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# Fractional Stochastic Systems Driven by Fractional Brownian Motion: Existence, Uniqueness, and Approximate Controllability with Generalized Memory Effects

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**Abstract**

In this research work, we present findings on fractional stochastic systems characterized by fractional Brownian motion, which is defined by a Hurst parameter $\mathbb{H} \in \left(\frac{1}{2}, 1\right)$. These systems are crucial for modeling complex phenomena that diverge from Markovian behavior and exhibit long-range dependence, particularly in areas such as financial engineering and statistical physics. We utilize the fixed-point iteration method to demonstrate the existence and uniqueness (Ex-Un) of mild solutions. Additionally, we investigate the approximate controllability of the system. We establish all results within the framework of the $\mu$-Caputo fractional derivative. This study makes a meaningful contribution to the existing body of literature by rigorously establishing the existence, uniqueness, and approximate controllability of mild solutions to generalized Caputo fractional stochastic differential equations driven by fractional Brownian motion.

**Keywords:** generalized fractional derivatives; Picard iteration technique; fractional stochastic systems; approximate controllability; existence and uniqueness

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## 1. Introduction

Classical derivatives show how quickly a quantity changes at a certain time, while fractional derivatives take into account the memory and other features of a process. Fractional derivatives are good for modeling things that change over time and space. Viscoelasticity, complex media exhibiting atypical diffusion patterns, and biological systems showcasing diverse dynamics are examples of these processes. These applications demonstrate the adaptability of fractional derivatives in encapsulating the intricacies of systems that conventional calculus fails to depict accurately. Consequently, scholars from various disciplines are progressively adopting fractional calculus to investigate complex subjects.

In fractional calculus, there are many operators, each with its own kernel and analytical structure. The Riemann–Liouville, Caputo, Grünwald–Letnikov, Caputo–Fabrizio, conformable, Hadamard, and Katugampola derivatives are some of the most well-known fractional derivatives [1–4]. Choose one of these operators based on the specific problem at hand, not randomly. This choice depends on how well the model shows memory effects and inherited traits, how clear the initial conditions are from a physical perspective, and

how well it follows the mathematical rules for being well-posed. The operator is an important part of the model because it is meant to capture the system's underlying physics or dynamics.

In 1967, Michele Caputo introduced the Caputo fractional derivative (CFD), which has since become one of the most widely used operators in physics and engineering. A significant reason for its popularity is that the Riemann–Liouville derivative cannot accommodate classical initial conditions such as $g(0)$ and $f'(0)$. However, the Caputo operator features a fixed power-law kernel, $(\kappa - \vartheta)^{\aleph-1}$, which limits its effectiveness in simulating systems with complex or time-varying memory characteristics.

In 2017 [5], Ricardo Almeida officially introduced the $\mu$-Caputo fractional derivative, referred to as $\mu$-CFD. This development indicates that the integral kernel now depends on more than just the difference $(\kappa - \vartheta)$. The primary concept involves transforming the variable $t$ into a function $\mu(\kappa)$. The $\mu$-Caputo operator serves as a powerful mathematical tool for modeling systems that do not adhere to the conventional power law regarding memory.

The $\mu$-CFD is defined for $g(\kappa) \in C^1[a,b]$ and $g : [a,b] \to \mathbb{R}$ as follows [6]:

$$\mathbb{D}^{\aleph,\mu} g(\kappa) = \frac{1}{\Gamma(n-\aleph)} \int_0^{\kappa} \left(\mu(\kappa) - \mu(\vartheta)\right)^{n-1-\aleph} \mu'(\vartheta) \left(\frac{1}{\mu(\vartheta)} \frac{d}{d\vartheta}\right)^n g(\vartheta)\, d\vartheta, \quad n-1 < \aleph < n,\ n \in \mathbb{Z}^+,$$

where $\mu'(\kappa) > 0$.

**Remark 1.** *If $\mu(\kappa) = \kappa$, the $\mu$-CFD simplifies to the CFD. In the case where $\mu(\kappa) = \ln \kappa$, it yields the Hadamard fractional derivative (HFD). Furthermore, when $\mu(\kappa) = t^x$ with $\mu^x > 0$, it corresponds to the Katugampola fractional derivative (KFD).*

Figure 1 below illustrates this phenomenon.

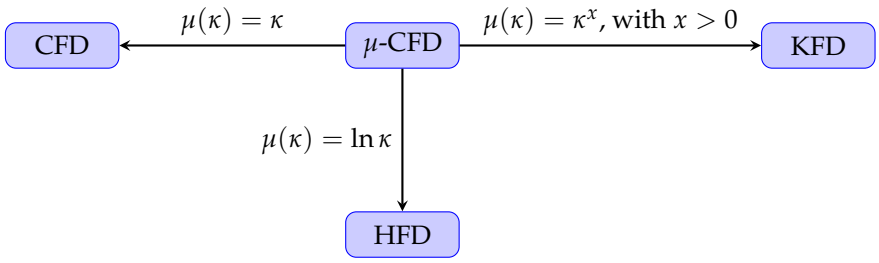


**Figure 1.** The behavior of $\mu$-CFD corresponds to $\mu(\kappa)$.

The $\mu$-Caputo fractional integral is defined as [6]:

$$\mathbb{I}_{\kappa}^{\aleph,\mu} g(\kappa) = \frac{1}{\Gamma(\aleph)} \int_0^{\kappa} \frac{\mu'(\vartheta) g(\vartheta)}{(\mu(\kappa) - \mu(\vartheta))^{1-\aleph}} d\vartheta.$$

Fractional stochastic differential equations (FSDEs) driven by fractional Brownian motion (fBm) provide a robust mathematical framework for modeling dynamical systems that exhibit significant memory effects and long-range dependence. Unlike classical Brownian motion, which features independent and identically distributed increments, fBm is a self-similar Gaussian process characterized by the Hurst parameter $\mathbb{H} \in (0,1)$. By incorporating fBm into FSDEs, researchers can capture hereditary dynamics and temporal coherence often observed in complex real-world phenomena where the Markovian assumption is not applicable. Consequently, these models have found extensive applications across various fields, including financial mathematics for modeling long-memory asset volatility, biophysics for elucidating anomalous subdiffusion in cellular environments, and geophysics for simulating persistent climatic patterns.

In the context of FSDEs, controllability refers to the capability to steer the system's trajectory from a specific initial state to a defined terminal state within a limited timeframe. Exact controllability means that the system can be driven precisely to the intended state, whereas approximate controllability indicates that the system can come very close to the target state. Achieving precise control is often challenging due to the non-local nature of fractional derivatives and the potentially significant impact of stochastic disturbances, particularly in infinite-dimensional spaces. As a result, both theoretical research and practical applications have focused on approximate controllability. The study of controllability in FSDEs is crucial in multiple domains, such as epidemic management in computational biology, resilient portfolio construction in quantitative finance, proactive vibration control in mechanical and aerospace engineering, and optimal resource allocation in systems affected by hereditary dynamics and stochastic fluctuations.

Numerous scholars have made substantial contributions to the development of FSDEs. For instance, the authors in [7] investigated the stability of such systems. The Ex–Un of solutions for FSDEs were demonstrated using Picard's iterative method [8], while the controllability of FSDEs involving the Hilfer fractional operator was examined in [9]. Further foundational progress in this area was presented in [10].

Building upon these findings, the study in [11] presented several qualitative results for FSDEs, whereas investigations involving specific fractional derivatives, such as the HFD, were also conducted [12]. The analysis was further extended in [13] to variable-order FSDEs, where new sufficient conditions were formulated, and solutions were obtained using Picard's iterative technique. A system of FSDEs with the Atangana–Baleanu operator was examined in [14], accompanied by numerical simulations and graphical illustrations. Moreover, theoretical advancements have been achieved through studies on the Ex-Un of impulsive FSDEs [15] and the stability of specific FSDE [16].

Additionally, the authors in [17] explored the Ex-Un of solutions for fractional stochastic system involving the Hilfer fractional operator. Controllability analyses by the Cap-FD were presented in [18], while their stability characteristics were discussed in [19]. Stochastic systems governed by the conformable fractional operator were investigated in [20]. Similarly, Ref. [21] examined the Ex-Un associated with the CFD. Xu and Li [22] established the Ex-Un for the mild solution of FSDEs with fBm. The authors of [23] presented various important results for fractional stochastic systems driven by fBm. In [24], the authors also worked on Ex-Un for mixed fractional stochastic systems. Han and Yan [25] established results regarding approximate controllability for FSDEs driven by fBm with the Caputo fractional operator. Dhayal et al. [26] also presented approximate controllability for fractional stochastic systems.

The contributions of this study are summarized as follows:

1. Most research on FSDEs has concentrated on standard Brownian motion. This study presents findings related to the Ex-Un for mild solutions of FSDEs driven by fBm. We utilize the Picard iteration method (PIM) of Ex-Un for mild solutions and also establish results concerning approximate controllability.
2. We derived all results about the $\mu$-CFD, which generalizes several recognized fractional operators.

We consider the following FSDEs with fBm:

$$\begin{cases} \mathbb{D}^{\aleph,\mu}\mathrm{g}(\kappa) = \mathcal{U}\mathrm{g}(\kappa) + \mathcal{V}\varrho(\kappa) + \chi(\kappa, \mathrm{g}(\kappa)) + \mathcal{M}(\kappa)\frac{d w^{\mathbb{H}}(\kappa)}{d\kappa}, \ \kappa \in I = [0, L], \\ \mathrm{g}(0) = \mathrm{g}_0, \end{cases} \tag{1}$$

where $\aleph \in \left(\frac{1}{2}, 1\right]$, $\mu'(\kappa) > 0$, $\kappa \in I$ and $\mathbb{D}^{\aleph,\mu}$ represent the $\mu$-CFD. The operator $\mathcal{U}$ is the infinitesimal generator of a strongly continuous semigroup $\{\Theta_\kappa\}_{\kappa \geq 0}$ defined on a real separable Hilbert space $\mathcal{Q}$. Let $w^{\mathbb{H}}$ denotes an fBm on another infinite-dimensional real separable Hilbert space $\mathcal{Y}$, with Hurst parameter $\mathbb{H} \in \left(\frac{1}{2}, 1\right)$. The initial condition $g_0$ is assumed to be independent with respect to $w^{\mathbb{H}}$ and has a finite second moment. We also consider the following operator $\mathcal{V} : \mathcal{L}^2_{\mathbb{F}}(I, \mathcal{B}) \to \mathcal{L}^2(I, \mathcal{Q})$ and, finally, we set $\chi : I \times \mathcal{Q} \to \mathcal{Q}$ and $\mathcal{M} : I \to \mathcal{L}^0_F(\mathcal{Y}, \mathcal{Q})$. These last two operators will be defined with further details in the next section.

**Remark 2.** *It is important to clarify the distinction between white noise and the formal derivative of Brownian motion, as both are often denoted by the symbolic expression* $\frac{dw(\kappa)}{d\kappa}$ *in the literature.*

*The Wiener process* $w(\kappa)$*, also called Brownian motion, is a continuous stochastic process that is almost surely nowhere differentiable. Therefore, its classical derivative* $\frac{dw(\kappa)}{d\kappa}$ *does not exist in the ordinary sense. To give meaning to this expression, we interpret white noise* $\xi(\kappa)$ *as the generalized (distributional) derivative of Brownian motion, defined through the action on smooth test functions* $\varphi(\kappa)$ *by*

$$\int_{-\infty}^{\infty} \xi(\kappa)\, \varphi(\kappa)\, d\kappa = -\int_{-\infty}^{\infty} w(\kappa)\, \varphi'(\kappa)\, d\kappa.$$

*Hence, white noise* $\xi(\kappa)$ *is not a conventional stochastic process but a generalized random distribution that exists only under an integral sign.*

*In stochastic calculus, the infinitesimal increment of Brownian motion* $dw(\kappa)$ *is used instead of* $\xi(\kappa)\, d\kappa$*. The quantity* $dw(\kappa)$ *represents a random increment of* $w(\kappa)$ *over a small time interval* $d\kappa$ *and satisfies*

$$\mathbb{E}[dw(\kappa)] = 0, \qquad \mathbb{E}[(dw(\kappa))^2] = d\kappa.$$

*Formally, one may write* $dw(\kappa) = \xi(t)\, d\kappa$*, indicating that Brownian motion is the time integral of white noise:*

$$w(\kappa) = \int_0^{\kappa} \xi(s)\, ds.$$

*In summary, white noise* $\xi(\kappa)$ *is a generalized stochastic distribution interpreted as the derivative of Brownian motion, while* $dw(\kappa)$ *is a well-defined stochastic increment used in stochastic differential equations. The relation* $dw(\kappa) = \xi(\kappa)\, d\kappa$ *should therefore be understood in a formal sense, rather than as an equality between ordinary functions.*

**Remark 3.** *It is important to clarify the meaning of the symbolic expression* $\frac{dw^{\mathbb{H}}(\kappa)}{d\kappa}$ *when* $w^{\mathbb{H}}(\kappa)$ *denotes a fBm with Hurst parameter* $\mathbb{H} \in (0, 1)$*. Since* $w^{\mathbb{H}}(\kappa)$ *is almost surely nowhere differentiable for every* $\mathbb{H}$*, its classical derivative does not exist. Thus, the expression* $\frac{dw^{\mathbb{H}}(\kappa)}{d\kappa}$ *cannot be interpreted as an ordinary function or as a classical stochastic process. To provide a rigorous meaning, the literature introduces fractional Gaussian noise, denoted by* $\dot{w}^{\mathbb{H}}(\kappa)$*, which is defined as the generalized (distributional) derivative of* $w^{\mathbb{H}}(\kappa)$*.*

The structure of this paper is as follows. The foundational components of our investigation are presented in Section 2. We then establish our core results on FSDEs within the $\mu$-CFD framework in Sections 3 and 4. The application of these theoretical findings is demonstrated through a numerical example in Section 5. The paper concludes with a summary in Section 6.

