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Constructive Conditions for a High-Dimensional Hilbert-Type Integral Inequality Involving Multivariate Variable Upper Limit Integral Functions and Optimal Constant Factors

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Abstract: Hilbert integral inequalities are beautiful inequalities with a symmetric structure, and have attracted much attention because of their important applications in the study of integral operators, and the Hilbert-type integral inequality involving variable upper limit integral functions is a generalized form of the Hilbert integral inequality. In this paper, we use a construction theorem of the Hilbert-type n-ple integral inequality with homogeneous kernels to discuss a high-dimensional Hilbert-type integral inequality involving multivariate variable upper limit integral functions, and obtain the sufficiently necessary conditions for constructing inequalities and the formulas for optimal constant factors, which improve and generalize the existing results.

Keywords: high-dimensional Hilbert-type integral inequality; multivariate variable upper limit integral function; homogeneous kernel; Gamma function; optimal constant factor

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1. Introduction

In 1934, Hardy et al. [1] derived a structurally elegant Hilbert integral inequality as follows:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{x+y}f(x)g(y)\mathrm{d}x\mathrm{d}y \le \frac{\pi}{\sin(\frac{\pi}{p})}\|f\|_p\|g\|_q.$$

Due to its significant applications in operator theory and various analytical disciplines, this inequality has received widespread attention. For instance, Salem and Xu et al. [2–5] introduced independent parameters and obtained some homogeneous kernel Hilbert-type inequalities; Krnic and Pecaric [6] considered the reverse Hilbert-type inequality; You and Guan [7] discussed a non-homogeneous kernel Hilbert-type inequality; Yang [8] explored the relationship between Hilbert's inequalities and Mulholland's inequalities; Hong and Rassias et al. [9,10] studied the conditions for optimal matching parameters in Hilbert-type inequalities. In 2019, Adiyasuren et al. [11] introduced partial sums $A_m = \sum_{i=1}^{m} a_i$ and $B_n = \sum_{i=1}^{n} b_i$, and discussed discrete Hilbert-type inequalities involving A_m and B_n:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{(m+n)^{\lambda}}a_m b_n \le M\|A_m\|_{p,\alpha}\|B_n\|_{q,\beta},$$

where $\frac{1}{p}+\frac{1}{q}=1\,(p>1)$. Mo and Yang [12] considered Hilbert-type integral inequalities involving variable upper limit integral functions $F(x)=\int_0^x f(t)\mathrm{d}t$ and $G(y)=\int_0^y g(t)\mathrm{d}t$.

Zhong and Yang [13] extended related results to multiple integral forms, and obtained Hilbert-type integral inequalities with optimal constant factors:

$$\int_0^{+\infty}\cdots\int_0^{+\infty}\frac{1}{(\sum_{i=1}^n x_i)^{\lambda}}\prod_{i=1}^n f_i(x_i)\mathrm{d}x_1\cdots\mathrm{d}x_n$$
$$\le\frac{1}{\Gamma(\lambda)}\prod_{i=1}^n\frac{\lambda}{r_i}\Gamma(\frac{\lambda}{r_i})\Big(\int_0^{+\infty}x_i^{-p_i\frac{\lambda}{r_i}-1}F_i^{p_i}(x_i)\mathrm{d}x_i\Big)^{\frac{1}{p_i}},\tag{1}$$

where $\sum_{i=1}^n\frac{1}{p_i}=1\,(p_i>1)$, $\sum_{i=1}^n\frac{1}{r_i}=1$, $F_i(x_i)=\int_0^{x_i}f_i(t)\mathrm{d}t$ (Note: The original text incorrectly wrote $\frac{1}{\Gamma(\lambda)}$ as $\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}$). Luo et al. [14] discussed Hilbert-type integral inequalities involving multiple variable upper limit integral functions

$$F_m(x)=\int_0^x\Big(\int_0^{t_{m-1}}\cdots\int_0^{t_1}f(t_0)\mathrm{d}t_0\cdots\mathrm{d}t_{m-2}\Big)\mathrm{d}t_{m-1},$$

$$G_n(y)=\int_0^y\Big(\int_0^{t_{n-1}}\cdots\int_0^{t_1}g(t_0)\mathrm{d}t_0\cdots\mathrm{d}t_{n-2}\Big)\mathrm{d}t_{n-1},$$

