

Article

On the Focal Geometry of Translation Surfaces: Flatness, Minimality, and Classification Results

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Abstract

In this study, we investigate the focal surfaces associated with translation surfaces in Euclidean 3-space from the viewpoint of differential geometry. We begin by defining the translation surface generated by two planar curves and derive the corresponding focal surfaces using the framework of the Frenet frame. Analytical conditions are obtained under which the focal surfaces exhibit minimality or flatness. Several theorems are proven to classify the focal images, supported by illustrative examples. The results provide insights into the curvature structure of translation surfaces and contribute to the broader understanding of their geometric behavior.

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1. Introduction and Preliminaries

Differential geometry has long been a fertile ground for the study of surfaces, leading to the development of numerous concepts such as curvature, minimality, geodesics, and focal structures. From classical minimal surfaces to modern applications in computer-aided geometric design, a wide variety of surfaces such as ruled, canal, developable, and translation types have been extensively analyzed from both theoretical and applied perspectives. These investigations aim to not only understand the intrinsic and extrinsic properties of surfaces but also explore their roles in disciplines such as physics, engineering, and architecture [1–9].

Among the various classes of surfaces investigated in differential geometry, translation surfaces form a distinguished subclass due to their geometric simplicity and practical relevance. Translation surfaces play a significant role in architectural design and construction due to their unique properties and structural capabilities. These surfaces are particularly useful in the engineering of various shell structures. Their applications extend to the design of elements such as membranes, domes, cable grids, vaults, and foldable structures. These applications leverage the geometric and structural efficiency of translation surfaces, making them a preferred choice in modern architecture and engineering for creating both aesthetically pleasing and structurally sound designs.

A translation surface is generally defined as a surface which is generated by translating one of two arbitrary space curves parallel to itself. This process ensures that each point on

the resulting surface gives a curve which is a translation of the other curve. A generalized form of a translation surface can be parametrized as follows:

$$\varphi(u, v) = \alpha_1(u) + \alpha_2(v),$$

where α_1 and α_2 are arbitrary space curves parametrized by u and v that may represent the arc-length parameters.

A notable type of translation surface is generated by two planar curves that lie in orthogonal planes. This type is referred to as a Monge-type translation surface (a translation surface of plane type) and is represented in the following form:

$$\varphi(u, v) = (u, 0, g(u)) + (0, v, h(v)), \quad (1)$$

where $g(u)$ and $h(v)$ are differentiable functions of the variables u and v , respectively.

Scherk's surface, proposed by H. Scherk in [10], is uniquely characterized as the only non-flat minimal surface that can be described as a translation surface. In addition, translation surfaces have been extensively studied in Euclidean spaces from various geometric perspectives by researchers in [11–15].

The concept of line congruences was first introduced in the field of visualization by Hagen et al. in 1991 [16]. Line congruences are, in fact, surfaces generated by transforming one surface into another through lines. The focal surface is an example of such congruences. For a given surface M with parameterization $\varphi(u, v)$, the line congruence is defined as follows:

$$C(u, v, z) = \varphi(u, v) + z\mathcal{E}(u, v). \quad (2)$$

Here, $\mathcal{E}(u, v)$ denotes a unit direction vector field. For each pair (u, v) , Equation (2) describes a line of congruence, called the generatrix. Each generatrix of C contains two points known as focal points, and the focal surface is the locus of these focal points. If $\mathcal{E}(u, v) = N(u, v)$, the unit normal vector field of the surface, then C is termed a normal congruence. In this case, the parametric formula of the focal surface $C = \varphi^*(u, v)$ corresponding to $\varphi(u, v)$ is given by the following:

$$\varphi_i^*(u, v) = \varphi(u, v) + k_i^{-1}N(u, v); \quad i = 1, 2, \quad (3)$$

where k_i s; ($i = 1, 2$) are the principal curvature functions of $\varphi(u, v)$ [17].

The geometry of focal curves and focal surfaces has been studied extensively; see, for instance, [17–22]. In particular, ref. [23] examines the focal surface of a surface of revolution in Galilean 3-space, while refs. [24,25] investigate the focal surfaces of tube surfaces in Euclidean and Galilean spaces. To the best of our knowledge, however, the focal surface of translation surfaces in Euclidean 3-space has not been developed. In this work, we close this gap by analyzing the focal surfaces of the cylindrical subclass of a Monge-type translation surface (a translation surface of plane type) in Euclidean 3-space, namely those generated by two planar curves with at least one generating curve being a straight line (so that the surface is a generalized cylinder). We derive explicit formulas in terms of the principal curvatures, establish conditions for flatness and minimality, and present classification results supported by illustrative examples.

