axioms

MDPI

Article

Conformal Solutions of Static Plane Symmetric Cosmological Models in Cases of a Perfect Fluid and a Cosmic String Cloud

Ragab M. Gad [1], Awatif Al-Jedani [2,*] and Shahad T. Alsulami [2]

[1] Mathematics Department, Faculty of Science, Minia University, El-Minia 61915, Egypt; ragab.ali@mu.edu.eg
[2] Department of Mathematics and Statistics, College of Science, University of Jeddah, Jeddah 23890, Saudi Arabia; salsulami0481.stu@uj.edu.sa
* Correspondence: amaljedani@uj.edu.sa

Abstract: In this work, we obtained exact solutions of Einstein's field equations for plane symmetric cosmological models by assuming that they admit conformal motion. The space-time geometry of these solutions is found to be nonsingular, non-vacuum and conformally flat. We have shown that in the case of a perfect fluid, these solutions have an energy-momentum tensor possessing dark energy with negative pressure and the energy equation of state is $\rho + p = 0$. We have shown that a fluid has acceleration, rotation, shear-free, vanishing expansion, and rotation. In the case of a cosmic string cloud, we found that the tension density and particle density decrease as the fluid moves along the direction of the strings, then vanish at infinity. We shown that the exact conformal solution for a static plane symmetric model reduces to the well-known anti-De Sitter space-time. We obtained that the space-time under consideration admits a conformal vector field orthogonal to the 4-velocity vector and does not admits a vector parallel to the 4-velocity vector. Some physical and kinematic properties of the resulting models are also discussed.

Keywords: conformal symmetries; Einstein's field equations; perfect fluid; cosmic strings cloud; dark energy

MSC: 83C05; 83C15

check for updates

Academic Editor: Irina Radinschi

Received: 14 November 2024
Revised: 17 January 2025
Accepted: 27 January 2025
Published: 2 February 2025

Citation: Gad, R.M.; Al-Jedani, A.; Alsulami, S.T. Conformal Solutions of Static Plane Symmetric Cosmological Models in Cases of a Perfect Fluid and a Cosmic String Cloud. *Axioms* **2025**, *14*, 117. https://doi.org/10.3390/axioms14020117

Copyright: © 2025 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

General relativity, a gravitational field theory, is described from the viewpoint of geometry and physics by Einstein's field equations, which are highly nonlinear. Because of this nonlinearity, it becomes very difficult to solve these equations unless we assume certain constraints, such as symmetries, on the space-time metric. However, finding exact solutions to such equations and their physical interpretations is sometimes more difficult. Despite these difficulties, there are many exact solutions to these equations. In addition to the exact solution, there are also non-exact solutions that describe certain physical systems.

One of the most successful ways of finding exact solutions to Einstein's field equations has been to consider that the space-time under study admits one of symmetries. Symmetries also provide us with deeper insights into the properties of space-time. Besides the interest of these symmetries from the geometric and physical aspects of space-time, they play an important role in simplifying Einstein's field equations and providing a classification of space-time according to the structure of the corresponding Lie algebra.

In the theory of general relativity and its equivalent theories, symmetries are studied on the basis of the geometry corresponding to each theory. In the context of Riemannian geometry, different types of symmetries such as isometric, homothetic motion, conformal

motion, matter collineations, and Ricci collineations . . . etc, have been extensively studied in the theory of general relativity.

Of course, the most studied type of these symmetries in general relativity is the Killing symmetry, and many examples and uses of it are known [1]. Killing symmetry is a special case of homothetic symmetry whose generalization is conformal symmetry. Duggal and Sharma [2] presented the characterizations and classifications of the space-times of general relativity admitting Killing, homothetic and conformal symmetries. A more detailed discussion of the different types of homothetic symmetry can be found in [3–14].

In a series of papers [15–19] Gad, Alofi and Al Mazrooei studied the homothetic symmetry using Lyra's geometry. In this case, space-times were classified according to admit of such symmetry. It turns out that in the case of a zero displacement vector field, the results obtained in the context of Lyra's geometry agree with those obtained previously in the theory of general relativity, using Riemannian's geometry. While in the case of a constant displacement vector field, it is not possible to compare the results obtained in the context of Lyra's geometry with those obtained in general relativity, using Riemannian's geometry. This showed that in Lyra's geometry, if the displacement vector field is taken to be constant, this does not give meaningful results. Killing and homothetic symmetries have also been studied in the theory of teleparallel gravity, using Weitzenböck's geometry [20–22]. In the context of Finsler's geometry, Sanjay et al. [23] investigated the charged gravastars with conformal motion. They examined charged gravitationally vacuum stars under the background of Finslerian gravity with the use of the conformal Killing vector and have considered charged stellar objects with three different regions and distinct equation of state parameters to analyze the structure of such objects.

