

Article

On Stability of Non-Surjective Coarse Isometries of Banach Spaces

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Abstract: In this paper, we establish the stability of the almost-surjective coarse isometries of Banach spaces by means of the weak stability condition. In addition, we also discuss the existence of coarse left-inverse operators in classical Banach spaces. Making use of them, we generalize several known results related to ε -isometries.

Keywords: coarse isometry; linear isometry; stability; coarse left-inverse; Banach space

MSC: 46B04; 46B20

1. Introduction

At the beginning, let us recall the definitions of isometry, ε -isometry and coarse isometry.

Definition 1. Assume that E and F are two Banach spaces, and $h : E \rightarrow F$ is a mapping. Put

$$\tau_h(w) = \sup_{\mu, \nu \in E, \|\mu - \nu\| \leq w} \{ \|h(\mu) - h(\nu)\| - \|\mu - \nu\| \} \quad \text{for } w \geq 0.$$

h is referred to as a coarse isometry if $\lim_{w \rightarrow \infty} \frac{\tau_h(w)}{w} = 0$. Specifically, we define h as an ε -isometry if $\varepsilon \equiv \lim_{t \rightarrow \infty} \tau_h(w) < \infty$; moreover, we say that h is an isometry if $\tau_h(w) = 0$ for all $w \geq 0$. For the sake of convenience, we say that h is standard if $h(0) = 0$.

The study of isometries has been a significant area of research since the publication of the Mazur–Ulam theorem ([1], 1932), which posits that any surjective standard isometry between normed linear spaces is necessarily linear. In the case of non-surjective isometries, Figiel ([2], 1968) established a significant finding: every standard isometry possesses a linear left-inverse. For more results on isometry and linear isometry, one can refer to [3–8].

In 1945, Hyers and Ulam [9] were the first to investigate the concept of ε -isometries and introduced a problem that can be restated as follows: for two Banach spaces, denoted as E and F , is there a constant $\gamma > 0$ such that for every surjective standard ε -isometry, $h : E \rightarrow F$, there exists a linear isometry, $L : E \rightarrow F$, satisfying the condition $\|h(\mu) - L\mu\| \leq \gamma\varepsilon$? After extensive research by numerous mathematicians (see, for example, [9–12]), a positive resolution to this problem, with the precise estimate of $\gamma = 2$, was ultimately provided by Omladič and Šemrl [13] in 1995 (also referenced in Benyamini and Lindenstauss ([14], Theorem 15.2).

Building upon Figiel’s theorem [2] and other significant findings related to non-surjective ε -isometries (as discussed in [15,16]), Cheng, Dong and Zhang formulated a



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profound theorem in 2013, referred to as the weak stability formula (see [17]). This theorem has been instrumental in advancing the understanding of the stability properties of ε -isometries (refer to [5,17–24]).

In 1985, Lindenstrauss and Szankowski [25] introduced a more comprehensive perturbation function for a mapping of $h : E \rightarrow F$, defined as follows:

$$v_h(w) = \sup_{\mu, \nu \in E, \|\mu - \nu\| \wedge \|h(\mu) - h(\nu)\| \leq w} \{ \|h(\mu) - h(\nu)\| - \|\mu - \nu\| \} \quad \text{for } w \geq 0.$$

They established an asymptotic stability result, demonstrating that if a standard surjective coarse isometry, $h : E \rightarrow F$, satisfies the condition $\int_1^\infty \frac{v_h(w)}{w^2} dw < \infty$, then there exists a surjective linear isometry, $L : E \rightarrow F$, such that

$$\|h(\mu) - L\mu\| = o(\|\mu\|) \quad \text{as } \|\mu\| \rightarrow \infty. \quad (1)$$

At the same time, they constructed a counterexample for two uniformly convex separable Banach spaces, E and F , and a coarse isometry, $h : E \rightarrow F$, with $\int_1^\infty \frac{v_h(w)}{w^2} dw = \infty$. Furthermore, Benyamini and Lindenstrauss [14] demonstrated that the assertion in Equation (1) may not hold when the integral $\int_1^\infty \frac{v_h(w)}{w^2} dw = \infty$, even in the context of the coarse isometries of the Euclidean plane $\mathbb{R}^2$ mapping onto itself. Additionally, Dolinar [26] observed that the condition for integral convergence, $\int_1^\infty \frac{v_h(w)}{w^2} dw < \infty$, can be replaced by the following condition:

$$\int_1^\infty \frac{\tau_h(w)}{w^2} dw < \infty. \quad (2)$$

Remark 1. The discussions above entail that the integral convergence condition (2) is essential for the asymptotical stability of coarse isometries. In other words, to guarantee that (1) is true, we cannot avoid assuming additional conditions. However, even if (2) holds, Lindenstrauss and Szankowski ([25], p. 359) also noted that the conclusion of (1) cannot be replaced by an estimate of the form

$$\|h(\mu) - L\mu\| \leq \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E, \quad (3)$$

where $\gamma > 0$.

Then, the following question is very natural.

Problem 1. What conditions are necessary to ensure that for every standard surjective coarse isometry, $h : E \rightarrow F$, there exists a surjective linear isometry, $L : E \rightarrow F$, such that the inequality (3) is satisfied?

In 2019, Cheng et al. [27] initiated an investigation into the non-surjective coarse isometry $h : E \rightarrow F$ and obtained a stability result, where F was a uniformly convex space of the power type p . Sun and Zhang's recent research (see [28,29]) examined the relationship between the stability and weak stability of the coarse isometries of $L_p(1 < p < \infty)$ spaces. Furthermore, they investigated the relationship between the coarse isometry and linear isometry of varieties of Banach spaces by means of coarse left-inverse operators (see [30]). The latest results regarding coarse isometries can be found in references [31,32].

For the study of coarse isometries, the asymptotical stability and other properties become more complicated than those of ε -isometries since the perturbation is no longer bounded. Owing to the importance of the weak stability formula for ε -isometries, we will

introduce the weak stability of coarse isometries. Then, the asymptotical stability and the existence of coarse left-inverse operators will be discussed.

This paper is organized as follows. In Section 2, we give the definition of the weak stability condition and some notation that will be used later. In Section 3, we first show that the answer to Problem 1 is affirmative if h is weakly stable and satisfies the following almost-surjective condition:

$$\liminf_{w \rightarrow \infty} d(wv, h(E))/|w| = 0 \quad \text{for all } v \in S_F, \quad (4)$$

which is much weaker than the surjectivity of h . In particular, when $\dim E = \dim F < \infty$, the almost-surjective assumption condition (4) can be removed. But, in general, we show that (4) cannot be dropped. As applications, we can deduce some known results of ε -isometries which were established by Cheng et al. [19] and Dilworth [33].