## 2. Preliminaries

We present the key definitions and lemmas that will be required for presenting our results.

Let $\mathcal{Q}, \mathcal{B}$, and $\mathcal{Y}$ be three real separable Hilbert spaces. Consider a complete probability space $(\Omega, \mathbb{F}, \mathbb{P})$, equipped with a normal filtration $\{\mathbb{F}_\kappa\}_{\kappa\in[0,L]}$ that satisfies the usual conditions, with $\mathbb{F}_L = \mathbb{F}$.

Let $\varrho$ be a control function in $\mathcal{L}^2_{\mathbb{F}}(I, \mathcal{B})$. This last space stands for the closed subspace of $\mathcal{L}^2(I \times \Omega, \mathcal{B})$ consisting of all $\mathbb{F}_\kappa$-adapted processes taking values in $\mathcal{B}$. Throughout this work, define $\varpi = \sup_{\kappa\in[0,+\infty)} |\Theta(\kappa)| < \infty$.

Let us also introduce the following Banach spaces: First, let $\mathcal{L}(\mathcal{Y}, \mathcal{Q})$ denote the collection of all bounded linear operators from the Hilbert space $\mathcal{Y}$ into $\mathcal{Q}$ with the usual operator norm, that is,

$$\mathcal{L}(\mathcal{Y}, \mathcal{Q}) := \{T : \mathcal{Y} \to \mathcal{Q} \,|\, T \text{ is a bounded linear operator}\}.$$

Second, we define $\mathcal{L}^2(\Omega, \mathcal{Q})$ as the set of all $\mathcal{Q}$-valued random variables g on $\Omega$ that are $\mathbb{F}$-measurable and square-integrable, given by

$$\mathcal{L}^2(\Omega, \mathcal{Q}) := \left\{ \mathrm{g} : \Omega \to \mathcal{Q} \,\middle|\, \mathrm{g} \text{ is } \mathbb{F}\text{-measurable and } \mathbf{E}\|\mathrm{g}\|^2 < \infty \right\}$$

endowed with its corresponding square norm.

Lastly, let $\mathcal{F}(I, \mathcal{L}^2(\Omega, \mathcal{Q}))$ be the space of all mappings $\mathrm{g} : I \to \mathcal{L}^2(\Omega, \mathcal{Q})$ such that

1. $\mathrm{g}(\kappa)$ is $\mathbb{F}_\kappa$-adapted for each $\kappa \in I$;
2. g is continuous with respect to $\kappa$;
3. and $\sup_{\kappa\in I} \mathbf{E}\|\mathrm{g}(\kappa)\|^2 < \infty$.

We denote this space by $A = \mathcal{F}(I, \mathcal{L}^2(\Omega, \mathcal{Q}))$ and we equipped it with the following norm in order to become a Banach space

$$\|\mathrm{g}\|_A = \left( \sup_{\kappa\in I} \mathbf{E}\|\mathrm{g}(\kappa)\|^2 \right)^{1/2}.$$

**Definition 1** ([27]). *The family of processes $\mathbb{W}^{\mathbb{H}} = \{\mathbb{W}^{\mathbb{H}}(\kappa),\ \kappa \in I\}$ is said to be a one-dimensional fBm characterized by a Hurst index $\mathbb{H} \in (0, 1)$. It is a zero-mean Gaussian process with continuous trajectories defined by the following covariance function:*

$$\mathcal{R}^{\mathbb{H}}(\kappa, \vartheta) = \mathbf{E}[\mathbb{W}^{\mathbb{H}}(\kappa)\mathbb{W}^{\mathbb{H}}(\vartheta)] = \frac{1}{2}(\kappa^{2\mathbb{H}} + \vartheta^{2\mathbb{H}} - |\kappa - \vartheta|^{2\mathbb{H}}), \ \text{with } \kappa, \vartheta \in I.$$

Throughout the rest of this paper, we consider the case where $\mathbb{H} \in \left(\frac{1}{2}, 1\right)$. The process $\mathbb{W}^{\mathbb{H}}$ admits the following representation:

$$\mathbb{W}^{\mathbb{H}}(\kappa) = \int_0^\kappa \mathbb{B}^{\mathbb{H}}(\kappa, \vartheta) d\mathbb{W}(\vartheta),$$

where $\mathbb{W} = \{\mathbb{W}(\kappa),\ \kappa \in I\}$ denotes a real-valued, one-dimensional standard Brownian motion, and

$$\mathbb{B}^{\mathbb{H}}(\kappa, \vartheta) = \lambda_{\mathbb{H}} \vartheta^{\frac{1}{2}-\mathbb{H}} \int_\vartheta^\kappa (\mathrm{e} - \vartheta)^{\mathbb{H}-\frac{3}{2}} \mathrm{e}^{\mathbb{H}-\frac{1}{2}} d\mathrm{e},\ \kappa > \vartheta, \tag{2}$$

where

$$\lambda_{\mathbb{H}} = \sqrt{\frac{\mathbb{H}(2\mathbb{H}-1)}{Be(2-2\mathbb{H},\ \mathbb{H}-\frac{1}{2})}}.$$

Let $\mathcal{G} \in \mathcal{L}^2(I)$ be a deterministic function.

Then

$$\int_0^L \mathcal{G}(\vartheta) d\mathbb{W}^{\mathbb{H}}(\vartheta) = \int_0^L (\mathbb{B}^*_L \mathcal{G})(\vartheta) d\mathbb{W}(\vartheta),$$

where

$$(\mathbb{B}_p^* \mathcal{G})(\vartheta) = \int_\vartheta^u \mathcal{G}(a) \frac{\partial \mathbb{B}^{\mathbb{H}}(a, \vartheta)}{\partial a} dz, \ a \in [0, L].$$

Next, we provide the definition of fBm in an infinite-dimensional setting along with its associated stochastic integral. Suppose that $\digamma \in \mathcal{L}(\mathcal{Y}, \mathcal{Y})$ is a self-adjoint, non-negative operator belonging to the trace class, such that

$$\digamma z_j = v_j z_j,$$

with $\{z_j\}_{j=1}^\infty$ being a complete orthonormal system in the Hilbert space $\mathcal{Y}$, and the respective eigenvalues $\{v_j\}_{j=1}^\infty \subseteq [0, \infty)$ satisfying

$$\mathrm{tr}(\digamma) = \sum_{j=1}^\infty v_j < \infty.$$

Then, a $\mathcal{Y}$-valued $\digamma$-cylindrical fBm on the probability space $(\Omega, \mathbb{F}, \mathbb{P})$, with covariance operator $\digamma$, is defined as

$$w^{\mathbb{H}}(\kappa) = \sum_{j=1}^\infty \digamma^{\frac{1}{2}} \mathcal{G}_j \mathbb{W}_j^{\mathbb{H}}(\kappa) = \sum_{j=1}^\infty \sqrt{v_j} \mathcal{G}_j \mathbb{W}_j^{\mathbb{H}}(\kappa),$$

where $\mathbb{W}_j^{\mathbb{H}}$ is one-dimensional fBm.

Let $\mathcal{L}_\digamma^0(\mathcal{Y}, \mathcal{Q})$ be the space of $\digamma$-Hilbert-Schmidt operators. If $\eta \in \mathcal{L}(\mathcal{Y}, \mathcal{Q})$ and

$$\|\eta\|^2_{\mathcal{L}_\digamma^0(\mathcal{Y}, \mathcal{Q})} = \sum_j^\infty \|\sqrt{v_j} \eta \mathcal{G}_j\|^2 < \infty,$$

then $\eta$ is referred to as a $\digamma$-Hilbert-Schmidt operator.

**Definition 2** ([28]). *If $\delta : I \to \mathcal{L}_\digamma^0(\mathcal{Y}, \mathcal{Q})$ such that $\delta$ describes a random operator-valued function that evolves over a domain $I$ and takes values in a special space of linear operators between two real separable Hilbert spaces $\mathcal{Q}$ and $\mathcal{Y}$, the following is satisfied:*

$$\sum_{j=1}^\infty \|\mathbb{B}_L^*(\delta \digamma^{\frac{1}{2}}) \mathcal{G}_J\|_{\mathcal{L}^2(I, \mathcal{Q})} < \infty, \tag{3}$$

*then the stochastic integral of $\delta(\vartheta)$ with respect to $w^{\mathbb{H}}(\vartheta)$ can be defined as*

$$\begin{aligned} \int_0^\kappa \delta(\vartheta) dw^{\mathbb{H}}(\vartheta) &= \sum_{j=1}^\infty \int_0^\kappa \delta(\vartheta) \digamma^{\frac{1}{2}} \mathcal{G}_j d\mathbb{W}_j^{\mathbb{H}}(\vartheta) \\ &= \sum_{j=1}^\infty \int_0^\kappa (\mathbb{B}_L^*(\delta \digamma^{\frac{1}{2}} \mathcal{G}_j))(\vartheta) d\mathbb{W}(\vartheta), \ \textit{for all } \kappa \in I. \end{aligned}$$

**Lemma 1** ([29]). *If $\delta : I \to \mathcal{L}_\digamma^0(\mathcal{Y}, \mathcal{Q})$ satisfies*

$$\sum_{j=1}^\infty \|\delta \digamma^{\frac{1}{2}} \mathcal{G}_j\|_{\mathcal{L}^{\frac{1}{\mathbb{H}}}(I, \mathcal{Q})} < \infty, \tag{4}$$

*then for all $0 \le \vartheta < \kappa \le L$, we have*

$$\mathbf{E} \left\| \int_\vartheta^\kappa \delta(p) dw^{\mathbb{H}}(p) \right\|_{\mathcal{Q}}^2 \le \lambda_{\mathbb{H}} (\kappa - \vartheta)^{2\mathbb{H}-1} \sum_{j=1}^\infty \int_\vartheta^\kappa \|\delta(p) \digamma^{\frac{1}{2}} \mathcal{G}_j\|_{\mathcal{Q}}^2 dp,$$

*with the constant $\lambda_{\mathbb{H}} > 0$ determined by the Hurst parameter $\mathbb{H}$.*

*In addition, if the series* $\sum_{j=1}^{\infty} \|\delta(\kappa)F^{\frac{1}{2}}z_j\|_{\mathcal{Q}}$ *is convergent, then*

$$\mathbf{E}\left\| \int_{\vartheta}^{\kappa} \delta(p)dw^{\mathbb{H}}(p)\right\|_X^2 \leq \lambda_{\mathbb{H}}(\kappa-\vartheta)^{2\mathbb{H}-1}\int_{\vartheta}^{\kappa}\|\delta(p)\|_{\mathcal{L}_F^0(\mathcal{Y},\mathcal{Q})}dp. \tag{5}$$

**Definition 3** ([30]). *We define the stochastic process* $\{\mathrm{g}(\kappa)\}_{\kappa\in I}$ *as a mild solution to system* (1) *if, for every control function* $\varrho \in \mathcal{L}^2_{\mathbb{F}}(I,\mathcal{B})$*, it satisfies the following integral formulation:*

$$\begin{aligned}\mathrm{g}(\kappa) &= \Theta_{\aleph,\mu}(\kappa)\mathrm{g}_0 + \int_0^{\kappa}\big(\mu(\kappa)-\mu(\vartheta)\big)^{\aleph-1}\Xi_{\aleph,\mu}\big(\mu(\kappa)-\mu(\vartheta)\big)[\chi(\vartheta,\mathrm{g}(\vartheta))+\mathcal{V}\varrho(\vartheta)]\mu'(\vartheta)d\vartheta \\ &\quad + \int_0^{\kappa}\big(\mu(\kappa)-\mu(\vartheta)\big)^{\aleph-1}\Xi_{\aleph,\mu}\big(\mu(\kappa)-\mu(\vartheta)\big)\mathcal{M}(\vartheta)\mu'(\vartheta)dw^{\mathbb{H}}(\vartheta),\ \kappa\in I,\end{aligned} \tag{6}$$

