axioms

MDPI

Article

Idempotent Triangular Matrices over Additively Idempotent Semirings: Decompositions into Products of Semicentral Idempotents

Dimitrinka Vladeva

Department of Algebra and Logic, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria; d_vladeva@abv.bg

Abstract: The explicit forms of idempotent and semicentral idempotent triangular matrices over an additively idempotent semiring are obtained. We define a diamond composition of idempotents and give a representation of an idempotent $n \times n$ matrix as an $(n-1)$ th degree of a sum of diamond compositions of semicentral idempotents. We construct a decomposition of a strictly upper matrix, a unitriangular matrix, and a nil-clean matrix by semicentral idempotents.

Keywords: additively idempotent semirings; matrix semirings; triangular matrices; idempotents; semicentral idempotents

MSC: 15A24; 15A80; 16S50; 16Y60; 17C27

1. Introduction and Preliminaries

In the second half of the past century, a number of mathematicians realized the useful features and the significant applications of a certain type of semiring structure. Semirings are structures that are associative and distributive on both sides but do not, in general, have additive inverses. These semiring structures are additively idempotent, like Boolean and fuzzy algebras.

There are prominent applications for additively idempotent semirings including optimization and control, differential equations, algebraic geometry, dynamic programming, mathematical physics and chemistry, mathematical biology, interval analysis, mathematical economics, game theory, computer technology, Petri nets, and cryptography (see [1–8]).

On the other hand, idempotents play an important role in ring theory and are closely related to matrices since the matrix units are idempotents. There is very extensive research dealing with idempotents. Following V. Drensky (see [9]), idempotents were introduced by Benjamin Peirce [10] in 1870.

In 2021, X. Hou (see [11]) provided a complete characterization of idempotent triangular matrices over a ring in which zero and one were the only idempotent elements.

In 2022, S. Wright (see [12]) characterized the idempotency of upper triangular matrices with entries from an arbitrary ring. Very recently, D. Vladeva (see [13,14]) considered the semicentral idempotents of upper triangular matrix rings.

The study of idempotents of additively idempotent semirings has a short history. In 2013, I. Trendafilov and D. Vladeva (see [15]) proved that in the endomorphism semiring of a finite chain, the set of all idempotents with s fixed points, $k_1, \ldots, k_s$, $1 \leq s \leq n-1$, is a semiring of order $\prod_{m=1}^{s-1}(k_{m+1} - k_m)$. In the main result of the paper the authors prove that

check for updates

Academic Editor: Hashem Bordbar

Received: 23 December 2024
Revised: 8 February 2025
Accepted: 12 February 2025
Published: 15 February 2025

Citation: Vladeva, D. Idempotent Triangular Matrices over Additively Idempotent Semirings: Decompositions into Products of Semicentral Idempotents. *Axioms* **2025**, *14*, 137. https://doi.org/10.3390/axioms14020137

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

the set of roots of a fixed idempotent of the endomorphism semiring of a finite chain is a semiring of the order $C_{k_{1,1}}\left(\prod_{i=1}^{\ell-1} C_{t_i}C_{s_i}\right)C_{n-1-k_{\ell,m_\ell}}$, where C_p is the p–th Catalan number.

In 2018, I. Chajda and H. Länger (see [16]) investigated a variety of commutative additively and multiplicatively idempotent semirings.

In 2020, D. Vladeva (see [17]) studied a class of the right ideals of a subsemiring of the endomorphism semiring of a finite chain and considered projections on these ideals, which were derivations. For one of these derivations, $d_{\ell,m}$, and for the fixed endomorphism α_0, the set of endomorphisms α such that $d_{\ell,m}(\alpha) = \alpha_0$ is denoted by $\int \alpha_0 d_{\ell,m}$. The last set is a semiring if and only if α_0 is an idempotent. The number of the semirings, $\int \alpha_0 d_{\ell,m}$, where $1 \leq m \leq n-1$, is equal to F_{2m}, which is the $2m$ th Fibonacci number.

In 2023, D. Vladeva (see [18]) suggested the following definitions:

1. Let S and S_0 be additively idempotent semirings, S be a noncommutative semiring and an S_0-semimodule, S_0 be a commutative semiring, and $\alpha s = s\alpha$ for any $\alpha \in S_0$ and $s \in S$. Then, S is called an S_0-semialgebra.
2. Let S be an S_0-semialgebra and $\mathcal{B}(S) = \{e_{ij}\},\ i,j = 1,\ldots,n$, a basis of S (relative to S_0) with the following properties:
   - $e_{ij}e_{kl} = \begin{cases} e_{il}, & \text{if } j \leq k \\ 0, & \text{if } j > k \end{cases}, i,j,k,l = 1,\ldots,n.$
   - $\sum_{i=1}^{n} e_{ii} = 1_S$, where 1_S is the identity of S.
   - $e_{ij} > e_{ik}$ for all $k > j$ and $e_{ij} > e_{hj}, h < i$.

Then, $\mathcal{B}(S)$ is called an e-basis. The basis elements e_{ii} are idempotents, and since $e_{ii}e_{jj} = e_{ij}$ but $e_{jj}e_{ii} = 0$, if $i < j$, they do not commute. In this paper, we prove the following: *Let S be an S_0-semialgebra and $\mathcal{B}(S) = \{e_{ij}\},\ i,j = 1,\ldots,n$ an e-basis of S. Then, S is isomorphic to a matrix subsemiring of $M_n(S_0)$.* From this theorem, it follows that if $\mathcal{B}(S)$ is a triangular e-basis, then S is isomorphic to a subsemiring of the semiring of upper triangular matrices.

Throughout the present paper, all semirings considered shall be assumed to be additively idempotent and possess an identity and a zero. Basic facts about additively idempotent semirings can be found in the seminal book of J. Golan (cf. [19]).

The present paper contains only new results and is arranged as follows. In Section 2, we obtain the explicit form of an arbitrary idempotent triangular matrix over an additively idempotent semiring. The next section deals with the left and right semicentral idempotents of an additively idempotent semiring. A new type of semicentral idempotent, called Jordan, specific for an additively idempotent semiring, is studied. We define a diamond composition, $e_1 \diamond e_2 = e_1 + e_2 + e_1e_2$, where e_1 and e_2 are idempotents and $e_2e_1 = 0$, which plays a crucial role in the decompositions of many triangular matrices. In Section 4, we find the explicit form of an arbitrary left and right semicentral idempotent triangular matrix. In the last four sections, we consider the decompositions of idempotent, strictly upper, unitriangular, and nil-clean triangular matrices by semicentral idempotent matrices.

2. Results on Idempotents of Triangular Matrix Semiring

In the following, S is an additively idempotent semiring and $T_n(S)$ is the semiring of $n \times n$ triangular matrices over S.

Example 1. *It is well known that all the entries on the main diagonal of an idempotent triangular matrix are idempotents. Let* $\begin{pmatrix} e_1 & x_{12} \\ 0 & e_2 \end{pmatrix}^2 = \begin{pmatrix} e_1 & x_{12} \\ 0 & e_2 \end{pmatrix}$, *where* $e_1, e_2, x_{12} \in S$ *and* $e_i^2 = e_i$,

$i = 1, 2$. *Then,* $e_1x_{12} + x_{12}e_2 = x_{12}$, *which implies that* $e_1x_{12} + e_1x_{12}e_2 = e_1x_{12}$ *and* $x_{12} = e_1x_{12} + e_1x_{12}e_2 + x_{12}e_2$. *For an arbitrary* $\alpha_{12} \in S$, *it follows that*

$$E(e_1, e_2) = \begin{pmatrix} e_1 & e_1\alpha_{12} + e_1\alpha_{12}e_2 + \alpha_{12}e_2 \\ 0 & e_2 \end{pmatrix} \tag{1}$$

is an idempotent matrix. Thus, (1) *represents the idempotent matrices of* $T_2(S)$.

Note that when $\alpha_{12} = e_1\beta_{12}e_2$, *we obtain the idempotent matrix* $\begin{pmatrix} e_1 & e_1\beta_{12}e_2 \\ 0 & e_2 \end{pmatrix}$, *and this matrix is a particular case of* (1). *Moreover, we have* $\begin{pmatrix} e_1 & e_1\alpha_{12}e_2 \\ 0 & e_2 \end{pmatrix}\begin{pmatrix} e_1 & e_1\beta_{12}e_2 \\ 0 & e_2 \end{pmatrix} = \begin{pmatrix} e_1 & e_1\alpha_{12}e_2 \\ 0 & e_2 \end{pmatrix} + \begin{pmatrix} e_1 & e_1\beta_{12}e_2 \\ 0 & e_2 \end{pmatrix}$; *thus, idempotent matrices of this form commute.*

Example 2. *Let* $X = \begin{pmatrix} e_1 & x_{12} & x_{13} \\ 0 & e_2 & x_{23} \\ 0 & 0 & e_3 \end{pmatrix}$, *where* $e_i, x_{ij} \in S$, $i, j = 1, 2, 3$, $i < j$, *and* $e_i^2 = e_i$, *be an idempotent matrix. As in the previous example, we have* $x_{12} = e_1x_{12} + e_1x_{12}e_2 + x_{12}e_2$, $x_{23} = e_2x_{23} + e_2x_{23}e_3 + x_{23}e_3$, *and* $x_{13} = e_1x_{13} + e_1x_{13}e_3 + x_{13}e_3 + x_{12}x_{23}$.

For arbitrary $\alpha_{12}, \alpha_{13}, \alpha_{23} \in S$, *we denote* $\delta_{ij} = e_i\alpha_{ij} + e_i\alpha_{ij}e_j + \alpha_{ij}e_j$ *(this concept should not be confused with the Kroneker symbol). Thus,* $x_{12} = \delta_{12}$, $x_{23} = \delta_{23}$, *and* $x_{13} = \delta_{13} + \delta_{12}\delta_{23}$. *Conversely, for arbitrary* $\alpha_{12}, \alpha_{13}, \alpha_{23} \in S$, *we consider* $X = \begin{pmatrix} e_1 & \delta_{12} & \delta_{13} + \delta_{12}\delta_{23} \\ 0 & e_2 & \delta_{23} \\ 0 & 0 & e_3 \end{pmatrix}$, *where* $\delta_{ij} = e_i\alpha_{ij} + e_i\alpha_{ij}e_j + \alpha_{ij}e_j$. *We obtain that* $e_1\delta_{12} + \delta_{12}e_2 = \delta_{12}$ *and* $e_2\delta_{23} + \delta_{23}e_3 = \delta_{23}$. *Since* $e_i\delta_{ij} + \delta_{ij} = \delta_{ij}$ *and* $\delta_{ij}e_j + \delta_{ij} = \delta_{ij}$, *it follows that* $e_1\delta_{12}\delta_{23} + \delta_{12}\delta_{23} = \delta_{12}\delta_{23}$ *and* $\delta_{12}\delta_{23}e_3 + \delta_{12}\delta_{23} = \delta_{12}\delta_{23}$. *Then,* $e_1(\delta_{13} + \delta_{12}\delta_{23}) + \delta_{12}\delta_{23} + (\delta_{13} + \delta_{12}\delta_{23})e_3 = \delta_{13} + \delta_{12}\delta_{23}$ *and* X *is an idempotent matrix. Hence,*

$$E(e_1, e_2, e_3) = \begin{pmatrix} e_1 & \delta_{12} & \delta_{13} + \delta_{12}\delta_{23} \\ 0 & e_2 & \delta_{23} \\ 0 & 0 & e_3 \end{pmatrix}$$

is the general form of the idempotent matrix of $T_3(S)$.

The next result is similar to Theorem 3.1 of S. Wright (cf. [12]) but different as we work in an additively idempotent matrix semiring.

Theorem 1. *An* $n \times n$ *upper triangular matrix,* $X = (x_{ij})$, *over the additively idempotent semiring* S *is idempotent if and only if* $x_{ii} = e_i$, *where* $e_i^2 = e_i$ *for* $i = 1, \ldots, n$, *and* $x_{ij} = \delta_{ij} + \sum_{k=i+1}^{j-1} x_{ik}x_{kj}$, *where* $\delta_{ij} = e_i\alpha_{ij} + e_i\alpha_{ij}e_j + \alpha_{ij}e_j$, $\alpha_{ij} \in S$ *for* $i, j = 1, \ldots, n$ *and* $i + 1 < j$. *If* $j = i + 1$, *then* $x_{i\,i+1} = \delta_{i\,i+1}$.

Proof. Let $X = (x_{ij})$ be an idempotent matrix. Then, $x_{ii} = e_i$ are idempotents for $i = 1, \ldots, n$, and for $i + 1 < j$, it follows that $x_{ij} = \sum_{k=i}^{j} x_{ik}x_{kj} = e_ix_{ij} + x_{ij}e_j + \sum_{k=i+1}^{j-1} x_{ik}x_{kj}$. Since $e_ix_{ij} + x_{ij} = x_{ij}$ and $x_{ij}e_j + x_{ij} = x_{ij}$, we have $e_ix_{ij}e_j + x_{ij}e_j = x_{ij}e_j$ and $e_ix_{ij}e_j + e_ix_{ij} = e_ix_{ij}$. Therefore, $e_ix_{ij} + x_{ij}e_j = \delta_{ij}$. If $j = i + 1$, we have $x_{i\,i+1} = e_ix_{i\,i+1} + x_{i\,i+1}e_{i+1} = \delta_{i\,i+1}$.

Conversely, for the matrix $X = (x_{ij})$, we assume that $x_{ii} = e_i$, $i = 1, \ldots, n$ are idempotents, $x_{i\,i+1} = \delta_{i\,i+1}$ and $x_{ij} = \delta_{ij} + \sum_{k=i+1}^{j-1} x_{ik}x_{kj}$, for $i + 1 < j$. Our induction hypothesis is the following.

Suppose that for a fixed $j \leq n$, the matrices with the main diagonal $x_{ii}, \ldots, x_{kk}$, where i is an arbitrary positive integer and $i < k < j$, are idempotent.

Now, X is an idempotent matrix if and only if for the arbitrary element x_{ij}, where $i+1 < j$, we prove that $x_{ij} = e_i x_{ij} + x_{ij} e_j + \sum_{k=i+1}^{j-1} x_{ik} x_{kj}$. From the assumption of x_{ij}, it follows that the right side of the last equality is

$$e_i \left(\delta_{ij} + \sum_{k=i+1}^{j-1} x_{ik} x_{kj} \right) + \sum_{k=i+1}^{j-1} x_{ik} x_{kj} + \left(\delta_{ij} + \sum_{k=i+1}^{j-1} x_{ik} x_{kj} \right) e_j =$$

$$= e_i \delta_{ij} + \delta_{ij} e_j + \sum_{k=i+1}^{j-1} \left(e_i x_{ik} x_{kj} + x_{ik} x_{kj} e_j \right) + \sum_{k=i+1}^{j-1} x_{ik} x_{kj}.$$

From this inductive conjecture, we have $e_i x_{ik} + x_{ik} = x_{ik}$ and $x_{kj} e_j + x_{kj} = x_{kj}$ for k, $i < k < j$. Then, $e_i x_{ik} x_{kj} + x_{ik} x_{kj} = x_{ik} x_{kj}$ and $x_{ik} x_{kj} e_j + x_{ik} x_{kj} = x_{ik} x_{kj}$. Hence,

$$\sum_{k=i+1}^{j-1} \left(e_i x_{ik} x_{kj} + x_{ik} x_{kj} e_j \right) + \sum_{k=i+1}^{j-1} x_{ik} x_{kj} = \sum_{k=i+1}^{j-1} x_{ik} x_{kj}.$$

Since $e_i \delta_{ij} + \delta_{ij} e_j = \delta_{ij} = e_i x_{ij} + x_{ij} e_j$, it follows that X is an idempotent matrix. □

For matrices over a field, the root of the matrix X is the matrix Y, such that $Y^m = X$ for some positive integer, $m > 1$ (see F. R. Gantmacher [20], Ch. 8). In 1979, S. K. Jain et al. (see [21]) studied the nonnegative m th roots of a nonnegative 0-symmetric idempotent matrix. Now, let Y be a matrix over an additively idempotent semiring and $Y = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$. Since $Y^2 = X = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$, it follows that Y is a square root of X. It is easy to check that X is an idempotent matrix. Thus, the following question is reasonable:

Can we find a square root of an arbitrary 3×3 *idempotent triangular matrix?*

In the following, we will answer a more general question in the semiring of upper triangular matrices over an additively idempotent semiring, S.