Basic Concepts and Notation

For clarity and later reference, we record the standard definitions and notation for surfaces in Euclidean 3-space used in this work, including the first and the second fundamental forms, the unit normal, the Gaussian and the mean curvatures, principal curvatures, and asymptotic and geodesic curves.

Let M be a regular surface parametrized by $\varphi(u, v)$, where $(u, v) \in D \subset \mathbb{E}^2$ in $\mathbb{E}^3$. The tangent space $T_p M$ at an arbitrary point $p = \varphi(u, v)$ is spanned by the partial derivative vectors φ_u and φ_v . The coefficients of the first fundamental form, also known as the metric coefficients, are defined as follows:

$$E = \langle \varphi_u, \varphi_u \rangle, \quad F = \langle \varphi_u, \varphi_v \rangle, \quad G = \langle \varphi_v, \varphi_v \rangle, \quad (4)$$

where $\varphi_u = \frac{\partial \varphi}{\partial u}$ and $\varphi_v = \frac{\partial \varphi}{\partial v}$. Here, $W^2 = EG - F^2 \neq 0$, indicating that the surface patch $\varphi(u, v)$ is regular. With the aid of the first partial derivatives, the unit normal vector field of the surface M is given as

$$N = \frac{\varphi_u \times \varphi_v}{\|\varphi_u \times \varphi_v\|}. \quad (5)$$

The coefficients of the second fundamental form are given by

$$l = \langle \varphi_{uu}, N \rangle, \quad m = \langle \varphi_{uv}, N \rangle, \quad n = \langle \varphi_{vv}, N \rangle, \quad (6)$$

where $\varphi_{uu} = \frac{\partial^2 \varphi}{\partial u^2}$, $\varphi_{uv} = \frac{\partial^2 \varphi}{\partial u \partial v}$, and $\varphi_{vv} = \frac{\partial^2 \varphi}{\partial v^2}$. The Gaussian curvature and the mean curvature functions of the surface M are defined by

$$K = \frac{ln - m^2}{EG - F^2}, \quad (7)$$

and

$$H = \frac{En + Gl - 2Fm}{2(EG - F^2)}, \quad (8)$$

respectively [26].

The principal curvature functions k_i s; ($i = 1, 2$) of M are calculated with the help of the Gaussian curvature and the mean curvature as [27].

$$k_1 = H - \sqrt{H^2 - K} \quad k_2 = H + \sqrt{H^2 - K} \quad (9)$$

A unit speed curve $\gamma : I \rightarrow M$ on the surface M is classified as an asymptotic curve if its normal curvature vanishes in the direction of γ' . This condition holds if and only if γ'' is orthogonal to the surface's normal vector N . Moreover, γ is considered a geodesic curve on M , if the tangential component $(\gamma'')^T$ of its acceleration vanishes, as discussed in [26].

2. Translation Surfaces in $\mathbb{E}^3$

In this section, we consider translation surfaces of plane type in Euclidean 3-space and provide some results regarding these surfaces.

Definition 1. Let α_1 and α_2 be the plane curves in Euclidean 3-space given by

$$\begin{aligned} \alpha_1(u) &= (u, 0, g(u)), \\ \alpha_2(v) &= (0, v, h(v)). \end{aligned}$$

The sum of these curves corresponds to a surface M called a translation surface, which is presented by the Monge patch

$$M : \varphi(u, v) = (u, v, g(u) + h(v)). \quad (10)$$

Tangent vector fields of M are

$$\varphi_u = (1, 0, g'(u)), \quad \varphi_v = (0, 1, h'(v)). \quad (11)$$

Then, one can obtain the coefficients of the first fundamental form and the unit normal vector field N as

$$E = 1 + g'^2, \quad F = g'h', \quad G = 1 + h'^2$$

and

$$N = \frac{1}{\sqrt{g'^2 + h'^2 + 1}}(-g', -h', 1), \quad (12)$$

respectively. Also, the second partial derivatives are

$$\varphi_{uu} = (0, 0, g''(u)), \quad \varphi_{uv} = (0, 0, 0), \quad \varphi_{vv} = (0, 0, h''(v)). \quad (13)$$

Thus, the coefficients of the second fundamental form are given as

$$l = \frac{g''}{\sqrt{g'^2 + h'^2 + 1}}, \quad m = 0, \quad n = \frac{h''}{\sqrt{g'^2 + h'^2 + 1}}.$$

