to +\infty} \|(Qu)(t)\| = 0$ uniformly for $u \in \Omega$. For $t \ge 0$, we need to prove that $\{(Qu)(t) | u \in \Omega\}$ is relatively compact in X. Furthermore, for $u \in \Omega$, by (9) and (H_5), we obtain

$$\begin{aligned}
&\|(Qu)(t) - (Q_{\varepsilon,\eta} u)(t)\| \\
&\le \left\| \int_0^t \int_0^\eta q\theta (\psi(t) - \psi(s))^{q-1} \xi_q(\theta) S((\psi(t) - \psi(s))^q \theta) \psi'(s) f(s, u(s)) d\theta ds \right\| \\
&\quad + \left\| \int_{t-\varepsilon}^t \int_\eta^{+\infty} q\theta (\psi(t) - \psi(s))^{q-1} \xi_q(\theta) S((\psi(t) - \psi(s))^q \theta) \psi'(s) f(s, u(s)) d\theta ds \right\| \\
&\le ML_2 q B(q, 1-\beta) (\psi(t) - \psi(0))^{q-\beta} \int_0^\eta \theta \xi_q(\theta) d\theta \\
&\quad + ML_2 \frac{1}{\Gamma(q+2)} (\psi(t-\varepsilon) - \psi(0))^{-\beta} (\psi(t) - \psi(t-\varepsilon))^q \to 0 \text{ as } \varepsilon, \eta \to 0,
\end{aligned}$$

which implies that $\{(Q(u)(t)) | u \in \Omega\}$ is relatively compact in X, where $B(\cdot, \cdot)$ is the Beta function. By Lemma 8, we know that Q is relatively compact. Then, the Schauder fixed-point theorem implies that Equation (12) has a mild solution $u \in \Omega$, and u is attractive. □

Remark 1. *If $A = 0$, the corresponding integer-order equation to Equation (1) does not have one attractive mild solution [19]. This indicates that there is a significant difference between the attractivity of solutions in both fractional order and integer order equations.*

5. Examples

Example 1. *With $X = L^2([0,2], \mathbb{R})$, we consider the fractional order partial differential equations*

$$\begin{cases} \partial_t^{\frac{1}{5}, \frac{3}{7}; t^2} u(x,t) = \partial_x^2 u(x,t) + f(x,t,u(x,t)), \ (x,t) \in [0,2] \times (0, \frac{1}{2}], \\ u(0,t) = u(2,t) = 0, \ t \in (0, \frac{1}{2}], \\ u(x,0) = u_0(x), \ x \in [0,2], \end{cases} \tag{21}$$

where $d = \frac{1}{2}$, and $\partial_t^{\frac{1}{5}, \frac{3}{7}; t^2}$ is the partial regularized ψ-Hilfer derivative of order $\frac{1}{5}$ with type $\frac{3}{7}$, $u_0(x) \in X$.

Consider the differential operator $Au = u''$ with the domain specified as

$$D(A) = \{u(\cdot) \in X | u \text{ and } u' \text{are absolutely continuous}, u'' \in X, u = 0 \text{ on } \partial\Omega\}.$$

This operator A generates a compact, strongly continuous semigroup $\{S(t)\}_{t \ge 0}$. The semigroup is given by

$$S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t} (u, z_n) z_n,$$

where the normalized eigenvector set $z_n = sin(nx)$ *corresponds to the eigenvalue set* $n^2 (n \in \mathbb{N})$ *of* A. *Therefore, for all* $t \geq 0$, $S(t)$ *is compact, and its elements satisfy* $\|S(t)\| \leq e^{-t}$. *We have*

$$\omega(t^2) = \int_0^\infty \xi_{\frac{1}{5}}(\theta) S\left(t^{\frac{2}{5}}\theta\right) d\theta,$$

$$g(t^2) = \frac{1}{5} \int_0^\infty \theta \xi_{\frac{1}{5}}(\theta) S\left(t^{\frac{2}{5}}\theta\right) d\theta,$$

and

$$\|\omega(t^2)\| \leq 1,$$

$$\|g(t^2)\| \leq \frac{1}{\Gamma(\frac{1}{5})},$$

which satisfies Lemma 6.

By defining $u(t)(x) = u(x,t)$ *and* $f(t,u(t))(x) = f(x,t,u(x,t))$, *Equation (21) can be formulated as abstract equations,*

$$\begin{cases} {}^{C}\mathbb{D}_t^{\frac{1}{5},\frac{3}{7};t^2} u(t) = Au(t) + f(t,u(t)), \ t \in (0, \frac{1}{2}], \\ u(0) = u_0. \end{cases} \tag{22}$$

Let us define $f(t,u(t))(x) = \frac{1}{2} \sin u(t)$. (H_1)–(H_4), *and* (H_1') *are satisfied where* $L_1(t) = 1$ *and*

$$\begin{aligned} &\sup_{t \in [0,\frac{1}{2}]} \frac{2}{\Gamma(\frac{1}{5})} \int_0^t (t^2 - s^2)^{-\frac{4}{5}} s ds \\ &\leq \frac{1}{\Gamma(\frac{6}{5})} (\frac{1}{2})^{\frac{2}{5}} \\ &\approx 0.8254. \end{aligned}$$

Then, $0 < L = 0.8254 < 1$. *Obviously, by Theorem 1 or Theorem 2, Equation (22) has a unique mild solution.*

Example 2. *Similarly to Example 1, consider the same operator* $Au = u''$, $\{S(t)\}_{t \geq 0}$, $q = \frac{1}{5}$, $p = \frac{1}{4}$ *and* $\psi = t^2$. *We let* $d = +\infty$; (H_5) *and* (H_6) *are satisfied where* $\beta = \frac{1}{2}$, $\gamma = \frac{1}{6}$, *and* $L_2 = 2$. *Clearly, Equation (22) has an attractive mild solution on* $[0, +\infty)$.

6. Conclusions

The initial value problem involving a regularized ψ-Hilfer derivative has a clear physical meaning. The innovation of this paper lies in combining with the ψ-Laplace transform and providing a reasonable definition for the mild solution of such equations for the first time. This lays the foundation for further qualitative analysis of related equations and is of essential significance. Meanwhile, the techniques used in handling ψ when studying the existence, uniqueness, and attractiveness of solutions are also worth learning from.

In the future, we will further investigate the representation, existence, and regularity of solutions to the initial value problem of fractional diffusion equations involving the regularized ψ-Hilfer derivative.

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