Let us consider the matrix $\Delta = (\delta_{ij})$, $i, j = 1, \ldots, n$, $i \leq j$, where $\delta_{ii} = e_i$, $e_i^2 = e_i$, $\delta_{ij} = e_i \alpha_{ij} + e_i \alpha_{ij} e_j + \alpha_{ij} e_j$, and α_{ij} are arbitrary elements of S for $i, j = 1, \ldots, n$ and $i < j$. From Theorem 1, it follows that if X is an arbitrary idempotent triangular matrix, then $X + \Delta = X$. This implies that $X + \Delta X = X$ and $X + X\Delta = X$ and then from $(X + \Delta)^2 = X$ we have $X + \Delta^2 = X$. In the same way, it follows that $X + \Delta^m = X$, where m is a positive integer. Now, we will prove that $\Delta^{n-1} = \Delta^n$.

Theorem 2. *Let S be an additively idempotent semiring, $\Delta = (\delta_{ij}) \in T_n(S)$, $i, j = 1, \ldots, n$, $i \leq j$, where $\delta_{ii} = e_i$, $e_i^2 = e_i$ and $\delta_{ij} = e_i \alpha_{ij} + e_i \alpha_{ij} e_j + \alpha_{ij} e_j$, $\alpha_{ij} \in S$ for $i, j = 1, \ldots, n$ and $i < j$. Then, $\Delta^{n-1} = \Delta^n$ and Δ^{n-1} is an idempotent matrix.*

Proof. We obtain that $\Delta^2 = (\delta_{ij}^{(2)})$, where $\delta_{ii}^{(2)} = e_i$, $\delta_{i\,i+1}^{(2)} = \delta_{i\,i+1}$ and for $j \geq i+2$, $\delta_{ij}^{(2)} = \delta_{ij} + \sum_{k=i+1}^{j-1} \delta_{ik} \delta_{kj}$. Note that from the last equality, it follows that on the second super-diagonal all entries are $\delta_{i\,i+2} + \delta_{i\,i+1} \delta_{i+1\,i+2}$ and on the third super-diagonal all entries are $\delta_{i\,i+3} + \delta_{i\,i+1} \delta_{i+1\,i+3} + \delta_{i\,i+2} \delta_{i+2\,i+3}$. The proof is by induction. Let us assume

that for an integer, m, where $m \geq 2$, we have $\Delta^m = (\delta_{ij}^{(m)})$, where $\delta_{ii}^{(m)} = e_i$, $\delta_{ii+1}^{(m)} = \delta_{i\,i+1}$, $\delta_{i\,i+2}^{(m)} = \delta_{i\,i+2} + \delta_{i\,i+1}\delta_{i+1\,i+2}$, and for $j \geq i + m$

$$\delta_{ij}^{(m)} = \delta_{ij} + \sum_{k=i+1}^{j-1} \delta_{ik}\delta_{kj} + \cdots + \sum_{\substack{k_1,\ldots,k_{m-1}=i+1 \\ k_1<\cdots<k_{m-1}}}^{j-1} \delta_{ik_1} \ldots \delta_{k_{m-1}j}.$$

In order to find Δ^{m+1}, we obtain that

$$\begin{aligned}
&\delta_{ii}^{(m+1)} = e_i\delta_{ii}^{(m)} = e_i^2 = e_i,\\
&\delta_{ii+1}^{(m+1)} = e_i\delta_{ii+1}^{(m)} + \delta_{ii+1}e_{i+1} = \delta_{ii+1},\\
&\delta_{i\,i+2}^{(m+1)} = e_i\delta_{i\,i+2}^{(m)} + \delta_{i\,i+1}\delta_{i+1\,i+2}^{(m)} + \delta_{i\,i+2}e_{i+2} =\\
&= e_i\delta_{i\,i+2} + e_i\delta_{i\,i+1}\delta_{i+1\,i+2} + \delta_{i\,i+1}\delta_{i+1\,i+2} + \delta_{i\,i+2}e_{i+2} = \delta_{i\,i+2} + \delta_{i\,i+1}\delta_{i+1\,i+2},
\end{aligned}$$

For $j \geq i + m + 1$, it follows that

$$\begin{aligned}
\delta_{ij}^{(m+1)} &= e_i\delta_{ij}^{(m)} + \delta_{i\,i+1}\delta_{i+1\,j}^{(m)} + \delta_{i\,i+2}\delta_{i+2\,j}^{(m)} + \cdots + \delta_{i\,j-2}\delta_{j-2\,j}^{(m)} + \delta_{i\,j-1}\delta_{j-1\,j}^{(m)} + \delta_{ij}e_j =\\
&= e_i\delta_{ij} + e_i\left(\sum_{k=i+1}^{j-1} \delta_{ik}\delta_{kj} + \cdots + \sum_{\substack{k_1,\ldots,k_{m-1}=i+1 \\ k_1<\cdots<k_{m-1}}}^{j-1} \delta_{ik_1} \ldots \delta_{k_{m-1}j}\right) +\\
&+\delta_{i\,i+1}\left(\delta_{i+1\,j} + \sum_{k=i+2}^{j-1} \delta_{i+1\,k}\delta_{kj} + \cdots + \sum_{\substack{k_1,\ldots,k_{m-1}=i+2 \\ k_1<\cdots<k_{m-1}}}^{j-1} \delta_{i+1\,k_1} \ldots \delta_{k_{m-1}j}\right) +\\
&+\delta_{i\,i+2}\left(\delta_{i+2\,j} + \sum_{k=i+3}^{j-1} \delta_{i+2\,k}\delta_{kj} + \cdots + \sum_{\substack{k_1,\ldots,k_{m-1}=i+3 \\ k_1<\cdots<k_{m-1}}}^{j-1} \delta_{i+2\,k_1} \ldots \delta_{k_{m-1}j}\right) +\\
&+ \cdots + \delta_{i\,j-2}\delta_{j-2\,j} + \delta_{i\,j-2}\delta_{j-2\,j-1}\delta_{j-1\,j} + \delta_{i\,j-1}\delta_{j-1\,j} + \delta_{ij}e_j =\\
&= \delta_{ij} + \sum_{k=i+1}^{j-1} \delta_{ik}\delta_{kj} + \cdots + \sum_{\substack{k_1,\ldots,k_m=i+1 \\ k_1<\cdots<k_m}}^{j-1} \delta_{i\,k_1} \ldots \delta_{k_m j}.
\end{aligned}$$

Therefore,

$$\Delta^{n-1} = \begin{pmatrix}
e_1 & \delta_{12} & \delta_{13} + \delta_{12}\delta_{23} & \cdots & \delta_{1n} + \sum_{k=2}^{n-1} \delta_{1k}\delta_{kn} + \cdots + \delta_{12}\delta_{23} \ldots \delta_{n-1\,n}\\
0 & e_2 & \delta_{23} & \cdots & \delta_{2n} + \sum_{k=3}^{n-1} \delta_{2k}\delta_{kn} + \cdots + \delta_{23} \ldots \delta_{n-1\,n}\\
0 & 0 & e_3 & \cdots & \delta_{3n} + \sum_{k=4}^{n-1} \delta_{3k}\delta_{kn} + \cdots + \delta_{34} \ldots \delta_{n-1\,n}\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & 0 & \cdots & e_n
\end{pmatrix}.$$

Now, it is easy to check that $\Delta^n = \Delta^{n-1}$. As a consequence, it follows that $\Delta^{n-1} = \Delta^{n+1} = \cdots = (\Delta^{n-1})^2$. □

The next result describes all idempotent triangular matrices over an additively idempotent semiring.

Theorem 3. *An $n \times n$ upper triangular matrix, $X = (x_{ij})$, over the additively idempotent semiring S is idempotent if and only if $x_{ii} = e_i$, where $e_i^2 = e_i$ for $i = 1, \dots, n$, $x_{i\,i+1} = \delta_{i\,i+1}$ for $i = 1, \dots, n-1$, and*

$$x_{ij} = \delta_{ij} + \sum_{k=i+1}^{j-1} \delta_{ik}\delta_{kj} + \sum_{\substack{k,k_1=i+1 \\ k<k_1}}^{j-1} \delta_{ik}\delta_{kk_1}\delta_{k_1j} + \cdots + \delta_{ii+1}\dots\delta_{j-1j},$$

where $\delta_{ij} = e_i\alpha_{ij} + e_i\alpha_{ij}e_j + \alpha_{ij}e_j$, $\alpha_{ij} \in S$ for $i, j = 1, \dots, n$ and $i+1 < j$.

Proof. From Theorem 1, we know that the matrix $X = (x_{ij})$ is idempotent if and only if $x_{ii} = e_i$, where $e_i^2 = e_i$ for $i = 1, \dots, n$, $x_{i\,i+1} = \delta_{i\,i+1}$ and $x_{ij} = \delta_{ij} + \sum_{k=i+1}^{j-1} x_{ik}x_{kj}$ for $i+1 < j$. Successively applying this equality, we obtain $x_{ij} =$

$$\delta_{ij} + \sum_{k=i+1}^{j-1} \left(\delta_{ik} + \sum_{k_2=i+1}^{k-1} x_{ik_2}x_{k_2k}\right)\left(\delta_{kj} + \sum_{k_1=k+1}^{j-1} x_{kk_1}x_{k_1j}\right) = \delta_{ij} + \sum_{k=i+1}^{j-1} \delta_{ik}\delta_{kj} +$$

$$+ \sum_{k=i+1}^{j-1} \delta_{ik}\left(\sum_{k_1=k+1}^{j-1}\left(\delta_{kk_1} + \sum_{k_3=k1+1}^{k_1-1} x_{kk_3}x_{k_3k_1}\right)\left(\delta_{k_1j} + \sum_{k_4=k_1+1}^{j-1} x_{k_1k_4}x_{k_4j}\right)\right) +$$

$$+ \sum_{k=i+1}^{j-1}\left(\sum_{k_2=i+1}^{k-1}\left(\delta_{ik_2} + \sum_{k_5=i+1}^{k_2-1} x_{ik_5}x_{k_5k_2}\right)\left(\delta_{k_2k} + \sum_{k_6=k_2+1}^{k-1} x_{k_2k_6}x_{k_6k}\right)\right)\delta_{kj} =$$

$$= \delta_{ij} + \sum_{k=i+1}^{j-1} \delta_{ik}\delta_{kj} + \sum_{\substack{k,k_1=i+1 \\ k<k_1}}^{j-1} \delta_{ik}\delta_{kk_1}\delta_{k_1j} + \sigma.$$

Some summands are reproduced; for example, $\sum_{k_2=i+1}^{k-1} \delta_{ik_2}\delta_{k_2k}\delta_{kj}$ is a part of $\sum_{\substack{k,k_1=i+1 \\ k<k_1}}^{j-1} \delta_{ik}\delta_{kk_1}\delta_{k_1j}$. For the sum

$$\sigma = \sum_{k=i+1}^{j-1} \delta_{ik}\left[\sum_{k_1=k+1}^{j-1}\left(\delta_{kk_1}\sum_{k_4=k_1+1}^{j-1} x_{k_1k_4}x_{k_4j} + \sum_{k_3=k+1}^{k_1-1} x_{kk_3}x_{k_3k_1}\delta_{k_1j}\right) +\right.$$

$$\left. + \sum_{k_2=i+1}^{k-1}\left(\delta_{ik_2}\sum_{k_6=k_2+1}^{k-1} x_{k_2k_6}x_{k_6k} + \sum_{k_5=i+1}^{k_2-1} x_{ik_5}x_{k_5k_2}\delta_{k_2k}\right)\right]\delta_{kj},$$

we have the following:

σ contains $\sum_{\substack{k,k_1=i+1 \\ k<k_1}}^{j-1} \delta_{ik}\delta_{kk_1}\delta_{k_1j}$ and is part of this sum;

σ contains all products $\delta_{im_1}\dots\delta_{m_sj}$ where $m_1 < \cdots < m_s$ and consequently the product $\delta_{ii+1}\dots\delta_{j-1j}$.

Hence, the result follows. □

Remark 1. *We denote the equality from Theorem 3 briefly by*

$$x_{ij} = \delta_{ij} + \sum_{i}^{j} * \tag{2}$$

and use it in Section 4.

Immediately from Theorems 2 and 3, we have the following.

Corollary 1. *Each $n \times n$ upper triangular idempotent matrix $E(e_1, \ldots, e_n)$ over the additively idempotent semiring S has the form*

$$\begin{pmatrix} e_1 & \delta_{12} & \delta_{13} + \delta_{12}\delta_{23} & \cdots & \delta_{1n} + \sum_{k=2}^{n-1} \delta_{1k}\delta_{kn} + \cdots + \delta_{12}\delta_{23}\ldots\delta_{n-1\,n} \\ 0 & e_2 & \delta_{23} & \cdots & \delta_{2n} + \sum_{k=3}^{n-1} \delta_{2k}\delta_{kn} + \cdots + \delta_{23}\ldots\delta_{n-1\,n} \\ 0 & 0 & e_3 & \cdots & \delta_{3n} + \sum_{k=4}^{n-1} \delta_{3k}\delta_{kn} + \cdots + \delta_{34}\ldots\delta_{n-1\,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e_n \end{pmatrix}, \tag{3}$$

where $\delta_{ij} = e_i\alpha_{ij} + e_i\alpha_{ij}e_j + \alpha_{ij}e_j$, $\alpha_{ij} \in S$ for $i, j = 1, \ldots, n$ and $i < j$.

Each idempotent matrix, $E(e_1, \ldots, e_n)$, has roots, which are the matrices Δ^m, where $m = 1, \ldots, n-2$.

As a consequence of the theorems above, we can show a result similar to the main theorem of X. Hou (cf. [11]).

Corollary 2. *Let $\Delta = (a_{ij}) \in T_n(S)$ be a matrix with the following properties:*

(i) All diagonal entries of Δ are ones or zeroes;

(ii) If $a_{ii} = a_{jj} = 0$, then $a_{ij} = 0$, where $i, j = 1, \ldots, n$ and $i < j$.

Then, all idempotent triangular matrices over S with only zeroes and ones on the main diagonal are the matrices $E = (x_{ij})$ such that $x_{ii} = a_{ii}$, $i = 1, \ldots, n$, and

$$x_{ij} = a_{ij} + \sum_{k=i+1}^{j-1} a_{ik}a_{kj} + \sum_{\substack{k,k_1=i+1 \\ k<k_1}}^{j-1} a_{ik}a_{k\,k_1}a_{k_1\,j} + \cdots + a_{i\,i+1}\ldots a_{j-1\,j}, 1 \le i < j \le n.$$

Proof. If in the proof of Theorem 2 at least one of the idempotents e_i and e_j are equal to 1, it follows that $\delta_{ij} = a_{ij}$. When $e_i = e_j = 0$, we have $\delta_{ij} = 0$. Therefore, the matrix Δ is the same matrix Δ in the proof of Theorem 2 under the given conditions. This result follows from Theorem 3. □

Since the entries of the idempotent matrix in (3) have rather complex structures, we obtain idempotent matrices with the parameters α_{ij} of a special form. Such a matrix is $\begin{pmatrix} e_1 & e_1\alpha_{12}e_2 \\ 0 & e_2 \end{pmatrix}$, considered in Example 1. To develop this idea to its conclusion, we construct the matrix $\Gamma = (\gamma_{ij})$, where $\gamma_{ii} = e_i$, $e_i^2 = e_i$ for $i = 1, \ldots, n$ and $\gamma_{ij} = e_i\alpha_{ij}e_j$, $\alpha_{ij} \in S$ for $1 < i < j \le n$. Matrices of the form Γ which are idempotents are called special idempotents. It is not difficult to verify that for $n = 3$, the matrix $\Gamma = \begin{pmatrix} e_1 & e_1\alpha_{12}e_2 & e_1\alpha_{13}e_3 \\ 0 & e_2 & e_2\alpha_{23}e_3 \\ 0 & 0 & e_3 \end{pmatrix}$ is not an idempotent, but the matrix $\Gamma^2 = \begin{pmatrix} e_1 & e_1\alpha_{12}e_2 & e_1\alpha_{13}e_3 + e_1\alpha_{12}e_2\alpha_{23}e_3 \\ 0 & e_2 & e_2\alpha_{23}e_3 \\ 0 & 0 & e_3 \end{pmatrix}$ is an idempotent. Moreover, $\Gamma^2 = \begin{pmatrix} e_1 & e_1\beta_{12}e_2 & e_1\beta_{13}e_3 \\ 0 & e_2 & e_2\beta_{23}e_3 \\ 0 & 0 & e_3 \end{pmatrix}$, where $\beta_{12} = \alpha_{12}$, $\beta_{13} = \alpha_{13} + \alpha_{12}e_2\alpha_{23}$ and $\beta_{23} = \alpha_{23}$. . Thus, for $n = 3$, the matrix Γ^2 is a special idempotent matrix.

