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On Polynomials Associated with Finite Topologies

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Abstract: Let τ be a topology on the finite set X_n. We consider the open-set polynomial associated with the topology τ. Its coefficients are the cardinalities of sets $U_j = U_j(\tau)$ of open sets of size $j = 0, \ldots, n$. We prove that this polynomial has only real zeros only in the trivial case where τ is the discrete topology. Hence, we answer a question raised by J. Brown. We give a partial answer to the question: for which topology is this polynomial log-concave, or at least unimodal? More specifically, we prove that if the topology has a large number of open sets, its open polynomial is unimodal. The idea of degree of log-concavity is introduced and it is shown to be limited for polynomials of non-trivial topologies. Furthermore, the maximum-sized topologies that omit open sets of given sizes are derived. Moreover, all topologies over n points with at least $(3/8)2^n$ open sets are proved to be unimodal, completing previous results.

Keywords: finite topology; log-concave; polynomial; unimodal; zeros

MSC: 05A15; 05C31; 11B65; 11B83; 12D10

1. Introduction

Let $X = X_n$ be an n–element set. A topology $\tau \subseteq \mathcal{P}(X)$ on X is a family of subsets (called open sets) that contains X, ϕ, and is closed under union and finite intersection. A basis for τ is subset of τ such that each open set is a union of members of the basis. Let $T(n)$ be the total number of topologies one can define on X. Little is known about this sequence; in fact, $T(n)$ is known just for $n \leq 18$. For the history of the subject, see [1].

Finite topologies are associated with digraphs [2]; in fact, there is a one-to-one correspondence between transitive digraphs and finite topologies, this fact was proved by Evans et al . Krishnamurty see ([1]) expressed this fact using 0–1 matrices, and deduced the upper bound $T(n) \leq 2^{n(n-1)}$.

Another important topic linked to finite topologies is that of preorders (or quasiorders) which are reflexive and transitive relations. The one-to-one correspondence between topologies and quasiorders was already observed in the 30's of the last century by Alexandroff [3] and Birkhoff [4]. A proof of this fact may be found in [2,5]. For the sake of completeness, here is the explicit correspondence: Let τ be a topology on a finite set X. Define the relation on X as follows: We say $x \leq y$ whenever x is in every open set that contains y. It is easy to verify that $\leq$ thus defined is a preorder. Conversely, given a preorder defined on X, we obtain a topology on X as follows: a set $U \subseteq X$ is open for τ if there is no element in X which is less than any element in U. The set of such U's is obviously a topology on X. In this correspondence, the T_0 topologies correspond to the partial orders (that is, antisymmetric quasiorders). Recall that a topology is T_0 if, for every $x \neq y$, there is an open set containing exactly one of them.

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We say that a polynomial has a certain property if its coefficients have this property. For example, a polynomial of degree n is said to be symmetric or palindromic if $a_k = a_{n-k}$ holds for every $1 \leq k \leq \lfloor \frac{n}{2} \rfloor$.

A chief method in discrete mathematics is to derive or characterize properties of a discrete structure from properties of a polynomial $P(x)$ defined from it. For example, the independence polynomial of a graph, $I(G,x)$, (introduced by Gutman and Harary [6]), whose coefficients a_k are the numbers of stable sets (sets of pairwise non-adjacent vertices) having cardinality k. A famous conjecture due Allavi et al. [7] stated that the independence polynomial of any tree is unimodal (the sequence of its coefficients first increases, then decreases). Also this question has attracted the attention of several researchers, but no systematic answer is yet known. Maybe, the first significant contribution was made by Hamidoune [8] who proved that the independence polynomials of a large class of trees (claw-free trees) are unimodal. This result has been deduced recently as a consequence of a more general one [9]. For a general and an excellent survey of the subject of unimodality, log-concavity from all areas of mathematics, generalization of the notion, see [10]. For chemical purposes, Hosoya [11] defined a polynomial $H(G,x)$, now known as Hosoya polynomial as follows: the coefficient a_k of the Hosoya polynomial is the number of pairs of vertices whose distance is k, $1 \leq k \leq d$, where d is the diameter of the graph. Note that this polynomial has also been defined independently by other researchers and baptized the Weiner polynomial. The value $H'(G,1) = W(G)$, is the famous Weiner index, an important chemical invariant. All of these polynomials as well as their roots supply a lot of information about the graph itself.

In this paper, we consider the open-set polynomial $P(\tau,x)$ of a topology τ defined on a finite set X. Its coefficients are the number of open sets of cardinality k, $0 \leq k \leq n$, or, which is the same, the number of ideals of the corresponding quasiorder of cardinality k. The open-set polynomial is yet another one of these combinatorics-counting polynomials. Yet, the open-set polynomial has not received as much attention as the other ones (independence, chromatic and Hosoya polynomials), as remarked by Brown et al. [12]. Perhaps the first attempt to study $P(\tau,x)$ started with [12,13]. There, the focus is geometric, in that sense is deals with the distribution of the zeros of the polynomials.

At the best knowledge of the authors, this is the first time where the unimodality and log-concavity of the open set polynomials is considered in the literature.

For the sake of completeness, we included many papers investigating the cardinality of finite topologies and, thus, related to $P(\tau,x)$, while missed in [12].

The organization of the paper is as follows: the next section is devoted to some basic definitions and results needed in the sequel. In the third section, we answer Brown's question: which open-set polynomials have only real zeros? We then consider a weaker question than Brown's, namely, which open-set polynomials are log-concave or at least unimodal? A partial answer is given to this question by proving that unimodality is guaranteed in case the topology has cardinality $\geq 6.2^{n-4}$. The fourth section is devoted to a result, important in its own, giving a condition ensuring that all of the coefficients of the open polynomial are not zero. We end the paper by some open questions concerning the open-set polynomials.

2. Preliminaries

In this section, we recall some definitions and results concerning partitions, finite topologies, as well as the notions of unimodality and log concavity of real sequences.