In Section 4, by using the weak stability condition, we demonstrate the existence of coarse left-inverse operators of non-surjective coarse isometries by assuming one of the following conditions: (1) both E and F are L_p ($1 < p < \infty$) spaces; (2) for any Γ , $E = \ell_\infty(\Gamma)$; (3) $E = c_0$ and F is separable; and (4) E is a finite-dimensional Banach space. As an application, we obtain a result which was shown by Šemrl and Väisälä [16].

2. Preliminaries

In 2013, Cheng, Dong and Zhang [17] established the following weak stability formula for ε -isometries.

Theorem 1. *Let $h : E \rightarrow F$ be a standard ε -isometry. Then, for each $\mu^* \in E^*$, there exists $g \in F^*$ with $\|g\| = \|\mu^*\| = k$ such that*

$$|\langle \mu^*, \mu \rangle - \langle g, h(\mu) \rangle| \leq 4\varepsilon k \quad \text{for all } \mu \in E. \quad (5)$$

Remark 2. *Cheng and Dong [18] proved that the constant 4 in (5) can be improved to 3, and this is the best estimate in general Banach spaces. Cheng et al. [19] also showed that the constant 3 can be reduced to 2 when F^* is strictly convex.*

Since the weak stability formula has played an important role in perturbation isometry theory, we will next present the weak stability condition for coarse isometries.

Definition 2. *Let $h : E \rightarrow F$ be a standard coarse isometry. We say that h is weakly stable with a constant $\gamma > 0$ if for each $\mu^* \in S_{E^*}$, there exists $g \in S_{F^*}$ such that*

$$|\langle \mu^*, \mu \rangle - \langle g, h(\mu) \rangle| \leq \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E. \quad (6)$$

If $h : E \rightarrow F$ is a weakly stable standard coarse isometry with $\gamma > 0$, then for each nonzero element, $\mu^* \in E^*$, there exists $\varphi \in S_{F^*}$ such that

$$|\langle \frac{\mu^*}{\|\mu^*\|}, \mu \rangle - \langle \varphi, h(\mu) \rangle| \leq \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

Put $g = \|\mu^*\|\varphi$, then $\|g\| = \|\mu^*\|$ and

$$|\langle \mu^*, \mu \rangle - \langle g, h(\mu) \rangle| \leq \|\mu^*\| \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

Consequently, if a standard coarse isometry, $h : E \rightarrow F$, is weakly stable with $\gamma > 0$, then for every $\mu^* \in E^*$, there exists $g \in F^*$ with $\|g\| = \|\mu^*\|$ such that

$$|\langle \mu^*, \mu \rangle - \langle g, h(\mu) \rangle| \leq \|\mu^*\| \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E. \quad (7)$$

In the following, we will define a set-valued mapping, $\ell : E^* \rightarrow 2^{F^*}$. If a standard coarse isometry, $h : E \rightarrow F$, is weakly stable with $\gamma > 0$, let

$$\ell(\mu^*) = \{g \in F^* : \exists \alpha > 0 \text{ s.t. } |\langle \mu^*, \mu \rangle - \langle g, h(\mu) \rangle| \leq \alpha \tau_h(\|\mu\|) \quad \forall \mu \in E\}, \quad (8)$$

$$N_h = \{g \in F^* : \exists \alpha > 0 \text{ s.t. } |\langle g, h(\mu) \rangle| \leq \alpha \tau_h(\|\mu\|) \quad \forall \mu \in E\} \quad \text{and} \quad N = \overline{N_h}.$$

In this article, the symbols E and F represent real Banach spaces, while E^* and F^* denote their respective dual spaces. For a given real Banach space, E , we define S_E and B_E to be the unit sphere and the closed unit ball of E , respectively. The notation $\partial\|\cdot\| : E \rightarrow 2^{E^*}$ refers to the subdifferential mapping associated with the norm $\|\cdot\|$. Additionally, for a bounded linear operator, $T : E \rightarrow F$, the dual operator is represented as $T^* : F^* \rightarrow E^*$.

3. Asymptotical Stability of Almost-Surjective Coarse Isometries

In the following, we obtain a linear isometry from E^* to a quotient space of F^* by using the weak stability condition.

Theorem 2. Let $h : E \rightarrow F$ be a standard coarse isometry, and let ℓ , N_h and N be defined as in Section 2. If h is weakly stable with a constant $\gamma > 0$, then $Q : E^* \rightarrow F^*/N$, defined by $Q\mu^* = \ell(\mu^*) + N$, is a linear isometry.

Proof. Note that $\ell(\mu^*) \neq \emptyset$ for all $\mu^* \in E^*$, since h is weakly stable with $\gamma > 0$. We first prove that ℓ satisfies

$$\ell(s\mu^* + tv^*) = s\ell(\mu^*) + t\ell(v^*) \quad \text{for all } \mu^*, v^* \in E^* \text{ and } s, t \in \mathbb{R}. \quad (9)$$

Clearly, $\ell(se^*) = s\ell(e^*)$ for all $s \in \mathbb{R}$. In the following, we show that $\ell(\mu^* + v^*) = \ell(\mu^*) + \ell(v^*)$. Given $\mu^*, v^* \in E^*$, let $g \in \ell(\mu^*)$ and $\psi \in \ell(v^*)$ and there exist $\alpha_1, \alpha_2 > 0$ such that

$$|\langle \mu^*, \mu \rangle - \langle g, h(\mu) \rangle| \leq \alpha_1 \tau_h(\|\mu\|) \quad \text{and} \quad |\langle v^*, \mu \rangle - \langle \psi, h(\mu) \rangle| \leq \alpha_2 \tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

This implies that

$$|\langle \mu^* + v^*, \mu \rangle - \langle g + \psi, h(\mu) \rangle| \leq (\alpha_1 + \alpha_2) \tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

then $g + \psi \in \ell(\mu^* + v^*)$. Conversely, let $\psi \in \ell(\mu^* + v^*)$ and $g \in \ell(\mu^*)$ and there exist $\alpha_1, \alpha_2 > 0$ such that

$$|\langle \mu^* + v^*, \mu \rangle - \langle \psi, h(\mu) \rangle| \leq \alpha_1 \tau_h(\|\mu\|) \quad \text{and} \quad |\langle \mu^*, \mu \rangle - \langle g, h(\mu) \rangle| \leq \alpha_2 \tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