Theorem 4. *Let $\Gamma = (\gamma_{ij}) \in T_n(S)$, where $\gamma_{ii} = e_i$, $e_i^2 = e_i$, for $i = 1, \ldots, n$ and $\gamma_{ij} = e_i\alpha_{ij}e_j$, $\alpha_{ij} \in S$ for $1 < i < j \le n$. Then, Γ^{n-1} is a special idempotent matrix.*

Proof. Since $e_i\gamma_{ij} = \gamma_{ij}$ and $\gamma_{ij}e_j = \gamma_{ij}$, it follows that the proof of the statement that Γ^{n-1} is an idempotent matrix is the same as the proof of Theorem 2. Following the proof of this theorem, we obtain that $\Gamma^{n-1} = (\gamma^{(n-1)})$, where

$$\gamma_{ij}^{(n-1)} = \gamma_{ij} + \sum_{k=i+1}^{j-1} \gamma_{ik}\gamma_{kj} + \sum_{\substack{k,k_1=i+1 \\ k<k_1}}^{j-1} \gamma_{ik}\gamma_{kk_1}\gamma_{k_1j} + \cdots + \gamma_{ii+1}\cdots\gamma_{j-1j} =$$

$$= e_i \left(\alpha_{ij} + \sum_{k=i+1}^{j-1} \alpha_{ik}e_k\alpha_{kj} + \sum_{\substack{k,k_1=i+1 \\ k<k_1}}^{j-1} \alpha_{ik}e_k\alpha_{kk_1}e_{k_1}\alpha_{k_1j} + \right.$$

$$\left. + \cdots + \alpha_{i\,i+1}e_{i+1}\alpha_{i+1\,i+2}\ldots\alpha_{j-2\,j-1}e_{j-1}\alpha_{j-1j} \right) e_j$$

and this completes the proof. □

The set of special idempotent matrices is denoted by $\mathcal{SP}(S)$. This notation is used in Section 4.

3. Definitions and Results of Idempotents of Additively Idempotent Semiring

Let S be an additively idempotent semiring. Following G. F. Birkenmeier et al. (see [22]), we give the following definition: An idempotent, $\ell \in S$ $(resp., r \in S)$, is called left (resp., right) semicentral if $\ell x\ell = x\ell$ (resp., $rxr = rx$) for an arbitrary $x \in S$.

From this definition, we easily have the following.

Proposition 1. *The set of left (resp., right) semicentral idempotents of the semiring S is a multiplicative semigroup with an identity.*

The multiplicative semigroup of the left (resp., right) semicentral idempotents of the semiring S is denoted by $(\mathcal{L}(S), .)$ (resp., $(\mathcal{R}(S), .)$).

For $\ell_1, \ell_2 \in \mathcal{L}(S)$, we consider the Jordan product $\ell_1 \circ \ell_2 = \ell_1\ell_2 + \ell_2\ell_1$. Similarly, for $r_1, r_2 \in \mathcal{R}(S)$, we define $r_1 \circ r_2 = r_1r_2 + r_2r_1$. It is straightforward to prove the following.

Proposition 2. *The semigroup $\mathcal{L}(S)$ (resp., $\mathcal{R}(S)$) is closed under the Jordan product.*

Let $\mathcal{LR}(S)$ be the semigroup generated by $\mathcal{L}(S) \cup \mathcal{R}(S)$. For the products and Jordan products of elements of $\mathcal{LR}(S)$, we easily have the following.

Proposition 3. *Let $\ell \in \mathcal{L}(S)$ and $r \in \mathcal{R}(S)$. Then, we have the following:*

(i) $r\ell$ is an idempotent, but it is neither a left nor a right semicentral idempotent; also, $\ell rx\ell r = r\ell xr\ell$ for all $x \in R$ and then $(\ell r)^2 = r\ell$.

(ii) $(\ell \circ r)^2 = r\ell$.

(iii) $r \circ r\ell = \ell \circ r\ell = r\ell$.

(iv) $\ell \circ \ell r = r \circ \ell r = \ell r \circ r = \ell(\ell \circ r) = \ell \circ (\ell \circ r) = r \circ (\ell \circ r) = \ell \circ r$.

In order to keep this exposition self-contained, we present some definitions very similar to those in [14]. Let $\ell_1, \ell_2 \in \mathcal{L}(S)$. We define

$$\ell_2 \succeq \ell_1 \text{ if and only if } \ell_2x\ell_1 = x\ell_1 \text{ for all } x \in S.$$

Then, it follows that $1 \succeq \ell \succeq 0$ for $\ell \in \mathcal{L}(S)$. Since $\ell_2 x \ell_1 = \ell_2 \ell_1 x \ell_1$, it follows that, if $\ell_2 \ell_1 = \ell_1$, then $\ell_2 \succeq \ell_1$. On the other hand, if $\ell_2 \succeq \ell_1$, then $\ell_2 \ell_1 = \ell_1$. Hence,

$$\ell_2 \succeq \ell_1 \iff \ell_2 \ell_1 = \ell_1. \tag{4}$$

Let $\ell_0, \ell_1, \ell_2 \in \mathcal{L}(S)$ and $\ell_2 \succeq \ell_1$. From (4), we have $\ell_2 \ell_0 \ell_1 \ell_0 = \ell_2 \ell_1 \ell_0 = \ell_1 \ell_0$. Thus, $\ell_2 \succeq \ell_1 \Rightarrow \ell_2 \ell_0 \succeq \ell_1 \ell_0$. Similarly, for $r_1, r_2 \in \mathcal{R}(S)$, we define

$$r_2 \succeq r_1 \text{ if and only if } \ r_1 x r_2 = r_1 x \ \text{ for all } x \in S.$$

It immediately follows that $1 \succeq r \succeq 0$ for $r \in \mathcal{R}(S)$. Since $r_1 x r_2 = r_1 x r_1 r_2$, it follows that $r_1 r_2 = r_1$, which implies that $r_2 \succeq r_1$. Conversely, if $r_2 \succeq r_1$, we obtain $r_1 r_2 = r_1$. Consequently,

$$r_2 \succeq r_1 \iff r_1 r_2 = r_1. \tag{5}$$

Let $r_0, r_1, r_2 \in \mathcal{L}(S)$ and $r_2 \succeq r_1$. From (5), it follows that $r_0 r_1 r_0 r_2 = r_0 r_1 r_2 = r_0 r_1$, which implies $r_2 \succeq r_1 \Rightarrow r_0 r_2 \succeq r_0 r_1$.

We define an equivalence relation on $\mathcal{L}(R)$ as follows. From the definition, we have the following:

(i) $\ell_1 \succeq \ell_1$ for an arbitrary $\ell_1 \in \mathcal{L}(R)$;

(ii) $\ell_2 \succeq \ell_1$ and $\ell_3 \succeq \ell_2$ implies that $\ell_3 \succeq \ell_1$.

Then, we state that $\ell_1 \sim_{\text{left}} \ell_2$ if and only if $\ell_1 \succeq \ell_2$ and $\ell_2 \succeq \ell_1$. Thus, $\sim_{\text{left}}$ is an equivalence relation on $\mathcal{L}(R)$.

The definition of an equivalence relation on $\mathcal{R}(R)$ is similar. We have the following:

(i) $r_1 \succeq r_1$ for an arbitrary $r_1 \in \mathcal{R}(R)$.

(ii) $r_2 \succeq r_1$ and $r_3 \succeq r_2$ implies that $r_3 \succeq r_1$.

If we define $r_1 \sim_{\text{right}} r_2$ if and only if $r_1 \succeq r_2$ and $r_2 \succeq r_1$, it follows that $\sim_{\text{right}}$ is an equivalence relation on $\mathcal{R}(R)$.

Let $e \in S$ be an idempotent. Since the additively idempotent semiring S is partially ordered ($a + b = a \iff a \geq b$), we can consider two cases:

Case 1. Let $e \leq 1$. Then, e is called left (*resp.*, right) if $ex \geq xe$ (*resp.*, $xe \geq ex$) for an arbitrary $x \in S$ (see D. Vladeva [23]). If e is a left idempotent, it follows that $exe \geq xe$. But, from $e \leq 1$, we have $exe \leq xe$. Thus, $exe = xe$, i.e., e is a left semicentral idempotent. Similarly, if e is a right idempotent, we obtain that it is a right semicentral idempotent.

Note that if e is a left or a right semicentral idempotent, then $ex + xe \geq exe$.

Case 2. Let $e \geq 1$. If e is a left (*resp.*, right) idempotent, it follows that $exe \geq ex \geq xe$ (*resp.*, $exe \geq xe \geq ex$). Therefore, $exe \geq ex + xe$.

The underlying idea of the last inequality is to construct a new specific class of idempotents. An idempotent, $\jmath \in S$, is called Jordan semicentral if we have the following:

(a) $\jmath \geq 1$;

(b) $\jmath x \jmath = \jmath \circ x = \jmath x + x\jmath$ for an arbitrary $x \in S$.

In order to find examples of such idempotents, we consider $1 + e$, where e is an idempotent of S. Since $1 + e$ is an idempotent, it follows that $1 + \ell$ and $1 + r$, where $\ell \in \mathcal{L}(S)$ and $r \in \mathcal{R}(S)$ are idempotents.

We obtain $(1 + \ell)x(1 + \ell) = x + \ell x + x\ell + \ell x \ell = x + \ell x + x\ell = (1 + \ell)x + x(1 + \ell)$ for any $x \in S$ and $\ell \in \mathcal{L}(S)$. Similarly, $(1 + r)x(1 + r) = (1 + r)x + x(1 + r)$ for any $x \in S$ and $r \in \mathcal{R}(S)$. Moreover, if e is an arbitrary idempotent of S, then $1 + e$ is a Jordan semicentral idempotent. Thus, we have enough examples for Jordan semicentral idempotents.

As in ring theory, the idempotent c from the centre of the semiring S is called a central idempotent. If we denote by $(\mathcal{C}(S), .)$ the semigroup of central idempotents, then

$\mathcal{C}(S) = \mathcal{L}(S) \cap \mathcal{R}(S)$. On the other hand, if we denote by $\mathcal{J}(S)$ the set of Jordan semicentral idempotents, it follows that if $c \in \mathcal{C}(S)$ and $c \geq 1$, then $c \in \mathcal{J}(S)$.

Proposition 4. *Let $\jmath \in \mathcal{J}(S)$. If $c \in \mathcal{C}(S)$, then $\jmath c \in \mathcal{J}(S)$.*

Proof. From $\jmath x\jmath = \jmath x + x\jmath$ for all $x \in S$, it follows that $\jmath cx\jmath c = (\jmath cx\jmath)c = (\jmath cx + cx\jmath)c = \jmath cx + x\jmath c$. □

Proposition 5. *Let $\ell \in \mathcal{L}(S)$, $r \in \mathcal{R}(S)$ and $\jmath, \jmath_1 \in \mathcal{J}(S)$. Then, we have the following:*
(1) $(\jmath\ell)^2 = \jmath\ell$; (2) $(\ell\jmath)^2 = \ell \circ \jmath$; (3) $(r\jmath)^2 = r\jmath$; (4) $(\jmath r)^2 = r \circ \jmath$;
(5) $(\jmath \circ \ell)^2 = \jmath \circ \ell$; (6) $(\jmath \circ r)^2 = \jmath \circ r$; (7) $(r\jmath\ell)^2 = r\jmath\ell$;
(8) $(\jmath \circ \ell r)^2 = (\jmath \circ r\ell)^2 = (\jmath \circ r)(\jmath \circ \ell)$;
(9) $((\ell \circ \jmath) \circ r)(\ell \circ (\jmath \circ r)) = (\jmath \circ r)(\jmath \circ \ell)$;
(10) $(\jmath \circ \jmath_1)^2 = \jmath \circ \jmath_1$.

Proof. The proof is routine from these definitions. □

Proposition 6. *Let e_1 and e_2 be idempotents of S. Then, $e_1 + e_2 + e_1 \circ e_2$ is an idempotent in the following cases:*
(a) $e_1e_2 = 0$; (b) $e_1, e_2 \in \mathcal{L}(S)$; (c) $e_1, e_2 \in \mathcal{R}(S)$; (d) $e_1, e_2 \in \mathcal{J}(S)$;
(e) $e_1 \in \mathcal{L}(S), e_2 \in \mathcal{R}(S)$; (f) $e_1 \in \mathcal{L}(S), e_2 \in \mathcal{J}(S)$; (g) $e_1 \in \mathcal{R}(S), e_2 \in \mathcal{J}(S)$.

Proof. The proof is routine from these definitions. □

Proposition 7. *If $\jmath, \jmath_1 \in \mathcal{J}(S)$, then $(\jmath\jmath_1)^2 = (\jmath_1\jmath)^2$.*

Proof. We obtain $(\jmath\jmath_1)^2 = \jmath\jmath_1\jmath\jmath_1 = \jmath(\jmath_1\jmath + \jmath\jmath_1) = \jmath\jmath_1\jmath + \jmath\jmath_1 = \jmath\jmath_1 + \jmath_1\jmath = \jmath \circ \jmath_1$. Since $(\jmath_1\jmath)^2 = \jmath_1 \circ \jmath$, the result follows. □

Let $\jmath, \jmath_1 \in \mathcal{J}(S)$. We write $\jmath \succeq \jmath_1$ if $\jmath x\jmath_1 = \jmath x + x\jmath_1$ for an arbitrary $x \in S$. Thus, when $\jmath \succeq \jmath_1$, we have $\jmath\jmath_1 = \jmath + \jmath_1$ and then $\jmath$ and $\jmath_1$ commute.

We now define $\jmath \sim \jmath_1$ if $\jmath \succeq \jmath_1$ and $\jmath_1 \succeq \jmath$. Note that for Jordan semicentral idempotents, the relation "$\succeq$" is not transitive and the relation "$\sim$" is not an equivalence relation.

From $\jmath \geq 1$, it follows that $\jmath \succeq 1$ and $1 \succeq \jmath$; thus, $\jmath \sim 1$ for any $\jmath \in \mathcal{J}(S)$.

It is easy to see that if $\jmath_1 = 1 + \ell_1$, $\jmath_2 = 1 + \ell_2$, and $\ell_1 \succeq \ell_2$, then $\jmath_1 \succeq \jmath_2$, where $\ell_1, \ell_2 \in \mathcal{L}(S)$. Similarly, if $\jmath_1 = 1 + r_1$, $\jmath_2 = 1 + r_2$ and $r_2 \succeq r_1$, then $\jmath_1 \succeq \jmath_2$, where $r_1, r_2 \in \mathcal{R}(S)$.

Therefore, if $\ell_1 \sim_{\text{left}} \ell_2$ for $\jmath_1 = 1 + \ell_1$ and $\jmath_2 = 1 + \ell_2$, it follows that $\jmath_1 \sim \jmath_2$ and, similarly, if $r_1 \sim_{\text{right}} r_2$, then $\jmath_1 = 1 + r_1$ and $\jmath_2 = 1 + r_2$ implies $\jmath_1 \sim \jmath_2$.

For a fixed $\jmath \in \mathcal{J}(S)$, we consider the set $\mathcal{J}_\jmath = \{\jmath_1 | \jmath_1 \sim \jmath\}$.

Proposition 8. *For any $\jmath \in \mathcal{J}(S)$, the set $\mathcal{J}_\jmath$ is a commutative semigroup.*

Proof. Let $\jmath, \jmath_1 \in \mathcal{J}(S)$ and $\jmath \sim \jmath_1$. Then, for an arbitrary $x \in S$

$$(\jmath + \jmath_1)x(\jmath + \jmath_1) = \jmath x\jmath + \jmath x\jmath_1 + \jmath_1 x\jmath + \jmath_1 x\jmath_1 =$$

$$= \jmath x\jmath + \jmath x + x\jmath_1 + \jmath_1 x + x\jmath + \jmath_1 x\jmath_1 = \jmath x\jmath + \jmath_1 x\jmath_1 = (\jmath + \jmath_1)x + x(\jmath + \jmath_1).$$

Hence, $\jmath\jmath_1 x\jmath\jmath_1 = \jmath\jmath_1 x + x\jmath\jmath_1$. □

Remark 2. *In the next section, we give examples of left semicentral idempotent triangular matrices, ℓ_1 and ℓ_2, such that $\ell_1 \sim_{left} \ell_2$ and there is similar for right and Jordan semicentral idempotent triangular matrices.*

Now, let e_1 and e_2 be arbitrary idempotents of S. It is straightforward to prove the following.