In this work, our focus will be on studying the conformal symmetry of a static plane symmetric cosmological model and finding exact solutions to the Einstein's field equations without assuming any restrictions either on the variables or on the physical properties of a given space-time, as is common in the literature. We will only assume that the model under study admits a conformal motion.

One of the important properties of conformal symmetries is they preserve the causal character of space-time. That is, if there is a conformal vector field in a space-time which, if the metric is dragged along it, the causal structure of space-time remains constant. One small drawback of these symmetries lies in the fact that, unlike isometry and homothetic symmetries, do not leave the Einstein tensor constant, and in this respect, they can be considered non-natural or accidental. However, although this drawback some solutions with conformal symmetries are known.

Many researchers have studied spherically symmetric perfect-fluid space-times admitting a conformal Killing vector field. Gad [24] studied these solutions and derived a different coordinate representation of the solutions obtained in [25,26]. Exact solutions of Einstein's field equations are found in the case when the conformal Killing vector field is parallel to the 4-velocity vector field u^a. Recently, non-static of these space-times have studied in [27]. Several families of exact analytical solutions are found for different choices of the conformal vector field in both the dissipative and the adiabatic regime. Recent and old literature also provides some important results using conformal Killing vector field (see for example the references [28–34]).

The contents of the paper are organized as follows: In the next Section, the physical and geometric parameters for a static plane symmetric space-time are given. We solve conformal equations for this space-time and get the conformal Killing vector, conformal factor and the relation between the coefficients of the metric. The results obtained must be satisfied Einstein's field equations, and this is what we will do in the two subsequent sections of Section 2. In Section 3, we consider the matter represented by a perfect fluid.

In Section 4, we consider the case of a cosmic strings could. In Section 5, we discus that the conformal vector field orthogonal to the 4-velocity vector and does not admits a vector parallel to the 4-velocity vector. In Section 5, we study a vector field that is orthogonal to the 4-velocity vector and a vector that is parallel to the 4-velocity vector. We discuss which vector fields are allowed by the space-time under study and which are not. In Section 6, some physical and kinematic properties of the resulting models are also discussed. Finally, in Section 7, concluding remarks are given.

2. Version of Model and Conformal Vector Field

Let M be a four-dimensional, Hausdorff, smooth manifold with a non-degenerate metric tensor g with signature $(+,-,-,-)$.

A vector field ζ on a space-time (M,g) is said to be conformal vector if the following is satisfied

$$\pounds_\zeta g_{ab} = \zeta_{a;b} + \zeta_{b;a} = 2\psi(t,x,y,z)g_{ab} \qquad \Leftrightarrow \qquad \zeta_{a;b} = \psi g_{ab} + F_{ab}, \tag{1}$$

where the conformal factor $\psi = \psi(x^a)$ is a scalar function, $\pounds_\zeta$ denotes a Lie derivative operator relative to ζ and semi-colon (;) denotes a covariant derivative w. r. t. the metric connection. If $\psi_{,a} = 0$, that is, ψ is constant on M, then the conformal vector field ζ is called homothetic (proper homothetic vector field if $\psi = const. \neq 0$ on M). If $\psi = 0$ on M, ζ is called a Killing vector field. In components form, the first equation in (1) takes the following form

$$g_{ab,c}\zeta^c + g_{ac}\zeta^c_{,b} + g_{cb}\zeta^c_{,a} = 2\psi g_{ab}. \tag{2}$$

A general plane symmetric space-time can always be written in the following form

$$ds^2 = e^{2A(t,x)}dt^2 - e^{2C(t,x)}dx^2 - e^{2B(x)}(dy^2 + dz^2)$$

For a static plane symmetric cosmological models, the coefficients of the metric will be independent of time t. In this case x can be redefined to get rid of the coefficient of dx^2, in the above equation, which now reduces to the standard representation and is given by [1]

$$ds^2 = e^{2A(x)}dt^2 - dx^2 - e^{2B(x)}(dy^2 + dz^2), \tag{3}$$

with the convention $x^0 = t$ (cosmic time), $x^1 = x$, $x^2 = y$ and $x^3 = z$ and the scale factors $A(x)$ and $B(x)$ are functions of x only.

As shown in [1], the above space-time (3) admits four independent Killing vector fields which are as follows:

$$\frac{\partial}{\partial t}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z},$$

$$y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}.$$

The physical and geometric parameters of the space-time (3) are determined by the following [35]:

1. The only non-vanishing component of the 4-acceleration vector, $\dot{u}_a = u_{a;b}u^b$, is

$$\dot{u}_1 = -A', \tag{4}$$

2. The only non-vanishing components of the rotation, $\omega_{ab} = u_{[a;b]} + \dot{u}_{[a}u_{b]}$, are

$$\omega_{01} = -\omega_{10} = A'e^A, \tag{5}$$

where prime denotes to the derivative w. r. t. x.