Put $\varphi = \psi - g$. Then,

$$|\langle v^*, \mu \rangle - \langle \varphi, h(\mu) \rangle| = |(\langle \mu^* + v^*, \mu \rangle - \langle \psi, h(\mu) \rangle) - (\langle \mu^*, \mu \rangle - \langle g, h(\mu) \rangle)| \leq (\alpha_1 + \alpha_2) \tau_h(\|\mu\|), \quad \mu \in E.$$

It follows that $\varphi \in \ell(v^*)$ and $\psi = g + \varphi \in \ell(\mu^*) + \ell(v^*)$. Consequently,

$$\ell(s\mu^* + tv^*) = s\ell(\mu^*) + t\ell(v^*) \quad \text{for all } \mu^*, v^* \in E^* \text{ and } s, t \in \mathbb{R}.$$

So,

$$Q(s\mu^* + tv^*) = \ell(s\mu^* + tv^*) + N = s\ell(\mu^*) + N + t\ell(v^*) + N = sQ(\mu^*) + tQ(v^*),$$

because N is a subspace of F^* , i.e., Q is linear.

In what follows, we will prove that

$$\|Q\mu^*\| = \|\mu^*\| \quad \text{for all } \mu^* \in E^*.$$

Note that $\ell(\mu^*) = g + N_h$ where $g \in \ell(\mu^*)$. In fact, by the definition of ℓ , $N_h = \ell(0)$. Given $\mu^* \in E^*$ and $g \in \ell(\mu^*)$, using (9), we obtain

$$\ell(\mu^*) = \ell(\mu^* + 0) = \ell(\mu^*) + \ell(0) \supset g + \ell(0) = g + N_h.$$

Conversely, for each $g, \psi \in \ell(\mu^*)$, there exists $\alpha, \alpha_2 > 0$ such that

$$|\langle \mu^*, \mu \rangle - \langle g, h(\mu) \rangle| \leq \alpha_1 \tau_h(\|\mu\|) \quad \text{and} \quad |\langle \mu^*, \mu \rangle - \langle \psi, h(\mu) \rangle| \leq \alpha_2 \tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

Then,

$$|\langle \psi - g, h(\mu) \rangle| \leq (\alpha_1 + \alpha_2) \tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

This entails that $\psi - g \in \ell(0) = N_h$, and then $\ell(\mu^*) \subset g + N_h$. Thus, $\ell(\mu^*) = g + N_h$ where $g \in \ell(\mu^*)$. It follows that $Q\mu^* = \ell(\mu^*) + N = g + N$ where $g \in \ell(\mu^*)$. Then,

$$\begin{aligned} \|Q\mu^*\| &= \inf\{\|g - u\| : g \in \ell(\mu^*), u \in N\} \\ &= \inf\{\|g - u\| : g \in \ell(\mu^*), u \in N_h\} \\ &= \inf\{\|g\| : g \in \ell(\mu^*)\}. \end{aligned}$$

It remains to be proven that

$$\|\mu^*\| = \inf\{\|g\| : g \in \ell(\mu^*)\}.$$

Since h is weakly stable and due to (7), we have

$$\|\mu^*\| \geq \inf\{\|g\| : g \in \ell(\mu^*)\}.$$

Conversely, let $g \in \ell(\mu^*)$ and there exist $\alpha > 0$ such that

$$|\langle \mu^*, \mu \rangle - \langle g, h(\mu) \rangle| \leq \alpha \tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

Given $\delta > 0$, we choose $\mu_0 \in S_E$ such that $\langle \mu^*, \mu_0 \rangle > \|\mu^*\| - \delta$. Substitute $n\mu_0$ by μ and divide both sides by n in the inequality above, then

$$|\langle \mu^*, \mu_0 \rangle - \langle g, \frac{h(n\mu_0)}{n} \rangle| \leq \frac{\alpha \tau_h(\|n\mu_0\|)}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Note that

$$\left| \frac{\|h(n\mu_0)\|}{n} - 1 \right| = \frac{|\|h(n\mu_0)\| - n\|\mu_0\||}{n} \leq \frac{\tau_h(n)}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Therefore,

$$\|\mu^*\| - \delta < \langle \mu^*, \mu_0 \rangle = \lim_{n \rightarrow \infty} \langle g, \frac{h(n\mu_0)}{n} \rangle \leq \|g\| \lim_{n \rightarrow \infty} \frac{\|h(n\mu_0)\|}{n} = \|g\|.$$

The arbitrariness of δ entails that $\|g\| \geq \|\mu^*\|$. Consequently,

$$\|\mu^*\| = \inf\{\|g\| : g \in \ell(\mu^*)\} = \|Q\mu^*\| \quad \text{for all } \mu^* \in E^*.$$

Thus, Q is a linear isometry. $\square$

By using Theorem 2, we obtain the following asymptotical stability result for coarse isometries.

Theorem 3. *Let $h : E \rightarrow F$ be a standard coarse isometry with an almost-surjective condition.*

$$\liminf_{w \rightarrow \infty} d(wv, h(E))/|w| = 0 \quad \text{for all } v \in S_F. \quad (10)$$

If h is weakly stable with a constant $\gamma > 0$, then there is a surjective linear isometry, $L : E \rightarrow F$, such that

$$\|h(\mu) - L\mu\| \leq \gamma\tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

Proof. Firstly, we prove that $N = \overline{N_h} = \{0\}$. For each $v \in S_F$, due to (10), there exist two sequences, $\{\mu_n\} \subset E$ and $\{w_n\} \subset \mathbb{N}$, with $\lim_{n \rightarrow \infty} |w_n| = \infty$ such that $\lim_{n \rightarrow \infty} \frac{\|w_n v - h(\mu_n)\|}{|w_n|} = 0$. This entails that $\lim_{n \rightarrow \infty} \frac{h(\mu_n)}{w_n} = v$. Since h is a coarse isometry, we have $\lim_{n \rightarrow \infty} \frac{\|h(\mu_n)\|}{\|\mu_n\|} = 1$ and then $\lim_{n \rightarrow \infty} \frac{\|\mu_n\|}{|w_n|} = 1$. Given $g \in N_h$, there exists $\alpha > 0$ so that

$$|\langle g, f \rangle| = \lim_{n \rightarrow \infty} \left| \left\langle g, \frac{h(\mu_n)}{w_n} \right\rangle \right| \leq \alpha \lim_{n \rightarrow \infty} \frac{\tau_h(\|\mu_n\|)}{|w_n|} = \alpha \lim_{n \rightarrow \infty} \frac{\tau_h(\|\mu_n\|)}{\|\mu_n\|} = 0.$$

It follows that $N = \overline{N_h} = \{0\}$. According to Theorem 2, $Q : E^* \rightarrow F^*/N = F^*$, defined by $Qe^* = \ell(\mu^*) + N = g$, is a single-valued linear isometry where g satisfies (7).