Proposition 9. *If $e_1 \leq 1$, then $e_1 + e_2$ is an idempotent.*

Similarly, it follows that $\sum_{i=1}^{n} e_i$ is an idempotent if $e_1, \ldots, e_n$ are idempotents such that $e_i \leq 1$ for $i = 1, \ldots, n-1$.

Proposition 10. *Let e be an idempotent of the semiring S, with $\ell \in \mathcal{L}(S)$, $r \in \mathcal{R}(S)$, and $\jmath \in \mathcal{J}(S)$. Then, $e\ell e \in \mathcal{L}(eSe)$, $ere \in \mathcal{R}(eSe)$, and $e\jmath e \in \mathcal{J}(eSe)$.*

Proof. The proof is routine from these definitions. □

Let e be a fixed idempotent of the semiring S. We include some basic results related to idempotents of the semiring eS.

(i) $e\ell$ is a left semicentral idempotent of eS for any $\ell \in \mathcal{L}(S)$.

(ii) Since $(er)^2 = ere$ for any $r \in \mathcal{R}(S)$, from Proposition 10, it follows that er is a square root of an idempotent.

(iii) $(er)(e\ell)$ is an idempotent where $\ell \in \mathcal{L}(S)$ and $r \in \mathcal{R}(S)$. Moreover, $e\ell r$ and $er\ell$ are square roots of $(er)(e\ell)$. Similarly, $(er)(e\ell e)$ is an idempotent.

(iv) Since $((e\ell)(ere))^2 = (er)(e\ell e)$, we have that $(e\ell)(ere)$ is a root of an idempotent.

(v) $(e\jmath)(e\ell)$ is an idempotent where $\ell \in \mathcal{L}(S)$ and $\jmath \in \mathcal{J}(S)$. Similarly, $(e\jmath)(e\ell e)$ is an idempotent.

(vi) $(er)(e\jmath e)$ is an idempotent where $r \in \mathcal{R}(S)$ and $\jmath \in \mathcal{J}(S)$.

(vii) $(er)(e\jmath)(e\ell)$ is an idempotent where $\ell \in \mathcal{L}(S)$, $r \in \mathcal{R}(S)$, and $\jmath \in \mathcal{J}(S)$.

(viii) $(e\jmath)^2 = e\jmath e + e\jmath$ and $e\jmath e + e\jmath$ are idempotents where $\jmath \in \mathcal{J}(S)$. Consequently, $(e\jmath)^4 = (e\jmath)^2$.

Now, we describe the dual results related to idempotents of the semiring Se:

(i) re is a right semicentral idempotent of Se for any $r \in \mathcal{R}(S)$.

(ii) Since $(\ell e)^2 = e\ell e$ for any $\ell \in \mathcal{L}(S)$, from Proposition 10, it follows that ℓe is a root of an idempotent.

(iii) $(re)(\ell e)$ is an idempotent where $\ell \in \mathcal{L}(S)$ and $r \in \mathcal{R}(S)$. Moreover, $r\ell e$ and ℓre are square roots of $(re)(\ell e)$. Similarly, $(ere)(\ell e)$ is an idempotent.

(iv) Since $((\ell e)(ere))^2 = (ere)(\ell e)$, we have that $(\ell e)(ere)$ is a root of an idempotent.

(v) $(re)(\jmath e)$ is an idempotent where $r \in \mathcal{R}(S)$ and $\jmath \in \mathcal{J}(S)$. Similarly, $(ere)(\jmath e)$ is an idempotent.

(vi) $(e\jmath e)(\ell e)$ is an idempotent where $\ell \in \mathcal{L}(S)$ and $\jmath \in \mathcal{J}(S)$.

(vii) $(re)(\jmath e)(\ell e)$ is an idempotent where $\ell \in \mathcal{L}(S)$, $r \in \mathcal{R}(S)$, and $\jmath \in \mathcal{J}(S)$.

(viii) $(\jmath e)^2 = \jmath e + e\jmath e$ and $\jmath e + e\jmath e$ are idempotents where $\jmath \in \mathcal{J}(S)$. Consequently, $(\jmath e)^4 = (\jmath e)^2$.

Let e_1 and e_2 be idempotents of S and $e_2 e_1 = 0$. Note that, by Proposition 6a, it follows that $e_1 + e_2 + e_1 e_2$ is an idempotent.

We denote $e_1 \diamond e_2 = e_1 + e_2 + e_1 e_2$. (The reader can obtain the same construction in ring theory, related to non-unital rings, in [24].)

The next proposition motivates the results in APPLICATION A.

Proposition 11. *Let e be an idempotent of the semiring S, with $\ell \in \mathcal{L}(S)$ and $r \in \mathcal{R}(S)$. If $r\ell = 0$, then (a) $\ell \diamond r$ and (b) $e\ell \diamond re$ is an idempotent of S.*

Proof. Part (a) follows immediately from Proposition 6 (a). Since $(re)(el) = re\ell = r\ell e\ell = 0$, (b) holds. □

Remark 3. *In the next section, we will show natural examples of a left semicentral idempotent, ℓ, and a right semicentral idempotent, r, such that $r\ell = 0$. So, it will be interesting to describe the semiring closure (the minimal subsemiring of S) generated by the fixed ℓ and r. The semiring closure consists of the nilpotent ℓr, since $(\ell r)^2 = r\ell = 0$, the square root $\ell + r$, since $(\ell + r)^2 = \ell + r + \ell r$, and five idempotents: ℓ, r, $\ell + \ell r$, $r + \ell r$, and $\ell + r + \ell r$.*

Using a left semicentral idempotent, ℓ, and a right semicentral idempotent, r, such that $r\ell = 0$, we can construct a new Jordan semicentral idempotent.

Proposition 12. *If $\ell \in \mathcal{L}(S)$, $r \in \mathcal{R}(S)$, and $rl = 0$, then it follows that $\jmath = 1 + \ell r$ is a Jordan semicentral idempotent.*

Proof. We obtain $(1 + \ell r)x(1 + \ell r) = x + \ell rx + x\ell r + \ell rx\ell r = (1 + \ell r)x + x(1 + \ell r) + rx\ell$, where $x \in S$. Since $rx\ell = rxr\ell = 0$, we have that $\jmath x\jmath = \jmath x + x\jmath$. □

Proposition 13. *If $\ell \in \mathcal{L}(S)$ and $r \in \mathcal{R}(S)$, then for the Jordan semicentral idempotents $\jmath_\ell = 1 + \ell$ and $\jmath_r = 1 + r$, it follows that $\jmath_\ell \diamond \jmath_r = 1 + \ell \diamond r$.*

Proof. It is easy to check that $1 + e$ is an idempotent for any idempotent e of S. We know that $\jmath_\ell = 1 + \ell$ and $\jmath_r = 1 + r$ are Jordan semicentral idempotents.

Consequently, $\jmath_\ell \diamond \jmath_r = (1 + \ell) \diamond (1 + r) = 1 + \ell + r + \ell r = 1 + \ell \diamond r$. □

We can extend the last result as follows.

Proposition 14. *Let $e \leq 1$ be an idempotent of S, with $\ell \in \mathcal{L}(S)$ and $r \in \mathcal{R}(S)$. Then, it follows that $\jmath_1 = 1 + e\ell$ and $\jmath_2 = 1 + re$ are Jordan semicentral idempotents and $\jmath_1 \diamond \jmath_2 = 1 + e\ell \diamond re$.*

Proof. As in the proof of Proposition 13, we obtain that $(1 + e\ell) \diamond (1 + re) = 1 + e\ell \diamond re$. When $e \leq 1$ for an arbitrary $x \in S$, it follows that $\jmath_1 x \jmath_1 = (1 + e\ell)x(1 + e\ell) = x + e\ell x + xe\ell + e\ell xe\ell = (1 + e\ell)x + x(1 + e\ell) = \jmath_1 x + x\jmath_1$. Similarly, we observe that $\jmath_2 = 1 + re$ is a Jordan semicentral idempotent. □

4. Results on Semicentral Idempotents of Triangular Matrix Semiring

For the proofs of the following two important theorems, we need another lemma (similar to Proposition 3.1 of [14] for trangular matrices over a ring).

Lemma 1. *Let $X = (x_{ij}) \in T_n(S)$ be an arbitrary matrix. If $P = (\pi_{ij}) \in T_n(S)$ is a left (resp., right or Jordan) semicentral idempotent of $T_n(S)$, then π_{ii}, $i = 1, \ldots, n$, are left (resp., right or Jordan) semicentral idempotents of S.*

Proof. We obtain the diagonal entries of the following matrices:

$x_{ii}\pi_{ii}$ of the matrix XP;
$\pi_{ii}x_{ii}$ of the matrix PX;
$\pi_{ii}x_{ii} + x_{ii}\pi_{ii}$ of the matrix $PX + XP$;
$\pi_{ii}x_{ii}\pi_{ii}$ of the matrix PXP for $i = 1, \ldots, n$.

Now, if P is a left semicentral idempotent of $T_n(S)$, we have $\pi_{ii}x_{ii}\pi_{ii} = x_{ii}\pi_{ii}$; if P is a right semicentral idempotent of $T_n(S)$, we obtain $\pi_{ii}x_{ii}\pi_{ii} = \pi_{ii}x_{ii}$; and if P is a Jordan semicentral idempotent of $T_n(S)$, then $\pi_{ii}x_{ii}\pi_{ii} = \pi_{ii}x_{ii} + x_{ii}\pi_{ii}$. Thus, the result follows. □

Theorem 5. *Each left semicentral idempotent triangular $n \times n$ matrix over an additively idempotent semiring, S, without zero divisors is of the form $\ell_k = \begin{pmatrix} \Lambda_k & \Lambda_k A \\ 0 & 0 \end{pmatrix}$, where $\Lambda_k = \operatorname{diag}(\lambda_1, \ldots, \lambda_k)$, $\lambda_i \in \mathcal{L}(S)$, for $i = 1, \ldots, k$, $\lambda_{i-1} \succeq \lambda_i$, $i = 2, \ldots k$, and $A = (\alpha_{ij}) \in M_{k\times(n-k)}$ for $k = 1, \ldots, n-1$, or is of the form $\ell_n = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_i \in \mathcal{L}(S)$ for $i = 1, \ldots n$ and $\lambda_{i-1} \succeq \lambda_i$ for $i = 2, \ldots, n$.*

Proof. From Lemma 1, it follows that diagonal entries of the left semicentral idempotent matrix Λ are left semicentral idempotents of S. Thus, from the notations in (2) and from (3), it follows that

$$\Lambda = \begin{pmatrix} \lambda_1 & \delta_{12} & \cdots & \delta_{1k} + \sum_1^k * & \cdots & \delta_{1n} + \sum_1^n * \\ 0 & \lambda_2 & \cdots & \delta_{2k} + \sum_2^k * & \cdots & \delta_{2n} + \sum_2^n * \\ \vdots & \vdots & \ddots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_k & \cdots & \delta_{kn} + \sum_k^n * \\ \vdots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \cdots & \lambda_n \end{pmatrix},$$

where $\lambda_i \in \mathcal{L}(S)$ for $i = 1, \ldots n$, is a left semicentral idempotent matrix. Note that $\delta_{im} + \sum_i^m *$, $1 \le i \le k < m \le n$ is just the (i, m)-entry of the matrix from (3) for $e_i = \lambda_i$. Since $\Lambda X \Lambda = X\Lambda$ for every triangular matrix, X, we can choose $X = e_{k-1\,k}$, where $e_{k-1\,k}$ is a matrix unit for $1 < k \le n$.

Then, we obtain $X\Lambda = \lambda_k e_{k-1\,k} + \delta_{k\,k+1}e_{k-1\,k+1} + \cdots + (\delta_{kn} + \sum_k^n *)e_{k-1\,n}$ and

$$\Lambda X \Lambda = \left(\delta_{1\,k-1} + \sum_1^{k-1} *\right)\lambda_k e_{1k} + \left(\delta_{1\,k-1} + \sum_1^{k-1} *\right)\delta_{k\,k+1}e_{1\,k+1} +$$

$$+ \cdots + \left(\delta_{1\,k-1} + \sum_1^{k-1} *\right)\left(\delta_{kn} + \sum_k^n *\right)e_{1n} + \cdots +$$

$$+\delta_{k-2\,k-1}\lambda_k e_{k-2\,k} + \delta_{k-2\,k-1}\delta_{k\,k+1}e_{k-2\,k+1} + \cdots + \delta_{k-2\,k-1}\left(\delta_{kn} + \sum_k^n *\right)e_{k-2\,n} +$$

$$+\lambda_{k-1}\lambda_k e_{k-1\,k} + \lambda_{k-1}\delta_{k\,k+1}e_{k-1\,k+1} + \cdots + \lambda_{k-1}\left(\delta_{kn} + \sum_k^n *\right)e_{k-1\,n}.$$

Comparing the corresponding elements, we find $\lambda_{k-1}\lambda_k = \lambda_k$ which is $\lambda_{k-1} \succeq \lambda_k$. Therefore, $\lambda_1 \succeq \lambda_2 \succeq \cdots \succeq \lambda_n$.

As a consequence, we have that $\lambda_{k-1}\delta_{k\,k+1} = \delta_{k\,k+1}, \ldots, \lambda_{k-1}(\delta_{kn} + \sum_k^n *) = \delta_{kn} + \sum_k^n *$ are identities for every $k = 2, \ldots, n-1$. Since $\lambda_i \succeq \lambda_j$ for $i < j$, it follows that $\delta_{ij} = \lambda_i\alpha_{ij} + \lambda_i\alpha_{ij}\lambda_j + \alpha_{ij}\lambda_j = \lambda_i\alpha_{ij} + \alpha_{ij}\lambda_j$. If $i < j < k$, we have $\delta_{ij}\lambda_k = (\lambda_i\alpha_{ij} + \alpha_{ij}\lambda_j)\lambda_k = \alpha_{ij}\lambda_k$. From $\left(\delta_{1\,k-1} + \sum_1^{k-1} *\right)\lambda_k = 0$, it follows that $\delta_{1\,k-1}\lambda_k = 0$ (since an additively idempotent semiring $a + b = 0$ implies that $a = 0$) and then $\alpha_{1\,k-1}\lambda_k = 0$, where $k \ge 3$. Similarly, from $\left(\delta_{2\,k-1} + \sum_2^{k-1} *\right)\lambda_k = 0$, we find $\alpha_{2\,k-1}\lambda_k = 0$ where $k \ge 4$.

Thus, using $\lambda_1 \succeq \cdots \succeq \lambda_k$, we obtain $\alpha_{ij}\lambda_k = 0$ for $1 \le i < j < k \le n$.

Now, we choose matrix $X = e_{kk}$. Then, $X\Lambda = \lambda_k e_{kk} + \delta_{k\,k+1} e_{k\,k+1} + \cdots + (\delta_{kn} + \sum_k^n *) e_{kn}$. Next, we obtain $\Lambda X \Lambda = \left(\delta_{1k} + \sum_1^k *\right)\lambda_k e_{1k} + \left(\delta_{1k} + \sum_1^k *\right)\delta_{k\,k+1} e_{1\,k+1} +$

$$+ \cdots + \left(\delta_{1k} + \sum_1^k *\right)\left(\delta_{kn} + \sum_k^n *\right) e_{1n} + \cdots +$$

$$+ \delta_{k-1\,k}\lambda_k e_{k-1\,k} + \delta_{k-1\,k}\delta_{k\,k+1} e_{k-1\,k+1} + \cdots + \delta_{k-1\,k}\left(\delta_{kn} + \sum_k^n *\right) e_{k-1\,n} +$$

$$+ \lambda_k e_{kk} + \lambda_k \delta_{k\,k+1} e_{k\,k+1} + \cdots + \lambda_k \left(\delta_{kn} + \sum_k^n *\right) e_{kn}.$$

We compare the corresponding elements and find that $\delta_{1k}\lambda_k = 0$ which implies that $\alpha_{1k}\lambda_k = 0$. Similarly, we obtain $\alpha_{2k}\lambda_k = 0, \ldots, \alpha_{k-1\,k}\lambda_k = 0$.