3. The expansion scalar
$$\Theta = u^a_{;a} = 0, \tag{6}$$
4. The shear scalar
$$\sigma^2 = \sigma_{ab}\sigma^{ab},$$
where
$$\sigma_{ab} = u_{(a;b)} + \dot{u}_{(a}u_{b)} - \frac{1}{3}\Theta(g_{ab} + u_a u_b).$$
For the space-time (3), we get
$$\sigma_{00} = \sigma_{11} = \sigma_{22} = \sigma_{33} = 0, \quad \sigma^2 = 0. \tag{7}$$
That is, the space-time (3) is shear free.

Study of conformal vector fields, $\zeta = \zeta^a(t,x,y,z)^3_{a=0}$, on a static plane symmetric model (3) is based on solving the ten reduced equations (due to the symmetry of the metric g_{ab}) obtained from the first equation of (1). For the space-time (3), the corresponding conformal equations are given by the following system of equations:
$$\zeta^0_{,0} + A'\zeta^1 = \psi(x), \tag{8}$$
$$e^{2A}\zeta^0_{,1} - \zeta^1_{,0} = 0, \tag{9}$$
$$e^{2A}\zeta^0_{,2} - e^{2B}\zeta^2_{,0} = 0, \tag{10}$$
$$e^{2A}\zeta^0_{,3} - e^{2B}\zeta^3_{,0} = 0, \tag{11}$$
$$\zeta^1_{,1} = \psi(x), \tag{12}$$
$$e^{2B}\zeta^2_{,1} + \zeta^1_{,2} = 0, \tag{13}$$
$$e^{2B}\zeta^3_{,1} + \zeta^1_{,3} = 0, \tag{14}$$
$$\zeta^2_{,2} + B'\zeta^1 = \psi(x), \tag{15}$$
$$\zeta^2_{,3} + \zeta^3_{,2} = 0, \tag{16}$$
$$\zeta^3_{,3} + B'\zeta^1 = \psi(x). \tag{17}$$
where the commas denote partial derivatives w. r. t. the coordinate indicated.

Integrating (12) w. r. t. x, we get
$$\zeta^1 = \int \psi(x)dx + F^1(t,y,z). \tag{18}$$
Using this result back into Equations (8)–(17), taking into account that $\psi = \psi(x)$, and after some algebraic calculations, the above system of equations gives the following components of the conformal vector field and constraint relation
$$\begin{aligned} \zeta^0 &= const. = c_0 \\ \zeta^1 &= \frac{\psi}{B'}, \\ \zeta^2 &= const. = c_2, \\ \zeta^3 &= const. = c_3, \\ F^1(t,y,z) &= const. = c_1 \end{aligned} \tag{19}$$

$$A(x) = B(x). \tag{20}$$

where $c_a, a = 0, 1, 2, 3$ are constants of integration.

Inserting the above results back into Equation (18) and integrating the obtained results, we get

$$\psi = A'e^A = B'e^B. \tag{21}$$

After the previous discussion, the following result can be established.

Theorem 1. *A plane symmetric space-time described by the metric ansatz (3) admits a conformal Killing vector if the following conditions are satisfied*

$$A = B$$

$$\psi = A'e^A.$$

The conformal Killing vector corresponding to this case takes the following form

$$\zeta = c_0\frac{\partial}{\partial t} + e^A\frac{\partial}{\partial x} + c_2\frac{\partial}{\partial y} + c_3\frac{\partial}{\partial z}. \tag{22}$$

According to the previous theory, in order to a static plane symmetric model (3) admits conformal motion, it must be the well-known anti-De Sitter space-time, which takes the usual form

$$ds^2 = e^{2A(x)}(dt^2 - dy^2 - dz^2) - dx^2. \tag{23}$$

To find the unknown coefficients of the metric (3), we need to solve the Einstein's field equations. This will be done in the next section.

The covariant components, $\zeta_a = g_{ab}\zeta^b$, of the conformal vector are

$$\begin{aligned} \zeta_0 &= c_0e^{2A}, \\ \zeta_1 &= -e^A, \\ \zeta_2 &= -c_2e^{2B}, \\ \zeta_3 &= -c_3e^{2B}. \end{aligned} \tag{24}$$

It is clear from Equations (22) and (24) that the obtained conformal vector is non-null conformal vector field, where $\zeta^a\zeta_a \neq 0$.