and obtained

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y)^{\lambda}}f(x)g(y)\mathrm{d}x\mathrm{d}y$$
$$\le\frac{\Gamma(\lambda+m+n)}{\Gamma(\lambda)}B^{\frac{1}{p}}(\lambda_2+n,\lambda+m-\lambda_2)B^{\frac{1}{q}}(\lambda_1+m,\lambda+n-\lambda_1)$$
$$\times\Big[\int_0^{+\infty}x^{p(1-\tilde{\lambda}_1-m)-1}F_m^p(x)\mathrm{d}x\Big]^{\frac{1}{p}}\Big[\int_0^{+\infty}y^{q(1-\tilde{\lambda}_2-n)-1}G_n^q(y)\mathrm{d}y\Big]^{\frac{1}{q}},\tag{2}$$

where $\frac{1}{p}+\frac{1}{q}=1\,(p>1)$, $\tilde{\lambda}_1=\frac{\lambda-\lambda_2}{p}+\frac{\lambda_1}{q}$, $\tilde{\lambda}_2=\frac{\lambda-\lambda_1}{q}+\frac{\lambda_2}{p}$. They also considered conditions for the constant factor to be optimal.

Obviously, the parameter structures in (1) and (2) are relatively complex and entail univariate functions and the conditions that render optimal constants. In this paper, we will consider multivariate variable upper limit integral functions with multivariate functions. Based on the constructive theorem of Hilbert-type multiple integral inequalities with homogeneous kernels, we will discuss the parametric conditions for the validity of Hilbert-type multiple integral inequalities involving multivariate variable upper limit integral functions, and obtain constructive theorems for such inequalities and formulas for calculating optimal constant factors.

Next, we introduce some notation: Let $m\in\mathbb{N}_+$, $x=(x_1,x_2,\cdots,x_m)\in\mathbb{R}_+^m$, $\|x\|_m=\sum_{i=1}^m x_i$. If $p>1$, $\alpha\in\mathbb{R}$, we define

$$L_p^{\alpha}(\mathbb{R}_+^m)=\Big\{f(x):\|f\|_{p,\alpha}=\Big(\int_{\mathbb{R}_+^m}\|x\|_m^{\alpha}|f(x)|^p\mathrm{d}x\Big)^{\frac{1}{p}}<+\infty\Big\}$$

as the weighted Lebesgue space. If we denote $\tilde{F}_0(x)=f(x)$, and define inductively

$$\tilde{F}_k(x)=\int_0^{x_k}\tilde{F}_{k-1}(x_1,\cdots,x_{k-1},t_k,x_{k+1},\cdots,x_m)\mathrm{d}t_k,$$

where $k=1,2,\cdots,m$, then

$$\frac{\partial}{\partial x_k}\tilde{F}_k(x)=\tilde{F}_{k-1}(x_1,x_2,\cdots,x_m)=\tilde{F}_{k-1}(x),$$

$$\begin{aligned}\tilde{F}_m(x) &= \int_0^{x_m} \tilde{F}_{m-1}(x_1, \cdots, x_{m-1}, t_m)\mathrm{d}t_m \\ &= \int_0^{x_m}\int_0^{x_{m-1}} \tilde{F}_{m-2}(x_1, \cdots, x_{m-2}, t_{m-1}, t_m)\mathrm{d}t_{m-1}\mathrm{d}t_m \\ &= \cdots\cdots \\ &= \int_0^{x_m}\cdots\int_0^{x_2}\int_0^{x_1} \tilde{F}_0(t_1, \cdots,, t_{m-1}, t_m)\mathrm{d}t_1\cdots\mathrm{d}_{m-1}\mathrm{d}t_m \\ &= \int_0^{x_m}\cdots\int_0^{x_2}\int_0^{x_1} f(t_1, \cdots,, t_{m-1}, t_m)\mathrm{d}t_1\cdots\mathrm{d}t_{m-1}\mathrm{d}t_m.\end{aligned}$$

We call $\tilde{F}_m(x)$ the multivariate variable upper limit integral function of the multivariate function $f(x_1, x_2, \cdots, x_m)$.