In the following, we show that $Q : E^* \rightarrow F^*$ is w^* -to- w^* -continuous. Using the Krein-Šmulian theorem, it suffices to prove that it is w^* -to- w^* -continuous on B_{E^*} . Let $(\mu_\alpha^*) \subset B_{E^*}$ be a net w^* converging to $\mu^* \in B_{E^*}$. Since f is weakly stable with a constant $\gamma > 0$, there is a net $(\phi_\alpha) \subset B_{F^*}$ such that

$$|\langle \mu_\alpha^*, \mu \rangle - \langle \phi_\alpha, h(\mu) \rangle| \leq \gamma\tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

The w^* -relative compactness of (ϕ_α) entails that there is a w^* cluster point, $g \in F^*$, of (ϕ_α) so that

$$|\langle \mu_\alpha^*, \mu \rangle - \langle g, h(\mu) \rangle| \leq \gamma\tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

This implies that $g = Q\mu^*$. It follows that g is a unique w^* cluster point of $(\phi_\alpha) \in F^*$. Then,

$$Q\mu_\alpha^* = \phi_\alpha \xrightarrow{w^*} g = Q\mu^*$$

Therefore, $Q : B_{E^*} \rightarrow F^*$ is w^* -to- w^* -continuous.

Finally, we will prove that there is a surjective linear isometry, $L : E \rightarrow F$, such that

$$\|h(\mu) - L\mu\| \leq \gamma\tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

Since $Q : E^* \rightarrow F^*$ is a w^* -to- w^* -continuous linear isometry, there is a surjective linear operator, $V : F \rightarrow E$, with $\|V\| = 1$ such that $V^* = Q$. Thus, for each $\mu^* \in S_{E^*}$,

$$\begin{aligned} |\langle \mu^*, \mu \rangle - \langle \mu^*, Vh(\mu) \rangle| &= |\langle \mu^*, \mu \rangle - \langle V^* \mu^*, h(\mu) \rangle| \\ &= |\langle \mu^*, \mu \rangle - \langle Q \mu^*, h(\mu) \rangle| \\ &= |\langle \mu^*, \mu \rangle - \langle g, h(\mu) \rangle| \\ &\leq \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E. \end{aligned}$$

This implies that

$$\|Vh(\mu) - \mu\| = \sup_{\mu^* \in S_{E^*}} |\langle \mu^*, \mu \rangle - \langle \mu^*, Vh(\mu) \rangle| \leq \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E. \quad (11)$$

In the following, we prove that $V : F \rightarrow E$ is a surjective linear isometry; it suffices to show that Q is surjective. Otherwise, $Q(E^*)$ is a proper w^* -closed subspace of F^* . Using the separation theorem, we can find $\psi \in S_{F^*} \setminus Q(E^*)$ and $\nu \in S_F$ such that

$$\langle \psi, \nu \rangle > 0 \quad \text{and} \quad \langle g, \nu \rangle = 0 \quad \text{for all } g \in Q(E^*). \quad (12)$$

Due to (10), we can find two sequences, $\{\mu_n\} \subset E$ and $\{w_n\} \subset \mathbb{R}$, with $\lim_{n \rightarrow \infty} |w_n| = \infty$ such that

$$\lim_{n \rightarrow \infty} \frac{\|w_n \nu - h(\mu_n)\|}{|w_n|} = 0. \quad (13)$$

This entails that $\lim_{n \rightarrow \infty} \frac{\|h(\mu_n)\|}{|w_n|} = 1$. For the sequence $\{\mu_n\}$ above, let $\mu_n^* \in S_{E^*}$ with $\langle \mu_n^*, \mu_n \rangle = \|\mu_n\|$, and let $g_n = Q \mu_n^*$. It follows from (11) and (12) that

$$\begin{aligned} \tau_h(\|\mu_n\|) &\geq \|Vh(\mu_n) - \mu_n\| \geq \langle \mu_n^*, \mu_n \rangle - \langle g_n, h(\mu_n) \rangle \\ &\geq \|\mu_n\| - \langle g_n, h(\mu_n) - w_n \nu \rangle - \langle g_n, w_n \nu \rangle \\ &\geq \|\mu_n\| - \|h(\mu_n) - w_n \nu\|. \end{aligned}$$

This implies that

$$\frac{\tau_h(\|\mu_n\|)}{\|\mu_n\|} \geq 1 - \frac{\|h(\mu_n) - w_n \nu\|}{\|\mu_n\|}.$$

Since $\lim_{n \rightarrow \infty} \frac{\|h(\mu_n)\|}{\|\mu_n\|} = 1$, we have $\lim_{n \rightarrow \infty} \frac{\|\mu_n\|}{|w_n|} = 1$. Putting $n \rightarrow \infty$ in the inequality above, (13) entails that

$$0 \geq 1 - \lim_{n \rightarrow \infty} \frac{\|h(\mu_n) - w_n \nu\|}{\|\mu_n\|} = 1 - \lim_{n \rightarrow \infty} \frac{\|h(\mu_n) - w_n \nu\|}{|w_n|} = 1.$$

This contradiction indicates that the mapping Q is surjective. Therefore, $Q : E^* \rightarrow F^*$ is a surjective linear isometry. As a result, the mapping $V : F \rightarrow E$ is likewise a surjective linear isometry. Letting $L = V^{-1}$, we finish the proof by referencing Equation (11). $\square$

The following examples show that the almost-surjective condition (10) in Theorem 3 is essential.