*where* $\Theta_{\aleph,\mu}(\kappa) = \int_0^{\infty}\eta_{\aleph}(\hbar)\Theta\big((\mu(\kappa)-\mu(0))^{\aleph}\hbar\big)d\hbar$,
$\Xi_{\aleph,\mu}(\kappa) = \aleph\int_0^{\infty}\hbar\eta_{\aleph}(\hbar)\Theta\big(\big(\mu\kappa-\mu(\vartheta)\big)^{\aleph}\hbar\big)d\hbar$, *and*

$$\eta_{\aleph}(\hbar) = \frac{1}{\aleph}\hbar^{-(1+\frac{1}{\aleph})}\overline{\omega}_{\aleph}(\hbar^{-\frac{1}{\aleph}}),\ \overline{\omega}_{\aleph}(\hbar) = \sum_{j=1}^{\infty}(-1)^{j-1}\hbar^{-j\aleph-1}\frac{\Gamma(j\aleph+1)}{\pi j!}\sin(j\pi\aleph),\ \hbar\in(0,\infty),$$

*We recall that* $\eta_{\aleph}$ *is a probability density function that satisfies*

$$\eta_{\aleph}(\hbar)\geq 0,\ \hbar\in(0,\infty)\ \text{and}\ \int_0^{\infty}\eta_{\aleph}(\hbar)d\hbar = 1.$$

**Remark 4.** *We briefly describe the process required for obtaining this mild solution as in [30] First, we apply the Laplace transform to the system* (1)*, converting it into an algebraic form in the frequency domain while formally preserving the linear structure of both deterministic and stochastic components. Using resolvent operator theory, we then solve the transformed equation for the state variable, which reveals the fundamental solution structure through the operators* $\Theta_{\aleph,\mu}(\kappa)$ *and* $\Xi_{\aleph,\mu}(\kappa)$ *with* $\kappa\geq 0$*. Finally, we recover the time-domain solution by applying the inverse Laplace transform.*

**Lemma 2** ([31]). *The families of operators* $\{\Theta_{\aleph,\mu}(\kappa)\}_{\kappa\geq 0}$ *and* $\{\Xi_{\aleph,\mu}(\kappa)\}_{\kappa\geq 0}$ *have the following properties.*

1. *For all* $\kappa\geq 0$, $\Theta_{\aleph,\mu}(\kappa)$ *and* $\Xi_{\aleph,\mu}(\kappa)$ *are bounded linear operators, that is*

$$\|\Theta_{\aleph,\mu}(\kappa)\mathrm{g}\|\leq\varpi\|\mathrm{g}\|,\ \mathrm{g}\in X\ \text{and}\ \|\Xi_{\aleph,\mu}(\kappa)\mathrm{g}\|\leq\frac{\aleph\varpi}{\Gamma(\aleph+1)}\|\mathrm{g}\|,\ \text{for all}\ \mathrm{g}\in\mathcal{Q}.$$

2. *The families of operators* $\{\Theta_{\aleph,\mu}(\kappa),\kappa\geq 0\}$ *and* $\{\Xi_{\aleph,\mu}(\kappa),\kappa\geq 0\}$ *are strongly continuous semigroups of operators; see [32].*
3. *If* $\Theta(\kappa)$ *is compact for every* $\kappa>0$*, it follows that* $\Theta_{\aleph,\mu}(\kappa)$ *and* $\Xi_{\aleph,\mu}(\kappa)$ *are compact operators as well for all* $\kappa>0$*.*

**Definition 4.** *The set*

$$\begin{aligned}\mathbb{B}_L(\chi) = \{\mathrm{g}(L) : \mathrm{g}(L)\ &\text{denotes the mild solution of the considered system}\\ &\text{at time } L \text{ associated with the control input } \varrho\}\end{aligned}$$

*is referred to as the reachable set of* (1)*, the set of all possible states that can be attained by the system.*

*If* $\chi=0$*, then* (1) *reduces to a linear control system, given by*

$$\begin{cases}\mathbb{D}^{\aleph,\mu}\mathrm{g}(\kappa) = \mathcal{U}\mathrm{g}(\kappa)+\mathcal{V}\varrho(\kappa)+\mathcal{M}(\kappa)\dfrac{dw^{\mathbb{H}}(\kappa)}{d\kappa},\quad \kappa\in I=[0,L],\\ \mathrm{g}(0)=\mathrm{g}_0.\end{cases} \tag{7}$$

*In addition, we refer to* $\mathbb{B}_L(0)$ *as the reachable set corresponding to system* (7)*.*

**Definition 5.** *The system* (1) *is said to be approximately controllable on $I$ if the closure of the set $\mathbb{B}_L(\chi)$, denoted by $\overline{\mathbb{B}_L(\chi)}$, is equal to $\mathcal{L}^2(\Omega, \mathcal{Q})$. In other words, for every $\eta \in \mathcal{L}^2(\Omega, \mathcal{Q})$ and any $\varepsilon > 0$, there exists a control function $\varrho \in \mathcal{L}^2_{\mathbb{F}}(I, \mathcal{B})$ such that*

$$\mathbb{E}\|\varrho(L) - \eta\|^2 < \varepsilon.$$

*In the same manner, the system* (7) *is approximately controllable if $\overline{\mathbb{B}_L(0)} = \mathcal{L}^2(\Omega, \mathcal{Q})$.*

**Remark 5.** *We present several assumptions that are the foundations of our established results in this research work. Specifically, we clarify that (H4), the compactness of the semigroup $\Theta(\kappa)$ for $\kappa > 0$, ensures the compactness of the associated solution operators, which is essential for applying fixed point techniques in the existence and controllability analysis. We also explicitly acknowledge that (H4) restricts the framework to systems generated by compact or sectorial operators and does not cover certain hyperbolic systems or PDEs on unbounded domains.*
*Furthermore, we explain that (H5) provides a natural sufficient condition for approximate controllability, as it guarantees that the range of the evolution operator can be approximated by the action of the control operator. This density condition is standard in controllability theory and ensures that the control has enough influence on the state space.*

## 3. Existence and Uniqueness

We establish the Ex-Un results for the solution of the FSDEs. We need to state the following hypotheses:

**Hypothesis 1.** *The function $\chi : I \times \mathcal{Q} \to \mathcal{Q}$ is measurable and there is $\lambda_1 > 0$ such that for each $\mathrm{g} \in \mathcal{Q}$ and any $\kappa \in I$,*

$$\|\chi(\kappa, \mathrm{g})\|^2 \leq \lambda_1(1 + \|\mathrm{g}\|^2).$$

**Hypothesis 2.** *There exists a constant $\lambda_2 > 0$ such that for all $\mathrm{g}_1, \mathrm{g}_2 \in \mathcal{Q}$, $\kappa \in I$ ,*

$$\|\chi(\kappa, \mathrm{g}_1) - \chi(\kappa, \mathrm{g}_2)\|^2 \leq \lambda_2\|\mathrm{g}_1 - \mathrm{g}_2\|^2.$$

**Hypothesis 3.** *The function $\mathcal{M} : I \to \mathcal{L}^0_F(\mathcal{Y}, \mathcal{Q})$ is Lebesgue measurable, and there exists a real constant $\lambda_3 > 0$ such that*

1. $\sup_{0 \leq \vartheta \leq L} \|\mathcal{M}(\vartheta)\|^2_{\mathcal{L}^0_F(\mathcal{Y}, \mathcal{Q})} \leq \lambda_3$;
2. $\sum_{j=1}^{\infty} \|\mathcal{M}F^{\frac{1}{2}}e_j\|_{\mathcal{L}^{\frac{1}{\mathbb{H}}}(I, \mathcal{Q})} < \infty$;
3. $\sum_{j=1}^{\infty} \|\mathcal{M}(\kappa)F^{\frac{1}{2}}e_j\|_X$ is uniformly convergent for $\kappa \in I$.

**Hypothesis 4.** *For all $\kappa > 0$, the operator $\Theta(\kappa)$ is compact.*

**Remark 6.** *The compactness of the semigroup $\Theta(\kappa)$ for $\kappa > 0$ (Assumption (H4)) is a strong requirement and that it does not hold for certain PDEs on unbounded domains or for systems with hyperbolic character. In our framework, however, this assumption is essential for ensuring the compactness of the associated solution operators, which is required to apply the fixed point arguments used to establish existence and controllability results. Without compactness, these arguments cannot be applied in a straightforward manner.*

Now, let us define the operator $U : A \to A$ by

$$\begin{aligned}(U\mathrm{g})(\kappa) =& \Theta_{\aleph,\mu}(\kappa)\mathrm{g}_0 + \int_0^{\kappa} (\mu(\kappa) - \mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa) - \mu(\vartheta))[\chi(\vartheta, \mathrm{g}(\vartheta)) + \mathcal{V}\varrho(\vartheta)]\mu'(\vartheta)d\vartheta \\ &+ \int_0^{\kappa} (\mu(\kappa) - \mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa) - \mu(\vartheta))\mathcal{M}(\vartheta)\mu'(\vartheta)dw^{\mathbb{H}}(\vartheta). \end{aligned} \tag{8}$$

**Lemma 3.** *Let us assume that conditions* $(H1)$, $(H3)$, *and* $(H4)$ *are satisfied. For every* $\mathrm{g} \in A$, *the mapping* $\kappa \to (U\mathrm{g})(\kappa)$ *is continuous on* $I$ *in the* $\mathcal{L}^2(\Omega, \mathcal{Q})$*-sense.*

**Proof.** For every $\mathrm{g} \in A$ and all $0 \le \kappa_1 < \kappa_2 \le L$, it holds that

$$\begin{aligned}
&\mathbf{E}\|(U\mathrm{g})(\kappa_2) - (U\mathrm{g})(\kappa_1)\|^2 \le 4\mathbf{E}\|\Theta_{\aleph,\mu}(\kappa_2)\mathrm{g}_0 - \Theta_{\aleph,\mu}(\kappa_1)\mathrm{g}_0\|^2 \\
&+ 4\mathbf{E}\Big\| \int_0^{\kappa_2} (\mu(\kappa_2) - \mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa_2) - \mu(\vartheta))\chi(\vartheta, \mathrm{g}(\vartheta))\mu'(\vartheta)d\vartheta - \\
&\int_0^{\kappa_1} (\mu(\kappa_1) - \mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa_1) - \mu(\vartheta))\chi(\vartheta, \mathrm{g}(\vartheta))\mu'(\vartheta)d\vartheta\Big\|^2 \\
&+ 4\mathbf{E}\Big\| \int_0^{\kappa_2} (\mu(\kappa_2) - \mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa_2) - \mu(\vartheta))\mathcal{V}\varrho(\vartheta)\mu'(\vartheta)d\vartheta - \\
&\int_0^{\kappa_1} (\mu(\kappa_1) - \mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa_1) - \mu(\vartheta))\mathcal{V}\varrho(\vartheta)\mu'(\vartheta)d\vartheta\Big\|^2 \\
&+ 4\mathbf{E}\Big\| \int_0^{\kappa_2} (\mu(\kappa_2) - \mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa_2) - \mu(\vartheta))\mathcal{M}(\vartheta)\mu'(\vartheta)dw^{\mathbb{H}}(\vartheta) - \\
&\int_0^{\kappa_1} (\mu(\kappa_1) - \mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa_1) - \mu(\vartheta))\mathcal{M}(\vartheta)\mu'(\vartheta)dw^{\mathbb{H}}(\vartheta)\Big\|^2 \\
&= \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4.
\end{aligned} \tag{9}$$

Due to the strong continuity of $\Theta_{\aleph,\mu}(\kappa)$, it follows that

$$\lim_{\kappa_2 \to \kappa_1} \|\Theta_{\aleph,\mu}(\kappa_2)\mathrm{g}_0 - \Theta_{\aleph,\mu}(\kappa_1)\mathrm{g}_0\| = 0. \tag{10}$$

In view of Lemma 2, it holds that

$$\|\Theta_{\aleph,\mu}(\kappa_2)\mathrm{g}_0 - \Theta_{\aleph,\mu}(\kappa_1)\mathrm{g}_0\| \le 2\varpi\|\mathrm{g}_0\| \ \text{ for } \mathrm{g}_0 \in \mathcal{L}^2(\Omega, \mathbb{R}^+). \tag{11}$$

In addition, in view of the Lebesgue Dominated Convergence Theorem, we conclude that

$$\lim_{\kappa_2 \to \kappa_1} \Psi_1 = 0.$$

Moreover,

$$\begin{aligned}
\Psi_2 &\le 12\mathbf{E}\Big\| \int_0^{\kappa_1} \Big[(\mu(\kappa_2) - \mu(\vartheta))^{\aleph-1} - (\mu(\kappa_1) - \mu(\vartheta))^{\aleph-1}\Big]\Xi_{\aleph,\mu}(\mu(\kappa_2) - \mu(\vartheta))\chi(\vartheta, \mathrm{g}(\vartheta))\mu'(\vartheta)d\vartheta\Big\|^2 \\
&+ 12\mathbf{E}\Big\| \int_0^{\kappa_1} (\mu(\kappa_1) - \mu(\vartheta))^{\aleph-1}\Big[\Xi_{\aleph,\mu}(\mu(\kappa_2) - \mu(\vartheta)) - \Xi_{\aleph,\mu}(\mu(\kappa_1) - \mu(\vartheta))\Big]\chi(\vartheta, \mathrm{g}(\vartheta))\mu'(\vartheta)d\vartheta\Big\|^2 \\
&+ 12\mathbf{E}\Big\| \int_{\kappa_1}^{\kappa_2} (\mu(\kappa_2) - \mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa_2) - \mu(\vartheta))\chi(\vartheta, \mathrm{g}(\vartheta))\mu'(\vartheta)d\vartheta\Big\|^2 \\
&= \Psi_{21} + \Psi_{22} + \Psi_{23}.
\end{aligned} \tag{12}$$