3. Solutions of Einstein's Field Equations for a Perfect Fluid

In this section, we will assume that the space-time under study admits a conformal vector field (conformal motion) and then solve the Einstein's field equations by considering that the matter in this space-time is represented by a perfect fluid. In general, the Einstein's field equations are of the following form:

$$R_{ab} - \frac{1}{2}Rg_{ab} = \kappa T_{ab}, \tag{25}$$

where R_{ab} is the Ricci tensor, R the Ricci scalar and T_{ab} the stress energy tensor, which describes the matter field in the space-time. In Equation (25), κ is the coupling constant defined by $\kappa = \frac{8\pi G}{c^4}$, where G is a Newtons gravitational constant and c the speed of light in vacuum (For convenience, we assumed that natural units $c = 8\pi G = 1$). In the case of a perfect fluid the energy momentum tensor, T_{ab}, is

$$T_{ab} = (\rho + p)u_au_b - pg_{ab}, \tag{26}$$

where p is the pressure, ρ the energy density and u_a the 4-velocity vector. The contravariant and covariant components of the 4-velocity vector, for the space-time (3), can be defined by $u^a = (e^{-A}, 0, 0, 0), u_a = (e^A, 0, 0, 0)$, and they are verified $g_{ab}u^a u^b = 1$.

For the line element (3), Equations (25) and (26) give the following system of equations

$$3B'^2 + 2B'' = \rho \tag{27}$$

$$B'^2 + 2A'B' = -p \tag{28}$$

$$B'^2 + B'' + A'B' + A'^2 + A'' = -p \tag{29}$$

From Equations (28) and (29), we have

$$A'' + B'' - A'B' + A'^2 = 0. \tag{30}$$

Using the constraint relation given in Theorem 1, $A = B$, in the above equation, we obtain

$$A'' = B'' = 0. \tag{31}$$

Integrating this equation, we get

$$A = B = ax + b, \tag{32}$$

where a and b are constants of integration.

As a result, the exact conformal solution of the Einstein's field equations for a static plane symmetric space-time (3) is given by the following form

$$ds^2 = e^{2(ax+b)}dt^2 - dx^2 - e^{2(ax+b)}(dy^2 + dz^2), \tag{33}$$

and p and ρ (the physical variables) are

$$\rho = -p = 3a^2. \tag{34}$$

Now, we can conclude that in the case of a perfect fluid, the assumption of conformal symmetry reproduces the well-known static plane symmetric solutions (3) to give the anti-De Sitter in the following form

$$ds^2 = e^{2(ax+b)}(dt^2 - dy^2 - dz^2) - dx^2. \tag{35}$$

4. Field Equations and Their Solutions for a Cosmic Strings Cloud

In this section, we will study the gravitational effects for the space-time (3) of a cosmic strings cloud. To do this study we consider the Einstein's field Equation (25) in a mixed form and the stress tensor T^a_b takes the following form

$$T^a_b = \mu u^a u_b - \lambda X^a X_b, \tag{36}$$

where μ and λ are a rest energy density and a string cloud tension density for a string cloud with particles attached to it.

Here, u^a is a 4-velocity vector of particles, as defined before, and $\mathbf{X}$ a unit space-like vector representing the direction of strings orthogonal to a 4-velocity vector. the vector $\mathbf{X}$ must be taken along any of the three directions $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$. For the space-time under

consideration, we choose **X** to be parallel to $\frac{\partial}{\partial x}$, so that $X^0 = X^2 = X^3 = 0$, and $X^1 \neq 0$. The components of the vectors **u** and **X** satisfy the following conditions

$$u_a u^a = -X_a X^a = 1, \quad u_a X^a = 0. \tag{37}$$

For the space-time (3), in a comoving coordinate system, we get

$$X^a = (0, 1, 0, 0),$$

$$X_a = (0, -1, 0, 0).$$

If we define the particle density of the configuration by μ_p, then the relation between a rest energy density μ and a string cloud tension density λ is given by

$$\mu = \mu_p + \lambda. \tag{38}$$

Using Equations (36)–(38) in Einstein's field Equation (25), then for a plane symmetric space-time (3), we have the following equations

$$B'^2 + 2A'B' = \lambda, \tag{39}$$

$$B'^2 + B'' + A'B' + A'^2 + A'' = 0, \tag{40}$$

$$3B'^2 + 2B'' = \mu. \tag{41}$$

Before solving the above Einstein's field equations, we note that if we assumed that the direction of the strings is parallel to $\frac{\partial}{\partial y}$ (or $\frac{\partial}{\partial z}$), the left-hand side of the Equation (39) equals zero, therefore either $B = const.$ or $B' + 2A' = 0$. According to Theorem 1 ($A = B$), the later gives $A = B = const$. Therefore, the space-time under consideration (3) becomes flat. So the direction of the fluid is taken to be in the direction of $\frac{\partial}{\partial x}$.