2. Preliminary Lemmas

Lemma 1 ([15]). *Let $m, n \in \mathbb{N}_+$, $\frac{1}{p}+\frac{1}{q}=1\,(p>1)$, $K(u,v)$ be a σ-order homogeneous non-negative measurable function, $\alpha, \beta, \in \mathbb{R}$, $x=(x_1, x_2, \cdots, x_m) \in \mathbb{R}_+^m$, $y=(y_1, y_2, \cdots, y_n) \in \mathbb{R}_+^n$, and*

$$W_0 = \int_0^{+\infty} K(t,1)t^{-\frac{\alpha+m}{p}+m-1}\mathrm{d}t < +\infty.$$

(i) If and only if $\frac{\alpha+m}{p}+\frac{\beta+n}{q}=m+n+\sigma$, there is a constant $M>0$ such that

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m} K(\|x\|_m, \|y\|_n)f(x)g(y)\mathrm{d}x\mathrm{d}y \le M\|f\|_{p,\alpha}\|g\|_{q,\beta}. \tag{3}$$

(ii) When $\frac{\alpha+m}{p}+\frac{\beta+n}{q}=m+n+\sigma$, the best constant factor of (3) is

$$M_1 = \frac{1}{\Gamma^{\frac{1}{q}}(m)\Gamma^{\frac{1}{p}}(n)}W_0.$$

Lemma 2. *Let $m, n \in \mathbb{N}_+$, $\frac{1}{p}+\frac{1}{q}=1\,(p>1)$, $\alpha, \beta, \in \mathbb{R}$, $\lambda > m-\frac{\alpha+m}{p} > 0$, $x=(x_1, x_2, \cdots, x_m) \in \mathbb{R}_+^m$, $y=(y_1, y_2, \cdots, y_n) \in \mathbb{R}_+^n$.*

(i) If and only if $\frac{\alpha+m}{p}+\frac{\beta+n}{q}=m+n-\lambda$, there is a constant $M>0$ such that

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m} \frac{1}{(\|x\|_m+\|y\|_n)^\lambda}f(x)g(y)\mathrm{d}x\mathrm{d}y \le M\|f\|_{p,\alpha}\|g\|_{q,\beta}. \tag{4}$$

(ii) When $\frac{\alpha+m}{p}+\frac{\beta+n}{q}=m+n-\lambda$, the best constant factor of (4) is

$$M_2 = \frac{1}{\Gamma(\lambda)\Gamma^{\frac{1}{q}}(m)\Gamma^{\frac{1}{p}}(n)}\Gamma(m-\frac{\alpha+m}{p})\Gamma(n-\frac{\beta+n}{q}).$$

Proof. Let $K(u,v)=\frac{1}{(u+v)^\lambda}$. Then $K(u,v)$ is a homogeneous non-negative function of $-\lambda$ order. When $\lambda > m-\frac{\alpha+m}{p} > 0$, we have

$$\begin{aligned}W_0 &= \int_0^{+\infty} K(t,1)t^{-\frac{\alpha+m}{p}+m-1}\mathrm{d}t = \int_0^{+\infty}\frac{1}{(t+1)^\lambda}t^{-\frac{\alpha+m}{p}+m-1}\mathrm{d}t \\ &= B(m-\frac{\alpha+m}{p}, \lambda+\frac{\alpha+m}{p}-m).\end{aligned}$$

If $\frac{\alpha+m}{p}+\frac{\beta+n}{q}=m+n-\lambda$, then

$$W_0=B(m-\frac{\alpha+m}{p},n-\frac{\beta+n}{q}).$$

In summary, according to Lemma 1, Lemma 2 holds. □

3. Main Theorems

Theorem 1. *Let* $m,n\in\mathbb{N}_+$, $\frac{1}{p}+\frac{1}{q}=1\,(p>1)$, $\alpha,\beta\in\mathbb{R}$, $0<m-\frac{\alpha+m}{p}<\lambda+m+n$, $x=(x_1,x_2,\cdots,x_m)\in\mathbb{R}_+^m$, $y=(y_1,y_2,\cdots,y_n)\in\mathbb{R}_+^n$, $\|x\|_m=\sum_{i=1}^m x_i$, $\|y\|_n=\sum_{i=1}^n y_i$, $\tilde{F}_k(x_1,x_2,\cdots,x_m)=o(e^{tx_k})\,(x_k\to+\infty,t>0)$, $\tilde{G}_k(y_1,y_2,\cdots,y_n)=o(e^{ty_k})\,(y_k\to+\infty,t>0)$, *and*

$$\tilde{F}_m(x)=\int_0^{x_m}\cdots\int_0^{x_2}\int_0^{x_1}f(t_1,\cdots,,t_{m-1},t_m)\mathrm{d}t_1\cdots\mathrm{d}t_{m-1}\mathrm{d}t_m,$$

$$\tilde{G}_n(y)=\int_0^{y_n}\cdots\int_0^{y_2}\int_0^{y_1}g(t_1,\cdots,,t_{m-1},t_m)\mathrm{d}t_1\cdots\mathrm{d}t_{m-1}\mathrm{d}t_n.$$