Based on $(H1)$ and Hölder inequality (Höl-In), it can be shown that

$$\begin{aligned}
\Psi_{21} &\le \frac{12\varpi^2}{\Gamma^2(\aleph)}\Big(\int_0^{\kappa_1} \Big[(\mu(\kappa_1) - \mu(\vartheta))^{\aleph-1} - (\mu(\kappa_2) - \mu(\vartheta))^{\aleph-1}\Big]^2 (\mu'(\vartheta))^2 d\vartheta\Big)\int_0^{\kappa_1} \mathbf{E}\|\chi(\vartheta, \mathrm{g}(\vartheta))\|^2 d\vartheta \\
&\le \frac{12\varpi^2}{\Gamma^2(\aleph)}\Big(\sup_{0\le\vartheta\le L}(\mu'(\vartheta))\Big)\int_0^{\kappa_1} \Big[(\mu(\kappa_1) - \mu(\vartheta))^{2\aleph-2} - (\mu(\kappa_2) - \mu(\vartheta))^{2\aleph-2}\Big](\mu'(\vartheta))^2 d\vartheta \times \int_0^{\kappa_1} \lambda_1(1 + \mathbf{E}\|\mathrm{g}(\vartheta)||^2)d\vartheta \\
&\le \mathcal{Q}\frac{12\varpi^2\lambda_1 t_1(1 + \sup_{\vartheta\in I}\mathbf{E}\|\mathrm{g}(\vartheta)\|^2)}{\Gamma^2(\aleph)(2\aleph - 1)} \times \Big[(\mu(\kappa_1) - \mu(0))^{2\aleph-1} + (\mu(\kappa_2) - \mu(\kappa_1))^{2\aleph-1} - (\mu(\kappa_2) - \mu(0))^{2\aleph-1}\Big],
\end{aligned} \tag{13}$$

where $Q = \sup_{0\le\vartheta\le L}\left(\mu'(\vartheta)\right)$. Then, we have

$$\Psi_{22} \le 12\mathbf{E}\left(\int_0^{\kappa_1}(\mu(\kappa_1)-\mu(\vartheta))^{\aleph-1}\left\|\Xi_{\aleph,\mu}(\mu(\kappa_2)-\mu(\vartheta))-\Xi_{\aleph,\mu}(\mu(\kappa_1)-\mu(\vartheta))\right\|\|\chi(\vartheta,\mathrm{g}(\vartheta))\|\mu'(\vartheta)d\vartheta\right)^2 \tag{14}$$
$$\le Q\frac{12(\mu(\kappa_1)-\mu(0))^{2\aleph-1}\lambda_1(1+\sup_{\vartheta\in I}\mathbf{E}\|\mathrm{g}(\vartheta)\|^2)}{(2\aleph-1)}\left(\sup_{\vartheta\in[0,\kappa_1]}\left\|\Xi_{\aleph,\mu}(\mu(\kappa_2)-\mu(\vartheta))^{\aleph-1}-\Xi_{\aleph,\mu}(\mu(\kappa_1)-\mu(\vartheta))\right\|\right)^2.$$

Based on Lemma 2 once more and on condition $(H4)$, it follows that $\Xi_{\aleph,\mu}(\kappa)$ is continuous with respect to $\kappa > 0$ in the uniform operator topology. Consequently, we obtain that $\lim_{\kappa_2\to\kappa_1}\Psi_{21} = \lim_{\kappa_2\to\kappa_1}\Psi_{22} = 0$.

Further,

$$\begin{aligned}\Psi_{23} &\le \frac{12\varpi^2}{\Gamma^2(\aleph)}\left(\int_{\kappa_1}^{\kappa_2}(\mu(\kappa_2)-\mu(\vartheta))^{2\aleph-2}(\mu'(\vartheta))^2d\vartheta\right)\left(\int_{\kappa_1}^{\kappa_2}\mathbf{E}\|\chi(\vartheta,\mathrm{g}(\vartheta))\|^2d\vartheta\right)\\ &\le Q\frac{12\varpi^2\lambda_1(\mu(\kappa_2)-\mu(\kappa_1))^{2\aleph-1}(\kappa_2-t_1)(1+\sup_{\vartheta\in I}\mathbf{E}\|\mathrm{g}(\vartheta)\|^2)}{\Gamma^2(\aleph)(2\aleph-1)}\\ &\to 0 \text{ as } \kappa_2\to\kappa_1.\end{aligned} \tag{15}$$

A similar computation yields

$$\begin{aligned}\Psi_3 &\le 12\mathbf{E}\left\|\int_0^{\kappa_1}\left[(\mu(\kappa_2)-\mu(\vartheta))^{\aleph-1}-(\mu(\kappa_1)-\mu(\vartheta))^{\aleph-1}\right]\Xi_{\aleph,\mu}(\mu(\kappa_2)-\mu(\vartheta))\mathcal{V}_\varrho(\vartheta)\mu'(\vartheta)d\vartheta\right\|^2\\ &+12\mathbf{E}\left\|\int_0^{\kappa_1}(\mu(\kappa_1)-\mu(\vartheta))^{\aleph-1}\left[\Xi_{\aleph,\mu}(\mu(\kappa_2)-\mu(\vartheta))-\Xi_{\aleph,\mu}(\mu(\kappa_1)-\mu(\vartheta))\right]\mathcal{V}_\varrho(\vartheta)\mu'(\vartheta)d\vartheta\right\|^2\\ &+12\mathbf{E}\left\|\int_{\kappa_1}^{\kappa_2}(\mu(\kappa_2)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa_2)-\mu(\vartheta))\mathcal{V}_\varrho(\vartheta)\mu'(\vartheta)d\vartheta\right\|^2\\ &=\Psi_{31}+\Psi_{32}+\Psi_{33}.\end{aligned} \tag{16}$$

Similarly,

$$\begin{aligned}\Psi_{31} &\le Q\frac{12\varpi^2\|\mathcal{V}_\varrho\|_{\mathcal{L}^2(I,\mathcal{Q})}\left[\left(\mu(\kappa_1)-\mu(0)\right)^{2\aleph-1}+\left(\mu(\kappa_2)-\mu(\kappa_1)\right)^{2\aleph-1}-\left(\mu(\kappa_2)-\mu(0)\right)^{2\aleph-1}\right]}{\Gamma^2(\aleph)(2\aleph-1)}\to 0 \text{ as } \kappa_2\to\kappa_1,\\ \Psi_{32} &\le Q\frac{12(\mu(\kappa_1)-\mu(0))^{2\aleph-1}\|\mathcal{V}_\varrho\|_{\mathcal{L}^2(I,\mathcal{Q})}}{2\aleph-1}\left(\sup_{\vartheta\in[0,\kappa_1]}\left\|\Xi_{\aleph,\mu}(\mu(\kappa_2)-\mu(\vartheta))^{\aleph-1}-\Xi_{\aleph,\mu}(\mu(\kappa_1)-\mu(\vartheta))\right\|\right)^2\to 0 \text{ as } \kappa_2\to\kappa_1,\\ \Psi_{33} &\le Q\frac{12\varpi^2(\mu(\kappa_2)-\mu(\kappa_1))^{2\aleph-1}\|\mathcal{V}_\varrho\|_{\mathcal{L}^2(I,\mathcal{Q})}}{\Gamma^2(\aleph)(2\aleph-1)}\to 0 \text{ as } \kappa_2\to\kappa_1.\end{aligned} \tag{17}$$

Along the same lines, one obtains

$$\begin{aligned}\Psi_4 &\le 12\mathbf{E}\left\|\int_0^{\kappa_1}\left[(\mu(\kappa_2)-\mu(\vartheta))^{\aleph-1}-(\mu(\kappa_1)-\mu(\vartheta))^{\aleph-1}\right]\Xi_{\aleph,\mu}(\mu(\kappa_2)-\mu(\vartheta))\mathcal{M}(\vartheta)\mu'(\vartheta)dw^{\mathbb{H}}(\vartheta)\right\|^2\\ &+12\mathbf{E}\left\|\int_0^{\kappa_1}(\mu(\kappa_1)-\mu(\vartheta))^{\aleph-1}\left[\Xi_{\aleph,\mu}(\mu(\kappa_2)-\mu(\vartheta))-\Xi_{\aleph,\mu}(\mu(\kappa_1)-\mu(\vartheta))\right]\mathcal{M}(\vartheta)\mu'(\vartheta)dw^{\mathbb{H}}(\vartheta)\right\|^2\\ &+12\mathbf{E}\left\|\int_{\kappa_1}^{\kappa_2}(\mu(\kappa_2)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa_2)-\mu(\vartheta))\mathcal{M}(\vartheta)\mu'(\vartheta)dw^{\mathbb{H}}(\vartheta)\right\|^2\\ &=\Psi_{41}+\Psi_{42}+\Psi_{43}.\end{aligned} \tag{18}$$

Combining Lemma 1 and $(H3)$, we have

$$\begin{aligned}\Psi_{41} &\leq 12\lambda_{\mathbb{H}}\big(\mu(\kappa_1)-\mu(0)\big)^{2\mathbb{H}-1}\int_0^{\kappa_1}\left\|\left[(\mu(\kappa_2)-\mu(\vartheta))^{\aleph-1}-\big(\mu(\kappa_1)-\mu(\vartheta)\big)^{\aleph-1}\right]\Xi_{\aleph,\mu}(\mu(\kappa_2)-\mu(\vartheta))\mu'(\vartheta)\mathcal{M}(\vartheta)\right\|^2_{\mathcal{L}_F^0(\mathcal{Y},\mathcal{Q})}d\vartheta\\ &\leq \frac{12\lambda_{\mathbb{H}}(\mu(\kappa_1)-\mu(0))^{2\mathbb{H}-1}\lambda_3\varpi^2}{(\Gamma(\aleph))^2}\int_0^{\kappa_1}\left[(\mu(\kappa_1)-\mu(\vartheta))^{2\aleph-2}-(\mu(\kappa_2)-\mu(\vartheta))^{2\aleph-2}\right](\mu'(\vartheta))^2d\vartheta\\ &\leq Q\frac{12\lambda_{\mathbb{H}}(\mu(\kappa_1)-\mu(0))^{2\mathbb{H}-1}\lambda_3\varpi^2}{(2\aleph-1)(\Gamma(\aleph))^2}\left[\big(\mu(\kappa_1)-\mu(0)\big)^{2\aleph-1}+\big(\mu(\kappa_2)-\mu(\kappa_1)\big)^{2\aleph-1}-\big(\mu(\kappa_2)-\mu(0)\big)^{2\aleph-1}\right]\\ &\to 0 \text{ as } \kappa_2\to\kappa_1,\end{aligned}\tag{19}$$

$$\begin{aligned}\Psi_{42} &\leq 12\lambda_{\mathbb{H}}\big(\mu(\kappa_1)-\mu(0)\big)^{2\mathbb{H}-1}\int_0^{\kappa_1}\left\|(\mu(\kappa_1)-\mu(\vartheta))^{\aleph-1}\left[\Xi_{\aleph,\mu}\big(\mu(t_2)-\mu(\vartheta)\big)-\Xi_{\aleph,\mu}(\mu(\kappa_1)-\mu(\vartheta))\right]\mu'(\vartheta)\mathcal{M}(\vartheta)\right\|^2_{\mathcal{L}_F^0(\mathcal{Y},\mathcal{Q})}d\vartheta\\ &\leq 12\lambda_{\mathbb{H}}\big(\mu(\kappa_1)-\mu(0)\big)^{2\mathbb{H}-1}\lambda_3\sup_{\vartheta\in[0,\kappa_1]}\left\|\Xi_{\aleph,\mu}(\mu(\kappa_2)-\mu(\vartheta))-\Xi_{\aleph,\mu}\big(\mu(\kappa_1)-\mu(\vartheta)\big)\right\|^2\int_0^{\kappa_1}(\mu(\kappa_1)-\mu(\vartheta))^{2\aleph-2}(\mu'(\vartheta))^2d\vartheta\\ &\leq Q\frac{12\lambda_{\mathbb{H}}\big(\mu(\kappa_1)-\mu(0)\big)^{2\mathbb{H}+2\aleph-2}\lambda_3}{(2\aleph-1)}\sup_{\vartheta\in[0,\kappa_1]}\left\|\Xi_{\aleph,\mu}(\mu(\kappa_2)-\mu(\vartheta))-\Xi_{\aleph,\mu}(\mu(\kappa_1)-\mu(\vartheta))\right\|^2\\ &\to 0 \text{ as } \kappa_2\to\kappa_1,\end{aligned}\tag{20}$$

$$\begin{aligned}\Psi_{43} &\leq 12\lambda_{\mathbb{H}}\big(\mu(\kappa_2)-\mu(\kappa_1)\big)^{2\mathbb{H}-1}\int_{\kappa_1}^{\kappa_2}\left\|\big(\mu(\kappa_2)-\mu(\vartheta)\big)^{\aleph-1}\mu'(\vartheta)\Xi_{\aleph,\mu}(\mu(\kappa_2)-\mu(\vartheta))\mathcal{M}(\vartheta)\right\|^2_{\mathcal{L}_F^0(\mathcal{Y},\mathcal{Q})}d\vartheta\\ &\leq Q\frac{12\lambda_{\mathbb{H}}\varpi^2\lambda_3\big(\mu(\kappa_2)-\mu(\kappa_1)\big)^{2\mathbb{H}+2\aleph-2}}{(2\aleph-1)(\Gamma(\aleph))^2}\\ &\to 0 \text{ as } \kappa_2\to\kappa_1,\end{aligned}\tag{21}$$