Definition 1. *A partition* $\pi = (A_1, A_2, \ldots, A_l)$ *of the set* X *is a collection of subsets of* X *such that all the* A_i *are non-empty,* $A_i \bigcap A_j = \emptyset,$ for $i \neq j$, *and* $X = \bigcup_{i=1}^{l} A_i$.

Definition 2. *A partition* $\pi = (A_1, A_2, \ldots, A_l)$ *of the finite set* X_n *is called type* $(\alpha_1, \alpha_2, \ldots, \alpha_l)$ *if it contains* α_i *subsets of cardinality* i, *(note that* $\sum_i i\alpha_i = n$ *).*

A partition topology is given in the following definition.

Definition 3. *A partition topology* τ *on* X *is a topology that is induced by a partition of* X.

For example, the topology

$$\tau = \{\emptyset, \{a, b\}, \{c, d\}, \{e\}, \{a, b, c, d\}, \{a, b, e\}, \{c, d, e\}, X\}$$

on $X = \{a, b, c, d, e\}$ is induced by the partition $\pi = (\{a, b\}, \{c, d\}, \{e\})$. In general, the union of two topologies on a set is not a topology. However, there is a kind of union to remedy to this lack: the "disjoint union topology".

Definition 4. *Let* X, Y *be two topological spaces, such that* $X \bigcap Y = \emptyset$. *The disjoint union topology on* $Z = X \bigcup Y$, *is defined by*

$$\tau = \Big\{U \textit{ such that } U \in \tau_X \text{ or } U \in \tau_Y \text{ or } U = U_X \bigcup U_Y, \Big\},$$

such that $U_X \in \tau_X$ *and* $U_Y \in \tau_Y$.

When the space X is finite and is a disjoint union of two spaces Y_1 and Y_2, then its topology τ satisfies $|\tau| = |\tau_{Y_1}| \cdot |\tau_{Y_2}|$. This result will be helpful for us in the next sections.

Let τ be a topology on X_n. Let us designate by u_j the number of open sets in τ having cardinality j, and consider the polynomial $P(x) = \sum_{j=0}^{n} u_j x^j$, called the *open–set polynomial* of τ.

In [13], Brown raised the following question: *When are all the roots of an open-set polynomial real?*

In the next section, we answer this question by showing that the only case where this happens is for the discrete topology. Before doing this, we recall some facts about polynomials and log-concave sequences.

A real positive sequence $(a_j)_{j=0}^{n}$ is said to be *unimodal* if there exist integers k_0, k_1, $k_0 \leq k_1$ (integers $k_0 \leq j \leq k_1$ are the modes of the sequence), such that

$$a_0 \leq a_1 \leq \ldots < a_{k_0} = a_{k_0+1} = \ldots = a_{k_1} > a_{k_1+1} \geq \ldots \geq a_n.$$

It is *log-concave* if $a_j^2 \geq a_{j-1}a_{j+1}$, for $1 \leq j \leq n-1$. A real sequence $(a_j)_{j=0}^{n}$ is said to be *with no internal zeros* (NIZs), if $i < j, a_i \neq 0,\ a_j \neq 0$ then $a_l \neq 0$ for every $l, i \leq l \leq j$. A NIZ log-concave sequence is obviously unimodal, but the converse is not true. The sequence 1, 1, 4, 5, 4, 2,1 is unimodal but not log-concave. Note the importance of NIZs: the sequence 0, 1, 0, 0, 2, 1 is log-concave but not unimodal. A real polynomial is unimodal (log-concave, symmetric, respectively) provided that the sequence of its coefficients is unimodal (log-concave, symmetric, respectively).

If inequalities in the log-concavity definition are strict, then the sequence is called strictly log-concave (SLC for short), and in this case, it has at most two consecutive modes. The following result may be helpful in proving unimodality:

Theorem 1 (I. Newton)**.** *If the polynomial* $\sum_{j=0}^{n} a_j x^j$ *associated with the sequence* $(a_j)_{j=0}^{n}$ *has only real zeros, then*

$$a_j^2 \geq \frac{j+1}{j}\frac{n-j+1}{n-j}a_{j-1}a_{j+1}, \text{ for } 1 \leq j \leq n-1 \tag{1}$$

When dealing with sequences, we manipulate their sum and product. So, we may ask about the preservation of these properties. Recall that the convolution product of the sequences $(a_j)_{j=0}^{n}$ and $(b_j)_{j=0}^{n}$ is the sequence $(c_k)_{k=0}^{n}$, such that $c_k = \sum_{i+j=k} a_j b_i$, $0 \leq k \leq n$. It is known that the convolution of two log-concave sequences is a log-concave one, but the convolution of two unimodal sequences is not necessarily unimodal. A proof of the following result may be found in [14].

Proposition 1. *The convolution of two log-concave sequences is a log-concave sequence. The convolution of a unimodal and a log-concave sequence is a unimodal one.*

3. Open-Set Polynomials with Only Real Zeros

Our first result is the following.

Theorem 2. *The open-set polynomial* $\sum_{j=0}^{n} u_j x^j$ *has only real zeros if and only if* τ *is the discrete topology on* X_n*.*

Proof. If the topology is discrete on X_n, then $\sum_{j=0}^{n} u_j x^j = (1+x)^n$. Now, suppose that there exists a topology τ, such that its open polynomial has only real zeros. We are concerned with real polynomials with positive coefficients, then the simple facts:

1. If there is $a_j = 0$ for a certain $j, 1 \leq j \leq n-1$, the polynomial cannot have all its zeros real.
2. If the inequalities (1) fail for a certain $j, 1 \leq j \leq n-1$, then the zeros of the polynomial cannot be all real.

The coefficients u_j of the polynomial $\sum_{j=0}^{n} u_j x^j$ satisfy

(1) $u_0 = u_n = 1$.
(2) $0 \leq u_j \leq \binom{n}{j}$, $1 \leq j \leq n-1$.