Since the scale factors A and B appearing in the left hand-sides of Equations (39)–(41) are functions of x only, then λ and μ must be functions of x only.

As we previously indicated in the introduction, we will consider that the space-time under study admits a conformal motion. Considering the current case and using Theorem 1, the previous Equations (39)–(41) reduce to the following equations

$$\lambda = 3B'^2, \tag{42}$$

$$3B'^2 + 2B'' = 0, \tag{43}$$

$$3B'^2 + 2B'' = \mu. \tag{44}$$

From Equations (43) and (44), we get

$$\mu = 0.$$

Integrating Equation (43), using Theorem 1, we obtain

$$B = \ln(\frac{3}{2}x + c_4)^{\frac{2}{3}} + c_5 = A, \tag{45}$$

where c_4 and c_5 are constant of integration.

Inserting Equation (45) into Equation (42), (assuming $c_5 = 0$), we get

$$\lambda = \frac{3}{(\frac{3}{2}x + c_4)^2}. \tag{46}$$

Since $\mu = \mu_p + \lambda$, then

$$\mu_p = -\frac{3}{(\frac{3}{2}x + c_4)^2}. \tag{47}$$

From the Equations (46) and (47) one can see that the tension density and particle density in the strings decrease as the fluid moves along the x-axis and the two densities vanish as $x \to \infty$.

As a result, in case of a cosmic strings cloud, the exact conformal solution of the Einstein's field equations for a static plane symmetric space-time (3) is given by the following form

$$ds^2 = \left(\frac{3}{2}x + c_4\right)^{\frac{4}{3}}(dt^2 - dy^2 - dz^2) - dx^2. \tag{48}$$

As in the case of perfect fluid the metric (3) reduces to the anti-De Sitter metric.

5. Orthogonal and Parallel Conformal Vector Fields

Discussion of conformal vector fields orthogonal or parallel to the 4-speed vector field gives some physical properties of a given space-time. In this section, we study the two cases of the space-time (3).

Conformal Vector Orthogonal to u^a

In this case

$$\zeta^a u_a = 0.$$

From the definition of the 4-velocity vector, we get

$$\zeta^0 = 0.$$

Using these results, Equations (8)–(17) are reduced to the following equations:

$$A'\zeta^1 = \psi(x), \tag{49}$$

$$\zeta^1_{,0} = \zeta^2_{,0} = \zeta^3_{,0} = 0, \tag{50}$$

$$\zeta^1_{,1} = \psi(x), \tag{51}$$

$$e^{2B}\zeta^2_{,1} + \zeta^1_{,2} = 0, \tag{52}$$

$$e^{2B}\zeta^3_{,1} + \zeta^1_{,3} = 0, \tag{53}$$

$$\zeta^2_{,2} + B'\zeta^1 = \psi(x), \tag{54}$$

$$\zeta^2_{,3} + \zeta^3_{,2} = 0, \tag{55}$$

$$\zeta^3_{,3} + B'\zeta^1 = \psi(x). \tag{56}$$

From Equations (49) and (51), we get

$$\zeta^1 = \frac{\psi(x)}{A'}, \qquad \zeta^1 = \int \psi(x)dx + F(y,z), \tag{57}$$

where $f(y,z)$ is an arbitrary function to be determined. Using the results (57) back into the above equations, we get $F(y,z)$= constant, without loss of generality, we take it equal zero. Integrating the result obtained from (57), we get

$$A = \ln(\int \psi(x)dx) + c, \tag{58}$$

where c is a constant of integration. Using (58) in Equations (49)–(56), we have

$$\zeta^1 = \zeta^1(x), \zeta^2 = c_2, \zeta^3 = c_3,$$

where c_2 and c_3 are constants of integration, which will be taken as zeros. We have also

$$A' = B' \quad \Rightarrow \quad A = B. \tag{59}$$

Consequently, the conformal vector field orthogonal to the 4-velocity vector is

$$\zeta_\perp = \frac{\psi(x)}{A'} \frac{\partial}{\partial x}. \tag{60}$$

To verify that the resulting vector is a proper conformal vector, that is, we prove that the conformal factor are function in the coordinate x, we use the above results in Einstein's field equations.