Adding all of these estimations, we have

$$\lim_{\kappa_2\to\kappa_1}\mathbb{E}\|(U\mathrm{g})(\kappa_2)-(U\mathrm{g})(\kappa_1)\|^2=0,$$

which shows that the mapping $\kappa\to(U\mathrm{g})(\kappa)$ is continuous on $I$ in the $\mathcal{L}^2(\Omega,\mathcal{Q})$-sense. □

**Lemma 4.** *Under* $(H1)$, $(H3)$ *and* $(H4)$, *the operator* $U$ *sends* $A$ *into* $A$.

**Proof.** For all $\mathrm{g}\in A$, we have

$$\begin{aligned}&E\|(U\mathrm{g})(\kappa)\|^2\\ &\leq 4\mathbf{E}\|\Theta_{\aleph,\mu}(\kappa)\mathrm{g}_0\|^2+4\mathbf{E}\left\|\int_0^{\kappa}(\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa)-\mu(\vartheta))\chi(\vartheta,\mathrm{g}(\vartheta))\mu'(\vartheta)d\vartheta\right\|^2\\ &+4\mathbf{E}\left\|\int_0^{\kappa}(\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa)-\mu(\vartheta))\mathcal{V}\varrho(\vartheta)\mu'(\vartheta)d\vartheta\right\|^2\\ &+4\mathbf{E}\left\|\int_0^{\kappa}(\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}\big(\mu(\kappa)-\mu(\vartheta)\big)\mathcal{M}(\vartheta)\mu'(\vartheta)dw^{\mathbb{H}}(\vartheta)\right\|^2\\ &=r_1+r_2+r_3+r_4.\end{aligned}\tag{22}$$

According to Lemma 2, one can obtain

$$r_1\leq 4\varpi^2\mathbf{E}\|\mathrm{g}_0\|^2.\tag{23}$$

By $(H1)$ and Höl-In, it follows that

$$\begin{aligned} r_2 &\leq \frac{4\varpi^2}{\Gamma^2(\aleph)}\mathbf{E}\bigg(\int_0^\kappa \big(\mu(\kappa)-\mu(\vartheta)\big)^{\aleph-1}\|\chi(\vartheta,\mathrm{g}(\vartheta))\|\mu'(\vartheta)d\vartheta\bigg)^2 \\ &\leq \frac{4\varpi^2}{\Gamma^2(\aleph)}\bigg(\int_0^\kappa(\mu(\kappa)-\mu(\vartheta))^{2\aleph-2}(\mu'(\vartheta))^2d\vartheta\bigg)\bigg(\int_0^\kappa \mathbf{E}\|\chi(\vartheta,\mathrm{g}(\vartheta))\|^2d\vartheta\bigg) \\ &\leq Q\frac{4\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}L\lambda_1(1+\sup_{\vartheta\in I}\mathbf{E}\|\mathrm{g}(\vartheta)\|^2)}{(2\aleph-1)\Gamma^2(\aleph)}, \end{aligned} \tag{24}$$

$$\begin{aligned} r_3 &\leq \frac{4\varpi^2}{\Gamma^2(\aleph)}\bigg(\int_0^\kappa(\mu(\kappa)-\mu(\vartheta))^{2\aleph-2}(\mu'(\vartheta))^2d\vartheta\bigg)\bigg(\int_0^\kappa \|\mathcal{V}\varrho(\vartheta)\|^2d\vartheta\bigg) \\ &\leq QL\frac{4\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}\|\mathcal{V}\varrho\|_{\mathcal{L}^2(I,\mathcal{Q})}}{(2\aleph-1)\Gamma^2(\aleph)}. \end{aligned} \tag{25}$$

Combining Lemma 1, $(H3)$ and Höl-In, we have

$$\begin{aligned} r_4 &\leq 4\lambda_{\mathbb{H}}\big(\mu(\kappa)-\mu(0)\big)^{2\mathbb{H}-1}\int_0^\kappa\Big\|(\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\mu'(\vartheta)\Xi_{\aleph,\mu}(\mu(\kappa)-\mu(\vartheta))\mathcal{M}(\vartheta)\Big\|^2_{\mathcal{L}^0_r(\mathcal{Y},\mathcal{Q})}d\vartheta \\ &\leq \frac{4\lambda_{\mathbb{H}}\varpi^2(\mu(\kappa)-\mu(0))^{2\mathbb{H}-1}}{\Gamma^2(\aleph)}\int_0^\kappa\big(\mu(\kappa)-\mu(\vartheta)\big)^{2\aleph-2}\|\mathcal{M}(\vartheta)\|^2_{\mathcal{L}^0_r(\mathcal{Y},\mathcal{Q})}(\mu'(\vartheta))^2d\vartheta \\ &\leq Q\frac{4\lambda_{\mathbb{H}}\varpi^2(\mu(\kappa)-\mu(0))^{2\mathbb{H}+2\aleph-2}\lambda_3}{(2\aleph-1)\Gamma^2(\aleph)}. \end{aligned} \tag{26}$$

Therefore, $\|U\mathrm{g}\|_a^2=\sup_{\kappa\in I}\mathbf{E}\|(U\mathrm{g})(\kappa)\|^2<\infty$. The $\mathcal{L}^2(\Omega,\mathcal{Q})$-continuity of $(U\mathrm{g})(\kappa)$ on $I$ (Lemma 3) ensures that $U(A)\subseteq A$. □

**Theorem 1.** *If the hypotheses $(H1)$–$(H4)$ are satisfied, we can ensure that the system (1) has a unique mild solution on $A$.*

**Proof.** The existence of mild solutions is demonstrated using PIM. For all $j\geq 0$, let

$$\begin{cases} \mathrm{g}_{j+1}(\kappa)=(U\mathrm{g}_j)(\kappa),\ j=0,1,2,\cdots, \\ \mathrm{g}_0(\kappa)=\mathrm{g}_0. \end{cases} \tag{27}$$

By Lemma 4, we have $\mathrm{g}_j\in a$, $j=0,1,2,\cdots$. Moreover, from Lemma 2 and $(H2)$, we have

$$\begin{aligned} &\mathbf{E}\|\mathrm{g}_{j+1}(\kappa)-\mathrm{g}_j(\kappa)\|^2 \\ &= \mathbf{E}\bigg\|\int_0^\kappa(\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa)-\mu(\vartheta))[\chi(\vartheta,\mathrm{g}_j(\vartheta))-\chi(\vartheta,\mathrm{g}_{j-1}(\vartheta))]\mu'(\vartheta)d\vartheta\bigg\|^2 \\ &\leq \frac{\varpi^2}{\Gamma^2(\aleph)}\bigg(\int_0^\kappa(\mu(\kappa)-\mu(\vartheta))^{2\aleph-2}(\mu'(\vartheta))^2d\vartheta\bigg)\bigg(\int_0^\kappa\mathbf{E}\|\chi(\vartheta,\mathrm{g}_j(\vartheta))-\chi(\vartheta,\mathrm{g}_{j-1}(\vartheta))\|^2d\vartheta\bigg) \\ &\leq Q\frac{\varpi^2\lambda_2(\mu(\kappa)-\mu(0))^{2\aleph-1}}{(2\aleph-1)\Gamma^2(\aleph)}\int_0^\kappa\mathbf{E}\|\mathrm{g}_j(\vartheta)-\mathrm{g}_{j-1}(\vartheta)\|^2d\vartheta \\ &\leq \bigg(\frac{\varpi^2\lambda_2(\mu(\kappa)-\mu(0))^{2\aleph-1}Q}{(2\aleph-1)\Gamma^2(\aleph)}\bigg)^2\int_0^\kappa\int_0^\vartheta\mathbf{E}\|\mathrm{g}_{j-1}(\vartheta_1)-\mathrm{g}_{j-2}(\vartheta_1)\|^2d\vartheta_1d\vartheta \\ &\leq \cdots \\ &\leq \bigg(\frac{\varpi^2\lambda_2(\mu(\kappa)-\mu(0))^{2\aleph-1}Q}{(2\aleph-1)\Gamma^2(\aleph)}\bigg)^j\int_0^\kappa\int_0^\vartheta\cdots\int_0^{\vartheta_{j-2}}\mathbf{E}\|\mathrm{g}_1(\vartheta_{j-1})-\mathrm{g}_0(\vartheta_{j-1})\|^2d\vartheta_{j-1}\ldots d\vartheta_1d\vartheta. \end{aligned} \tag{28}$$

Now we start with the nested integral:

$$\int_0^{\kappa}\int_0^{\vartheta}\cdots\int_0^{\vartheta_{j-2}} \mathbf{E}\|\mathrm{g}_1(\vartheta_{j-1})-\mathrm{g}_0(\vartheta_{j-1})\|^2\, d\vartheta_{j-1}\ldots d\vartheta_1\, d\vartheta. \tag{29}$$

The expectation term is bounded by its maximum over $[0,\kappa]$:

$$\mathbf{E}\|\mathrm{g}_1(\vartheta_{j-1})-\mathrm{g}_0(\vartheta_{j-1})\|^2 \le \sup_{\vartheta\in[0,\kappa]} \mathbf{E}\|\mathrm{g}_1(\vartheta)-\mathrm{g}_0(\vartheta)\|^2 =: M. \tag{30}$$

By factoring out the supremum, we have

$$\begin{aligned}&\int_0^{\kappa}\int_0^{\vartheta}\cdots\int_0^{\vartheta_{j-2}} \mathbf{E}\|\mathrm{g}_1(\vartheta_{j-1})-\mathrm{g}_0(\vartheta_{j-1})\|^2\, d\vartheta_{j-1}\ldots d\vartheta_1\, d\vartheta \\ &\le M\int_0^{\kappa}\int_0^{\vartheta}\cdots\int_0^{\vartheta_{j-2}} 1\, d\vartheta_{j-1}\ldots d\vartheta_1\, d\vartheta.\end{aligned} \tag{31}$$

Evaluate the nested integral using the Dirichlet formula as follows:

$$\int_0^{\kappa}\int_0^{\vartheta}\cdots\int_0^{\vartheta_{j-2}} 1\, d\vartheta_{j-1}\ldots d\vartheta_1\, d\vartheta = \frac{\kappa^j}{j!}. \tag{32}$$

By combining bounds,

$$\int_0^{\kappa}\int_0^{\vartheta}\cdots\int_0^{\vartheta_{j-2}} \mathbf{E}\|\mathrm{g}_1(\vartheta_{j-1})-\mathrm{g}_0(\vartheta_{j-1})\|^2\, d\vartheta_{j-1}\ldots d\vartheta_1\, d\vartheta \le \frac{\kappa^j}{j!}\, M \tag{33}$$

$$= \frac{\kappa^j}{j!} \sup_{\vartheta\in[0,\kappa]} \mathbf{E}\|\mathrm{g}_1(\vartheta)-\mathrm{g}_0(\vartheta)\|^2. \tag{34}$$

which implies that

$$\sup_{\kappa\in I}\mathbf{E}\|\mathrm{g}_{j+1}(\kappa)-\mathrm{g}_j(\kappa)\|^2 \le \left(\frac{\varpi^2\lambda_2(\mu(\kappa)-\mu(0))^{2\aleph-1}Q\kappa}{(2\aleph-1)\Gamma^2(\aleph)}\right)^j \frac{\sup_{\vartheta\in I}\mathbf{E}\|\mathrm{g}_1(\vartheta)-\mathrm{g}_0(\vartheta)\|^2}{j!}. \tag{35}$$

Consequently, the sequence $\{\mathrm{g}_j(\kappa)\}_{j\ge 0}\subseteq\mathcal{L}^2(\Omega,\mathcal{Q})$ is identified as a Cauchy sequence. Therefore, one can conclude the existence of $\mathrm{g}\in\mathcal{L}^2(\Omega,\mathcal{Q})$ as

$$\sup_{\kappa\in I}\mathbf{E}\|\mathrm{g}_j(\kappa)-\mathrm{g}(\kappa)\|^2 = 0. \tag{36}$$

Letting $j\to\infty$ in (27), we deduce that a mild solution exists.