Therefore, the Gaussian curvature and the mean curvatures of M are obtained as follows:

$$K = \frac{g''h''}{(g'^2 + h'^2 + 1)^2} \quad (14)$$

and [28].

$$H = \frac{g''(1 + h'^2) + h''(1 + g'^2)}{2(g'^2 + h'^2 + 1)^{\frac{3}{2}}}. \quad (15)$$

Theorem 1. Let M be a translation surface given with the parametrization (10) in Euclidean 3-space. Then, the principal curvature functions of M are, respectively, given as

$$k_1 = \frac{-\sqrt{\{g''(1 + h'^2) + h''(1 + g'^2)\}^2 - 4g''h''(g'^2 + h'^2 + 1)} + g''(1 + h'^2) + h''(1 + g'^2)}{2(g'^2 + h'^2 + 1)^{\frac{3}{2}}} \quad (16)$$

and

$$k_2 = \frac{\sqrt{\{g''(1 + h'^2) + h''(1 + g'^2)\}^2 - 4g''h''(g'^2 + h'^2 + 1)} + g''(1 + h'^2) + h''(1 + g'^2)}{2(g'^2 + h'^2 + 1)^{\frac{3}{2}}}, \quad (17)$$

where $g(u)$ and $h(v)$ are differentiable functions.

Proof. Assume that M is a translation surface given with the parametrization (10) in $\mathbb{E}^3$. Then, by substituting Equations (14) and (15) in (9), we obtain the result. $\square$

Moreover, we give the following theorems that characterize the parameter curves of the translation surface.

Theorem 2. Let M be a translation surface given with the parametrization (10) in Euclidean 3-space. Then,

- (i) u —parameter curves of the surface M are asymptotic curves if and only if the curve α_1 is congruent to a straight line.
- (ii) v —parameter curves of the surface M are asymptotic curves if and only if the curve α_2 is congruent to a straight line.

Proof. Assume that M is a translation surface with the parametrization (10) in $\mathbb{E}^3$.

- (i) According to the definition of an asymptotic curve and using (12) and (13), we get $\langle \varphi_{uu}, N \rangle = 0$ if and only if $g'' = 0$, which means that $g(u) = au + b$, $a, b = \text{const.}$ i.e., α_1 is congruent to a straight line.

(ii) Similar to the above proof, using (12) and (13), we get $\langle \varphi_{vv}, N \rangle = 0$ if and only if $h'' = 0$, which means that $h(v) = cv + d$, $c, d = \text{const.}$ i.e., α_2 is congruent to a straight line. $\square$

Corollary 1. *The surface M is flat if one of its parametric curves is a straight line.*

Theorem 3. *Let M be a translation surface given with the parametrization (10) in Euclidean 3-space. Then,*

(i) *u —parameter curves of the surface M are geodesic curves if and only if at least one of the following statements holds:*

- (a) α_1 is congruent to a straight line.
- (b) α_1 is congruent to a straight line that is parallel to the x -axis.
- (c) α_1 and α_2 are congruent to straight lines that are parallel to the x and y axes, respectively.

(ii) *v —parameter curves of the surface are geodesic curves if and only if at least one of the following statements holds:*

- (a) α_2 is congruent to a straight line.
- (b) α_2 is congruent to a straight line that is parallel to the y -axis.
- (c) α_1 and α_2 are congruent to straight lines that are parallel to the x and y axes, respectively.

Proof. Assume that M is a translation surface with the parametrization (10) in $\mathbb{E}^3$.

(i) From (12) and (13), we get $\varphi_{uu} \wedge N = 0$ if and only if $g''h' = 0$ and $g''g' = 0$, which means that either $g'' = 0$ or $g' = 0$ or both $h' = 0$ and $g' = 0$.

(ii) Again using the same equations, we obtain $\varphi_{vv} \wedge N = 0$ if and only if $h''h' = 0$ and $h''g' = 0$, which means that either $h'' = 0$ or $h' = 0$ or both $g' = 0$ and $h' = 0$. $\square$

Corollary 2. *The surface M is flat if one of its parametric curves is a straight line.*

3. Focal Surface of a Translation Surface in $\mathbb{E}^3$

In this section, we investigate the focal surface corresponding to the Monge-type translation surface (a translation surface of plane type), restricting to the cases treated in Sections 3.1 and 3.2 where one generating curve is a straight line, i.e., where the surface is a generalized (possibly right) cylinder.