(i) If and only if $\frac{\alpha+m}{p}+\frac{\beta+n}{q}=-\lambda$, *there is a constant* $M>0$, *independent of* $f(x)$ *and* $g(y)$, *such that the following Hilbert-type integral inequality holds:*

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m}\frac{1}{(\|x\|_m+\|y\|_n)^\lambda}f(x)g(y)\mathrm{d}x\mathrm{d}y\le M\|\tilde{F}_m\|_{p,\alpha}\|\tilde{G}_n\|_{\beta,q}. \tag{5}$$

(ii) When $\frac{\alpha+m}{p}+\frac{\beta+n}{q}=-\lambda$, *the best constant factor of (5) is*

$$M_0=\frac{1}{\Gamma(\lambda)\Gamma^{\frac{1}{q}}(m)\Gamma^{\frac{1}{p}}(n)}\Gamma(m-\frac{\alpha+m}{p})\Gamma(n-\frac{\beta+n}{q}).$$

Proof. (i) When $a\neq0$, by the definition of the Gamma function, we have

$$\int_0^{+\infty}t^{\lambda-1}e^{-at}\mathrm{d}t=\frac{1}{a^\lambda}\int_0^{+\infty}u^{\lambda-1}e^{-u}\mathrm{d}u=\frac{1}{a^\lambda}\Gamma(\lambda),$$

thereby obtaining

$$\frac{1}{(\|x\|_m+\|y\|_n)^\lambda}=\frac{1}{\Gamma(\lambda)}\int_0^{+\infty}t^{\lambda-1}e^{-t(\|x\|_m+\|y\|_n)}\mathrm{d}t.$$

From this, we get

$$\begin{aligned}&\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m}\frac{1}{(\|x\|_m+\|y\|_n)^\lambda}f(x)g(y)\mathrm{d}x\mathrm{d}y\\&=\frac{1}{\Gamma(\lambda)}\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m}\left(\int_0^{+\infty}t^{\lambda-1}e^{-t(\|x\|_m+\|y\|_n)}\mathrm{d}t\right)f(x)g(y)\mathrm{d}x\mathrm{d}y\\&=\frac{1}{\Gamma(\lambda)}\int_0^{+\infty}t^{\lambda-1}\left(\int_{\mathbb{R}_+^m}e^{-t\sum_{i=1}^m x_i}f(x)\mathrm{d}x\right)\left(\int_{\mathbb{R}_+^n}e^{-t\sum_{i=1}^n y_i}g(y)\mathrm{d}y\right)\mathrm{d}t.\end{aligned}$$

Since $\tilde{F}_k(x_1,x_2,\cdots,x_m)=o(e^{tx_k})\,(x_k\to+\infty,t>0)$, using integration by parts, we have