Now, since the polynomial has only real zeros, its coefficients satisfy Newton's inequalities (Relation (1)), which may be written as follows:

$$\frac{u_j}{u_{j-1}} \geq \frac{j+1}{j}\frac{n-j+1}{n-j}\frac{u_{j+1}}{u_j}, \quad 1 \leq j \leq n-1.$$

By induction, we deduce

$$\frac{u_1}{u_0} \geq n^2\frac{u_n}{u_{n-1}}.$$

Remembering that $u_0 = u_n = 1$, we obtain

$$u_1 u_{n-1} \geq n^2.$$

However, we also have

$$u_1 u_{n-1} \leq n^2.$$

The numbers $u_k,\ 0 \leq k \leq n$ are integers; it follows that $u_1 = u_{n-1} = n$. This means that the topology has n open sets of cardinality one, or all the n singletons are open sets. This means that the topology is the discrete one. □

Definition 5. *Let $d > 1$ be a real number. The sequence $(a_j)_{j=0}^{n}$ is said to be d-log-concave if*

$$a_j^2 \geq d a_{j-1} a_{j+1}, \text{ for } 1 \leq j \leq n-1 \tag{2}$$

We can state the following theorem using the d-log-concavity property conclusion as Theorem 2.

Theorem 3. *If the coefficients of the open-set polynomial $\sum_{j=0}^{n} u_j x^j$ $(n \geq 2)$ satisfy $u_j^2 \geq d u_{j+1} u_{j-1}$ for $d > 1$, and τ is not the discrete topology on X_n, then $d < n^{\frac{2}{n-1}}$.*

Proof. The inequalities may be written,

$$\frac{u_j}{u_{j-1}} \geq d \frac{u_{j+1}}{u_j}, \text{ for } 1 \leq j \leq n-1,$$

then

$$\frac{u_1}{u_0} \geq d^{n-1} \frac{u_n}{u_{n-1}}.$$

As $u_0 = u_n = 1$, and $n > u_1,\ n > u_{n-1}$, this yields $d^{n-1} < n^2$, or equivalently,

$$d < n^{\frac{2}{n-1}}.$$

□

The natural question after this last result is a weak version of Brown's question:

Which topologies have their open-set polynomials log-concave or at least unimodal?

In order to investigate this question, we have to know, at first, the topologies for which all the coefficients of the open polynomial are non zero. To begin resolving this question, it may help to know which topologies have open polynomials will all non-zero coefficients. Unfortunately, a complete answer to this, like for the first question, is unknown. What we can do is to determine the open-set polynomials explicitly when the topology has a cardinality greater than $\geq (3/8)2^n$. In studying the unimodality of the open-set polynomials, we encounter the following questions:

- Which topologies have a log-concave open-set polynomial?
- Which have unimodal open-set polynomials?
- Which have all $u_k \neq 0$?
- Which fail to be unimodal because $u_k = 0$?
- Which have a cardinality l satisfying $C_1 2^n \leq l \leq C_2 2^n$?

The last section in this paper gives a partial answer to the first and second questions. There are no general known solutions for questions 1 and 2.

It is known that if the topology is T_0, then $u_k \neq 0$ for all $1 \leq k \leq n$ (see [15]). However, this does not guarantee unimodality. For the fourth question, we can say that the topology is not T_0. A partial answer of the last question is to be found in the studies [16,17].

Definition 6. *Let τ be a topology on the finite set X. The non-empty open set A is called a minimal open set if either $A \bigcap U = \emptyset$ or $A \bigcap U = A$ for every $U \in \tau$.*

Since a topology is a lattice, the previous definition means that the open set A is an atom, that is, an element covering the least element. The following lemma is obvious but important.

Lemma 1. *If τ is a topology on X_n, then so is $\tau^c = \{U^c \ / \ U \in \tau\}$, called the cotopology of τ.*

It may happen that $\tau = \tau^c$. For example, let $X_4 = \{a, b, c, d\}$ and $\tau = \{\varnothing, \{a, b\}, \{c, d\}, X_4\}$, then $\tau = \tau^c$. We may ask for which topology this is true. The next lemma characterizes topologies with this property.

Lemma 2. *Let τ be a topology on X_n, then*

$$\tau = \tau^c \Longleftrightarrow \tau \quad \textit{is induced by a partition}$$

Proof. If τ is induced by a partition, then it is obvious that $\tau^c = \tau$. Now suppose that $\tau^c = \tau$. Consider the minimal open sets $A_i,\ 1 \le i \le l$ of τ. We have for $i \ne j$, $A_i \bigcap A_j = \varnothing$ and we will show that $X_n = \bigcup_{i=1}^{l} A_i$. If $X_n \ne \bigcup_{i=1}^{l} A_i$, then $\varnothing \ne X_n - \bigcup_{i=1}^{l} A_i = C = A_{l+1} \in \tau$ is another minimal open set. A contradiction. So, $(A_1,\ A_2,\ \ldots,\ A_l)$ is a partition of X_n. □

In the case where τ is induced by a partition, its open polynomial is given explicitly.

Theorem 4. *Let τ be a topology induced by the partition π of type $(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_l)$, then*

$$P_\tau(x) = \prod_{i=1}^{l} \left(x^i + 1\right)^{\alpha_i}$$

Proof. One base of the topology is the partition π which contains α_i subsets of cardinality i. It follows that the open sets of cardinality j are all disjoint unions of members of the base π, so their number is the coefficient of x^j in the polynomial on the right-hand side of the previous formula. □

For example, the open polynomial associated with the topology induced by the partition $\pi = (\{a,\ b\},\ \{c,\ d\},\ \{e\})$ is $P(x) = (x+1)(x^2+1)^2 = 1 + x + 2x^2 + 2x^3 + x^4 + x^5$, while the polynomial associated with the partition $\pi = (\{a,\ b\},\ \{c,\ d,\ e\})$ is $P(x) = (x^2+1)(x^3+1) = 1 + x^2 + x^3 + x^5$. In the last example, there are some missing coefficients. Let the open polynomial $\sum_{j=0}^{n} u_j x^j$ be associated with a topology induced by a partition, when have we $u_j \ne 0$ for all $0 \le j \le n$? It is obvious that α_1 must be ≥ 1. So, we will suppose this condition.