Definition 2. *Suppose that a translation surface M is given by the parameterization (10). Then, focal surfaces of this surface are represented by*

$$\varphi_{1,2}^*(u, v) = (u, v, g(u) + h(v)) + \frac{k_{1,2}^{-1}}{\sqrt{g'^2 + h'^2 + 1}}(-g', -h', 1), \quad (18)$$

where the principal curvatures k_1 and k_2 are specified in (16) and (17).

3.1. In Case the First Curve α_1 Is a Straight Line (M Is a Generalized Cylinder)

Suppose that the translation surface M consists of the curves $\alpha_1(u)$ and $\alpha_2(v)$, and the curve $\alpha_1(u)$ is congruent to a straight line. Then, taking $g''(u) = 0$ ($g(u) = au + b$, $a, b = \text{const.}$, $a \neq 0$) in (16) and (17), we obtain

$$k_1(u, v) = \frac{(1 + a^2)(h''(v) - |h''(v)|)}{2(1 + a^2 + (h'(v))^2)^{3/2}}, \quad k_2(u, v) = \frac{(1 + a^2)(h''(v) + |h''(v)|)}{2(1 + a^2 + (h'(v))^2)^{3/2}}, \quad (19)$$

where primes denote derivatives with respect to v . In particular, exactly one principal curvature vanishes. More precisely,

$$\text{if } h''(v) \geq 0, \text{ then } k_1(u, v) = 0 \text{ and } k_2(u, v) = \frac{(1 + a^2) h''(v)}{(1 + a^2 + (h'(v))^2)^{3/2}},$$

and

$$\text{if } h''(v) \leq 0, \text{ then } k_2(u, v) = 0 \text{ and } k_1(u, v) = \frac{(1 + a^2) h''(v)}{(1 + a^2 + (h'(v))^2)^{3/2}}.$$

It is obvious that no focal surface can be obtained by using $k_1(u, v) = 0$ or $k_2(u, v) = 0$. Thus, we have only one focal surface for the principal curvature $\frac{h''(1+a^2)}{(1+a^2+h'^2)^{\frac{3}{2}}}$. Hence, using (18), we obtain the parametrization of the focal surface to be

$$\varphi^*(u, v) = \left(u - a \frac{1 + a^2 + h'^2(v)}{h''(v)(1 + a^2)}, v - h'(v) \frac{1 + a^2 + h'^2(v)}{h''(v)(1 + a^2)}, au + b + h(v) + \frac{1 + a^2 + h'^2(v)}{h''(v)(1 + a^2)} \right). \quad (20)$$

The first tangent vector is

$$\varphi_u^* = (1, 0, a) \quad (21)$$

and the second tangent vector is

$$\varphi_v^* = (\lambda_1(v), \lambda_2(v), \lambda_3(v)), \quad (22)$$

where

$$\begin{aligned} \lambda_1(v) &= \frac{a(-2h'(v)h''^2(v) + h'''(v)(1 + a^2 + h'^2(v)))}{h''^2(v)(1 + a^2)}, \\ \lambda_2(v) &= \frac{h'(v)(-3h'(v)h''^2(v) + h'''(v)(1 + a^2 + h'^2(v)))}{h''^2(1 + a^2)}, \\ \lambda_3(v) &= \frac{(3 + a^2)h'(v)h''^2(v) - h'''(v)(1 + a^2 + h'^2(v))}{h''^2(v)(1 + a^2)}. \end{aligned} \quad (23)$$

The coefficients of the first fundamental form are

$$\begin{aligned} E^* &= 1 + a^2, \\ F^* &= \lambda_1 + \lambda_3 a, \\ G^* &= \lambda_1^2 + \lambda_2^2 + \lambda_3^2. \end{aligned} \quad (24)$$

By using the vector product of (21) and (22), one can obtain the unit normal vector field as

$$N^* = \frac{1}{W^*}(-a\lambda_2, -\lambda_3 + a\lambda_1, \lambda_2), \quad (25)$$

where

$$\begin{aligned} W^{*2} &= E^*G^* - F^{*2}, \\ &= \lambda_2^2(1 + a^2) + (-\lambda_3 + a\lambda_1)^2. \end{aligned}$$

The second partial derivatives are

$$\begin{aligned} \varphi_{uu}^* &= (0, 0, 0), \\ \varphi_{uv}^* &= (0, 0, 0), \\ \varphi_{vv}^* &= ((\lambda_1)_v, (\lambda_2)_v, (\lambda_3)_v). \end{aligned} \quad (26)$$