Case I: Perfect fluid case

In this case, using Equations (58) and (59) in Einstein's field Equations (27)–(29), we get the following equations

$$\frac{\psi^2(x)}{(\int \psi(x)dx)^2} + \frac{2\psi'(x)}{\int \psi(x)dx} = \rho \tag{61}$$

$$\frac{3\psi^2(x)}{(\int \psi(x)dx)^2} = -p \tag{62}$$

$$\frac{3\psi^2(x)}{(\int \psi(x)dx)^2} + 2\frac{\psi'(x)\int \psi(x)dx - \psi^2(x)}{(\int \psi(x)dx)^2} = -p \tag{63}$$

From Equations (62) and (63), we have

$$\psi'(x)\int \psi(x)dx - \psi^2(x) = 0. \tag{64}$$

The solution of this equation is

$$\psi(x) = e^{mx}, \tag{65}$$

where m is a constant of integration. Using the above equation in (58), the scale factor A can be written as follows

$$A = ax + b = B. \tag{66}$$

Now the previous discussion can be summarized in the following:

Proposition 1. *All perfect fluid solutions described by the metric ansatz (3) admit a conformal Killing vector field,* $\zeta_\perp = \frac{e^{mx}}{a}\frac{\partial}{\partial x}$*, orthogonal to the 4-velocity vector,* $\mathbf{u}$*, if the scale factors are*

$$A = B = ax + b,$$

and the conformal factor is

$$\psi(x) = e^{mx}.$$

According to the above proposition, the dynamical variables are

$$\rho = -p = 3m^2.$$

Case II: Cosmic string cloud case.

Inserting (58) and (59) into (42)–(44), we get $\mu = 0$ and

$$3A'^2 + 2A'' = 0$$

Using (58) in the above equation and integrating the obtain results, we obtain the conformal factor and scale factors, respectively, as follows

$$\psi(x) = \left(\frac{2q}{3x}\right)^{\frac{1}{3}},$$

$$A = \ln\left(mx^{\frac{2}{3}}\right) + n = B.$$

where q and n are constants of integration and $m = \left(\frac{9q}{4}\right)^{\frac{1}{3}}$.

The previous discussion can be summarized as the following.

Proposition 2. *All cosmic string cloud solutions described by the metric (3) admit a conformal Killing vector field, $\zeta_\perp = \frac{x^2}{4q}\frac{\partial}{\partial x}$, orthogonal to the 4-velocity vector* **u** *if the conformal factor and scale factors are, respectively*

$$\psi(x) = \left(\frac{2q}{3x}\right)^{\frac{1}{3}}, \quad A = B = \ln\left(mx^{\frac{2}{3}}\right) + n.$$

Conformal Vector Parallel to u^a ($\zeta_\parallel$)

In this case

$$\zeta^a \propto u^a,$$

then

$$\zeta^a = \ell u^a,$$

where ℓ is a constant of proportionality. From the definition of u^a for the space-time (3), the above equation gives

$$\zeta^1 = \zeta^2 = \zeta^3 = 0.$$

Then the conformal Equations (8)–(17) reduce to the following

$$\zeta^0_{,0} = \psi(x), \tag{67}$$

$$\zeta^0_{,1} = \zeta^0_{,2} = \zeta^0_{,3} = 0.$$

Then the parallel conformal vector field is

$$\zeta_\parallel = \zeta_\parallel(t).$$

Therefore, Equation (67) gives ψ =constant or $\psi = 0$, that is, $\zeta_\parallel = 0$ or equal constant. So the space-time (3) does not admit a conformal vector field parallel to the 4-velocity vector.

6. Physical Properties

In the previous three sections, we discussed a static plane symmetric space-time (3) and attempted to obtain exact solutions to the Einstein's field Equation (25). To do this, in addition to considering the space-time under study admits conformal symmetry, we assumed that the matter is represented by a perfect fluid, as in Section 3, and solved the field equations for a cosmic string cloud in Section 4. To discuss the physical behavior of the obtained conformal solutions given by the metrics (33) and (48), we need to find the

following physical and kinematical parameters of the model which are very important to give us a deeper insight into the properties of the cosmology.

For the metric (33), we find the following parameters:

1. The non-vanishing component of the 4-acceleration is

$$\dot{u}_1 = const. = -a$$

2. The non-vanishing component of the rotation is

$$\omega_{01} = -\omega_{10} = ae^{ax+b}.$$

For the metric (48), the above components are

$$\dot{u}_1 = -\frac{1}{2\left(\frac{3}{2}x + c_4\right)},$$

$$\omega_{01} = -\omega_{10} = \frac{1}{2\left(\frac{3}{2}x + c_4\right)^{\frac{2}{3}}}.$$

7. Discussion and Conclusions

One of the most common attempts to obtain exact solutions to Einstein's field equations is to assume symmetries in space-time. These symmetries are defined by operating the Lie derivative of the considered tensor, such as, $g_{ab}, \Gamma^a_{bc}, T_{ab}, R_{ab}$, ... etc, with respect to space, time, or null vector. The resulting geometric objects created by these operators are tensors with the same index or zero.