Theorem 5. *Let τ be a topology induced by the partition π of type $(\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_l)$, with $\alpha_1 \ge 1$, then the coefficients of the polynomial P_τ:*

$$P_\tau(x) = \prod_{i=1}^{l} \left(x^i + 1\right)^{\alpha_i} = \sum_{m=0}^{n} u_m x^m,$$

are given by

$$u_m = \sum_{j_1 + 2j_2 + \ldots + l j_l = m} \binom{\alpha_1}{j_1} \cdots \binom{\alpha_l}{j_l}.$$

The coefficient u_m is $\ne 0$ if and only if there exists j_i, $0 \le j_i \le \alpha_i$ for $1 \le i \le l$ such that $j_1 + 2j_2 + \ldots + l j_l = m$.

Proof. The coefficient u_j is obtained by an identification between the left and right expression. □

Unfortunately, even when the topology has a number of open sets $\geq C2^n$, $0 < C < 1$, this does not guarantee the unimodality. For example, consider the topology τ on X_n.

$$\tau = \mathcal{P}(X_{n-2}) \bigcup \{\{x_1, x_2, \ldots, x_{n-2}, x_{n-1}\}, \{x_1, x_2, \ldots, x_{n-2}, x_n\}, X_n\}$$

We have $|\tau| = 2^{n-2} + 3$ and its open polynomial

$$P(x) = (x+1)^{n-2} + 2x^{n-1} + x^n$$

is not unimodal.

In conclusion of this section, we list some cases where all the coefficients of the polynomial $\sum_{j=0}^{n} u_j x^j$ are non zero .

(1) If the topology τ is T_0, since a T_0 topology contains open sets of all size from 0 to n.
(2) If the topology is induced by a partition, and the number of open sets, which are singletons, is $\geq \frac{n}{2}$.

It is easy to see that if the topology τ is T_0 and $|\tau| = n+1$, or $n+2$, then the sequence is unimodal. In this case, the topology is either a chain or a boolean algebra of four elements (Figure 1), inserted in a chain of $n-2$ elements .

Finally, note that if the $|\tau| \leq n$, then the polynomial cannot be unimodal, since there is at least one $0 < j_0 < n$, such that $u_{j_0} = 0$.

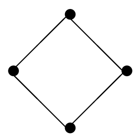

Figure 1. Boolean Algebra of Four Elements.

As we remarked above, the presence of the open sets of all cardinality k, $0 \leq k \leq n$ is necessary for the study of the unimodality. The next section is devoted to this fact. We give the bound where all the coefficients of the polynomial associated with the topology are non zero.

4. Maximum-Sized Topologies That Omit Open Sets of Given Sizes

The main result of this section is interesting in itself. We will introduce the notation $\mathcal{T}(n, k_1, k_2)$ for some classes of topologies for which the open-set polynomial is not unimodal because its k_1 through k_2 coefficients are zero, and then establish the least upper bounds on the sizes of such topologies. This implies that for $1 \leq k \leq n-1$, if the cardinality of a topology exceeds our upper bound, then it must have some open sets with cardinality k. For this purpose, we introduce some definitions and notations. Since we are studying a cardinality problem, the considered topologies can be viewed up to one-to-one correspondence, that is, they have the same cardinality.

Let k_1, k_2, be two integers such that $1 \leq k_1 \leq k_2 \leq n-1$. We denote by $\mathcal{T}(n, k_1, k_2)$ the class of topologies τ defined on X_n such that any open set $O \in \tau$ satisfies: $|O| \leq k_1 - 1$, or $|O| \geq k_2 + 1$, i.e., topologies that do not contain any open set of cardinality k, with $k_1 \leq k \leq k_2$.

Let τ be a collection of subsets of X_n. We say that τ is a basic collection if it contains the empty and the ground sets. Let $Y \subseteq X_n$ such that $Y \cap X = \varnothing$ for any $X \in \tau$. We denote by $Y + \tau$ the collection of subsets of the form $Y \cup X$ for all $X \in \tau$. It is called the Y-shift of τ, or a shifted τ for short. In particular, τ is a Ø-shift of itself, and is homeomorphic to any of its shifts.

Definition 7. *Let A and B be two subsets of X_n such that $|A| = k_1 - 1$, $|B| = n - k_2 - 1$ and $A \subseteq B$ or $A \supseteq B$ according to which set is bigger. Let*

$$\tau_w = 2^A \cup (B^c + 2^B),$$

where 2^X is the collection of all subsets of X. τ_w is called the w-topology corresponding to $\mathcal{T}(n, k_1, k_2)$ and is denoted by $\tau(n, k_1, k_2)$.

It is clear that $|\tau_w| = 2^{|A|} + 2^{|B|} = 2^{k_1-1} + 2^{n-k_2-1} = (2^{-(n-k_1+1)} + 2^{-(k_2+1)})2^n$. Moreover, if we denote by τ^c the collection of subsets $X_n - X$ for all $X \in \tau$, then $(2^Z)^c = 2^Z$, for any $Z \subseteq X_n$. This yields $\tau_w^c = (A^c + 2^A) \cup 2^B = 2^B \cup (A^c + 2^A)$.

Roadmap (Theorem 6 proof). The initial objective of this section was to obtain an upper bound for the cardinality of a topology that lacks open sets with k elements, $1 \leq k \leq n-1$ (Corollary 4). It seems that a proof analysing the whole case is very long and may be unreachable. So, the idea is to use induction. The proof strategy demonstrates an example of a theorem for which a generalization is easier to prove because it provides a stronger induction hypothesis. Thus, we prove a more general upper bound for topologies that lack open sets for some successive cardinalities from k_1 to k_2 with $1 \leq k_1 \leq k_2 \leq n-1$. The upper bound $(2^{-(n-k_1+1)} + 2^{-(k_2+1)})2^n$ is natural because it is the cardinality of the disjoint union of a discrete topology with a shifted one, that is, a w-topology. Lemma 16 states this for the extreme cases when $k_1 = 1$ or $k_2 = n-1$, which are the induction bases of the main result proof. To use induction, we need some operations that keep the properties of topologies and w-topologies, as well as reduce the cardinality of the ground set. The contraction and deletion operations are the heart of the arsenal used to achieve this induction proof. The idea behind the contraction and deletion operations is to "partition" a topology into two subtopologies. After deleting or contracting a ground set element $e \in X_n$, the result is two new topologies on $X_n \backslash \{e\}$. Their open sets, except sometimes for Ø and $X_n \backslash \{e\}$, correspond to the original open sets, respectively, not containing and containing e.