$$
\begin{aligned}
&\int_{\mathbb{R}_+^m} e^{-t\sum_{i=1}^m x_i} f(x)\mathrm{d}x \\
&= \int_{\mathbb{R}_+^{m-1}} e^{-t\sum_{i\neq 1}^m x_i}\left(\int_0^{+\infty} e^{-tx_1}\tilde{F}_0(x_1,x_2,\cdots,x_m)\mathrm{d}x_1\right)\mathrm{d}x_2\cdots\mathrm{d}x_m \\
&= \int_{\mathbb{R}_+^{m-1}} e^{-t\sum_{i\neq 1}^m x_i}\left(\lim_{x_1\to+\infty} e^{-tx_1}\tilde{F}_1(x_1,x_2,\cdots,x_m)\right. \\
&\quad \left.+ t\int_0^{+\infty} e^{-tx_1}\tilde{F}_1(x_1,x_2,\cdots,x_m)\mathrm{d}x_1\right)\mathrm{d}x_2\cdots\mathrm{d}x_m \\
&= t\int_{\mathbb{R}_+^{m-1}} e^{-t\sum_{i\neq 1}^m x_i}\left(\int_0^{+\infty} e^{-tx_1}\tilde{F}_1(x_1,x_2,\cdots,x_m)\mathrm{d}x_1\right)\mathrm{d}x_2\cdots\mathrm{d}x_m \\
&= t\int_{\mathbb{R}_+^{m}} e^{-t\sum_{i=1}^m x_i}\tilde{F}_1(x_1,x_2,\cdots,x_m)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_m \\
&= t\int_{\mathbb{R}_+^{m-1}} e^{-t\sum_{i\neq 2}^m x_i}\left(\int_0^{+\infty} e^{-tx_2}\tilde{F}_1(x_1,x_2,\cdots,x_m)\mathrm{d}x_2\right)\mathrm{d}x_1\mathrm{d}x_3\cdots\mathrm{d}x_m \\
&= t\int_{\mathbb{R}_+^{m-1}} e^{-t\sum_{i\neq 2}^m x_i}\left(\lim_{x_2\to+\infty} e^{-tx_2}\tilde{F}_2(x_1,x_2,\cdots,x_m)\right. \\
&\quad \left.+ t\int_0^{+\infty} e^{-tx_2}\tilde{F}_2(x_1,x_2,\cdots,x_m)\mathrm{d}x_2\right)\mathrm{d}x_1\mathrm{d}x_3\cdots\mathrm{d}x_m \\
&= t^2\int_{\mathbb{R}_+^{m}} e^{-t\sum_{i=1}^m x_i}\tilde{F}_2(x_1,x_2,\cdots,x_m)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_m \\
&= \cdots\cdots \\
&= t^m\int_{\mathbb{R}_+^{m}} e^{-t\sum_{i=1}^m x_i}\tilde{F}_m(x_1,x_2,\cdots,x_m)\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_m \\
&= t^m\int_{\mathbb{R}_+^{m}} e^{-t\|x\|_m}\tilde{F}_m(x)\mathrm{d}x.
\end{aligned}
$$

Similarly, we can obtain

$$\int_{\mathbb{R}_+^{n}} e^{-t\sum_{i=1}^n y_i} g(y)\mathrm{d}y = t^n\int_{\mathbb{R}_+^{n}} e^{-t\|y\|_n}\tilde{G}_n(y)\mathrm{d}y.$$

Thus,

$$
\begin{aligned}
&\int_{\mathbb{R}_+^{n}}\int_{\mathbb{R}_+^{m}} \frac{1}{(\|x\|_m+\|y\|_n)^{\lambda}} f(x)g(y)\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{\Gamma(\lambda)}\int_0^{+\infty} t^{\lambda+m+n-1}\left(\int_{\mathbb{R}_+^{m}} e^{-t\|x\|_m}\tilde{F}_m(x)\mathrm{d}x\right)\left(\int_{\mathbb{R}_+^{n}} e^{-t\|y\|_n}\tilde{G}_n(y)\mathrm{d}y\right)\mathrm{d}t \\
&= \frac{1}{\Gamma(\lambda)}\int_{\mathbb{R}_+^{n}}\int_{\mathbb{R}_+^{m}} \tilde{F}_m(x)\tilde{G}_n(y)\left(\int_0^{+\infty} t^{\lambda+m+n-1}e^{-t(\|x\|_m+\|y\|_n)}\mathrm{d}t\right)\mathrm{d}x\mathrm{d}y \\
&= \frac{\Gamma(\lambda+m+n)}{\Gamma(\lambda)}\int_{\mathbb{R}_+^{n}}\int_{\mathbb{R}_+^{m}} \frac{1}{(\|x\|_m+\|y\|_n)^{\lambda+m+n}}\tilde{F}_m(x)\tilde{G}_n(y)\mathrm{d}x\mathrm{d}y.
\end{aligned}
$$

Since $0 < m - \frac{\alpha+m}{p} < \lambda+m+n$, according to Lemma 2(i), if and only if $\frac{\alpha+m}{p} + \frac{\beta+n}{q} = m+n-(\lambda+m+n) = -\lambda$, there is a constant $\bar{M} > 0$ such that

$$\int_{\mathbb{R}_+^{n}}\int_{\mathbb{R}_+^{m}} \frac{1}{(\|x\|_m+\|y\|_n)^{\lambda+m+n}}\tilde{F}_m(x)\tilde{G}_n(y)\mathrm{d}x\mathrm{d}y \le \bar{M}\|\tilde{F}_m\|_{p,\alpha}\|\tilde{G}_n\|_{q,\beta}. \tag{6}$$