This work is devoted to studying one of these symmetries, in particular conformal symmetry, of a plane static symmetric model in the framework of general relativity. We focused on this type of symmetries because a space-time admitting it preserves the causal character of space-time, so it is in an important physical form one. For a static plane symmetric space-time, we solved the conformal equations and obtained the conformal vector field that the space-time admits. Furthermore, solving these equations helped us to obtain a relationships between the metric coefficients. We have used these relationship to simplifying Einstein's field equations and got the energy density (ρ) and pressure (p) (dynamical variables), which depend on the coordinate x. We obtained new exact solutions of the Einstein's field equations for a static plane symmetric space-times by considering that they admit conformal symmetry. In the case of a perfect fluid, the resulting solutions have negative pressure, which represents a possible example of a dark energy star, and the energy equation of state is $\rho + p = 0$. Moreover, we have shown that these solutions reduce to the well-known anti-De Sitter space-times, when the energy-momentum tensor is represented by a perfect fluid or cosmic strings cloud. In the case of a cosmic string cloud, we found that the tension density and particle density decrease as the fluid moves along the direction of the strings, and then vanish at infinity.

For the solutions obtained, all coefficients of the metric are well defined so there is no singularity present. They have acceleration, rotation, shear-free, vanishing expansion, and rotation. We have discussed the orthogonal and parallel conformal vector fields and obtained that the space-time under consideration admits a conformal vector field orthogonal to the 4-velocity vector but does not admits a vector parallel to the 4-velocity vector.

Author Contributions: Methodology, R.M.G. and A.A.-J.; writing-original draft preparation, R.M.G. and S.T.A.; writing-review and editing, A.A.-J. and S.T.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by the University of Jeddah, grant number (UJ-24-FR-2299-1).

Data Availability Statement: Our manuscript has no associated data.

Acknowledgments: This work was funded by the University of Jeddah, Jeddah, Saudi Arabia, under grant No. (UJ-24-FR-2299-1). Therefore, the authors thank the University of Jeddah for its technical and financial support.

Conflicts of Interest: The authors declare that they have no conflicts of interest.