All intermediate results from Lemma 3 to Corollary 3 prepare the ground for the proof of the main result (Theorem 6). We think that piecewise proofs give more clarity than one block one. So we have many lemmas and corollaries in this section. To close this section, we state a generalization result of Theorem 6 that gives an upper bound for the cardinality of a topology lacking open sets of some cardinalities whether they are successive or not. The obtained upper bound is the cardinality of the disjoint union of many shifted discrete topologies. It can be conducted by induction on the number of what we call discrete intervals of cardinalities of lacking open sets.

From the above definitions, we deduce the following lemma.

Lemma 3. *Let k_1, k_2, n be three integers such that $1 \leq k_1 \leq k_2 \leq n-1$, $\tau \in \mathcal{T}(n, k_1, k_2)$ and τ_w its corresponding w-topology. Then the following assertions hold.*

(i) $\tau^c \in \mathcal{T}(n, n-k_2, n-k_1)$;

(ii) *If $k_1 = 1$ or $k_2 = n-1$ then $|\tau| \leq |\tau_w|$.*

Proof. We prove only (ii) when $k_1 = 1$. The case $k_2 = n-1$ is deduced by using τ^c.

Consider the collection $\mathcal{A}(\tau)$ of the containment-wise minimal non-empty subsets in a topology τ with no singleton sets. Since each is minimal, the intersection I between any two sets $X \neq Y$ in $\mathcal{A}$ must be empty, otherwise $I \subset X$ or $I \subset Y$ ($\subset$ denotes strict containment). Each τ over X_n with a given $\mathcal{A}$ is isomorphic to the topology obtained by for each $X \in \mathcal{A}$, identifying the points in X (to replace them with one point). The number of points in this isomorphic topology is $n' = n - \sum\{|X| - 1 : X \in \mathcal{A}\}$, so $|\tau| \leq 2^{n'}$. Since no $X \in \mathcal{A}$ has $|X| = k_1 = 1$, and $\mathcal{A} \neq \emptyset$, the maximum size of topology obtained this way is 2^{n-1} for which $\mathcal{A} = \{\{x_1, x_2\}\}$. □

Other properties of the w-topology are given below.

Lemma 4. *Let k_1, k_2, n be three integers such that $1 \leq k_1 \leq k_2 \leq n-1$. Then we have the following:*

(i) $\tau_w^c = \tau(n, n-k_2, n-k_1)$;

(ii) $\tau_w \in \mathcal{T}(n, k_1, k_2)$.

Now let τ be a basic collection on X_n. For $e \in X_n$, we denote by $\tau_{\bar{e}}$ (τ_e, respectively) the collection of subsets of τ that do not contain (contain, respectively) e. We call $\tau \backslash e = \{X_n \backslash \{e\}\} \cup \tau_{\bar{e}}$ ($\tau / e = \{\emptyset\} \cup \{O \backslash \{e\}$, respectively) the deletion (contraction, respectively) of e (applied to τ). We need the following lemma for the next results.

Lemma 5. *Let τ be a topology defined on X_n, and $e \in X_n$. The following assertions hold.*

(i) $\{X_n\} \cup \tau_{\bar{e}}$ *is a topology defined on* X_n;

(ii) $\tau \backslash e$ *is a topology defined on* $X_n \backslash \{e\}$;

(iii) $\{\emptyset\} \cup \tau_e$ *is a topology defined on* X_n;

(iv) τ / e *is a topology defined on* $X_n \backslash \{e\}$.

Proof. It is clear that (i) and (iii) are equivalent by using the cotopology. Moreover, (ii) and (iv), respectively, are direct consequences of (i) and (iii), respectively. So it suffices to prove (i).

The empty set and the ground set are members of our collection. Furthermore, given any two subsets from $\tau_{\bar{e}}$, their intersection as well as their union do not contain e. This means that they are members of $\tau_{\bar{e}}$, and we are finished. □

In the following corollary, the cardinality of τ is given by means of the cardinalities of $\tau \backslash e$ and τ / e .

Corollary 1. *Let τ be a basic collection of X_n, and $e \in X_n$. Then*

$$|\tau \backslash e| + |\tau / e| - 2 \leq |\tau| \leq |\tau \backslash e| + |\tau / e|.$$

Proof. The facts $|\tau| = |\tau_e| + |\tau_{\bar{e}}|$, $|\tau \backslash e| - 1 \leq |\tau_{\bar{e}}| \leq |\tau \backslash e|$ and $|\tau / e| - 1 \leq |\tau_e| \leq |\tau / e|$ imply the result. □

For a basic collection τ, we call τ^c the cocollection of τ. It is clear that τ is a basic collection if and only if τ^c is too. Moreover, the deletion and contraction operations keep the basic collection property. Some direct properties of these deletion and contraction operations combined with the cocollection are given in the following lemma.

Lemma 6. *Let τ be a basic collection of X_n, and $e, f \in X_n$. Then the following assertions hold.*

(i) $(\tau \backslash e)^c = \tau^c / e$;

(ii) $(\tau / e)^c = \tau^c \backslash e$;

(iii) $(\tau \backslash e) / f = (\tau / f) \backslash e$;

(iv) $(\tau \backslash e) \backslash f = (\tau \backslash f) \backslash e$;

(v) $(\tau/e)/f = (\tau/f)/e$.