For proving the uniqueness of the solution, let us establish that system (1) has a unique solution.

$$\begin{aligned}&\mathbf{E}\|\mathrm{g}(\kappa)-\mathrm{g}'(\kappa)\|^2 \\ &= \mathbf{E}\left\|\int_0^{\kappa}(\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa)-\mu(\vartheta))[\chi(\vartheta,\mathrm{g}(\vartheta))-\chi(\vartheta,\mathrm{g}'(\vartheta))]\mu'(\vartheta)d\vartheta\right\|^2 \\ &\le \frac{\varpi^2}{\Gamma^2(\aleph)}\left(\int_0^{\kappa}(\mu(\kappa)-\mu(\vartheta))^{2\aleph-2}(\mu'(\vartheta))^2 d\vartheta\right)\left(\int_0^{\kappa}\mathbf{E}\|\chi(\vartheta,\mathrm{g}(\vartheta))-\chi(\vartheta,\mathrm{g}'(\vartheta))\|^2 d\vartheta\right) \\ &\le \frac{\varpi^2\lambda_2(\mu(\kappa)-\mu(0))^{2\aleph-1}Q}{(2\aleph-1)\Gamma^2(\aleph)}\int_0^{\kappa}\mathbf{E}\|\mathrm{g}(\vartheta)-\mathrm{g}'(\vartheta)\|^2 d\vartheta.\end{aligned} \tag{37}$$

With the help of Gronwall's lemma, we conclude

$$\sup_{\kappa\in I}\mathbf{E}\|\mathrm{g}(\kappa)-\mathrm{g}'(\kappa)\|^2 = 0. \tag{38}$$

□

## 4. Controllability Results

This section is devoted to examining the approximate controllability of system (1). Describe an operator $\mathbb{Y} : A \to \mathcal{L}^2(I, \mathcal{Q})$ by

$$(\mathbb{Y}g)(\kappa) = \chi(\kappa, g(\kappa)),\ \kappa \in I.$$

The linear operator $\mathcal{E} : \mathcal{L}^2(I, \mathcal{Q}) \to X$ is given by

$$\mathcal{E}(\ell) = \int_0^L \big(\mu(L) - \mu(\vartheta)\big)^{\aleph-1} \Xi_{\aleph,\mu}\big(\mu(L) - \mu(\vartheta)\big)\ell(\vartheta)\mu'(\vartheta)d\vartheta,\ \ell \in \mathcal{L}^2(I, \mathcal{Q}).$$

The kernel of $\mathcal{E}$ is represented by $M_0(\mathcal{E})$. It is straightforward to verify that $M_0(\mathcal{E})$ is a closed subspace of $\mathcal{L}^2(I, \mathcal{Q})$, and its orthogonal complement is denoted by $M_0^{\perp}(\mathcal{E})$. Furthermore, $\mathcal{L}^2(I, \mathcal{Q})$ can be decomposed uniquely as $\mathcal{L}^2(I, \mathcal{Q}) = M_0(\mathcal{E}) \bigoplus M_0^{\perp}(\mathcal{E})$.

Define $Ran(\mathcal{V})$ as the range of the operator $\mathcal{V}$. The following assumption is also required:

**Hypothesis 5.** *For all $\ell \in \mathcal{L}^2(I, \mathcal{Q})$, there exists a function $x \in \overline{Ran(\mathcal{V})}$ such that $\mathcal{E}_\ell = \mathcal{E}_x$.*

Moreover, $Ran(\mathcal{E}) = X$.

Start with any $\ell \in \mathcal{L}^2(I, \mathcal{Q})$.

From assumption ($H5$), there exists $x \in \overline{Ran(\mathcal{V})}$ such that

$$\mathcal{E}_\ell = \mathcal{E}_x.$$

This implies

$$\mathcal{E}(\ell - x) = 0 \quad \Rightarrow \quad \ell - x \in M_0(\mathcal{E}).$$

Therefore, we can write

$$\ell = (\ell - x) + x,$$

where $\ell - x \in M_0(\mathcal{E})$ and $x \in \overline{Ran(\mathcal{V})}$.

If

$$M_0(\mathcal{E}) \cap \overline{Ran(\mathcal{V})} = \{0\},$$

then the sum is direct and from ($H5$), we obtain

$$\mathcal{L}^2(I, \mathcal{Q}) = M_0(\mathcal{E}) \bigoplus \overline{Ran(\mathcal{V})}.$$

We introduce a linear and continuous operator

$$\Lambda : M_0^{\perp}(\mathcal{E}) \to \overline{Ran(\mathcal{V})} \tag{39}$$

defined by $\Lambda s^* = x^*$, where $x^*$ is the unique element in the set $\{s^* + M_0(\mathcal{E})\} \cap \overline{Ran(\mathcal{V})}$ having the minimal norm, that is,

$$\|\Lambda s^*\| = \|x^*\| = \min\Big\{\|\theta\| :\ \theta \in \{s^* + M_0(\mathcal{E})\} \cap \overline{Ran(\mathcal{V})}\Big\}. \tag{40}$$

According to ($H5$), it follows that for every $s^* \in M_0^{\perp}(\mathcal{E})$, the intersection $s^* + M_0(\mathcal{E})\} \cap \overline{Ran(\mathcal{V})}$ is non-empty, and each element $s \in \mathcal{L}^2(I, \mathcal{Q})$ admits a unique representation of the form $s = h + h^*$. As a result, the operator $\Lambda$ is well-defined, and its norm is bounded by a constant $\tau$, i.e., $\|\Lambda\| \leq \tau$ [33].

**Remark 7.** *Assumption (H5) is indeed central for establishing approximate controllability, and it reflects the standard density requirement commonly used in controllability theory. The condition ensures that the control operator $\mathcal{V}$ can approximate the action of the evolution operator $\mathcal{E}$, which is essential for driving the system toward any desired state.*
*This assumption is not overly restrictive and can be verified for several important classes of operators, such as systems generated by sectorial operators, compact resolvent operators, and admissible boundary control operators.*

**Lemma 5** ([34]). *For each $s \in \mathcal{L}^2(I, \mathcal{Q})$ and its corresponding element $h \in M_0(\mathcal{E})$, there exists a constant $K > 0$ such that*

$$\|h\|_{\mathcal{L}^2(I,\mathcal{Q})} \leq (1+K)\|s\|_{\mathcal{L}^2(I,\mathcal{Q})}.$$

*We define the operator $\mathbb{C} : \mathcal{L}^2(I, \mathcal{Q}) \to \mathcal{L}^2(I, \mathcal{Q})$ by*

$$(\mathbb{C}p)(\kappa) = \int_0^\kappa (\mu(\kappa) - \mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa) - \mu(\vartheta))p(\vartheta)\mu'(\vartheta)d\vartheta.$$

*Let*

$$\mathbb{X}_0 = \{b \in \mathcal{L}^2(I, \mathcal{Q}) : b(\kappa) = (\mathbb{C}h)(\kappa),\ h \in M_0(\mathcal{E}),\ \kappa \in I\}.$$

*It is clear that for all $b \in \mathbb{X}_0$, $b(L) = (\mathbb{C}h)(L) = 0$.*

*Suppose that $\mathrm{g}(\cdot)$ represents a mild solution of system (7). We define the operator $\mathbb{G}_\mathrm{g} : \mathbb{X}_0 \to \mathbb{X}_0$ as*

$$(\mathbb{G}_\mathrm{g}b)(\kappa) = (\mathbb{C}h)(\kappa), \quad \kappa \in I,$$

*where $h$ is determined by the unique decomposition.*

$$\mathbb{Y}(\mathrm{g} + b) = h + x,\ h \in M_0(\mathcal{E}),\ x \in \overline{Ran(\mathcal{V})}.$$

**Theorem 2.** *Assume that the conditions $(H1)$-$(H5)$ are met. Then, system (7) is approximately controllable, that is, $\overline{\mathbb{B}_L(0)} = \mathcal{L}^2(\Omega, \mathcal{Q})$.*

**Proof.** For every $\eta \in \mathcal{L}^2(\Omega, \mathcal{Q})$, the expression

$$\eta - \Theta_{\aleph,\mu}(L)\mathrm{g}_0 - \int_0^L (\mu(L) - \mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(L) - \mu(\vartheta))\mathcal{M}(\vartheta)\mu'(\vartheta)dw^{\mathbb{H}}(\vartheta)$$

belongs to $\mathcal{L}^2(\Omega, \mathcal{Q})$. In particular, for almost every $u \in \Omega$, the quantity

$$\eta - \Theta_{\aleph,\mu}(L)\mathrm{g}_0 - \int_0^L (\mu(L) - \mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(L) - \mu(\vartheta))\mathcal{M}(\vartheta)\mu'(\vartheta)dw^{\mathbb{H}}(\vartheta)$$

lies in $X$. By condition $(H5)$, there exists an element $\ell \in \mathcal{L}^2(I, \mathcal{Q})$ such that

$$\begin{aligned}\eta - \Theta_{\aleph,\mu}(L)\mathrm{g}_0 - \int_0^L \big(\mu(L) - \mu(\vartheta)\big)^{\aleph-1}\Xi_{\aleph,\mu}\big(\mu(L) - \mu(\vartheta)\big)\mathcal{M}(\vartheta)\mu'(\vartheta)dw^{\mathbb{H}}(\vartheta) \\ = \int_0^L \big(\mu(L) - \mu(\vartheta)\big)^{\aleph-1}\Xi_{\aleph,\mu}\big(\mu(L) - \mu(\vartheta)\big)\ell(\vartheta)\mu'(\vartheta)d\vartheta.\end{aligned} \tag{41}$$

By applying condition $(H5)$ once more, we obtain the existence of an element $x \in \overline{Ran(\mathcal{V})}$ such that

$$\int_0^L \big(\mu(L) - \mu(\vartheta)\big)^{\aleph-1}\Xi_{\aleph,\mu}\big(\mu(L) - \mu(\vartheta)\big)\ell(\vartheta)\mu'(\vartheta)d\vartheta = \int_0^L \big(\mu(L) - \mu(\vartheta)\big)^{\aleph-1}\Xi_{\aleph,\mu}\big(\mu(L) - \mu(\vartheta)\big)x(\vartheta)\mu'(\vartheta)d\vartheta. \tag{42}$$

Therefore,

$$\eta = \Theta_{\aleph,\mu}(L)g_0 + \int_0^L \big(\mu(L)-\mu(\vartheta)\big)^{\aleph-1}\Xi_{\aleph,\mu}\big(\mu(L)-\mu(\vartheta)\big)\mathcal{M}(\vartheta)\mu'(\vartheta)dw^{\mathbb{H}}(\vartheta) + \int_0^L \big(\mu(L)-\mu(\vartheta)\big)^{\aleph-1}\Xi_{\aleph,\mu}\big(\mu(L)-\mu(\vartheta)\big)x(\vartheta)\mu'(\vartheta)d\vartheta. \tag{43}$$

Notice that $x \in \overline{Ran(\mathcal{V})}$. Then, for any given $\varepsilon > 0$, one can find $\varrho_\varepsilon$ as

$$\sup_{\kappa\in I}\mathbf{E}\|\mathcal{V}\varrho_\varepsilon(\kappa)-x(\kappa)\|^2 < \frac{\Gamma^2(\aleph)(2\aleph-1)\varepsilon}{\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}Q}. \tag{44}$$

Define

$$\eta_\varepsilon = \Theta_{\aleph,\mu}(L)g_0 + \int_0^L \big(\mu(L)-\mu(\vartheta)\big)^{\aleph-1}\Xi_{\aleph,\mu}\big(\mu(L)-\mu(\vartheta)\big)\mathcal{M}(\vartheta)\mu'(\vartheta)dw^{\mathbb{H}}(\vartheta) + \int_0^L \big(\mu(L)-\mu(\vartheta)\big)^{\aleph-1}\Xi_{\aleph,\mu}\big(\mu(L)-\mu(\vartheta)\big)\mathcal{V}\varrho_\varepsilon(\vartheta)\mu'(\vartheta)d\vartheta. \tag{45}$$

Thus, $\eta_\varepsilon \in \mathbb{B}_L(0)$. Moreover,

$$\begin{aligned} &E\|\eta-\eta_\varepsilon\|^2 \\ &= \mathbf{E}\bigg\|\int_0^L \big(\mu(L)-\mu(\vartheta)\big)^{\aleph-1}\Xi_{\aleph,\mu}\big(\mu(L)-\mu(\vartheta)\big)[\mathcal{V}\varrho_\varepsilon(\vartheta)-x(\vartheta)]\mu'(\vartheta)d\vartheta\bigg\|^2 \\ &\le \frac{\varpi^2}{\Gamma^2(\aleph)}\mathbf{E}\bigg(\int_0^L \big(\mu(L)-\mu(\vartheta)\big)^{\aleph-1}\|\mathcal{V}\varrho_\varepsilon - x\|\mu'(\vartheta)d\vartheta\bigg)^2 \\ &\le \frac{\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}LQ}{\Gamma^2(\aleph)(2\aleph-1)}\sup_{\vartheta\in I}\mathbf{E}\|\mathcal{V}\varrho_\varepsilon(\vartheta)-x(\vartheta)\|^2 \\ &< \varepsilon, \end{aligned} \tag{46}$$

indicating that system (7) possesses approximate controllability. □

**Lemma 6.** *Assume that conditions* $(H1)$ *through* $(H4)$ *hold. Then, the operator* $\mathbb{G}_g$ *admits a fixed point* $b_0 \in \mathbb{X}_0$*, provided that*

$$\frac{4\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}Q(1+K)^2\lambda_2}{\Gamma^2(\aleph)(2\aleph-1)} < 1. \tag{47}$$