Example 1. Define $h : \mathbb{R} \rightarrow (\mathbb{R}^2, \|\cdot\|_\infty)$ by $h(e) = (e, |e|)$. Clearly, $h(0) = 0$ and $\tau_h(w) = 0$ for all $w \geq 0$, but for any linear isometry, $L : \mathbb{R} \rightarrow (\mathbb{R}^2, \|\cdot\|_\infty)$, we have $h \neq L$.

Example 2 ([16]). Let $\varepsilon > 0$ and $h : \mathbb{R} \rightarrow \mathbb{R}^2$, given by $h(\mu) = (\mu, \sqrt{2\varepsilon|\mu|})$. Note that $h(0) = 0$ and $\lim_{t \rightarrow \infty} \tau_h(w) = \varepsilon < \infty$, i.e., h is a standard ε -isometry, but for any linear isometry, $L : \mathbb{R} \rightarrow \mathbb{R}^2$,

$$\sup_{\mu \in \mathbb{R}} \|h(\mu) - L\mu\| = \infty.$$

Example 3 ([26]). Let $0 < p < 1$, $E = \mathbb{R}$ and $F = \mathbb{R}^2$, and let $h : E \rightarrow F$ be defined by

$$h(\mu) = \begin{cases} (\mu, 0), & \mu \leq 0, \\ (\mu, \max\{\mu^p, \mu^{(1+p)/2}\}), & \mu > 0. \end{cases}$$

Then, h is a standard coarse isometry and weakly stable with a constant $\gamma > 0$, but there is no linear isometry, $L : E \rightarrow F$, such that

$$\|h(\mu) - L\mu\| \leq \gamma \tau_h(|\mu|) \quad \text{for all } \mu \in E.$$

Proof. It was shown in [26], Lemma 2, that $\|h(\mu) - h(\nu)\| - |\mu - \nu| \leq |\mu - \nu|^p$ for all $\mu, \nu \in E$. Clearly, $h(0) = 0$ and $\tau_h(w) \leq w^p$ for all $w \geq 0$. This implies that h is a standard coarse isometry. Note that for $\mu^* \in \{\pm 1\} = S_{E^*}$, if we let $g = (\pm 1, 0) \in S_{F^*}$, then $\langle \mu^*, \mu \rangle - \langle g, h(\mu) \rangle = 0$. Assume that there exist $\gamma > 0$ and a linear isometry, $L : E \rightarrow F$, such that

$$\|h(\mu) - L\mu\| \leq \gamma \tau_h(|\mu|) \leq \gamma |\mu|^p \quad \text{for all } \mu \in E. \quad (14)$$

Inequality (14) shows that $L\mu = (\mu, 0)$ for all $\mu \leq 0$. Then,

$$\|h(\mu) - L\mu\| = \mu^{(1+p)/2} \quad \text{for all } \mu \geq 1.$$

This is in contradiction with (14). Consequently, h is weakly stable with any constant $\gamma > 0$, but there is no linear isometry, $L : E \rightarrow F$, such that

$$\|h(\mu) - L\mu\| \leq \gamma \tau_h(|\mu|) \quad \text{for all } \mu \in E.$$

□

Remark 3. From Remark 1 and the two examples above, we determine that both the weak stability condition and the almost-surjective condition in Theorem 3 are indispensable.

It is worth mentioning that when $\dim E = \dim F < \infty$, condition (10) in Theorem 3 can be removed.

Theorem 4. Let $h : E \rightarrow F$ be a standard coarse isometry with $\dim E = \dim F < \infty$. If h is weakly stable with a constant $\gamma > 0$, then there is a surjective linear isometry, $L : E \rightarrow F$, such that

$$\|h(\mu) - L\mu\| \leq \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

Proof. According to Theorem 2, $Q : E^* \rightarrow F^*/N$, defined by $Q\mu^* = g + N$, is a linear isometry where $g \in \ell(\mu^*)$. Note that $\dim E^* = \dim F^* < \infty$; this entails that $N = N_h = \{0\}$ and $Q : E^* \rightarrow F^*$ is a surjective linear isometry. Then, $Q\mu^* = g$ where g satisfies (7). Letting

$V = Q^* : F \rightarrow E$, we determine that V is also a surjective linear isometry and for each $\mu^* \in S_{E^*}$.

$$\begin{aligned} |\langle \mu^*, \mu \rangle - \langle \mu^*, Vh(\mu) \rangle| &= |\langle \mu^*, \mu \rangle - \langle \mu^*, Q^*h(\mu) \rangle| \\ &= |\langle \mu^*, \mu \rangle - \langle Q\mu^*, h(\mu) \rangle| \\ &= |\langle \mu^*, \mu \rangle - \langle g, h(\mu) \rangle| \\ &\leq \gamma\tau_h(\|\mu\|) \quad \text{for all } \mu \in E. \end{aligned}$$

This implies that

$$\|Vh(\mu) - \mu\| = \sup_{\mu^* \in S_{E^*}} |\langle \mu^*, \mu \rangle - \langle \mu^*, Vh(\mu) \rangle| \leq \gamma\tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

Putting $L = V^{-1} : X \rightarrow Y$, we finish the proof using the inequality above. $\square$

In particular, when $h : E \rightarrow F$ is an ε -isometry, in 2015, Cheng et al. [19] obtained an asymptotical stability result when $h(E)$ contained a sublinear growth net of F . A subset, C , in a metric space (X, d) is called a sublinear growth net if for every $x_0 \in X$,

$$\lim_{d(x, x_0) \rightarrow \infty} \frac{d(x, C)}{d(x, x_0)} = 0. \quad (15)$$

Note that for every standard coarse isometry, $h : E \rightarrow F$, if $h(E)$ contains a sublinear growth net, C , of F , then condition (10) in Theorem 3 holds. In fact, with a fixed $x_0 = 0 \in E$, from (15), we determine that for all $v \in S_F$

$$\lim_{d(wv, 0) \rightarrow \infty} \frac{d(wv, h(E))}{d(wv, 0)} \leq \lim_{d(wv, 0) \rightarrow \infty} \frac{d(wv, C)}{d(wv, 0)} = 0.$$

This implies that

$$\liminf_{w \rightarrow \infty} d(wv, h(E))/|w| = \lim_{d(v, 0) \rightarrow \infty} \frac{d(wv, h(E))}{d(wv, 0)} = 0 \quad \text{for all } v \in S_F.$$

We also note that ([34], Proposition 2) for a standard ε -isometry, $h : E \rightarrow F$, if there is a surjective linear isometry, $L : E \rightarrow F$, such that $\|h(\mu) - L\mu\| = o(\|\mu\|)$ as $\|\mu\| \rightarrow \infty$, then $\|h(\mu) - L\mu\| \leq 2\varepsilon$ for all $\mu \in E$.