Proof. To show (i)–(ii), it suffices to see that the cocollection of $\{X_n\} \cup \tau_{\bar{e}}$ ($\{\emptyset\} \cup \tau_e$, respectively) is $\{\emptyset\} \cup (\tau^c)_e$ ($\{X_n\} \cup (\tau^c)_{\bar{e}}$, respectively), and vice versa.

For (iii)–(v), it suffices to see that the subsets containing (not containing, respectively) f after removing e are the subsets containing (not containing, respectively) e after removing f. The empty and the ground sets belong to the obtained collection in all cases. □

The assertions (iii)–(v) justify the following notation for any basic collection τ, and two disjoints subsets Y, Z of X_n: $\tau \backslash Y/Z = \tau/Z\backslash Y$.

Corollary 2. *Let $2 \leq k_1 \leq k_2 \leq n-2$, $\tau \in \mathcal{T}(n, k_1, k_2)$ and $e \in X_n$. Then the following assertions hold.*

(i) *If $X_n \backslash \{e\} \in \tau$, then $\tau \backslash e \in \mathcal{T}(n-1, k_1-1, k_2)$;*
(ii) *If $\{e\} \in \tau$, then $\tau \backslash e \in \mathcal{T}(n-1, k_1-1, k_2)$;*
(iii) *If $X_n \backslash \{e\} \in \tau$, then $\tau/e \in \mathcal{T}(n-1, k_1-1, k_2)$;*
(iv) *If $\{e\} \in \tau$, then $\tau/e \in \mathcal{T}(n-1, k_1-1, k_2)$.*

Proof. To prove (ii), it suffices to see that any open set of $X \in \tau$ with $|X| = k_1 - 1$, $e \in X$. Otherwise, by adding $X \cup \{e\} \in \tau$ since $\{e\} \in \tau$, and $|X \cup \{e\}| = k_1$, a contradiction. Hence, every $X \in \tau \backslash e$ has either $|X| \leq k_1 - 2$ or $|X| \geq k_2 + 1$.
To prove (iii), it suffices to see that $(\tau/e)^c = \tau^c \backslash e \in \mathcal{T}(n-1, n-k_2-1, n-k_1)$, according to Lemmas 3 (i), 6 (ii), and (ii) of the present corollary.
To prove (iv), we use what we have remarked in (ii)-proof, that is, any open set of cardinality $k_1 - 1$ must contain e.
Finally, to prove (i), we proceed similarly for (iii), by using (i) of Lemma 6. □

Next, we prove an important property of the deletion and contraction operations under a given condition.

Lemma 7. *Let τ be a basic collection on X_n and $e \in X_n$. If $\{e\}$ and $X_n \backslash \{e\}$ are both open sets of τ, then each of the following assertions holds.*

(i) $|\tau| = |\tau \backslash e| + |\tau/e|$.
(ii) $\tau \backslash e = \tau/e$.
(iii) $|\tau^c| = |\tau^c \backslash e| + |\tau^c/e|$.
(iv) $\tau^c \backslash e = \tau^c/e$.

Proof. (i) Since $\{e\} \in \tau$, $|\tau/e| = |\tau_e|$. Similarly, $|\tau \backslash e| = |\tau_{\bar{e}}|$ because $X_n \backslash \{e\} \in \tau$. Since $|\tau| = |\tau/e| + |\tau \backslash e|$, this permits us to conclude.

(ii) Since $\{e\} \in \tau$, $X \in \tau \backslash e$ implies that $X \cup \{e\} \in \tau$, and then $X \in \tau/e$. This means that $\tau \backslash e \subseteq \tau/e$.

Similarly, if $X \in \tau/e$, then $X \cup \{e\} \in \tau$. Thus, $X = (X \cup \{e\}) \cap (X_n \backslash \{e\}) \in \tau$ since $X_n \backslash \{e\} \in \tau$. In other words, $\tau/e \subseteq \tau \backslash e$, and we are finished.

(iii)–(iv) Similarly. □

These properties imply a more refined analogous of Corollary 1 for a w-topology.

Corollary 3. *Let k_1, k_2, n be three integers such that $2 \leq k_1 \leq k_2 \leq n-2$ and $\tau_w = \tau(n, k_1, k_2)$. Then there exists $e \in X_n$ such that each of the following assertions holds.*

(i) $|\tau_w| = |\tau_w \backslash e| + |\tau_w/e|$;
(ii) $|\tau_w^c| = |\tau_w^c \backslash e| + |\tau_w^c/e|$;
(iii) $\tau_w \backslash e = \tau_w/e = \tau(n-1, k_1-1, k_2)$.

Proof.

(i) Let A and B be two finite subsets so $\tau_w = 2^A \cup (B^c + 2^B)$ according to Definition 7. Therefore, $e \in A \cap B$ satisfies the condition of Lemma 7 that $\{e\}$ and $X_n \backslash \{e\}$ are open sets of τ_w. Applying Lemma 7 (i) gives the requested result.

(ii) Similarly for (i).

(iii) Lemma 7 (ii) guarantees the first equality. For the second one, it suffices to see that $\tau_w \backslash e = 2^{A\backslash\{e\}} \cup (B^c + 2^{B\backslash\{e\}})$ since $(2^X)\backslash e = 2^{X\backslash\{e\}}$.

□

Now, we can prove the main result of this section.

Theorem 6. *Let k_1, k_2, n be three integers such that $1 \leq k_1 \leq k_2 \leq n-1$, and $\tau \in \mathcal{T}(n, k_1, k_2)$. Then $|\tau| \leq (2^{-(n-k_1+1)} + 2^{-(k_2+1)})2^n$.*

Proof. Without loss of generality we can suppose that $n - k_2 \geq k_1$, that is, $A \subseteq B$ if $\tau_w = 2^A \cup (B^c + 2^B)$ is the corresponding w-topology of $\mathcal{T}(n, k_1, k_2)$ (see Definition 7). We use induction on k_1 to prove that $|\tau| \leq |\tau_w|$ (if $n - k_2 \leq k_1$, then we use induction on k_2). The induction base ($k_1 = 1$) has been proved in Lemma 3 (ii).