**Proof.** For any $\nu > 0$, define the set $\varsigma_\nu = \{s \in \mathbb{X}_0 : \|s\|_{\mathcal{L}^2(I,\mathcal{Q})} \le \nu\}$. We now aim to show that $\mathbb{G}_g(\varsigma_\nu) \subseteq \varsigma_\nu$. Suppose, on the contrary, that this is not the case. Then, for every $\nu > 0$, there exists some $b \in \varsigma_\nu$ such that $\|\mathbb{G}_g(b)\|^2_{\mathcal{L}^2(I,\mathcal{Q})} > \nu$. Consequently,

$$\nu^2 < \|\mathbb{G}_g(b)\|^2_{\mathcal{L}^2(I,\mathcal{Q})} = \|\mathbb{C}h\|^2_{\mathcal{L}^2(I,\mathcal{Q})}. \tag{48}$$

In fact, using Lemma 5, we have

$$\begin{aligned} &\|(\mathbb{C}h)(\kappa)\|^2 = \bigg\|\int_0^\kappa (\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa)-\mu(\vartheta))h(\vartheta)\mu'(\vartheta)d\vartheta\bigg\|^2 \\ &\le \frac{\varpi^2}{\Gamma^2(\aleph)}\bigg(\int_0^\kappa (\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\|h(\vartheta)\|\mu'(\vartheta)d\vartheta\bigg)^2 \\ &\le \frac{\varpi^2}{\Gamma^2(\aleph)}\bigg(\int_0^\kappa (\mu(\kappa)-\mu(\vartheta))^{2\aleph-2}(\mu'(\vartheta))^2d\vartheta\bigg)\bigg(\int_0^\kappa \|h(\vartheta)\|^2d\vartheta\bigg) \\ &\le \frac{\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}Q}{\Gamma^2(\aleph)(2\aleph-1)}\|h\|^2_{\mathcal{L}^2(I,\mathcal{Q})} \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}Q(K+1)^2}{\Gamma^2(\aleph)(2\aleph-1)}\|\mathbb{Y}(\mathrm{g}+b)\|^2_{\mathcal{L}^2(I,\mathcal{Q})} \\
&\leq \frac{\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}Q(K+1)^2}{\Gamma^2(\aleph)(2\aleph-1)}\left(\int_0^L \|\chi(\kappa,(\mathrm{g}+b)(\kappa))-\chi(\kappa,0)+\chi(\kappa,0)\|^2 d\kappa\right) \\
&\leq \frac{2\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}Q(K+1)^2}{\Gamma^2(\aleph)(2\aleph-1)}\left(\int_0^L \|\chi(\kappa,(\mathrm{g}+b)(\kappa))-\chi(\kappa,0)\|^2+\|\chi(\kappa,0)\|^2 d\kappa\right) \\
&\leq \frac{2\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}Q(K+1)^2}{\Gamma^2(\aleph)(2\aleph-1)}\left[\int_0^L (\lambda_2\|(\mathrm{g}+b)(\kappa)\|^2+\tau_\chi^2)d\kappa\right] \\
&\leq \frac{2\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}Q(K+1)^2}{\Gamma^2(\aleph)(2\aleph-1)}\left[2\lambda_2\left(\int_0^L \|\mathrm{g}(\kappa)\|^2 d\kappa+\nu^2\right)+\tau_\chi^2 L\right],
\end{aligned}
\tag{49}
$$

where $\tau_\chi = \max_{\kappa\in I}\|\chi(\kappa,0)\|$. Hence,

$$
\begin{aligned}
\nu^2 <&\|\mathbb{G}_{\mathrm{g}}(b)\|^2_{\mathcal{L}^2(I,\mathcal{Q})} \\
&= \|\mathbb{C}n\|^2_{\mathcal{L}^2(I,\mathcal{Q})} \\
&\leq \frac{2\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}Q(K+1)^2}{\Gamma^2(\aleph)(2\aleph-1)}\left[2\lambda_2\int_0^L \|\mathrm{g}(\kappa)\|^2 d\kappa+2\lambda_2\nu^2+\tau_\chi^2 L\right].
\end{aligned}
\tag{50}
$$

Dividing by $\nu^2$ on both sides of (50) and taking limitation as $\nu\to\infty$, it follows that

$$
\frac{4\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}Q(K+1)^2}{\Gamma^2(\aleph)(2\aleph-1)} \geq 1, \tag{51}
$$

which contradicts (47). Therefore, the operator $\mathbb{G}_{\mathrm{g}}$ maps $\varsigma_\nu$ into itself.

Now, from assumption $(H4)$ and Lemma 2, it follows that $\Xi_{\aleph,\mu}(\kappa)$ is a compact operator. Consequently, $\mathbb{G}_{\mathrm{g}}$ is also compact. Then, by Schauder's theorem, there exists a fixed point $b_0\in\mathbb{X}_0$ such that

$$
\mathbb{G}_{\mathrm{g}}(b_0) = \mathbb{C}\mathrm{h}_0 = b_0.
$$

□

**Theorem 3.** *Assume that the conditions $(H1)$ to $(H5)$ are satisfied and that (47) holds. Then, the system described by (1) is approximately controllable.*

**Proof.** We define $\mathrm{g}(\cdot)$ as the mild solution of system (7), formally expressed as

$$
\begin{aligned}
\mathrm{g}(\kappa) =&\Theta_{\aleph,\mu}(\kappa)\mathrm{g}_0+\int_0^\kappa(\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa)-\mu(\vartheta))\mathcal{V}\varrho(\vartheta)\mu'(\vartheta)d\vartheta \\
&+\int_0^\kappa(\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa)-\mu(\vartheta))\mathcal{M}(\vartheta)\mu'(\vartheta)dw^{\mathbb{H}}(\vartheta).
\end{aligned}
\tag{52}
$$

Recalling that $\mathbb{G}_{\mathrm{g}}(b_0) = \mathbb{C}\mathrm{h}_0 = b_0$, then we get

$$
\mathbb{Y}(\mathrm{g}+b_0)(\kappa) = \mathrm{h}_0(\kappa)+x_0(\kappa). \tag{53}
$$

Applying $\mathbb{C}$ yields

$$
\mathbb{C}\mathbb{Y}(\mathrm{g}+b_0)(\kappa) = \mathbb{C}\mathrm{h}_0(\kappa)+\mathbb{C}x_0(\kappa) = b_0(\kappa)+\mathbb{C}x_0(\kappa). \tag{54}
$$

Hence,

$$
\mathrm{g}(\kappa)+\mathbb{C}\mathbb{Y}(\mathrm{g}+b_0)(\kappa) = \mathrm{g}(\kappa)+b_0(\kappa)+\mathbb{C}x_0(\kappa). \tag{55}
$$

Denote $\mathrm{g}'(\kappa) = \mathrm{g}(\kappa)+b_0(\kappa)$, then

$$
\mathrm{g}(\kappa)+\mathbb{C}\mathbb{Y}\mathrm{g}'(\kappa) = \mathrm{g}'(\kappa)+\mathbb{C}x_0(\kappa). \tag{56}
$$

Hence,

$$\begin{aligned}
&\mathrm{g}'(\kappa) = \mathrm{g}(\kappa) + \mathbb{C}\mathbb{Y}\mathrm{g}'(\kappa) - \mathbb{C}x_0(\kappa)\\
&= \Theta_{\aleph,\mu}(\kappa)\mathrm{g}_0 + \int_0^\kappa (\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa)-\mu(\vartheta))[\mathcal{V}\varrho(\vartheta) - x_0(\vartheta)]\mu'(\vartheta)d\vartheta\\
&+ \int_0^\kappa (\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa)-\mu(\vartheta))\chi(\vartheta,\mathrm{g}'(\vartheta))\mu'(\vartheta)d\vartheta\\
&+ \int_0^\kappa (\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa)-\mu(\vartheta))\mathcal{M}(\vartheta)\mu'(\vartheta)dw^{\mathbb{H}}(\vartheta).
\end{aligned} \tag{57}$$

Consequently, for $\widetilde{\mathrm{g}} = \mathrm{g} + b_0$, we have

$$\begin{cases}
\mathbb{D}^{\aleph,\mu}\widetilde{\mathrm{g}}'(\kappa) = A\widetilde{\mathrm{g}}(\kappa) + (\mathcal{V}\varrho - x_0)(\kappa) + \chi(\kappa,\widetilde{\mathrm{g}}'(\kappa)) + \mathcal{M}(\kappa)\frac{dw^{\mathbb{H}}(\kappa)}{d\kappa},\ \kappa\in[0,L],\\
\widetilde{\mathrm{g}}(0) = \mathrm{g}_0.
\end{cases} \tag{58}$$

By the definition of $\mathbb{C}$ and $h_0 \in M_0(\mathcal{E})$, we have $b_0(0) = b_0(L) = 0$. Furthermore,

$$\begin{aligned}
&\widetilde{\mathrm{g}}(0) = \mathrm{g}(0) + b_0(0) = \mathrm{g}_0,\\
&\widetilde{\mathrm{g}}(L) = \mathrm{g}(L) + b_0(L) = \mathrm{g}(L) \in \mathbb{C}_L(0).
\end{aligned} \tag{59}$$

Next, we will prove $\mathbb{C}_L(0) \subseteq \mathbb{C}_L(\chi)$ . Since $x_0 \in \overline{Ran(\mathcal{V})}$, there exists $p \in \mathcal{L}^2_{\mathbb{F}}(I,\mathcal{B})$ such that

$$\sup_{\kappa\in I}\mathbf{E}\|\mathcal{V}p - x_0\|^2 < \varepsilon. \tag{60}$$

Let $\tilde{\varrho} = \varrho - p$, and assume that $\mathrm{g}_{\tilde{\varrho}}$, so

$$\begin{cases}
\mathbb{D}^{\aleph,\mu}\phi(\kappa) = \mathcal{V}\phi(\kappa) + \mathcal{V}\tilde{\varrho}(\kappa) + \chi(\kappa,\phi(\kappa)) + \mathcal{M}(\kappa)\frac{dw^{\mathbb{H}}(\kappa)}{d\kappa},\ \kappa\in[0,L],\\
\phi(0) = \mathrm{g}_0.
\end{cases} \tag{61}$$

Thus,

$$\begin{aligned}
\mathrm{g}_{\tilde{\varrho}}(\kappa) =&\Theta_{\aleph,\mu}(\kappa)\mathrm{g}_0 + \int_0^\kappa (\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}\big(\mu(\kappa)-\mu(\vartheta)\big)[\chi(\vartheta,\mathrm{g}_{\tilde{u}}(\vartheta)) + \mathcal{V}\tilde{\varrho}(\vartheta)]\mu'(\vartheta)d\vartheta\\
&+ \int_0^\kappa (\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa)-\mu(\vartheta))\mathcal{M}(\vartheta)\mu'(\vartheta)dw^{\mathbb{H}}(\vartheta),\ \kappa\in[0,L],
\end{aligned} \tag{62}$$

and $\mathrm{g}_{\tilde{\varrho}}(L) \in \mathbb{C}_L(\chi)$.