References

1. Stephani, H.; Kramer, D.; MacCallum, M.A.H.; Hoenselears, C.; Herlt, E. *Exact Solutions of Einstein's Field Equations*; Cambridge University Press: Cambridge, UK, 2003.
2. Duggal, K.L.; Sharma, R. *Symmetries of Spacetimes and Riemannian Manifolds*; Kluwer Academic Publishers: Dordrecht, The Netherlands, 1999; p. 487.
3. Henriksen, R.N.; Wesson, P.S. Self-similar space-times I: Three Solutions. *Astrophys. Space Sci.* **1978**, *53*, 429. [CrossRef]
4. Cahill, M.E.; Taub, A.H. Spherically symmetric similarity solutions of the Einstein field equations for a perfect fluid. *Commun. Math. Phys.* **1971**, *21*, 1. [CrossRef]
5. Eardley, D.M. Self-similar spacetimes: Geometry and dynamics. *Commun. Math. Phys.* **1974**, *37*, 287. [CrossRef]
6. Carter, B.; Henriksen, R.N. A systematic approach to self-similarity in Newtonian space-time. *J. Math. Phys.* **1991**, *32*, 2580. [CrossRef]
7. Wainwright, J. Asymptotic Self-similarity Breaking in Cosmology. *Gen. Relativ. Gravit.* **2000**, *32*, 1041. [CrossRef]
8. Wesson, P.S. An exact solution to Einsteins equations with a stiff equation of state. *J. Math. Phys.* **1978**, *19*, 2283. [CrossRef]
9. Collins, M.E.; Lang, J.M. A class of self-similar perfect-fluid spacetimes, and a generalisation. *Class. Quantum Gravity* **1987**, *4*, 61. [CrossRef]
10. Gad, R.M.; Hassan, M.M. On the Geometrical and Physical Properties of Spherically Symmetric Non-Static Space-Times: Self-Similarity. *IL Nuovo C.* **2003**, *118B*, 759. [CrossRef]
11. Gad, R.M.; Al-Jedani, A. Self-Similar Solutions of a Bianchi Type-III Model with a Perfect Fluid and Cosmic String Cloud in Riemannian Geometry. *Symmetry* **2023**, *15*, 1703. [CrossRef]
12. Gad, R.M. On spherically symmetric non-static space-times admitting homothetic motions. *Il Nuovo C.* **2009**, *124B*, 61. [CrossRef]
13. Carr, B.J.; Coley, A.A. Self-similarity in general relativity. *Class. Quantum Gravity* **1999**, *16*, R31. [CrossRef]
14. Carr, B.J.; Coley, A.A. The Similarity Hypothesis in General Relativity. *Gen. Relativ. Gravit.* **2005**, *37*, 2165. [CrossRef]
15. Gad, R.M.; Alofi, A.S. Homothetic vector field in plane symmetric Bianchi type-I cosmological model in Lyra geometry. *Mod. Phys. Lett. A* **2014**, *22*, 1450116. [CrossRef]
16. Gad, R.M. Homothetic Motion in a Bianchi Type-I Model in Lyra Geometry. *Int. J. Theor. Phys.* **2015**, *54*, 2932. [CrossRef]
17. Alofi, A.S.; Gad, R.M. Homothetic vector fields in a specially homogenous Bianchi type-I cosmological model in Lyra geometry. *Can. J. Phys.* **2015**, *93*, 1397. [CrossRef]
18. Gad, R.M.; Al Mazrooei, A.E. On axially symmetric space-times admitting homothetic vector fields in Lyra's geometry. *Can. J. Phys.* **2016**, *94*, 1148. [CrossRef]
19. Gad, R.M.; Alkhateeb, S.A.; Alharbi, H.D. Self-Similar Solutions of the Kantowski-Sachs Model with a Perfect Fluid in General Relativity. *J. App. Math. Phys.* **2021**, *9*, 3165. [CrossRef]
20. Sharif, M.; Majeed, B. Teleparallel Killing Vectors of Spherically Symmetric Spacetimes. *Commun. Theor. Phys.* **2009**, *52*, 435. [CrossRef]
21. Shabbir, G.; Ali, A.; Khan, S. A note on teleparallel Killing vector fields in Bianchi type VIII and IX spacetimes in teleparallel theory of gravitation. *Chin. Phys. B* **2011**, *20*, 070401. [CrossRef]
22. Sharif, M.; Amir, M.J. Teleparallel Killing Vectors of the Einstein Universe. *Mod. Phys. Lett. A* **2008**, *23*, 963. [CrossRef]
23. Sanjay, T.; Narasimhamurthy, S.K.; Nekouee, Z.; Manjunatha, H.M. Charged Gravastars with Conformal Motion in the Finslerian Space-Time. *Eur. Phys. J. C* **2024**, *84*, 393. [CrossRef]
24. Gad, R.M. On spherically symmetric perfect-fluid solutions admitting conformal motions. *Il Nuovo C.* **2002**, *117B*, 533. Available online: https://www.sif.it/riviste/sif/ncb/econtents/2002/117/05 (accessed on 26 January 2025).
25. Herrera, L.; Ponce de León, J. Perfect Fluid Spheres Admitting a One-Parameter Group of Conformal Motions. *J. Math. Phys.* **1985**, *26*, 778–784. [CrossRef]
26. Kitamura, S. On spherically symmetric perfect fluid solutions with shear. *Class. Quantum Gravity* **1994**, *11*, 195. [CrossRef]
27. Herrera, L.; Di Prisco, A.; Ospino, J. Non-Static Fluid Spheres Admitting a Conformal Killing Vector: Exact Solutions. *Universe* **2022**, *8*, 296. [CrossRef]
28. Saifullah, K.; Yazdan, S.E. Conformal motions in plane symmetric static spacetimes. *Int. J. Mod.* **2009**, *D18*, 71. [CrossRef]

29. Duggala, K.L.; Sharmab, R. Conformal killing vector fields on spacetime solutions of Einsteins equations and initial data. *Nonlinear Anal.* **2005**, *63*, 447. [CrossRef]
30. de Cesare, M.; Moffat, J.W.; Sakellariadou1, M. Local conformal symmetry in non-Riemannian geometry and the origin of physical scales. *Eur. Phys. J. C* **2017**, *77*, 605. [CrossRef] [PubMed]
31. Keane, A.J.; Tupper, B.O.J. Conformal symmetry classes for pp-wave spacetimes. *Class. Quantum Gravity* **2004**, *21*, 2037. [CrossRef]
32. Kühnel, W.; Rademacher, H.B. Conformal Geometry of Gravitational Plane Waves. *Geom. Dedicata* **2004**, *109*, 175. [CrossRef]
33. Camci, U.; Saifullah, K. Conformal Symmetries of the Energy Momentum Tensor of Spherically Symmetric Static Spacetimes. *Symmetry* **2022**, *14*, 647. [CrossRef]
34. Khan, S.; Hussain, T.; Bokhari, A.H.; Khan, G.A. Conformal killing vectors of plane symmetric four dimensional lorentzian manifolds. *Eur. Phys. J. C* **2015**, *75*, 523. [CrossRef]
35. Raychaudhuri, A.K. *Theoretical Cosmology*; Clarendon: Oxford, UK, 1979.

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.