Hence, $\alpha_{ij}\lambda_k = 0$ for $1 \leq i < j \leq k \leq n$. As a consequence, if $k = n$ and $\lambda_n \neq 0$, since there are no zero divisors in S, it follows that all $\alpha_{ij} = 0$ where $1 \leq i < j \leq n$. Thus, $\Lambda = \ell_n = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$, where $\lambda_1 \succeq \lambda_2 \succeq \cdots \succeq \lambda_n$.

Now, assume that k is the largest number such that $\lambda_k \neq 0$. Then, all $\alpha_{ij} = 0$ where $1 \leq i < j \leq k$. Since $\lambda_{k+1} = \cdots = \lambda_n = 0$, it follows that the nonzero entries of the matrix Λ are $\lambda_1, \ldots, \lambda_k$ and, generally, the entries of the $k \times (n-k)$ matrix on the right upper corner of Λ. These entries are $\delta_{ij} + \sum_i^j *$ for $i = 1, \ldots, k$ and $j = k+1, \ldots, n$. From $\alpha_{ij} = 0$, for $1 \leq i < j \leq k$, it follows that $\sum_i^j * = 0$. So, the enties of the matrix are $\delta_{ij} = \lambda_i \alpha_{ij} + \alpha_{ij}\lambda_j = \lambda_i \alpha_{ij}$, where $i = 1, \ldots, k$, $j = k+1, \ldots, n$. Hence, $\Lambda = \ell_k$.

Conversely, it is easily checked that the matrix ℓ_k is a left semicentral idempotent. Let $X \in M_{k\times k}$, $Y \in M_{k\times(n-k)}$, and $Z \in M_{(n-k)\times(n-k)}$. Then, $\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}\ell_k = \begin{pmatrix} X\Lambda_k & X\Lambda_k A \\ 0 & 0 \end{pmatrix}$ and $\ell_k \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix}\ell_k = \begin{pmatrix} \Lambda_k X \Lambda_k & \Lambda_k X \Lambda_k A \\ 0 & 0 \end{pmatrix}$.

Since the matrix Λ_k is a left semicentral idempotent of $T_k(S)$, it follows that ℓ_k is a left semicentral idempotent of $T_n(S)$, and the result follows. □

The number k, $1 \leq k \leq n$, in Theorem 5 is a type of the left semicentral idempotent ℓ_k. For a fixed k, the set of all left semicentral idempotent matrices of the type k is denoted by $\mathcal{L}_k$.

Remark 4. *Recall that $\mathcal{SP}(S)$ is the set of special idempotent matrices over the semiring S. From Theorem 5, it is easy to see that for an arbitrary number, k, $1 \leq k \leq n$, it follows that $\mathcal{SP}(S) \cap \mathcal{L}_k = 0$. Therefore, $\mathcal{SP}(S) \cap \mathcal{L}(T_n(S)) = 0$.*

Proposition 15. *The set $\mathcal{L}_k$ is a subsemigroup of $\mathcal{L}(T_n(S))$ with an identity.*

Proof. Let ℓ_k with the diagonal entries $\lambda_1, \ldots, \lambda_k \in \mathcal{L}(S)$ and ℓ'_k with the diagonal entries $\lambda'_1, \ldots, \lambda'_k \in \mathcal{L}(S)$ be left semicentral idempotents of the same type. It is easy to verify that

$$\ell_k \ell'_k = \begin{pmatrix} \lambda_1\lambda'_1 & \cdots & 0 & \lambda_1\lambda'_1\alpha_{1\,k+1} & \cdots & \lambda_1\lambda'_1\alpha_{1n} \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \lambda_k\lambda'_k & \lambda_k\lambda'_k\alpha_{k\,k+1} & \cdots & \lambda_k\lambda'_k\alpha_{kn} \\ 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Thus, for a fixed k, $1 \leq k \leq n$, the set $\mathcal{L}_k$ is a noncommutative semigroup. It is easy to verify that the identity element of $\mathcal{L}_k$ is the matrix $1_k^\ell = \begin{pmatrix} I_k & A \\ 0 & 0 \end{pmatrix}$, where $I_k = \sum_{i=1}^{k} e_{ii}$ and $A \in M_{k\times(n-k)}(S)$. □

The matrix 1_k^ℓ is called a monic left semicentral idempotent of the type k.

In the papers [13,14], the author considered some properties of monic semicentral idempotents.

Note that $\ell_k = \ell_n 1_k^\ell$, where $\ell_n = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$, for $1 \leq k \leq n-1$.

Remark 5. *Let $k, 1 \leq k \leq n-1$ be a fixed number and $A, B \in M_{k\times(n-k)}(S)$. The matrices $\ell_A = \begin{pmatrix} I_k & A \\ 0 & 0 \end{pmatrix}$ and $\ell_B = \begin{pmatrix} I_k & B \\ 0 & 0 \end{pmatrix}$ are monic left semicentral idempotent matrices. It is easy to see that $\ell_A \sim_{left} \ell_B$. Thus, 1_k^ℓ is an equivalence class under the relation $\sim_{left}$.*

For right semicentral idempotent triangular matrices, we prove a result quite similar to Theorem 5.

Theorem 6. *Each right semicentral idempotent triangular $n \times n$ matrix over an additively idempotent semiring, S, without zero divisors is of the form $r_k = \begin{pmatrix} 0 & A\mathfrak{R}_k \\ 0 & \mathfrak{R}_k \end{pmatrix}$, where $\mathfrak{R}_k = \operatorname{diag}(\rho_{n-k+1}, \ldots, \rho_n)$, $\rho_i \in \mathcal{R}(S)$, for $i = n-k+1, \ldots, n$, $\rho_{i+1} \succeq \rho_i$, $i = n-k+1, \ldots, n-1$, and $A = (\alpha_{ij}) \in M_{(n-k)\times k}$ for $k = 1, \ldots, n-1$, or is of the form $r_n = \operatorname{diag}(\rho_1, \ldots, \rho_n)$, where $\rho_i \in \mathcal{R}(S)$ for $i = 1, \ldots n$ and $\rho_{i+1} \succeq \rho_i$ for $i = 1, \ldots, n-1$.*

Proof. Again using Lemma 1, we obtain that diagonal entries of the right semicentral idempotent matrix $\mathfrak{R}$ are right semicentral idempotents of S. From the notations in (2) and from (3), it follows that

$$\mathfrak{R} = \begin{pmatrix} \rho_1 & \cdots & \delta_{1\,k-1} + \sum_1^{k-1} * & \delta_{1k} + \sum_1^{k} * & \cdots & \delta_{1n} + \sum_1^{n} * \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \rho_{k-1} & \delta_{k-1\,k} & \cdots & \delta_{k-1\,n} + \sum_{k-1}^{n} * \\ 0 & \cdots & 0 & \rho_k & \cdots & \delta_{kn} + \sum_{k}^{n} * \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \rho_n \end{pmatrix},$$

where $\rho_i \in \mathcal{R}(S)$, for $i = 1, \ldots n$, is a right semicentral idempotent matrix and $\delta_{im} + \sum_i^m *$, $1 \leq i \leq k < m \leq n$, is just the (i, m)-entry of the matrix from (3) for $e_i = \rho_i$.

Since $\mathfrak{R}X\mathfrak{R} = \mathfrak{R}X$ for every triangular matrix, X, we can choose $X = e_{k-1\,k}$, where $e_{k-1\,k}$ is a matrix unit for $1 < k \leq n$.

We obtain $\mathfrak{R}X = \left(\delta_{1\,k-1} + \sum_1^{k-1} *\right) e_{1k} + \cdots + \rho_{k-1} e_{k-1\,k}$ and

$$\mathfrak{R}X\mathfrak{R} = \left(\delta_{1\,k-1} + \sum_1^{k-1} *\right)\rho_k e_{1k} + \left(\delta_{1\,k-1} + \sum_1^{k-1} *\right)\delta_{k\,k+1} e_{1\,k+1} +$$

$$+ \cdots + \left(\delta_{1\,k-1} + \sum_1^{k-1} *\right)\left(\delta_{kn} + \sum_k^{n} *\right) e_{1n} + \cdots +$$

$$+ \rho_{k-1}\rho_k e_{k-1\,k} + \rho_{k-1}\delta_{k\,k+1} e_{k-1\,k+1} + \cdots + \rho_{k-1}\left(\delta_{kn} + \sum_k^{n} *\right) e_{k-1\,n}.$$

By comparing the coefficients of $e_{k-1\,k}$, we find $\rho_{k-1}\rho_k = \rho_{k-1}$ that is $\rho_k \succeq \rho_{k-1}$ (see (5)). Therefore, $\rho_n \succeq \rho_{n-1} \succeq \cdots \succeq \rho_1$.

Also, we have that

$$\left(\delta_{1\,k-1} + \sum_1^{k-1} *\right)\delta_{k\,k+1} = 0, \left(\delta_{1\,k-1} + \sum_1^{k-1} *\right)\left(\delta_{kn} + \sum_k^{n} *\right) = 0, \ldots,$$

$$\rho_{k-1}\delta_{k\,k+1} = 0, \ldots, \rho_{k-1}\left(\delta_{kn} + \sum_k^{n} *\right) = 0.$$

Since $\rho_j \succeq \rho_i$ for $i < j$, it follows that $\delta_{ij} = \rho_i\alpha_{ij} + \rho_i\alpha_{ij}\rho_j + \alpha_{ij}\rho_j = \rho_i\alpha_{ij} + \alpha_{ij}\rho_j$.

Then, if $p < i < j$, we have $\rho_p\delta_{ij} = \rho_p\alpha_{ij}$.

From $\rho_{k-1}(\delta_{kn} + \sum_k^n *) = 0$, it follows that $\rho_{k-1}\alpha_{ij} = 0$ for $1 \le k < i < j \le n$.

Now, for $X = e_{kk}$, we have $\mathfrak{R}X = \left(\delta_{1\,k} + \sum_1^k *\right)e_{1k} + \cdots + \delta_{k-1\,k}e_{k-1\,k} + \rho_k e_{kk}$ and then obtain

$$\mathfrak{R}X\mathfrak{R} = \left(\delta_{1k} + \sum_1^{k} *\right)\rho_k e_{1k} + \left(\delta_{1k} + \sum_1^{k} *\right)\delta_{k\,k+1}e_{1\,k+1} +$$

$$+ \cdots + \left(\delta_{1k} + \sum_1^{k} *\right)\left(\delta_{kn} + \sum_k^{n} *\right)e_{1n} + \cdots +$$

$$+\delta_{k-1\,k}\rho_k e_{k-1\,k} + \delta_{k-1\,k}\delta_{k\,k+1}e_{k-1\,k+1} + \cdots + \delta_{k-1\,k}\left(\delta_{kn} + \sum_k^{n} *\right)e_{k-1\,n} +$$

$$+\rho_k e_{kk} + \rho_k\delta_{k\,k+1}e_{k\,k+1} + \cdots + \rho_k\left(\delta_{kn} + \sum_k^{n} *\right)e_{kn}.$$

It is easy to see that $\left(\delta_{1k} + \sum_1^k *\right)\rho_k = \delta_{1k} + \sum_1^k *, \ldots, \delta_{k-1\,k}\rho_k = \delta_{k-1\,k}$ are identities. It follows that

$$\left(\delta_{1k} + \sum_1^{k} *\right)\delta_{k\,k+1} = 0, \ldots, \left(\delta_{1k} + \sum_1^{k} *\right)\left(\delta_{kn} + \sum_k^{n} *\right) = 0, \ldots, \delta_{k-1\,k}\delta_{k\,k+1} = 0,$$

$$\ldots, \delta_{k-1\,k}\left(\delta_{kn} + \sum_k^{n} *\right) = 0, \ldots, \rho_k\delta_{k\,k+1} = 0, \ldots, \rho_k\left(\delta_{kn} + \sum_k^{n} *\right) = 0.$$

Now, $\rho_k\delta_{k\,k+1} = 0$ implies that $\rho_k\alpha_{k\,k+1} = 0$ for all $1 \le k \le n-1$. Thus, if $\rho_1 \ne 0$, we have $\alpha_{12} = 0$, and from $\rho_1\alpha_{ij} = 0$, $1 < i < j \le n-1$, we obtain that $\rho_1\alpha_{ij} = 0$ for all $i < j$. Therefore, $\mathfrak{R} = r_n = \mathrm{diag}(\rho_1, \ldots, \rho_n)$.

Let $\rho_1 = \cdots = \rho_{n-k} = 0$ and $\rho_{n-k+1} \ne 0$. As shown above, it follows that $\alpha_{ij} = 0$ for all $n-k+1 \le i < j \le n$.

Since $\delta_{ij} = 0$ for $1 \le i < j \le n-k$ and also for $n-k+1 \le i < j \le n$, it follows that the (i,j)-entries, where $1 \le i \le n-k$ and $n-k+1 \le j \le n$, of the matrix are equal: $\delta_{ij} = \alpha_{ij}\rho_j$. Hence, $\mathfrak{R} = r_k$.

Conversely, by the same reasoning as in the proof of Theorem 5, we check that r_k is a right semicentral idempotent matrix, and the proof is completed. □

The number k, $1 \le k \le n$, in Theorem 6 is called a type of the right semicentral idempotent r_k. For a fixed k, the set of all right semicentral idempotent matrices of the type k is denoted by $\mathcal{R}_k$.

As in Remark 3, it follows that $\mathcal{SP}(S) \cap \mathcal{R}(T_n(S)) = 0$.

Proposition 16. *The set $\mathcal{R}_k$ is a subsemigroup of $\mathcal{R}(T_n(S))$ with an identity.*

Proof. The proof is similar to that of Proposition 15. □

The identity element of $\mathcal{R}_k$ is the matrix $1_k^r = \begin{pmatrix} 0 & A \\ 0 & I_{n-k+1} \end{pmatrix}$, where $I_{n-k+1} = \sum_{i=n-k+1}^{n} e_{ii}$ and $A \in M_{(n-k)\times k}(S)$. This matrix is called a monic right semicentral idempotent of the type k. In a similar way to in Remark 5, it follows that 1_k^r is an equivalence class under the relation $\sim_{\text{right}}$.

Remark 6. *In spite of some evident similarities with semicentral idempotent matrices over a ring, there is a significant difference. All non-diagonal left and right semicentral idempotent matrices over an additively idempotent semiring are singular.*

Further, we construct the matrix $1_{\ell_k}^J = I + 1_k^\ell = \begin{pmatrix} I_k & A \\ 0 & I_{n-k} \end{pmatrix}$ with only ones on the main diagonal and an arbitrary matrix, $A \in M_{k\times(n-k)}$, in the right upper corner. Then, it follows that $1_{\ell_k}^J$ is a Jordan semicentral idempotent for $k = 1, \ldots, n-1$ (see Section 3). Similarly, the matrix $1_{r_m}^J = I + 1_m^r = \begin{pmatrix} I_{n-m} & A \\ 0 & I_m \end{pmatrix}$ is a Jordan semicentral idempotent for $m = 1, \ldots, n-1$. Then, $1_{r_{n-k}}^J = 1_{\ell_k}^J$. These semicentral idempotents are called monic Jordan semicentral idempotent matrices.

Remark 7. *(i) Assume that k, where $1 \le k \le n-1$ is a fixed integer and $A, B \in M_{k\times(n-k)}(S)$. Now, the matrices $\jmath_A = \begin{pmatrix} I_k & A \\ 0 & I_{n-k} \end{pmatrix}$ and $\jmath_B = \begin{pmatrix} I_k & B \\ 0 & I_{n-k} \end{pmatrix}$ are monic Jordan semicentral idempotent matrices.*

Let $X \in M_{k\times k}$, $Y \in M_{k\times(n-k)}$ and $Z \in M_{(n-k)\times(n-k)}$.

We obtain $\jmath_A \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} = \begin{pmatrix} X & Y+AZ \\ 0 & Z \end{pmatrix}$ and $\begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \jmath_B = \begin{pmatrix} X & XB+Y \\ 0 & Z \end{pmatrix}$ and then $\jmath_A \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} + \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \jmath_B = \begin{pmatrix} X & XB+Y+AZ \\ 0 & Z \end{pmatrix} = \jmath_A \begin{pmatrix} X & Y \\ 0 & Z \end{pmatrix} \jmath_B$. Therefore, $\jmath_A \sim \jmath_B$.

(ii) There are no monic Jordan semicentral idempotent matrices that are special idempotent matrices.

The next result shows that there are other Jordan semicentral idempotents.