Suppose now $k_1 \geq 2$; then there exists $e \in A = A \cap B$ such that $\{e\} \in \tau$. According to Corollaries 1, 2 (ii)–(iv) and Lemma 3 (i)–(iii), induction yields $|\tau| \leq |\tau\backslash e| + |\tau/e| \leq |\tau_w\backslash e| + |\tau_w/e| = |\tau_w|$. □

Corollary 4. *Let n and k be two integers such that $1 \leq k \leq n-1$ and τ is a topology on X_n. If τ does not contain any open set of cardinality k, then $|\tau| \leq (2^{-(n-k+1)} + 2^{-(k+1)})2^n$.*

Proof. Direct consequence of Theorem 6 by taking $k_1 = k_2 = k$ □

Corollary 5. *Let n and k be two integers such that $1 \leq k \leq n-1$, and τ is a topology on X_n. If $|\tau| \geq 2^{n-2} + 2$ then τ contains open sets of all possible cardinalities k.*

Finally, we introduce some notations and definitions to state a generalization of Theorem 6. Let $I \subseteq \{1, 2, ..., n-1\}$. We denote by $\mathcal{T}(n, I)$ the class of topologies that do not contain any open set O of cardinality $|O| \in I$. We say that $J \subseteq I$ is a maximal discrete interval of I if $J = \{k, k+1, ..., k+j\}$ for some $k, j \in \{1, 2, ..., n-1\}$, and J is maximal for this property, that is, J is a maximal subset of I containing only successive numbers. We use the notations: $k_1(J) = k$ and $k_2(J) = k + j$. In this case, we have $I = \bigcup_{i=1}^{p} J_i$, where all J_is are maximal discrete intervals of I. We denote by $II = (k_{1,1}, k_{1,2}, ..., k_{1,p}, k_{2,p})$, where $k_{1,i} = k_1(J_i)$ and $k_{2,i} = k_2(J_i)$ for $i = 1, ..., p$. It is not difficult to prove the following generalization of Theorem 6 by using induction on p.

Theorem 7. *Let n be a positive integer, I a discrete interval of $\{1, ..., n-1\}$, $II = (k_{1,1}, k_{1,2}, ..., k_{1,p}, k_{2,p})$, and $\tau \in \mathcal{T}(n, I)$. Then $|\tau| \leq \sum_{i=0}^{p} 2^{k_{1,i+1} - k_{2,i} - 1}$, where $k_{2,0} = 0$ and $k_{1,p+1} = n$.*

5. Large Enough Topologies Have Unimodal Open Polynomials

In this section, we will determine explicitly the open-set polynomials of a class of topologies, namely, all topologies having cardinality $\geq (3/8)2^n$. For $n \leq 5$, and using the results given in [16], we may check by hand that these polynomials are log-concave. So, we assume in the sequel that $n > 5$. The explicit form of these polynomials and Proposition 1, allow us to show their log-concavity.

The symbol X_k will designate any finite set of cardinality k. If we consider a subset of X_n of cardinality $k \leq n-1$, the elements of $X_n \setminus X_k$ will be designated by $a, b, c, \ldots$ or $y_1,\ y_2,\ \ldots$ Those of X_k are denoted by $x_1,\ x_2,\ x_3,\ \ldots$

Consider the set $\mathcal{T}(n,k)$ of the topologies having k open sets. In order to compute its cardinality, Kolli [16] divided it into two disjoint subsets:

$$\mathcal{T}_1(n,k) = \left\{ \tau \in \mathcal{T}(n,k) : \left(\bigcap_{\emptyset \neq U \in \tau} U \right) \neq \emptyset \right\}, \text{ and}$$

$$\mathcal{T}_2(n,k) = \mathcal{T}(n,k) - \mathcal{T}_1(n,k).$$

The main result of this section is as follows.

Theorem 8. *If τ is a topology on the set $X_n,\ n \geq 6$, such that $|\tau| \geq (3/8)2^n$, then its open-set polynomial is log-concave.*

Since $(5/16) < (3/8)$, we can begin with topologies $\mathcal{T}_1(n,k)$ for $k \geq (5/16)2^n$ and then conclude by proving Theorem 8 only for $\mathcal{T}_2(n,k)$ and $k \geq (3/8)2^n$, which is the rest of $\mathcal{T}(n,k)$. Since the computations for the concluding step are similar, we give full details for the beginning step and just a sample for the rest.

Theorem 9. *Let $\tau \in \mathcal{T}_1(n,k)$ with $k \geq (5/16)2^n$ open sets. Then the open polynomial of τ is one of the three.*

$$\begin{aligned}
P_1(x) &= x(x+1)^{n-1} + 1 \\
P_2(x) &= x^2(x+1)^{n-2} + x(x+1)^{n-3} + 1 \\
P_3(x) &= x^3(x+1)^{n-3} + 2x^2(1+x)^{n-4} + x(1+x)^{n-4} + 1.
\end{aligned}$$

Proof. Let $\tau \in \mathcal{T}_1(n,k)$ and $k \geq \left(\frac{5}{16}\right) \cdot 2^n$, then τ is necessarily one of the three types of the following topologies (see [16]):

(1) The cotopology of $\tau = \mathcal{P}(X_{n-1}) \bigcup \{X_n\}$.
(2) The cotopology of $\tau = \mathcal{P}(X_{n-2}) \bigcup \{\{a, x\},\ X_n\}$.
(3) The cotopology of $\tau = \mathcal{P}(X_{n-3}) \bigcup \{\{a, x\}, \{b, x\},\ X_n\}$.