Consequently, according to Theorems 3 and 4, and the discussion above, we obtain the following results, which were established by Cheng et al. [19] and Dilworth [33], respectively.

Corollary 1 ([19], Theorem 3.1). *Let $h : E \rightarrow F$ be a standard ε -isometry, and let $h(E)$ contain a sublinear growth net of F . Then, there is a surjective linear isometry, $L : E \rightarrow F$, such that*

$$\|h(\mu) - L\mu\| \leq 2\varepsilon \quad \text{for all } \mu \in E.$$

Corollary 2 ([33], Theorem 1). *Let $h : E \rightarrow F$ be a standard ε -isometry with $\dim E = \dim F < \infty$. Then, there is a surjective linear isometry, $L : E \rightarrow F$, such that*

$$\|h(\mu) - L\mu\| \leq 2\varepsilon \quad \text{for all } \mu \in E.$$

4. Coarse Left-Inverse Operators of Non-Surjective Coarse Isometries

In this section, we investigate the existence of coarse left-inverse operators of non-surjective coarse isometries. Before our discussions, let us recall the existence of the ε -left-inverse operators of ε -isometries.

Qian ([15], 1995) was the first to explore the following inquiry: is there a constant $\gamma > 0$ that is contingent upon the spaces E and F such that for every standard ε -isometry, $h : E \rightarrow F$, one can identify a bounded linear operator, $T : F \rightarrow E$, satisfying the condition $\|Th(e) - e\| \leq \gamma\varepsilon$ for all $e \in E$? Here, we refer to T as an ε -left-inverse of h .

Qian demonstrated that the assertion holds true for L_p spaces, where $1 < p < \infty$, with $\gamma = 6$. Furthermore, Qian established that for any $\varepsilon > 0$, every separable Banach space, F , contains an uncomplemented subspace, E , for which there exists no ε -left-inverse of h . This revealed that the appropriate complementability assumption for the subspace of F related to the mapping $h : E \rightarrow F$ is essential for the Banach spaces E and F . In 2003, Šemrl and Väisälä [16] showed that for L_p spaces, $\gamma = 2$ is a best estimate.

For coarse isometries, the perturbation is no longer bounded. This makes the problem of coarse isometries more complicated than ε -isometries. In the following, by using the condition of weak stability (Definition 2), we first show the existence of coarse left-inverse operators for L_p spaces.

Theorem 5. Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two measure spaces and $1 < p < \infty$. Suppose that $E = L_p(\Omega_1, \Sigma_1, \mu_1)$, $F = L_p(\Omega_2, \Sigma_2, \mu_2)$ and that $h : E \rightarrow F$ is a standard coarse isometry. If f is weakly stable with a constant $\gamma > 0$, then there is a surjective bounded linear operator, $T : F \rightarrow E$, with $\|T\| = 1$ such that

$$\|Th(\mu) - \mu\| \leq \gamma\tau_h(\|\mu\|) \quad \text{for all } \mu \in E. \quad (16)$$

Proof. According to the proof presented in [28], Theorem 2.4, it can be established that there is a linear isometry, $L : E \rightarrow F$, so that the following holds for all $e \in E$:

$$L\mu = \lim_{r \rightarrow \infty} \frac{h(r\mu)}{r}.$$

For each $\mu^* \in E^* \setminus \{0\}$, let $\mu_0 \in \partial\|\mu^*\|$; according to the weak stability of h with a constant $\gamma > 0$, there exists $g \in F^*$ with $\|g\| = \|\mu^*\|$ such that

$$|\langle \mu^*, \mu_0 \rangle - \langle g, h(\mu_0) \rangle| \leq \|\mu^*\| \gamma\tau_h(\|\mu_0\|).$$

Then,

$$|\langle \mu^*, \mu_0 \rangle - \langle g, \frac{h(n\mu_0)}{n} \rangle| \leq \frac{\|\mu^*\| \gamma\tau_h(\|n\mu_0\|)}{n} = \frac{\|\mu^*\| \gamma\tau_h(n)}{n} \quad \text{for all } n \in \mathbb{N}.$$

By letting $n \rightarrow \infty$, we have

$$\|g\| = \|\mu^*\| = \langle \mu^*, \mu_0 \rangle = \langle g, L\mu_0 \rangle. \quad (17)$$

Since Y is smooth, g is unique in (17). It follows that for each $\mu^* \in E^*$, there is a unique $g \in F^*$ with $\|g\| = \|\mu^*\|$ such that

$$|\langle \mu^*, \mu \rangle - \langle g, h(\mu) \rangle| \leq \|\mu^*\| \gamma\tau_h(\|\mu\|) \quad \text{for all } \mu \in E. \quad (18)$$

We define $S : E^* \rightarrow F^*$ by

$$S(\mu^*) = g,$$

where g satisfies (18). Then, S is well defined and $\|S(\mu^*)\| = \|\mu^*\|$. Since $L : E \rightarrow F$ is a linear isometry, $L(E)$ is a 1-complemented subspace in F according to Theorem 3 in [35],

p. 162. Letting $M = L(E)$, there is a projection, $P : F \rightarrow M$, with $\|P\| = 1$. Define $S|_M : E^* \rightarrow M^*$ by

$$S|_M(\mu^*) = S(\mu^*)|_M = g|_M.$$

Claim 1: $S|_M : E^* \rightarrow M^*$ is a linear isometry.