Proposition 17. *If $\jmath_i \in \mathcal{J}(S)$, $i = 1, \ldots, n$ and $\jmath_i \succeq \jmath_k$ for $i < k$, the matrix $A = \operatorname{diag}(\jmath_1, \ldots, \jmath_n)$ is a Jordan semicentral idempotent.*

Proof. Let $X = (x_{ik}) \in T_n(S)$. Then, it follows that $AX = (\jmath_i x_{ik})$, $XA = (x_{ik}\jmath_k)$ and $AXA = (\jmath_i x_{ik} \jmath_k)$. Since $\jmath_i x_{ik} \jmath_k = \jmath_i x_{ik} + x_{ik}\jmath_k$ for any $i < k$, we have $AXA = AX + XA$. □

Remark 8. *The monic Jordan semicentral idempotent matrices and diagonal matrices considered in the last proposition are particular cases of the matrices $\begin{pmatrix} I_p & 0 & B \\ 0 & C & 0 \\ 0 & 0 & I_q \end{pmatrix}$, where $I_p = \sum_{i=1}^{p} e_{ii}$, $I_q = \sum_{i=n-q+1}^{n} e_{ii}$, $B \in M_{p\times q}(S)$, and $C = \operatorname{diag}(\jmath_{p+1}, \ldots, \jmath_{n-q})$, where $\jmath_i \in \mathcal{J}(S)$, $i = p+1, \ldots, n-q$, and $\jmath_i \succeq \jmath_k$ for $i < k$. When $p = q = 0$, the matrix is diagonal, and when $n = p+q$, all the diagonal entries of the matrix are ones. The proof of this fact is routine.*

In the next example, we describe idempotents of certain triangular matrix semirings.

Example 3. *Let $\mathbb{B} = (\{0,1\}, +, .)$ be a Boolean semiring. We present all idempotents as well as left (right and Jordan) semicentral idempotents, roots of idempotent matrices, and nilpotent matrices in the semiring $T_n(\mathbb{B})$ for $n = 2$ and $n = 3$.*

The matrices $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ are left semicentral idempotents, $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$ are right semicentral idempotent matrices, and $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is a Jordan semicentral idempotent matrix. The matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ is nilpotent. The identity matrix and the zero matrix can be considered a Jordan semicentral idempotent and nilpotent matrix, respectively.

Let $\varepsilon_{ij} = 1$ or $\varepsilon_{ij} = 0$ for $i, j = 1, \ldots, n$, with $i < j$. Then, every matrix of $T_n(\mathbb{B})$ has the form (ε_{ij}). For $n = 3$, it is easy to verify that the matrices $\ell_1 = e_{11} + \varepsilon_{12}e_{12} + \varepsilon_{13}e_{13}$ and $\ell_2 = \ell_1 + e_{22} + \varepsilon_{23}e_{23}$ are left semicentral idempotents. Therefore, we obtain 12 left semicentral idempotent matrices.

Similarly, the right semicentral idempotents are $r_1 = e_{33} + \varepsilon_{23}e_{23} + \varepsilon_{13}e_{13}$ and $r_2 = r_1 + e_{22} + \varepsilon_{12}e_{12}$. The Jordan semicentral idempotents are $\jmath = I + \ell_2 = I + \varepsilon_{12}e_{12} + \varepsilon_{13}e_{13} + \varepsilon_{23}e_{23}$. Thus, we obtain 32 semicentral idempotent matrices.

From Proposition 3, we find that $r_2\ell_2 = e_{22} + \varepsilon_{12}e_{12} + \varepsilon_{13}e_{13} + \varepsilon_{23}e_{23}$ are idempotents. Since $\ell_1 r_1 \leq r_1$, from Proposition 3 again, we obtain that $\ell_1 \circ r_1 = e_{11} + \varepsilon_{12}e_{12} + \varepsilon_{13}e_{13} + \varepsilon_{23}e_{23} + e_{33}$ are idempotent matrices. Therefore, we find 16 idempotent matrices that are not semicentral idempotents.

It is easy to see that $\varepsilon_{12}e_{12} + \varepsilon_{13}e_{13} + \varepsilon_{23}e_{23}$ are strictly nilpotent matrices.

All other matrices are square roots of semicentral idempotents. Thus, $e_{11} + \varepsilon_{12}e_{12} + \varepsilon_{13}e_{13} + e_{23}$ are roots of ℓ_1 and $e_{33} + \varepsilon_{23}e_{23} + \varepsilon_{13}e_{13} + e_{12}$ are roots of r_1.

5. Application A: Decomposition of Idempotent Triangular Matrices into Semicentral Idempotents

Example 4. *In Example 1, we obtain that all idempotent 2×2 matrices over S are $E(e_1, e_2) = \begin{pmatrix} e_1 & e_1\alpha_{12} + e_1\alpha_{12}e_2 + \alpha_{12}e_2 \\ 0 & e_2 \end{pmatrix}$, where $e_1, e_2, \alpha_{12} \in S$, and e_1 and e_2 are idempotents.*

Let us consider the monic left semicentral idempotent $1_1^\ell = \begin{pmatrix} 1 & \alpha_{12} \\ 0 & 0 \end{pmatrix}$ and the monic right semicentral idempotent $1_1^r = \begin{pmatrix} 0 & \alpha_{12} \\ 0 & 1 \end{pmatrix}$.

Then, we construct new idempotents as follows: $E_1^\ell = \operatorname{diag}(e_1, e_2)\, 1_1^\ell = \begin{pmatrix} e_1 & e_1\alpha_{12} \\ 0 & 0 \end{pmatrix}$ and $E_1^r = 1_1^r \operatorname{diag}(e_1, e_2) = \begin{pmatrix} 0 & \alpha_{12}e_2 \\ 0 & e_2 \end{pmatrix}$. Then, $E_1^\ell E_1^r = \begin{pmatrix} 0 & e_1\alpha_{12}e_2 \\ 0 & 0 \end{pmatrix}$. Therefore, $E(e_1, e_2) = E_1^\ell + E_1^r + E_1^\ell E_1^r = E_1^\ell \diamond E_1^r$.

We can now state another representation. The matrix $1^\jmath = \begin{pmatrix} 1 & \alpha_{12} \\ 0 & 1 \end{pmatrix}$ is a monic Jordan semicentral idempotent. We construct the matrices $E_1^{\jmath\,\text{left}} = \operatorname{diag}(e_1, e_2)\, 1^\jmath = \begin{pmatrix} e_1 & e_1\alpha_{12} \\ 0 & e_2 \end{pmatrix}$ and $E_1^{\jmath\,\text{right}} = 1^\jmath \operatorname{diag}(e_1, e_2) = \begin{pmatrix} e_1 & \alpha_{12}e_2 \\ 0 & e_2 \end{pmatrix}$. Thus, we have $E_1^{\jmath\,\text{left}} E_1^{\jmath\,\text{right}} = \begin{pmatrix} e_1 & e_1\alpha_{12}e_2 \\ 0 & e_2 \end{pmatrix}$ and then it follows that $E(e_1, e_2) = E_1^{\jmath\,\text{left}} + E_1^{\jmath\,\text{right}} + E_1^{\jmath\,\text{left}} E_1^{\jmath\,\text{right}} = E_1^{\jmath\,\text{left}} \diamond E_1^{\jmath\,\text{right}}$.

Example 5. *In Example 2, we find that all idempotent matrices of $T_3(S)$ are matrices of the form $E(e_1,e_2,e_3)=\begin{pmatrix} e_1 & \delta_{12} & \delta_{13}+\delta_{12}\delta_{13} \\ 0 & e_2 & \delta_{23} \\ 0 & 0 & e_3 \end{pmatrix}$, where $\delta_{ij}=e_i\alpha_{ij}+e_i\alpha_{ij}e_j+\alpha_{ij}e_j$, $e_i,\alpha_{ij}\in S$, $i,j=1,2,3$, $i<j$, and $e_i^2=e_i$.*

The matrices $1_1^\ell=\begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $1_2^\ell=\begin{pmatrix} 1 & 0 & \alpha_{13} \\ 0 & 1 & \alpha_{23} \\ 0 & 0 & 0 \end{pmatrix}$ are monic left semicentral idempotents. The matrices $1_1^r=\begin{pmatrix} 0 & 0 & \alpha_{13} \\ 0 & 0 & \alpha_{23} \\ 0 & 0 & 1 \end{pmatrix}$ and $1_2^r=\begin{pmatrix} 0 & \alpha_{12} & \alpha_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ are monic right semicentral idempotents, where $\alpha_{12},\alpha_{13},\alpha_{23}\in S$.

Now, we can construct $E_1^\ell=\operatorname{diag}(e_1,e_2,e_3)\,1_1^\ell=\begin{pmatrix} e_1 & e_1\alpha_{12} & e_1\alpha_{13} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $E_2^\ell=\operatorname{diag}(e_1,e_2,e_3)\,1_2^\ell=\begin{pmatrix} e_1 & 0 & e_1\alpha_{13} \\ 0 & e_2 & e_2\alpha_{23} \\ 0 & 0 & 0 \end{pmatrix}$. Similarly, $E_1^r=1_1^r\operatorname{diag}(e_1,e_2,e_3)=\begin{pmatrix} 0 & 0 & \alpha_{13}e_3 \\ 0 & 0 & \alpha_{23}e_3 \\ 0 & 0 & e_3 \end{pmatrix}$ and $E_2^r=1_2^r\operatorname{diag}(e_1,e_2,e_3)=\begin{pmatrix} 0 & \alpha_{12}e_2 & \alpha_{13}e_3 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$. Then, $E_1^\ell E_2^r=\begin{pmatrix} 0 & e_1\alpha_{12}e_2 & e_2\alpha_{13}e_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $E_2^\ell E_1^r=\begin{pmatrix} 0 & 0 & e_1\alpha_{13}e_3 \\ 0 & 0 & e_2\alpha_{23}e_3 \\ 0 & 0 & 0 \end{pmatrix}$. All these matrices are idempotents.

Next, let $\Delta_1=E_1^\ell\diamond E_2^r=\begin{pmatrix} e_1 & \delta_{12} & \delta_{13} \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$ and $\Delta_2=E_2^\ell\diamond E_1^r=\begin{pmatrix} e_1 & 0 & \delta_{13} \\ 0 & e_2 & \delta_{23} \\ 0 & 0 & e_3 \end{pmatrix}$, which are also idempotents. Then, we have $\Delta_1+\Delta_2=\begin{pmatrix} e_1 & \delta_{12} & \delta_{13} \\ 0 & e_2 & \delta_{23} \\ 0 & 0 & e_3 \end{pmatrix}=\Delta$ and, from Theorem 2, it follows that $E(e_1,e_2,e_3)=(\Delta_1+\Delta_2)^2$.

There is another way of describing the idempotent matrix $E(e_1,e_2,e_3)$. It is easy to see that $(E_1^\ell\diamond E_2^r)(E_2^\ell\diamond E_1^r)+(E_2^\ell\diamond E_1^r)(E_1^\ell\diamond E_2^r)=E(e_1,e_2,e_3)$.

From the monic Jordan semicentral idempotent $1_{\ell_1}^J=\begin{pmatrix} 1 & \alpha_{12} & \alpha_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we construct $E_{\ell_1}^{J\,\text{left}}=\begin{pmatrix} e_1 & e_1\alpha_{12} & e_1\alpha_{13} \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$ and $E_{\ell_1}^{J\,\text{right}}=\begin{pmatrix} e_1 & \alpha_{12}e_2 & \alpha_{13}e_3 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$. Then, $E_{\ell_1}^{J\,\text{left}}E_{\ell_1}^{J\,\text{right}}=\begin{pmatrix} e_1 & e_1\alpha_{12}e_2 & e_1\alpha_{13}e_3 \\ 0 & e_2 & 0 \\ 0 & 0 & e_3 \end{pmatrix}$. Therefore, $E_{\ell_1}^{J\,\text{left}}\diamond E_{\ell_1}^{J\,\text{right}}=\Delta_1$.

Similarly, $E_{r_1}^{J\,\text{left}}\diamond E_{r_1}^{J\,\text{right}}=\Delta_2$, where $1_{r_1}^J=\begin{pmatrix} 1 & 0 & \alpha_{13} \\ 0 & 1 & \alpha_{23} \\ 0 & 0 & 1 \end{pmatrix}$ is another monic Jordan semicentral idempotent. Thus, we can represent $E(e_1,e_2,e_3)$ only by the monic Jordan semicentral idempotents.

Theorem 7. *Let $E(e_1,\dots,e_n)$ be an arbitrary idempotent triangular $n\times n$ matrix over an additively idempotent semiring. Then,*

$$E(e_1,\dots,e_n)=\left(\sum_{k=1}^{n-1}E_k^{\ell}\diamond E_{n-k}^{r}\right)^{n-1}, \tag{6}$$

where $E_k^{\ell}=\operatorname{diag}(e_1,\dots,e_n)\,1_k^{\ell}$ and $E_{n-k}^{r}=1_{n-k}^{r}\operatorname{diag}(e_1,\dots,e_n)$, $k=1,\dots,n-1$.

Proof. We know that $1_k^{\ell}=\begin{pmatrix} I_k & A\\ 0 & 0\end{pmatrix}$, where $I_k=\sum_{i=1}^{k}e_{ii}$ and $A\in M_{k\times(n-k)}(S)$. Then, $E_k^{\ell}=\begin{pmatrix} E_k & E_kA\\ 0 & 0\end{pmatrix}$, where $E_k=\operatorname{diag}(e_1,\dots,e_k)$. Similarly, from $1_{n-k}^{r}=\begin{pmatrix} 0 & A\\ 0 & I_{n-k}\end{pmatrix}$, where $I_{n-k}=\sum_{i=k+1}^{n}e_{ii}$, it follows that $E_{n-k}^{r}=\begin{pmatrix} 0 & AE_{n-k}\\ 0 & E_{n-k}\end{pmatrix}$, where $E_{n-k}=\operatorname{diag}(e_{k+1},\dots,e_n)$. Now, we find $E_k^{\ell}E_{n-k}^{r}=\begin{pmatrix} 0 & E_kAE_{n-k}\\ 0 & 0\end{pmatrix}$ and $E_{n-k}^{r}E_k^{\ell}=0$.

Then, $\Delta_k=E_k^{\ell}\diamond E_{n-k}^{r}=I+D_k$, where I is the identity matrix and $D_k=(\delta_{ij})$, $i=1,\dots,k$, $j=k+1,\dots,n$.

Consequently,

$$\sum_{k=1}^{n-1}E_k^{\ell}\diamond E_{n-k}^{r}=\sum_{k=1}^{n-1}\Delta_k=\Delta$$

and by Theorem 2 and Corollary 1, the result follows. □

In a particular case when $E(e_1,\dots,e_n)=\operatorname{diag}(e_1,\dots,e_n)$, the formula in Theorem 7 is an identity. In order to represent the diagonal matrix $E(e_1,\dots,e_n)$ by left and right semicentral idempotents, we prove the next result.

Corollary 3. *The matrix $E(e_1,\dots,e_n)=\operatorname{diag}(e_1,\dots,e_n)$ can be presented in the form*

$$\sum_{i=1}^{n}e_i1_{n-i+1}^{r}1_i^{\ell}=e_1r_n\ell_1\diamond\cdots\diamond e_i1_{n-i+1}^{r}1_i^{\ell}\diamond\cdots\diamond e_nr_1\ell_n.$$

Proof. It follows that the matrix unit e_{ii}, $i=1,\dots,n$, can be presented in the form $1_{n-i+1}^{r}1_i^{\ell}$, where all entries of these semicentral idempotents over the main diagonal are zeroes. Note that $1_n^{r}=1_n^{\ell}=I$. Since $e_i1_{n-i+1}^{r}1_i^{\ell}$ are orthogonal idempotents, the sum is equal to a diamond composition on the right side. □

From the proof of Theorem 7, we obtain a new equality, similar to (6).

Corollary 4. *Let $E(e_1,\dots,e_n)$ be an arbitrary idempotent triangular $n\times n$ matrix over an additively idempotent semiring. Then,*

$$E(e_1,\dots,e_n)=\left(\sum_{k=1}^{n-1}E_{\ell_k}^{J}\diamond E_{r_{n-k}}^{J}\right)^{n-1},$$

where $E_{\ell_k}^{J}=\operatorname{diag}(e_1,\dots,e_n)1_{\ell_k}^{J}$, $E_{r_{n-k}}^{J}=1_{r_{n-k}}^{J}\operatorname{diag}(e_1,\dots,e_n)$ for $k=1,\dots,n-1$.