Denote $M = \frac{\Gamma(\lambda+m+n)}{\Gamma(\lambda)}\bar{M}$. It follows that

$$\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^m}\frac{1}{(\|x\|_m+\|y\|_n)^\lambda}f(x)g(y)\mathrm{d}x\mathrm{d}y \le M\|\tilde{F}_m\|_{p,\alpha}\|\tilde{G}_n\|_{q,\beta}.$$

(ii) When $\frac{\alpha+m}{p}+\frac{\beta+n}{q} = -\lambda = m+n-(\lambda+m+n)$, by Lemma 2(ii), the best constant factor of (6) is

$$\bar{M}_0 = \frac{1}{\Gamma(\lambda+m+n)\Gamma^{\frac{1}{q}}(m)\Gamma^{\frac{1}{p}}(n)}\Gamma(m-\frac{\alpha+m}{p})\Gamma(n-\frac{\beta+n}{q}).$$

Therefore, when $\frac{\alpha+m}{p}+\frac{\beta+n}{q} = -\lambda$, the best constant factor of (5) is

$$\begin{aligned} M_0 &= \frac{\Gamma(\lambda+m+n)}{\Gamma(\lambda)}\frac{1}{\Gamma(\lambda+m+n)\Gamma^{\frac{1}{q}}(m)\Gamma^{\frac{1}{p}}(n)}\Gamma(m-\frac{\alpha+m}{p})\Gamma(n-\frac{\beta+n}{q}) \\ &= \frac{1}{\Gamma(\lambda)\Gamma^{\frac{1}{q}}(m)\Gamma^{\frac{1}{p}}(n)}\Gamma(m-\frac{\alpha+m}{p})\Gamma(n-\frac{\beta+n}{q}). \end{aligned}$$

□

Taking $m = n = 1$ in Theorem 1, we obtain the following:

Corollary 1. *Suppose that* $\frac{1}{p}+\frac{1}{q} = 1\,(p>1)$, $\alpha,\beta\in\mathbb{R}$, $0<\frac{1}{q}-\frac{\alpha}{p}<\lambda+2$, $\frac{1}{p}-\frac{\beta}{q}>0$, $F(x) = \int_0^x f(t)\mathrm{d}t$, $G(y) = \int_0^y g(t)\mathrm{d}t$. *Then*

(i) if and only if $\frac{\alpha}{p}+\frac{\beta}{q} = -\lambda-1$, *there is a constant* $M>0$ *such that*

$$\int_0^{+\infty}\int_0^{+\infty}\frac{1}{(x+y)^\lambda}f(x)g(y)\mathrm{d}x\mathrm{d}y \le M\|F\|_{p,\alpha}\|G\|_{\beta,q}, \tag{7}$$

where $f\in L_p^\alpha(0,+\infty)$, $g\in L_q^\beta(0,+\infty)$.

(ii) When $\frac{\alpha}{p}+\frac{\beta}{q} = -\lambda-1$, *the best constant factor of (7) is*

$$M_0 = \frac{1}{\Gamma(\lambda)}\Gamma(\frac{1}{q}-\frac{\alpha}{p})\Gamma(\frac{1}{p}-\frac{\beta}{q}).$$

Proof. When $m = n = 1$, the term $\frac{\alpha+m}{p}+\frac{\beta+n}{q} = -\lambda$ in Theorem 1 simplifies to $\frac{\alpha}{p}+\frac{\beta}{q} = -\lambda-1$, and M_0 simplifies to $\frac{1}{\Gamma(\lambda)}\Gamma(\frac{1}{q}-\frac{\alpha}{p})\Gamma(\frac{1}{p}-\frac{\beta}{q})$. Thus, according to Theorem 1, it suffices to prove that, when $t>0$,

$$F(x) = o(e^{tx})\,(x\to+\infty),\quad G(y) = o(e^{ty})\,(y\to+\infty).$$

In fact, since $0<\frac{1}{q}-\frac{\alpha}{p}$, we have $(1-q)\alpha+1>0$. Applying Hölder's integral inequality yields