On the other hand,

$$\begin{aligned}
&E\|\mathrm{g}'(\kappa) - \mathrm{g}_{\tilde{\varrho}}(\kappa)\|^2\\
&= \mathbf{E}\bigg\| \int_0^\kappa (\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa)-\mu(\vartheta))[(\mathcal{V}p)(\vartheta) - x(\vartheta)]\mu'(\vartheta)d\vartheta\\
&+ \int_0^\kappa (\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa)-\mu(\vartheta))[\chi(\vartheta,\mathrm{g}'(\vartheta)) - \chi(\vartheta,\mathrm{g}_{\tilde{\varrho}}(\vartheta))]\mu'(\vartheta)d\vartheta\bigg\|^2\\
&\le 2\mathbf{E}\bigg\| \int_0^\kappa (\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa)-\mu(\vartheta))[(\mathcal{V}p)(\vartheta) - x(\vartheta)]\mu'(\vartheta)d\vartheta\bigg\|^2\\
&+ 2\mathbf{E}\bigg\| \int_0^\kappa (\mu(\kappa)-\mu(\vartheta))^{\aleph-1}\Xi_{\aleph,\mu}(\mu(\kappa)-\mu(\vartheta))[\chi(\vartheta,\mathrm{g}'(\vartheta)) - \chi(\vartheta,\mathrm{g}_{\tilde{u}}(\vartheta))]\mu'(\vartheta)d\vartheta\bigg\|^2\\
&\le \frac{2\varpi^2}{\Gamma^2(\aleph)}\bigg(\int_0^\kappa \big(\mu(\kappa)-\mu(\vartheta)\big)^{2\aleph-2}(\mu'(\vartheta))^2d\vartheta\bigg)\bigg(\int_0^\kappa \mathbf{E}\|\mathcal{V}p - x\|^2d\vartheta\bigg)\\
&+ \frac{2\varpi^2\tau_1^2}{\Gamma^2(\aleph)}\bigg(\int_0^\kappa (\mu(\kappa)-\mu(\vartheta))^{2\aleph-2}(\mu'(\vartheta))^2d\vartheta\bigg)\bigg(\int_0^\kappa \mathbf{E}\|\mathrm{g}'(\vartheta) - \mathrm{g}_{\tilde{\varrho}}(\vartheta)\|^2d\vartheta\bigg)\\
&\le \frac{2\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}LQ\varepsilon}{\Gamma^2(\aleph)(2\aleph-1)} + \frac{2\varpi^2\tau_1^2(\mu(\kappa)-\mu(0))^{2\aleph-1}Q}{\Gamma^2(\aleph)(2\aleph-1)}\int_0^\kappa \mathbf{E}\|\mathrm{g}'(\vartheta) - \mathrm{g}_{\tilde{\varrho}}(\vartheta)\|^2d\vartheta.
\end{aligned} \tag{63}$$

Define $\mathbb{A}(\kappa) = \mathbf{E}\|\mathrm{g}'(\kappa) - \mathrm{g}_{\tilde{\varrho}}(\kappa)\|^2$. By applying Gronwall's inequality, we obtain the following result:

$$\mathbf{E}\|\mathrm{g}'(\kappa) - \mathrm{g}_{\tilde{\varrho}}(\kappa)\|^2 \leq \frac{2\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}LQ\varepsilon}{\Gamma^2(\aleph)(2\aleph-1)} \exp\left\{\frac{2\varpi^2\tau_1^2(\mu(\kappa)-\mu(0))^{2\aleph-1}LQ}{\Gamma^2(\aleph)(2\aleph-1)}\right\}. \quad (64)$$

Moreover,

$$\begin{aligned}\mathbf{E}\|\widetilde{\mathrm{g}}(L) - \mathrm{g}_{\tilde{\varrho}}(L)\|^2 &\leq \sup_{\kappa\in I}\mathbf{E}\|\mathrm{g}'(\kappa) - \mathrm{g}_{\tilde{\varrho}}(\kappa)\|^2 \\ &\leq \frac{2\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}LQ\varepsilon}{\Gamma^2(\aleph)(2\aleph-1)} \exp\left\{\frac{2\varpi^2\tau_1^2(\mu(\kappa)-\mu(0))^{2\aleph-1}LQ}{\Gamma^2(\aleph)(2\aleph-1)}\right\}.\end{aligned} \quad (65)$$

Therefore,

$$\begin{aligned}\mathbf{E}\|\mathrm{g}(L) - \mathrm{g}_{\tilde{\varrho}}(L)\|^2 &= \mathbf{E}\|\widetilde{\mathrm{g}}(L) - \mathrm{g}_{\tilde{\varrho}}(L)\|^2 \\ &\leq \frac{2\varpi^2(\mu(\kappa)-\mu(0))^{2\aleph-1}LQ\varepsilon}{\Gamma^2(\aleph)(2\aleph-1)} \exp\left\{\frac{2\varpi^2\tau_1^2(\mu(\kappa)-\mu(0))^{2\aleph-1}LQ}{\Gamma^2(\aleph)(2\aleph-1)}\right\}.\end{aligned} \quad (66)$$

which implies that $\mathbb{C}_L(0) \subseteq \mathbb{C}_L(\chi)$. By Theorem 2, $\overline{\mathbb{C}_L(0)} = \mathcal{L}^2(\Omega, \mathcal{Q})$. Therefore, $\overline{\mathbb{C}_L(\chi)} = \mathcal{L}^2(\Omega, \mathcal{Q})$ and the proof is complete. □

In the next section, we present an example to illustrate the theoretical results we have established regarding Ex-Un and approximate controllability.

## 5. Example

Consider the following fractional stochastic control system:

$$\begin{cases}\mathbb{D}^{\aleph,\kappa^2}\mathrm{s}(\kappa,y) = \frac{\partial^2}{\partial y^2}\mathrm{s}(\kappa,y) + \chi(\kappa,\mathrm{s}(\kappa,y)) + \mathcal{V}\varrho(\kappa,y) + \mathcal{M}(\kappa)\frac{dw^{\mathbb{H}}(\kappa)}{d\kappa},\ \kappa\in[0,1],\ y\in(0,\pi),\\ \mathrm{s}(\kappa,0) = \mathrm{s}(\kappa,\pi) = 0,\ \kappa\in[0,1],\\ \mathrm{s}(0,y) = \mathrm{s}_0(y),\ y\in(0,\pi),\end{cases} \quad (67)$$

where $\mathbb{D}^{\aleph,\kappa^2}$ is the $\mu$-CFD.

Let $\mathcal{Q} = \mathcal{Y} = \mathcal{L}^2(0,\pi)$, $I = [0,1]$. Define the operator $\mathcal{U} : \mathbb{N}(\mathcal{U}) \subset \mathcal{Q} \to \mathcal{Q}$ by $\mathcal{U}s = \frac{\partial^2 s}{\partial y^2}$, where

$$\mathbb{N}(\mathcal{U}) = \left\{\mathrm{s}\in\mathcal{Q} : \mathrm{s}, \frac{\partial s}{\partial y} \text{ are absolutely continuous, } \frac{\partial^2\mathrm{s}}{\partial y^2}\in\mathcal{Q}, \mathrm{s}(0) = \mathrm{s}(\pi) = 0\right\}.$$

Define $\mathrm{g}_j(y) = \sqrt{\frac{2}{\pi}}\sin(jy)$ for $j = 1, 2, \ldots$. Observe that the sequence $\{\mathrm{g}_j\}_{j\geq 1}$ forms a complete orthonormal basis of eigenfunctions for the operator $A$. One can verify that the operator $A$ generates a strongly continuous semigroup $\{\Theta_\kappa\}_{\kappa\geq 0}$ which possesses compactness, analyticity, and self-adjointness properties [35]. Therefore, assumption $(H4)$ holds. Construct the infinite-dimensional space $\mathcal{B}$ via

$$\mathcal{B} = \left\{\varrho : \mathbb{R}\to\mathbb{R} : \varrho = \sum_{j=2}^{\infty}\varrho_j\mathrm{g}_j \text{ with } \sum_{j=2}^{\infty}\varrho_j^2 < \infty\right\}.$$

The norm in $\mathcal{B}$ is defined by $\|\varrho\|_{\mathcal{B}} = (\sum_{j=1}^{\infty} \varrho_j^2)^{\frac{1}{2}}$. Define the bounded linear operator $\mathcal{V} : \mathcal{B} \to \mathcal{Q}$ as follows:

$$\mathcal{V}\varrho = 2\varrho_2 \mathrm{g}_1 + \sum_{j=2}^{\infty} \varrho_j \mathrm{g}_j, \text{ for } \varrho = \sum_{j=2}^{\infty} \varrho_j \mathrm{g}_j \in \mathcal{B}.$$

Select a sequence of positive numbers $\{\xi_j\}_{j=1}^{\infty}$. Define the operator $\digamma : \mathcal{Y} \to \mathcal{Y}$ such that $\digamma \mathrm{g}_j = \aleph_j \mathrm{g}_j$. Suppose that

$$tr(\digamma) = \sum_{j=1}^{\infty} \sqrt{\xi_j} < \infty.$$

The process $w^{\mathbb{H}}(\kappa)$ is defined by

$$w^{\mathbb{H}}(\kappa) = \sum_{j=1}^{\infty} \sqrt{\xi_j} \mathbb{W}_j^{\mathbb{H}}(\kappa) \mathrm{g}_j, \ \kappa \geq 0, \frac{1}{2} < \mathbb{H} < 1,$$

with $\{\mathbb{W}_j^{\mathbb{H}}\}_{j \in \mathcal{N}}$ representing a collection of one-dimensional fBm processes that are statistically independent.

Let

$$\mathrm{g}(\kappa)(y) = \mathrm{s}(\kappa, y), \ \chi(\kappa, \mathrm{g}(\kappa))(y) = \chi(\kappa, \mathrm{s}(\kappa, y)), \ \varrho(\kappa)(y) = \varrho(\kappa, y).$$

Then, (67) can be reformulated as

$$\begin{cases} \mathbb{D}^{\aleph,\mu} \mathrm{g}(\kappa) = \mathcal{U}\mathrm{g}(\kappa) + \mathcal{V}\varrho(\kappa) + \chi(\kappa, \mathrm{g}(\kappa)) + \mathcal{M}(\kappa) \frac{dw^{\mathbb{H}}(\kappa)}{d\kappa}, \ \kappa \in [0,1], \\ \mathrm{g}(0) = \mathrm{g}_0. \end{cases} \tag{68}$$

Define $\chi(\kappa, \mathrm{s}(\kappa, y)) = \frac{\exp^{-t} |\mathrm{s}(\kappa,y)|}{(1+\exp^{\kappa})(1+|\mathrm{s}(\kappa,y)|)}$. Clearly, we have

$$\|\chi(\kappa, \mathrm{s}(\kappa, y))\| \leq |\mathrm{s}(\kappa, y)|, \tag{69}$$

and

$$\begin{aligned} &\|\chi(\kappa, \mathrm{s}_1(\kappa))(y) - \chi(\kappa, \mathrm{s}_2(\kappa))(y)\| \\ &= \frac{\exp^{-t} ||\mathrm{s}_1(\kappa, y)| - |\mathrm{s}_2(\kappa, y)||}{(1+\exp^{\kappa})(1+|\mathrm{s}_1(\kappa, y)|)(1+|\mathrm{s}_2(\kappa, y)|)} \\ &\leq \frac{\exp^{-t}}{1+\exp^{\kappa}} |\mathrm{s}_1(\kappa, y) - \mathrm{s}_2(\kappa, y)| \\ &\leq \frac{1}{2} |\mathrm{s}_1(\kappa, y) - \mathrm{s}_2(\kappa, y)|. \end{aligned} \tag{70}$$

Therefore, the conditions $(H1)$ and $(H2)$ hold. For the operators chosen in this study, assumption (H5) is indeed satisfied. Recall that the control operator $\mathcal{V} : \mathcal{B} \to \mathcal{Q}$ is defined by

$$\mathcal{V}\varrho = 2\varrho_2 \mathrm{g}_1 + \sum_{j=2}^{\infty} \varrho_j \mathrm{g}_j, \quad \varrho = \sum_{j=2}^{\infty} \varrho_j \mathrm{g}_j \in \mathcal{B},$$

where $\{\mathrm{g}_j\}_{j \geq 1}$ forms a complete orthonormal basis of $\mathcal{Q} = L^2(0, \pi)$. By construction,

1. The first mode $\mathrm{g}_1$ can be reached via $\varrho_2$;
2. All higher modes $\mathrm{g}_j$, $j \geq 2$, are directly included in the range of $\mathcal{V}$.

Hence, the closure of the range of $\mathcal{V}$ satisfies

$$\overline{Ran(\mathcal{V})} = \mathcal{Q}.$$

Next, consider the integral operator

$$\mathcal{E}_x = \int_0^T \Theta_{T-\kappa} x \, d\kappa,$$

where $\Theta_\kappa$ is the strongly continuous semigroup generated by $\mathcal{U}$. Since $\Theta_\kappa$ is analytic, compact, and self-adjoint, it densely maps $\mathcal{Q}$ into itself. Therefore, for any $\ell \in L^2(I, \mathcal{Q})$, there exists $x \in \overline{Ran(\mathcal{V})}$ such that

$$\mathcal{E}_\ell = \mathcal{E}_x,$$

and moreover, the range of $\mathcal{E}$ satisfies

$$Ran(\mathcal{E}) = \mathcal{Q}.$$

In conclusion, assumption (H5) is satisfied for the chosen operators $\mathcal{U}$ and $\mathcal{V}$. This provides a guarantee of the approximate controllability of the system within the considered framework.

## 6. Conclusions

This study investigates the controllability of FSDEs characterized by the $\mu$-CFD and influenced by fBm. The Ex-Un solutions are analyzed using the PIM within the framework of $\mu$-CFD. To derive sufficient conditions for controllability, we apply the Schauder fixed-point approach. To demonstrate the effectiveness of the theoretical findings, an illustrative example is presented.

Much of the current literature on approximate controllability is based on restrictive assumptions, such as the presence of a uniformly bounded nonlinear term and the approximate controllability of the related fractional linear system. In contrast, this study eliminates these constraints and explores the approximate controllability of fractional stochastic differential equations influenced by fBm.

## 7. Future Directions

Our future research will focus on developing a disease transmission model within the generalized FSDE framework, incorporating fBm and Poisson jumps. This model will account for long-memory effects, random fluctuations, and sudden external shocks to provide a more realistic representation of complex epidemic dynamics.

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