We first prove that $S|_M$ is linear. For every $\mu^*, \nu^* \in E^*$ and $\alpha, \beta \in \mathbb{R}$, we get

$$|\langle \alpha\mu^* + \beta\nu^*, \mu \rangle - \langle \alpha S(\mu^*) + \beta S(\nu^*), h(\mu) \rangle| \leq (|\alpha| \|\mu^*\| + |\beta| \|\nu^*\|) \gamma \tau_h(\|\mu\|),$$

and

$$|\langle \alpha\mu^* + \beta\nu^*, \mu \rangle - \langle S(\alpha\mu^* + \beta\nu^*), h(\mu) \rangle| \leq (\|\alpha\mu^* + \beta\nu^*\|) \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

Then,

$$|\langle \alpha S(\mu^*) + \beta S(\nu^*) - S(\alpha\mu^* + \beta\nu^*), h(\mu) \rangle| \leq 2(|\alpha| \|\mu^*\| + |\beta| \|\nu^*\|) \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E. \quad (19)$$

By substituting $n\mu$ for μ in (19) and subsequently dividing both sides of the inequality by n , we obtain

$$|\langle \alpha S(\mu^*) + \beta S(\nu^*) - S(\alpha\mu^* + \beta\nu^*), \frac{h(n\mu)}{n} \rangle| \leq \frac{2(|\alpha| \|\mu^*\| + |\beta| \|\nu^*\|) \gamma \tau_h(\|n\mu\|)}{n} \quad \text{for all } \mu \in E.$$

By taking the limit as $n \rightarrow \infty$ in the aforementioned inequality, we derive that

$$|\langle \alpha S(\mu^*) + \beta S(\nu^*) - S(\alpha\mu^* + \beta\nu^*), L\mu \rangle| = 0 \quad \text{for all } \mu \in E.$$

This implies that $(\alpha S(\mu^*) + \beta S(\nu^*))|_M = S(\alpha\mu^* + \beta\nu^*)|_M$. Then, $S|_M$ is linear.

Next, we show that $S|_M$ is norm-preserved. For each $\mu^* \in E^*$, there is a unique $g \in F^*$ which satisfies (18). Then, $\langle \mu^*, \mu \rangle = \langle g, L\mu \rangle$ for all $\mu \in E$. It follows that

$$\|\mu^*\| = \sup_{\mu \in S_E} |\langle \mu^*, \mu \rangle| = \sup_{\mu \in S_E} |\langle g, L\mu \rangle| = \sup_{\mu \in S_E} |\langle g|_M, L\mu \rangle| = \|g|_M\|.$$

Consequently, $S|_M$ is a linear isometry.

Claim 2: $S : E^* \rightarrow F^*$ is a linear isometry.

Since S is norm-preserved, it suffices to prove that S is linear. Given $\mu^* \in E^*$, let $\mu_0 \in \partial\|\mu^*\|$. Then, using (18), we have

$$\langle \mu^*, \mu_0 \rangle = \langle S(\mu^*), L\mu_0 \rangle = \langle S|_M(\mu^*), L\mu_0 \rangle = \langle S|_M(\mu^*)P, L\mu_0 \rangle.$$

This implies that $S(\mu^*) = S|_M(\mu^*)P$ since F is smooth. Thus, S is linear.

Let $T = S^*$; then, $T : F \rightarrow E$ is a surjective linear operator with $\|T\| = 1$ and

$$\begin{aligned} \|Th(\mu) - \mu\| &= \sup_{\mu^* \in S_{E^*}} |\langle \mu^*, \mu \rangle - \langle \mu^*, Th(\mu) \rangle| \\ &= \sup_{\mu^* \in S_{E^*}} |\langle \mu^*, \mu \rangle - \langle \mu^*, S^*h(\mu) \rangle| \\ &= \sup_{\mu^* \in S_{E^*}} |\langle \mu^*, \mu \rangle - \langle S(\mu^*), h(\mu) \rangle| \\ &= \sup_{\mu^* \in S_{E^*}} |\langle \mu^*, \mu \rangle - \langle g, h(\mu) \rangle| \\ &\leq \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E. \end{aligned}$$

□

Since for a Hilbert space, H , all of the closed subspace of H is 1-complemented in H and H is also uniformly convex and uniformly smooth, we have the following result.

Theorem 6. *Let H be a Hilbert space, and let $h : E \rightarrow H$ be a standard coarse isometry. If h is weakly stable with a constant $\gamma > 0$, then there is a surjective bounded linear operator, $T : H \rightarrow E$, with $\|T\| = 1$ such that*

$$\|Th(\mu) - \mu\| \leq \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

Based on Theorem 5 and Remark 2, we obtain the following stability result, which was demonstrated by Šemrl and Väisälä [16].

Corollary 3 ([16], Theorem 2.4). *Let $(\Omega_1, \Sigma_1, \mu_1)$ and $(\Omega_2, \Sigma_2, \mu_2)$ be two measure spaces and $1 < p < \infty$. Suppose that $E = L_p(\Omega_1, \Sigma_1, \mu_1)$, $F = L_p(\Omega_2, \Sigma_2, \mu_2)$ and that $h : E \rightarrow F$ is a standard ε -isometry. Then, there is a surjective bounded linear operator, $T : F \rightarrow E$, with $\|T\| = 1$ such that*

$$\|Th(\mu) - \mu\| \leq 2\varepsilon \quad \text{for all } \mu \in E.$$

Next, we show the existence of coarse left-inverse operators of non-surjective coarse isometries when X is one of the following three types of Banach spaces: (1) $E = \ell_\infty(\Gamma)$; (2) $E = c_0$; and (3) E is a finite-dimensional Banach space.

Theorem 7. *For any Γ , let $E = \ell_\infty(\Gamma)$ and $h : E \rightarrow F$ be a standard coarse isometry. If h is weakly stable with a constant $\gamma > 0$, then there is a bounded linear operator, $T : F \rightarrow E$, with $\|T\| = 1$ such that*

$$\|Th(\mu) - \mu\| \leq \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

Proof. Given $\alpha \in \Gamma$, let $\hat{e}_\alpha$ be the standard unit vector of X that has 1 as its α term and 0 as all other terms, and let δ_α be defined by

$$\delta_\alpha(\mu) = \mu(\alpha), \quad \mu = (\mu(\alpha))_{\alpha \in \Gamma} \in E.$$

Clearly, $\delta_\alpha \in S_{E^*}$. Since h is weakly stable with a constant $\gamma > 0$, there exists $g_\alpha \in S_{F^*}$ such that

$$|\langle \delta_\alpha, \mu \rangle - \langle g_\alpha, h(\mu) \rangle| \leq \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

Let $T : F \rightarrow E$ be defined by

$$Tf = (\langle g_\alpha, f \rangle \hat{e}_\alpha)_{\alpha \in \Gamma}.$$

Then, $\|T\| = 1$ and

$$\|Th(\mu) - \mu\| = \sup_{\alpha \in \Gamma} |\langle \delta_\alpha, \mu \rangle - \langle g_\alpha, h(\mu) \rangle| \leq \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