Proof. From Section 4, we know that $1_{\ell_k}^{J}=I+1_k^{\ell}$ and $1_{r_{n-k}}^{J}=I+1_{n-k}^{r}$ where $k=1,\dots,n-1$ are monic Jordan semicentral idempotents. Both matrices are equal to

$\begin{pmatrix} I_k & A \\ 0 & I_{n-k} \end{pmatrix}$, which has only ones on the main diagonal and a matrix, $A \in M_{k\times(n-k)}(S)$, is in the right upper corner.

Under the notations in the proof of Theorem 7, it follows that $E^J_{\ell_k} = \begin{pmatrix} E_k & E_kA \\ 0 & E_{n-k} \end{pmatrix}$, $E^J_{r_{n-k}} = \begin{pmatrix} E_k & AE_{n-k} \\ 0 & E_{n-k} \end{pmatrix}$ and $E^J_{\ell_k} E^J_{r_{n-k}} = \begin{pmatrix} E_k & E_kAE_{n-k} \\ 0 & E_{n-k} \end{pmatrix}$.

Hence, $E^J_{\ell_k} \diamond E^J_{r_{n-k}} = \Delta_k$, and the result follows. □

6. Application B: Decomposition of Strictly Upper Triangular Matrices into Semicentral Idempotents

Note that strictly upper triangular matrices (i.e., those with zeroes on the main diagonal) over an additively idempotent semiring, S, form a subsemiring of $T_n(S)$.

The following reasonings are very similar to those in [14,25] for triangular matrices over a ring.

By $NT_n(S)$, we denote the ring of strictly upper triangular (or nil-triangular) $n \times n$ matrices over the additively idempotent semiring S.

Example 6. *Let $A \in NT_2(S)$. Then, it follows that*

$$A = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = 1^\ell_1 1^r_1.$$

Let $A \in NT_3(S)$. As in the proof of Proposition 2.7 (cf. [25]), it follows that

$$A = \begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= 1^\ell_2 \begin{pmatrix} 1 & a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1^\ell_2 \begin{pmatrix} 1 & a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} 1^r_2.$$

On the other hand, for $1^\ell_1 = \begin{pmatrix} 1 & a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $1^r_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, we obtain that $1^\ell_1 1^r_1 = 0$, which implies that $1^\ell_1 \diamond 1^r_1 = 1^\ell_1 + 1^r_1 = \begin{pmatrix} 1 & a_{12} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Therefore, $A = 1^\ell_2(1^\ell_1 + 1^r_1)1^r_2$.

Theorem 8. *Let S be an additively idempotent semiring, $n \in \mathbb{N}$, and A a strictly upper triangular $n \times n$ matrix over S. Then,*

$$A = \prod_{k=0}^{n-1} \left(1^\ell_{n-1-k} + 1^r_k\right), \tag{7}$$

where $1^\ell_0 = 1^r_0 = 0$.

Proof. We can show that

$$A = \begin{pmatrix} B & C \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} I_{n-1} & C \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} = 1^\ell_{n-1}\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix},$$

where $B \in NT_{n-1}(S)$ and $C \in M_{(n-1)\times 1}(S)$.

If $B = \prod_k B_k$, where $B_k \in Tr_{n-1}(S)$, then $\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} = \prod_k \begin{pmatrix} B_k & 0 \\ 0 & 1 \end{pmatrix}$.

The proof proceeds by induction on n. By $1^{\ell}_k(p)$ (resp., $1^r_k(p)$) we denote a left (resp., right) semicentral idempotent $p \times p$ matrix where $p = 2, \ldots, n-1$.

For $n = 3$, using the first part of Example 4, we obtain

$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ 0 & 0 & a_{23} \\ 0 & 0 & 0 \end{pmatrix} = 1^{\ell}_2(3) \begin{pmatrix} 1^{\ell}_1(2)1^r_1(2) & 0 \\ 0 & 1 \end{pmatrix} = 1^{\ell}_2(3) \begin{pmatrix} 1^{\ell}_1(2) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1^r_1(2) & 0 \\ 0 & 1 \end{pmatrix} =$$

$$= 1^{\ell}_2(3) \left[\begin{pmatrix} 1^{\ell}_1(2) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] 1^r_2(3) = 1^{\ell}_2(3)(1^{\ell}_1(3) + 1^r_1(3))1^r_2(3) = \prod_{k=0}^{3} \left(1^{\ell}_{2-k} + 1^r_k \right).$$

We assume that $B = \prod_{k=0}^{n-2} \left(1^{\ell}_{n-2-k}(n-1) + 1^r_k(n-1) \right)$. Then, we find

$$\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} = \prod_{k=0}^{n-2} \begin{pmatrix} 1^{\ell}_{n-2-k}(n-1) + 1^r_k(n-1) & 0 \\ 0 & 1 \end{pmatrix} = \prod_{k=0}^{n-2} \left(1^{\ell}_{n-2-k}(n) + 1^r_{k+1}(n) \right).$$

Hence, $A = 1^{\ell}_{n-1}(n) \prod_{k=0}^{n-3} \left(1^{\ell}_{n-2-k}(n) + 1^r_{k+1}(n) \right) 1^r_{n-1}(n)$, which gives the desired conclusion. □

7. Application C: Decomposition of Unitriangular Matrices into Semicentral Idempotents

Recall that a matrix is called unitriangular if it is upper triangular and has only ones on the main diagonal. Each unitriangular matrix has the form $I + A$, where I is an identity matrix and A is a strictly upper triangular matrix. It is easy to see that unitriangular matrices over an additively idempotent semiring, S, form a subsemiring of $T_n(S)$.

First, let us consider the representation of a unitriangular idempotent matrix by left and semicentral idempotents. As a consequence of Theorem 7, we have the following.

Proposition 18. *Let U be a unitriangular idempotent $n \times n$ matrix over an additively idempotent semiring. Then, $U = \left(\sum_{k=1}^{n-1} 1^{\ell}_k \diamond 1^r_{n-k} \right)^{n-1}$.*

For some unitriangular idempotent matrices, which are "near" to the identity matrix, the representation by semicentral idempotents takes an especially simple form.

Proposition 19. *Let $U = (a_{ij}) \in T_n(S)$, where $a_{ii} = 1$, $a_{ij} = 0$ for all $i, j = 1, \ldots, n$, $i < j$, except $a_{i\,i+1}$ if i is an odd integer, and S be an additively idempotent semiring. Then, $U = \sum_{k=1}^{n-1} \left(1^{\ell}_k \diamond 1^r_{n-k} \right)$.*

Proof. It is enough to show that U is an idempotent matrix, and for each $k = 1, \ldots, n-1$, the matrix $1^{\ell}_k \diamond 1^r_{n-k}$ contains a $k \times (n-k)$ matrix which is a submatrix of U with all entries being zeroes except $a_{k\,k+1}$ if k is an odd integer. □

The next example elucidates that there are unitriangular idempotent matrices represented only by a diamond composition of left and right semicentral idempotents.

Example 7. *Let $A = (a_{ij})$ where $a_{ij} \in S$ is a unitriangular idempotent 4×4 matrix. Let $1_k^\ell = \begin{pmatrix} I_k & B \\ 0 & 0 \end{pmatrix}$, where $I_k = \sum_{i=1}^{k} e_{ii}$, $B = (a_{ij}) \in M_{k\times(4-k)}(S)$, and B is a submatrix of A, be a monic left semicentral idempotent for $k = 1, 2, 3$. Let $1_{4-k}^r = \begin{pmatrix} 0 & C \\ 0 & I_{4-k} \end{pmatrix}$, where $I_{4-k} = \sum_{i=k+1}^{4} e_{ii}$ and $C = (a_{ij}) \in M_{(4-k)\times k}(S)$ and C is a submatrix of A, be a monic right semicentral idempotent for $k = 1, 2, 3$. Then, we have*

$$A = (1_3^\ell \diamond 1_1^r) \diamond (1_2^\ell \diamond 1_2^r) \diamond (1_1^\ell \diamond 1_3^r).$$

Next, we study the notion of a diamond composition of the arbitrary triangular matrices A and B. Thus, $A \diamond B = A + B + AB$ is neither an idempotent matrix nor a nilpotent matrix.

Now, for the arbitrary triangular matrices A and B, it is easy to obtain that $(I + A) \diamond (I + B) = I + (A \diamond B)$, where I is the identity matrix. Consequently, the semiring of unitriangular matrices is closed under a diamond composition.

By induction (and using the associativity of a diamond composition), it follows that $\diamondsuit_k(I + A_k) = I + \diamondsuit_k A_k$, where A_k are arbitrary triangular matrices.

In the text between Remark 5 and Remark 6, we find, for Jordan semicentral idempotents, that $1^J_{\ell_{n-k}} = I + 1^\ell_{n-k}$ and $1^J_{r_k} = I + 1^r_k$.

Therefore, it follows that $1^J_{\ell_{n-k}} \diamond 1^J_{r_k} = I + 1^\ell_{n-k} \diamond 1^r_k$ which implies that $\diamondsuit_{k=1}^{n-1}\left(1^J_{\ell_{n-k}} \diamond 1^J_{r_k}\right) = I + \diamondsuit_{k=1}^{n-1}\left(1^\ell_{n-k} \diamond 1^r_k\right)$.

Theorem 9. *Let S be an additively idempotent semiring, $n \in \mathbb{N}$, and U a unitriangular $n \times n$ matrix over S. Then,*

$$U = \diamondsuit_{k=1}^{n-1}\left(1^\ell_{n-k} \diamond 1^r_k\right) = \diamondsuit_{k=1}^{n-1}\left(1^J_{\ell_{n-k}} \diamond 1^J_{r_k}\right). \tag{8}$$

Proof. Let A be the strictly upper triangular matrix from Theorem 8. The sum of each two corresponding (with the same number) entries on the main diagonals of the matrices 1^ℓ_{n-k} and 1^r_k is 1. The sum of all left and all right semicentral idempotent matrices (from the representation of A in Theorem 8) contains all entries of A. Moreover, the product $1^\ell_{n-k}1^r_k$ is a submatrix of A (using the reasoning in Section 4 before Lemma 1). Then, we have $\sum_{k=1}^{n-1}\left(1^\ell_{n-k} \diamond 1^r_k\right) = I + A$.

On the other hand, any product, $\prod_{k=p}^{q}\left(1^\ell_{n-k} \diamond 1^r_k\right)$, $0 \le p < q \le n-1$, is a submatrix of $I + A$. Consequently, $U = I + A = \diamondsuit_{k=1}^{n-1}\left(1^\ell_{n-k} \diamond 1^r_k\right) = \diamondsuit_{k=1}^{n-1}\left(1^J_{\ell_{n-k}} \diamond 1^J_{r_k}\right)$. □

From the proof of the last theorem, we have the following.

Corollary 5. *Let 1^ℓ_{n-k} and 1^r_k for $k = 1, \ldots, n-1$ be left and right semicentral idempotent matrices. Let all these semicentral idempotents be submatrices of the unitriangular matrix $A = (a_{ij}) \in T_n(S)$. Then,*

$$\sum_{k=1}^{n-1}\left(1^\ell_{n-k} \diamond 1^r_k\right) = \prod_{k=1}^{n-1}\left(1^\ell_{n-k} \diamond 1^r_k\right) = \diamondsuit_{k=1}^{n-1}\left(1^\ell_{n-k} \diamond 1^r_k\right).$$

D. I. Vladeva (see [26]) considered derivations of the semirings of triangular Toeplitz $n \times n$ matrices over an additively idempotent semiring. For basic facts and many applications of Toeplitz matrices, we refer the reader to R. Gray (see [27]). Note that any triangular Toeplitz $n \times n$ matrix (or diagonal-constant matrix) depends only on the entries $a_{11}, \ldots, a_{1n}$

placed on the diagonals of the matrix. As a consequence of the last theorem and (8), we obtain the following.

Corollary 6. *Let $A = (a_{ij}) \in T_n(S)$ be a Toeplitz matrix, where S is an additively idempotent semiring. If a_{11} is a unit of S, then $A = a_{11} \diamondsuit_{k=1}^{n-1} \left(1_{n-k}^{\ell} \diamond 1_k^r\right)$.*

8. Application D: Nil-Clean Triangular Matrices—Decomposition into Semicentral Idempotents

A classical definition of W. K. Nicholson (see [28,29]) says that a ring, R, is clean if every element of R can be written as a sum of an idempotent and a unit. Using this notion, A.J. Diesl, in [24], introduced nil-clean rings as rings in which every element can be written as the sum of an idempotent and a nilpotent. Diesl's work caught the attention of a number of mathematicians, and in the past ten years, there have been many investigations concerning variants of nil-clean rings (see [30–32]). Inspired by these articles, we introduce a nil-clean additively idempotent semiring as a semiring in which every element can be written as the sum of an idempotent and a nilpotent.

Since the sum of two elements, a and b, in an additively idempotent semiring is $\max\{a, b\}$, we need an example of a nil-clean semiring.

Example 8. *We consider the finite set $\mathcal{C}_n = \{0, 1, \ldots, n-1\}$ and define $x \vee y = \max\{x, y\}$, where $x, y \in \mathcal{C}_n$. Thus, $(\mathcal{C}_n, \vee)$ is a join-semilattice or a commutative idempotent semigroup. The map $\alpha : \mathcal{C}_n \to \mathcal{C}_n$ is called an endomorphism of the finite chain $\mathcal{C}_n$ if $\alpha(x \vee y) = \alpha(x) \vee \alpha(y)$. The set of endomorphisms with a fixed point, 0, is a semiring, denoted by $\mathcal{E}_{\mathcal{C}_n}^{(0)}$, with respect to the addition and multiplication defined by the following:*

$(\alpha + \beta)(x) = \alpha(x) \vee \beta(x)$ and $(\alpha\beta)(x) = \beta(\alpha(x))$, where $x \in \mathcal{C}_n$ and $\alpha, \beta \in \mathcal{E}_{\mathcal{C}_n}^{(0)}$.

We denote by $(0, 1, \ldots, n-1)$ and by $(0, 0, \ldots, 0)$ the identity and the zero of $\mathcal{E}_{\mathcal{C}_n}^{(0)}$, respectively. For more information regarding endomorphism semirings, the reader is referred to [17]. In [17], as well as in [18], we consider a semiring which is a subsemiring of the endomorphism semiring $\mathcal{E}_{\mathcal{C}_n}^{(0)}$, consisting of the endomorphisms α such that $\alpha(k) \leq k$ for all $k \in \mathcal{C}_n$. As this semiring is associated with Catalan numbers, we call it the Catalan endomorphism semiring and denote it by $\mathcal{CAT}_n$. We note the following properties:

(i) For an arbitrary $\alpha \in \mathcal{CAT}_n$, it follows that $\alpha \leq (0, 1, \ldots, n-1)$, where $\mathcal{CAT}_n$ is an incline (see [33]). It can be easily checked that this is the maximal subsemiring of $\mathcal{E}_{\mathcal{C}_n}^{(0)}$, which is an incline.

(ii) The order of $\mathcal{CAT}_n$ is the n-th Catalan number C_n. (For the definition and properties of Catalan numbers, the reader is referred to [34].)

(iii) The nilpotent endomorphisms of $\mathcal{E}_{\mathcal{C}_n}^{(0)}$ are the endomorphisms α such that $\alpha^m = (0, 0, \ldots, 0)$ for some positive integer, m, and they also have the property $\alpha(k) < k$ for all $k \in \mathcal{C}_n$. The set of nilpotent endomorphisms is an ideal of $\mathcal{CAT}_n$ of the order of the $(n-1)$-th Catalan number C_{n-1}.

Now, consider an arbitrary endomorphism, $\alpha \in \mathcal{CAT}_n$. If α has no fixed points, it follows that $\alpha(k) < k$ for all $k \in \mathcal{C}_n$. Consequently, α is a nilpotent. Since $\alpha = \alpha + (0, 0, \ldots, 0)$, we have that α is a sum of an idempotent and nilpotent.

Let k be the lowest number which is a fixed point of α. We consider the following possibilities.

Case 1. Let $k = 1$. Assume that $k_1 = k = 1 < k_2 < \cdots < k_s$ are all fixed points of α. Now, if $x \in \mathcal{C}_n$ and $1 < x < k_2$, then $\alpha(x) = 1$. Similarly, if $k_{s-1} < x < k_s$, then $\alpha(x) = k_{s-1}$.

Case 1.1. If $k_s = n - 1$, then α is an idempotent. It is reasonable to assume that $n \geq 3$. Then, the endomorphism $(0, \ldots, 0, 1)$ is a nilpotent. Now, from $\alpha = \alpha + (0, \ldots, 0, 1)$, we have that α is a sum of an idempotent and nilpotent.