$$
\begin{aligned}
|F(x)| &\le \int_0^x |f(t)|\mathrm{d}t = \int_0^x (t^{\frac{\alpha}{p}}|f(t)|)t^{-\frac{\alpha}{p}}\mathrm{d}t \\
&\le \left(\int_0^x t^{\alpha}|f(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^x t^{-\frac{q}{p}\alpha}\mathrm{d}t\right)^{\frac{1}{q}} \\
&\le \left(\int_0^{+\infty} t^{\alpha}|f(t)|^p\mathrm{d}t\right)^{\frac{1}{p}}\left(\int_0^x t^{(1-q)\alpha}\mathrm{d}t\right)^{\frac{1}{q}} \\
&= \|f\|_{p,\alpha}\frac{1}{(1-q)\alpha+1}x^{(1-q)\alpha+1}.
\end{aligned}
$$

It follows that for $t > 0$,

$$
\lim_{x\to+\infty}\left|\frac{F(x)}{e^{tx}}\right| \le \|f\|_{p,\alpha}\frac{1}{(1-q)\alpha+1}\lim_{x\to+\infty}\frac{x^{(1-q)\alpha+1}}{e^{tx}} = 0.
$$

Hence, $F(x) = o(e^{tx})\,(x \to +\infty)$. Similarly, $G(y) = o(e^{ty})\,(y \to +\infty)$. □

Corollary 2. *Let* $m, n \in \mathbb{N}_+$, $\frac{1}{p}+\frac{1}{q} = 1\,(p > 1)$, $\lambda > \max\{p(\lambda_1 - m), -q(\lambda_1 + n)\}$, $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}_+^m$, $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}_+^n$, $\|x\|_m = \sum_{i=1}^m x_i$, $\|y\|_n = \sum_{i=1}^n y_i$, $\tilde{F}_k(x_1, x_2, \cdots, x_m) = o(e^{tx_k})\,(x_k \to +\infty, t > 0)$, $\tilde{G}_k(y_1, y_2, \cdots, y_n) = o(e^{ty_k})\,(y_k \to +\infty, t > 0)$.

(i) If and only if $\lambda_1 + \lambda_2 = 0$, *there is a constant* $M > 0$ *such that*

$$
\int_{\mathbb{R}_+^n}\int_{\mathbb{R}_+^n}\frac{f(x)g(y)}{(\|x\|_m + \|y\|_n)^{\lambda}}\mathrm{d}x\mathrm{d}y \le M\|\tilde{F}_m\|_{p,p\lambda_1-m-\lambda}\|\tilde{G}_n\|_{q,q\lambda_2-n-\lambda}. \tag{8}
$$

(ii) When $\lambda_1 + \lambda_2 = 0$, *the best constant factor of (8) is*

$$
M_0 = \frac{1}{\Gamma(\lambda)\Gamma^{\frac{1}{q}}(m)\Gamma^{\frac{1}{p}}(n)}\Gamma(\frac{\lambda}{p}+m-\lambda_1)\Gamma(\frac{\lambda}{q}+n-\lambda_2).
$$

Proof. Denote $\alpha = p\lambda_1 - m - \lambda$, $\beta = q\lambda_2 - n - \lambda$. Then

$$
\frac{\alpha+m}{p}+\frac{\beta+n}{q} = \frac{p\lambda_1-m-\lambda+m}{p}+\frac{q\lambda_2-n-\lambda+n}{q} = (\lambda_1+\lambda_2)-\lambda.
$$

From this, we can deduce that the condition $\frac{\alpha+m}{p}+\frac{\beta+n}{q} = -\lambda$ in Theorem 1 is equivalent to $\lambda_1 + \lambda_2 = 0$.

Since $\lambda > \max\{p(\lambda_1 - m), -q(\lambda_1 + n)\}$, through simple calculations, we can obtain $0 < m - \frac{\alpha+m}{p} < \lambda + m + n$, and

$$
m - \frac{\alpha+m}{p} = m - \lambda_1 + \frac{\lambda}{p},\ \ n - \frac{\beta+n}{q} = n - \lambda_2 + \frac{\lambda}{q}.
$$

In summary, based on Theorem 1, we can conclude that Corollary 2 holds. □

4. Conclusions

In this paper, we generalize the traditional Hilbert-type double integral inequality from a new perspective and obtain sufficient and necessary conditions for constructing such inequalities. The parameter structure is concise and easy to apply, and the proof method has also been optimized. The conclusions laid a certain foundation for further exploring the application of Hilbert-type inequalities in operator theory and other disciplines.

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