So, let us determine the open polynomial of these topologies. For the first type of topologies, the total number of open sets of cardinality k is $u_k = \binom{k-1}{k}$ $0 \leq k \leq n-1$ and $u_n = 1$. It follows then that

$$Q_1(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} x^k + x^n = x^n + (1+x)^{n-1}.$$

A topology of the second type comes from a discrete topology on the set X_{n-2} of cardinal $n-2$ and the adjoining of the open set $\{\{a, x\},\ X_n\}$. The open set $\{\{a, x\}$ contributes $\binom{n-3}{k-2}$ open sets of cardinality $k,\ 0 \leq k \leq n-1$. This gives the total number of open sets of cardinality k, denoted u_k:

$$u_k = \binom{n-2}{k} + \binom{n-3}{k-2},\ 0 \leq k \leq n-1.$$

Furthermore, the open polynomial of the topology $\tau = \mathcal{P}(X_{n-2}) \bigcup \{\{a, x\},\ X_n\}$ is

$$
\begin{aligned}
Q_2(x) &= \sum_{k=0}^{n-1} \left(\binom{n-2}{k} + \binom{n-3}{k-2} \right) x^k + x^n \\
&= \sum_{k=0}^{n-1} \binom{n-2}{k} x^k + \sum_{k=0}^{n} \binom{n-3}{k-2} x^k + x^n \\
&= (1+x)^{n-2} + x^2(x+1)^{n-3} + x^n
\end{aligned}
$$

The open sets of the third topology $\tau = \mathcal{P}(X_{n-3}) \bigcup \{\{a, x\}, \{b, x\},\ X_n\}$ are the elements of $\mathcal{P}(X_{n-3})$ and sets obtained from the unions of $\{\{a, x\}, \{b, x\}\}$. More precisely, each element of $\{\{a, x\}, \{b, x\}\}$ contributes $\binom{n-4}{k-2}$, $0 \le k \le n-1$ and their union $\{a,\ b,\ x\}$ contributes $\binom{n-4}{k-3}$ open sets, $0 \le k \le n-1$. So, for this topology, the number of open sets of cardinality k is given by

$$
u_k = \binom{n-3}{k} + 2\binom{n-4}{k-2} + \binom{n-4}{k-3},\ 0 \le k \le n-1,\ u_n = 1.
$$

Its open polynomial $\sum_{k=0}^{n} u_k x^k$ is

$$
\begin{aligned}
Q_3(x) &= \sum_{k=0}^{n} u_k x^k = \sum_{k=0}^{n-1} u_k x^k + x^n \\
&= \sum_{k=0}^{n} \left(\binom{n-3}{k} + 2\binom{n-4}{k-2} + \binom{n-4}{k-3} \right) x^k + x^n \\
&= (x+1)^{n-3} + 2x^2(1+x)^{n-4} + x^3(1+x)^{n-4} + x^n.
\end{aligned}
$$

Note that if the polynomial associated with the topology τ is $\sum_{j=0}^{n} u_j x^j$, then $\sum_{j=0}^{n} u_{n-j} x^j$ (its reciprocal) is the open polynomial of τ^c. Thus, the wanted polynomials are just the reciprocal of these last ones, that is,

$$
P_1(x) = x(x+1)^{n-1} + 1, \quad P_2(x) = x^2(x+1)^{n-2} + x(x+1)^{n-3} + 1,
$$

and

$$
P_3(x) = x^3(x+1)^{n-3} + 2x^2(1+x)^{n-4} + x(1+x)^{n-4} + 1.
$$

This proves the theorem. □

Remark 1. *The polynomials P_1, P_2, P_3 are unimodal for every $n \ge 4$, but not log-concave. Note that in general a topology in $\tau_1(n,k)$ is never log-concave, unless it is a chain.*

The remaining of this section is devoted to the determination of the polynomials of the topologies in the set $\tau_2(n,k)$ and $k \ge (3/8)2^n$. Kolli [16] computed these topologies according to the number and cardinal of their minimal open sets. We use these details to determine u_j, then the open polynomial of the topology. Since there are so many cases, because of the number of minimal open sets (it varies from 1 to $n-1$) and the computations are similar, we give just some cases, for example the topologies having $(n-1)$ minimal open sets.

Theorem 10. *If τ is a topology having $(n-1)$ minimal open sets, at least one is not a singleton. Then, either τ is induced by a partition; thus its open-set polynomial is*

$$\begin{aligned} P_1(x) &= (1+x^2)(x+1)^{n-2} \\ &= (1+x+x^2+x^3)(x+1)^{n-3}. \end{aligned}$$

or $\tau = \mathcal{P}(X_{n-3}) \bigsqcup \{\varnothing,\ \{a,b\},\ \{a,b,c\}\}$ (the disjoint union topology) and its open polynomial is given by

$$\begin{aligned} P_2(x) &= (1+x^2+x^3)(1+x)^{n-3} \\ &= (1+x+x^2+2x^3+x^4)(1+x)^{n-4}. \end{aligned}$$

Proof. This topology (see [16]) is such that $|\tau| = 2^{n-1} \quad or \quad 6 \cdot 2^{n-4}$. In the first case, it is induced by a partition, whose blocks are all singletons, but one have cardinality two. So, its open polynomial is

$$\begin{aligned} P_1(x) &= (1+x^2)(1+x)^{n-2} \\ P_1(x) &= (1+x+x^2+x^3)(1+x)^{n-3} \end{aligned}$$

In the second case, τ is the disjoint union of the discrete topology on $(n-3)$ elements and the chain topology $\{\varnothing,\ \{a,b\},\ \{a,b,c\}\}$. The open polynomial is then

$$\begin{aligned} P_2(x) &= (1+x^2+x^3)(1+x)^{n-3} \\ P_2(x) &= (1+x^2+x^3)(1+x)^3(1+x)^{n-6} \\ P_2(x) &= \left(1+3x+4x^2+5x^3+6x^4+4x^5+x^6\right)(1+x)^{n-6} \end{aligned}$$

Now, since $P_1,\ P_2$ are products of two log-concave polynomials, it follows that they are log-concave too. □

6. Conclusions

We proved that if the topology has more than $(3/8)2^n$ open sets, then its open-set polynomial is log-concave. We ask if there is a precise and general characterization of topologies having log-concave open-set polynomials?

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