Case 1.2. Let $k_s < n-1$. If $\alpha(x) = k_s$ for all x such that $k_s \leq x \leq n-1$, then α is an idempotent. Assume that for some x, $k_s < x \leq n-1$, we have $\alpha(x) = p < x$ where $p \neq k_s$. Now, we construct a new endomorphism, β, such that $\beta(x) = 0$ for all x, $0 \leq x \leq k_s$, and $\beta(x) = \alpha(x)$ for x such that $k_s < x \leq n-1$. Then, β is a nilpotent. Additionally, we construct an endomorphism, γ, such that $\gamma(x) = \alpha(x)$ for all x, such that $0 \leq x \leq k_s$ and $\gamma(x) = k_s$ for x, $k_s \leq x \leq n-1$. Consequently, γ is an idempotent. It is easy to see that $\alpha = \beta + \gamma$.

Case 2. Let k, where $k > 1$, be the smallest fixed point of α. Then, for all x, $1 \leq x \leq k-1$, we have $\alpha(x) < x$. Let $\alpha(k-1) = p < k-1$. Now, we construct an endomorphism, β, such that $\beta(x) = \alpha(x)$ for all x, $0 \leq x \leq k-1$, and $\beta(x) = p$ for all x, $k \leq x \leq n-1$. Since β has no fixed points, it is a nilpotent. Assume that $k_1 = k < k_2 < \cdots < k_s$ are all fixed points of α. Now, for x such that $k < x < k_2$, we have $\alpha(x) = k$. Similarly, if $k_{s-1} < x < k_s$, then $\alpha(x) = k_{s-1}$.

Case 2.1. Let $k_s = n-1$. We construct an endomorphism, γ, where $\gamma(x) = 0$ f0r all x such that $0 \leq x \leq k-1$ and $\gamma(x) = \alpha(x)$ for x, $k \leq x \leq n-1$. Then, γ is an idempotent and $\alpha = \beta + \gamma$.

Case 2.2. Let $k_s < n-1$. We construct an endomorphism, β, such that $\beta(x) = \alpha(x)$ for all x, $0 \leq x \leq k-1$, $\beta(x) = \alpha(k-1)$ for all x, $k \leq x \leq k_s$, and $\beta(x) = \alpha(x)$ for all x, $k_s < x \leq n-1$. Since β has no fixed points, it is a nilpotent. Finally, we construct an endomorphism, γ, such that $\gamma(x) = 0$ for all x such that $0 \leq x \leq k-1$, $\gamma(x) = \alpha(x)$, for x, $k \leq x \leq k_s$, and $\gamma(x) = k_s$ for all x, such that $k_s \leq x \leq n-1$. Then, γ is an idempotent and $\alpha = \beta + \gamma$. Consequently, we prove that $\mathcal{CAT}_n$ is a nil-clean semiring.

Remark 9. *In the example above, we write $\alpha = \alpha + \nu$, where ν is a nilpotent element, for each idempotent α. We call this representation trivial. Are there nontrivial representations in the following sense: $e_1 = e + n$, where e_1 and e are idempotents, $e \neq e_1$, and n is a nilpotent element?*

In order to find such a representation, we consider an additive idempotent semiring, S, where ℓ (resp., r) is a left (resp., right) semicentral idempotent. Assume (as in Section 2) that $r\ell = 0$. Then, ℓr is a nilpotent element and $1 + \ell r$, $\ell + \ell r$ and $r + \ell r$ are nil-clean elements. It is easy to see that $1 + \ell r$, $\ell + \ell r$ and $r + \ell r$ are idempotents (moreover, $1 + \ell r$ is a Jordan semicentral idempotent; see Proposition 4). But, the subsemiring of S, generated by ℓ and r with the property $r\ell = 0$, is not a nil-clean semiring. It is interesting to know if there is a nil-clean semiring with the nontrivial representation of all idempotent elements. Such a nil-clean semiring will be considered in the remark after Proposition 22.

The next result shows why the notion of a nil-clean semiring is relevant to the study of triangular matrices.

Proposition 20. *The semiring $T_n(S)$ is nil-clean, provided that S is a nil-clean semiring.*

Proof. Let $A \in T_n(S)$, where S is a nil-clean semiring. Then, $A = (\varepsilon_{ij} + \eta_{ij})$, where ε_{ij} is an idempotent and η_{ij} is a nilpotent element for $i, j = 1, \ldots, n$, $i \leq j$. It follows that $A = E + N$, where $E = \operatorname{diag}(\varepsilon_{11}, \ldots, \varepsilon_{nn})$, and $N = (\eta_{ij}) + E_1$, where $E_1 = (\varepsilon_{ij})$ is a strictly upper triangular matrix. Since E is an idempotent matrix and N is a nilpotent matrix, it follows that $T_n(S)$ is a nil-clean semiring. □

In the next result, we represent a triangular matrix over a nil-clean semiring by left and right monic semicentral idempotent matrices.

Proposition 21. *Let $A \in T_n(S)$ where S is a nil-clean semiring. Then, there are the matrices $B, C \in T_n(S)$, such that we have the following:*

(i) $A = B + C$;

(ii) $B = \sum_{i=1}^{n} \varepsilon_{ii} 1^r_{n-i+1} 1^\ell_i$, where ε_{ii} are idempotents of S;

(iii) $C^m = \prod_{k=0}^{n-1}\left(1^{\ell}_{n-1-k} + 1^{r}_{k}\right)$ *for some positive integer,* m.

Proof. By the last proposition, it follows that $A = B + C$, where B is an idempotent diagonal matrix and C is a nilpotent matrix. Using Corollary 11, we have that $B = \sum_{i=1}^{n} \varepsilon_{ii} 1^{r}_{n-i+1} 1^{\ell}_{i}$ where ε_{ii} are its diagonal elements.

Since diagonal entries of C are nilpotent elements, there is a positive integer, say, m, such that (according to Theorem 8 and (7)) $C^m = \prod_{k=0}^{n-1}\left(1^{\ell}_{n-1-k} + 1^{r}_{k}\right)$. □

It is important to know if there are nil-clean triangular matrices over S and which entries are not nil-clean elements of S. In a similar vein, J. Šter (see [35]) has shown that $M_n(R)$ is a nil-clean ring if R is a Boolean ring.

Note that in Example 3, we describe all matrices of $T_3(\mathbb{B})$, where $\mathbb{B}$ is the Boolean semiring, as follows: left, right, and Jordan semicentral idempotents, idempotents which are not semicentral, and strictly upper and square roots of semicentral idempotent matrices.

Proposition 22. *The semiring* $T_n(\mathbb{B})$ *is nil-clean where* $\mathbb{B}$ *is the Boolean semiring.*

Proof. The proof is straightforward. □

Remark 10. *Every idempotent matrix,* $A \in T_n(\mathbb{B})$*, is a sum of its main diagonal, which is also an idempotent matrix and strictly upper matrix consisting of all entries over the main diagonal of* A*. Therefore, there is a nontrivial representation of* A *in the sense of Remark 8.*

Now, we represent a matrix in $T_n(\mathbb{B})$ by semicentral idempotent matrices.

Proposition 23. *Let* $A \in T_n(\mathbb{B})$*, where* $\mathbb{B}$ *is the Boolean semiring. Then,*

$$A = \varepsilon_{11} 1^{r}_{n} 1^{\ell}_{1} + \cdots + \varepsilon_{ii} 1^{r}_{n-i+1} 1^{\ell}_{i} + \cdots + \varepsilon_{nn} 1^{r}_{1} 1^{\ell}_{n} + \prod_{k=0}^{n-1}\left(1^{\ell}_{n-1-k} + 1^{r}_{k}\right),$$

where $\varepsilon_{ii} = 1$ *or* $\varepsilon_{ii} = 0$ *for* $i = 1, \ldots, n$.

Proof. The proof is similar to that of Proposition 20. □

By the same reasoning, we prove the next result.

Proposition 24. *Let* S_0 *be a nil-clean subsemiring of* $T_n(S)$ *where* S *is an additively idempotent semiring. Then, for every matrix,* $A \in S_0$*, there are* $B, C \in S_0$ *such that we have the following:*

(i) $A = B + C$.

(ii) B *is an idempotent matrix with the diagonal entries* e_i, $i = 1, \ldots, n$, *and* $B = \left(\sum_{k=1}^{n-1} E^{\ell}_{k} \diamond E^{r}_{n-k}\right)^{n-1}$, $E^{\ell}_{k} = \mathrm{diag}(e_1, \ldots, e_n) 1^{\ell}_{k}$, $E^{r}_{n-i} = 1^{r}_{n-k} \mathrm{diag}(e_1, \ldots, e_n)$, $k = 1, \ldots, n-1$.

(iii) $C^m = \prod_{k=0}^{n-1}\left(1^{\ell}_{n-1-k} + 1^{r}_{k}\right)$ *for some positive integer,* m.

There are enough nil-clean subsemirings of $T_n(S)$. Since every semicentral (not necessarily monic) idempotent matrix is nil-clean, we can add to such matrices an arbitrarily strictly upper matrix. Now, the matrices $1^{\ell}_{k} + N_k$ and $1^{r}_{m} + N_m$, where N_k and N_m are strictly upper, are nil-clean matrices. It is easy to see that these matrices form a semiring.

9. Conclusions

In this paper, we obtain the explicit form of an arbitrary idempotent triangular matrix over an additively idempotent semiring. The explicit forms of left and right semicentral

idempotents are given. A new type of semicentral idempotent, called Jordan, is studied. A diamond composition of two idempotent matrices is defined, which plays an important role in decompositions of many triangular matrices.

We consider this paper as a step towards the construction of a linear algebra on additively idempotent semirings, generalizing the idea of well-known max-algebra.

Funding: This research received no external funding.

Data Availability Statement: Data are contained within the article.

Acknowledgments: The author wish to tank the reviewers and The editors for helpful comments and suggestions that improved the quality of the article.

Conflicts of Interest: The author declare no conflicts of interest.

References

1. Bistarelli, S. *Semirings for Soft Constraint Solving and Programming*; Springer: Berlin/Heidelberg, Germany, 2004.
2. Brunsch, T.; Hardouin L.; Maiac, C.; Raisc, J. Duality and interval analysis over idempotent semirings. *Linear Algebra Appl.* **2012**, *437*, 2436–2454. [CrossRef]
3. Butkovič, P. *Max-Linear Systems: Theory and Algorithms*; Springer: London, UK, 2010.
4. Gaubert, S.; Katz, R. Reachability problems for products of matrices in semirings. *Int. J. Algebra Comp.* **2006**, *16*, 603–622. [CrossRef]
5. Gaubert, S.; Katz, R. Max-Plus Convex Geometry. In Proceedings of the Conference on Relational Methods in Computer Science, RelMiCS/AKA 2006, Manchester, UK, 29 August–2 September 2006.
6. Grigoriev, D.; Shpilrain, V. Tropical cryptography. *Commun. Algebra* **2014**, *42*, 2624–2632. [CrossRef]
7. Di Nola, A.; Gerla, B. Algebras of Łukasiewicz's Logic and their Semiring Reducts. In *Idempotent Mathematics and Mathematical Physics*; Contemporary Mathematics; American Mathematical Society: Providence, RI, USA, 2005; Volume 377, pp. 131–144.
8. Perri, T.; Rowen, L.H. Kernels in tropical geometry and a Jordan-Holder theorem. *J. Algebra Its Appl.* **2018**, *17*, 1850066. [CrossRef]
9. Drensky, V.S. Idempotents of 2×2 matrix rings over rings of formal power series. *Linear Multilinear Algebra* **2022**, *70*, 5207–5213. [CrossRef]
10. Peirce, B. Linear Associative Algebra. Washington, 1870. *Am. J. Math.* **1870**, *4*, 99–215. Available online: http://www.math.harvard.edu/history/peircealgebra/index.html (accessed on 1 February 2025).
11. Hou, X. Idempotents in triangular matrix rings. *Linear Multilinear Algebra* **2021**, *69*, 296–304. [CrossRef]
12. Wright, S.E. Triangular idempotent matrices over a general ring. *Linear Multilinear Algebra* **2022**, *70*, 1717–1731. [CrossRef]
13. Vladeva, D.I. The role of semicentral idempotents in triangular matrix rings. In *Communications in Algebra*; Taylor & Francis: Abingdon, UK, 2024; Volume53, pp. 1024–1038. [CrossRef]
14. Vladeva, D.I. Semicentral idempotents of upper triangular matrix ring. *J. Algebra Appl.* 2024, *Submitted 02.02.2024, Accepted 24.11.2024* (to appear).
15. Trendafilov, I.D.; Vladeva, D.I. Idempotent Elements of the Endomorphism Semiring of a Finite Chain. *C. R. l'Académie Bulg. Sci.* **2013**, *66*, 621–628.
16. Chajda, I.; Länger, H. The variety of commutative additively and multiplicatively idempotent semirings. *Semigroup Forum* **2018**, *96*, 409–415. [CrossRef]
17. Vladeva, D.I. Projections on right and left ideals of endomorphism semiring which are derivations. *J. Algebra Appl.* **2020**, *19*, 2050212-1–2050212-26. [CrossRef]
18. Vladeva, D.I. Derivations of some classes of additively idempotent semirings. *Commun. Algebra* **2023**, *51*, 3244–3265. [CrossRef]
19. Golan, J.S. *Semirings and Their Applications*; Kluwer Academic Press: Dordrecht, The Netherlands, 1999.
20. Gantmacher, F.R. *Matrix Theory Vol. 1*; Chelsea: New York, NY, USA, 1959.
21. Jain, S.K.; Goel, V.K.; Kwak, E.K. Nonnegative m th Roots of Nonnegative 0-Symmetric ldempotent Matrices. *Linear Algebra Its Appl.* **1979**, *23*, 37–51. [CrossRef]
22. Birkenmeier, G.F.; Park, J.K.; Rizvi, S.T. *Extensions of Rings and Modules*; Springer: Berlin/Heidelberg, Germany; Birkhäuser: Basel, Switzerland, 2013.
23. Vladeva, D.I. Derivations of skew Ore polynomial semirings. *Commun. Algebra* **2020**, *48*, 4718–4731. [CrossRef]
24. Diesl, A.J. Nil clean rings. *J. Algebra* **2013**, *383*, 197–211. [CrossRef]
25. Alahmadi, A.; Jain, S.K.; Leroy, A.; Sathaye, A. Decompositions into products of idempotents. *Electron. J. Linear Algebra* **2015**, *29*, 74–88. [CrossRef]

26. Vladeva, D.I. Derivatives of triangular, Toeplitz, circulant matrices and of matrices of other forms over semirings. In Proceedings of the 43rd International Conference Applications of Mathematics in Engineering and Economics, Sozopol, Bulgaria, 8–13 June 2017; Volume 1910, pp. 060006-1–060006-12.
27. Gray, R. *Toeplitz and Circulant Matrices: A Review*; Stanford University Press: Stanford, CA, USA, 2006.
28. Nicholson, W.K. Lifting idempotents and exchange rings. *Trans. Am. Math. Soc.* **1977**, *229*, 269–278. [CrossRef]
29. Nicholson, W.K.; Zhou, Y. *Clean Rings: A Survey, Advances in Ring Theory*; World Scientific Publication: Hackensack, NJ, USA, 2005; pp. 181–198.
30. Breaz, S.; Călugăreanu, G.; Danchev, P.; Micu, T. Nil-clean matrix rings. *Linear Algebra Appl.* **2013**, *439*, 3115–3119. [CrossRef]
31. Breaz, S.; Danchev, P.; Zhou, Y. Rings in which every element is either a sum or a difference of a nilpotent and an idempotent. *J. Algebra Appl.* **2016**, *15*, 1650148-1–1650148-13. [CrossRef]
32. Danchev, P.; Garcia, E.; Lozano, M. Decompositions of matrices into diagonalizable and square-zero matrices. *Linear Multilinear Algebra* **2022**, *70*, 4056–4070. [CrossRef]
33. Cao, Z.; Kim, K.; Roush, F. *Incline Algebra and Applications*; John Wiley & Sons: New York, NY, USA, 1984.
34. Stanley, R. *Catalan Numbers*; Cambridge University Press: New York, NY, USA; Cambridge, UK, 2015.
35. Šter, J. On expressing matrices over $\mathbb{Z}_2$ as the sum of an idempotent and a nilpotent. *Linear Algebra Appl.* **2015**, *544*, 339–349. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.