□

Theorem 8. *Let F be a separable Banach space, and let $h : c_0 \rightarrow F$ be a standard coarse isometry. If h is weakly stable with a constant $\gamma > 0$, then there is a bounded linear operator, $T : F \rightarrow c_0$, with $\|T\| \leq 2$ such that*

$$\|Th(\mu) - \mu\| \leq \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in c_0.$$

Proof. Let us consider the sequence $\{e_n\}_{n \in \mathbb{N}}$, which represents the canonical basis of the space c_0 . Correspondingly, we denote the standard biorthogonal functionals by $\{e_n^*\}_{n \in \mathbb{N}}$,

which are elements of the space ℓ_1 . According to the weak stability of h with $\gamma > 0$, for each $n \in \mathbb{N}$, we can find $g_n \in S_{F^*}$ such that

$$|\langle e_n^*, \mu \rangle - \langle g_n, h(\mu) \rangle| \leq \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in c_0. \quad (20)$$

Define

$$K = \{\varphi \in B_{F^*} : |\langle \varphi, h(\mu) \rangle| \leq \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in c_0\}.$$

Then, K is a non-empty w^* -compact subset of F^* according to the Banach–Alaoglu theorem. Since F is separable, (B_{F^*}, w^*) is metrizable. Let d be a metric such that (B_{F^*}, d) is isomorphic to (B_{F^*}, w^*) . Since (B_{F^*}, d) is a compact metric space, $\{\varphi_n\} \subset B_{F^*}$ has at least one sequential d cluster point. By $e_n^* \xrightarrow{w^*} 0$, (20) implies that every d cluster point of $\{g_n\}$ is in K . This further entails that

$$\lim_{n \rightarrow \infty} d(g_n, K) = 0.$$

Thus, there is a sequence, $\{\varphi_n\} \subset K$, so that $d(g_n, \varphi_n) \rightarrow 0$ as $n \rightarrow \infty$, or equivalently, $g_n - \varphi_n \xrightarrow{w^*} 0$. This implies that $\langle g_n - \varphi_n, \nu \rangle \rightarrow 0$ as $n \rightarrow \infty$ for all $\nu \in F$.

Define $T : F \rightarrow c_0$ by

$$T\nu = \sum_{k=1}^{\infty} \langle g_k - \varphi_k, \nu \rangle e_k \in c_0.$$

For each $m \in \mathbb{N}$,

$$\begin{aligned} & \left\| \sum_{k=1}^m \langle g_k - \varphi_k, h(\mu) \rangle e_k - \sum_{k=1}^m \langle e_k^*, \mu \rangle e_k \right\| \\ &= \left\| \sum_{k=1}^m (\langle g_k, h(\mu) \rangle - \langle e_k^*, \mu \rangle) e_k - \sum_{k=1}^m \langle \varphi_k, h(\mu) \rangle e_k \right\| \\ &\leq \left\| \sum_{k=1}^m (\langle g_k, h(\mu) \rangle - \langle e_k^*, \mu \rangle) e_k \right\| + \left\| \sum_{k=1}^m \langle \varphi_k, h(\mu) \rangle e_k \right\| \\ &= \sup_{1 \leq k \leq m} |\langle \varphi_k, h(\mu) \rangle - \langle e_k^*, \mu \rangle| + \sup_{1 \leq k \leq m} |\langle \varphi_k, h(\mu) \rangle| \\ &\leq \gamma \tau_h(\|\mu\|) + \gamma \tau_h(\|\mu\|) \\ &= 2\gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in c_0. \end{aligned}$$

Therefore, $\|T\| = \sup_{\nu \in S_F} \sup_{k \in \mathbb{N}} |\langle g_k - \varphi_k, \nu \rangle| \leq \sup_{k \in \mathbb{N}} \|g_k - \varphi_k\| \leq 2$ and

$$\begin{aligned} \|Th(\mu) - \mu\| &= \lim_{m \rightarrow \infty} \left\| \sum_{k=1}^m \langle g_k - \varphi_k, h(\mu) \rangle e_k - \sum_{k=1}^m \langle e_k^*, \mu \rangle e_k \right\| \\ &\leq 2\gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in c_0. \end{aligned}$$

□

Theorem 9. Let E be a finite-dimensional Banach space with $\dim E = n$, and let $h : E \rightarrow F$ be a standard coarse isometry. If h is weakly stable with a constant $\gamma > 0$, then there is a bounded linear operator, $T : F \rightarrow E$, with $\|T\| \leq n$ such that

$$\|Th(\mu) - \mu\| \leq n\gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

Proof. Since E is a finite-dimensional space, according to Auerbach's theorem, there exist n vectors of $\{e_k\}_{k=1}^n \subset S_E$ and n vectors of $\{e_k^*\}_{k=1}^n \subset S_{E^*}$ so that

$$\langle e_k^*, e_j \rangle = \delta_{kj} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases}$$

According to the weak stability of h with $\gamma > 0$, there are n linear functionals, $\{g_k\}_{k=1}^n \subset S_{F^*}$, such that for every $1 \leq k \leq n$,

$$|\langle e_k^*, \mu \rangle - \langle g_k, h(\mu) \rangle| \leq \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E.$$

Define $T : F \rightarrow E$ by

$$Tf = \sum_{k=1}^n \langle g_k, f \rangle e_k \quad \text{for all } f \in F.$$

Then, $\|T\| \leq n$ and

$$\begin{aligned} \|Th(\mu) - \mu\| &= \left\| \sum_{k=1}^n \langle g_k, h(\mu) \rangle e_k - \sum_{k=1}^n \langle e_k^*, \mu \rangle e_k \right\| \\ &\leq \sum_{k=1}^n |\langle e_k^*, \mu \rangle - \langle g_k, h(\mu) \rangle| \\ &\leq n \gamma \tau_h(\|\mu\|) \quad \text{for all } \mu \in E. \end{aligned}$$

□

5. Conclusions

Lindenstrauss and Szankowski [25] observed that a coarse isometry between infinite-dimensional uniformly convex spaces cannot be transformed into a linear isometry without imposing further conditions. Consequently, in the investigation of the stability of coarse isometries, it is customary to introduce an additional assumption. This paper explores the stability of coarse isometries under the assumption of the weak stability condition.

In terms of potential research avenues, we propose the following inquiries:

1. What specific category of coarse isometries satisfies the weak stability condition?
2. Given a coarse isometry, $h : E \rightarrow F$, are there other assumptions that ensure the existence of a linear isometry from E to F ?

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