Hence, using (25) and (26), we have the coefficients l^* , m^* , and n^* as

$$\begin{aligned} l^* &= m^* = 0, \\ n^* &= \frac{-a\lambda_2(\lambda_1)_v + (-\lambda_3 + a\lambda_1)(\lambda_2)_v + \lambda_2(\lambda_3)_v}{W^*}. \end{aligned} \quad (27)$$

Theorem 4. Suppose that M^* is the focal surface of M given with the parametrization (20). Then, the Gaussian curvature of the focal surface M^* vanishes, i.e., the focal surface M^* is flat.

Proof. By using Equations (7), (24) and (27), the desired result is obtained. $\square$

Corollary 3. Suppose that M^* is the focal surface of M given with the parametrization (20). Then, the focal surface M^* is minimal if and only if the second fundamental form coefficient n^* vanishes.

Proof. By using Equations (8), (24) and (27), the mean curvature of the focal surface M^* is yielded as

$$H^* = \frac{n^*(1+a^2)}{2W^{*2}}. \quad (28)$$

Thus, clearly, M^* is minimal $\Leftrightarrow n^* = 0$. This completes the proof. $\square$

Theorem 5. Let M^* be the focal surface of M given with the parametrization (20) in Euclidean 3-space. Then, M^* is minimal if and only if the differentiable function $h(v)$ is given by

$$h(v) = \pm \sqrt{\frac{c_1 - (1+a^2)^2(v+c_2)^2}{1+a^2}} + c_3, \quad (29)$$

where $c_i, i = 1, 2, 3$ are real constants.

Proof. Using (28), we know if the focal surface M^* is minimal, $n^* = 0$. That is,

$$-a\lambda_2(\lambda_1)_v + (-\lambda_3 + a\lambda_1)(\lambda_2)_v + \lambda_2(\lambda_3)_v = 0. \quad (30)$$

This means that

$$\lambda_2^2 \left(\left(\frac{\lambda_3}{\lambda_2} \right)_v - a \left(\frac{\lambda_1}{\lambda_2} \right)_v \right) = 0. \quad (31)$$

Thus, by substituting (23) into (31), we encounter the following differential equation

$$3h'(v)h''(v) = h'''(v)(1+a^2+h^2(v)),$$

which has the solution (29). This completes the proof. $\square$

Theorem 6. Let M^* be the focal surface of M given with the parametrization (20) in Euclidean 3-space. Then,

- (i) u —parameter curves of the focal surface M^* are always asymptotic curves.
- (ii) v —parameter curves of the focal surface M^* are asymptotic curves if and only if

$$h(v) = \pm \sqrt{\frac{c_1 - (1+a^2)^2(v+c_2)^2}{1+a^2}} + c_3,$$

where $c_i, i = 1, 2, 3$ are real constants.

Proof. Assume that M^* is the focal surface of M given with the parametrization (20) in $\mathbb{E}^3$.

(i) From Equations (25) and (26), the equality $\langle \varphi_{uu}^*, N^* \rangle = 0$ always holds. This completes the proof.

(ii) Again using Equations (25) and (26), we get $\langle \varphi_{vv}^*, N^* \rangle = 0$ if and only if (30) holds. By solving Equation (30), we obtain the result. $\square$

Theorem 7. Let M^* be the focal surface of M given with the parametrization (20) in Euclidean 3-space. Then,

- (i) u —parameter curves of the focal surface M^* are always geodesic curves.
- (ii) v —parameter curves of the focal surface M^* are geodesic curves if and only if the following equations hold:

$$\begin{aligned} \lambda_2(\lambda_2)_v - (\lambda_3)_v(a\lambda_1 - \lambda_3) &= 0, \\ -(\lambda_1)_v\lambda_2 - a(\lambda_3)_v\lambda_2 &= 0, \\ (\lambda_1)_v(a\lambda_1 - \lambda_3) + a\lambda_2(\lambda_2)_v &= 0. \end{aligned} \quad (32)$$

Proof. Assume that M^* is the focal surface of M given with the parametrization (20) in $\mathbb{E}^3$.

(i) From Equations (25) and (26), the equality $\varphi_{uu}^* \wedge N^* = 0$ always holds, which means that u —parameter curves are always geodesic.

(ii) Again using Equations (25) and (26), we get $\varphi_{vv}^* \wedge N^* = 0$ if and only if (32) holds. Thus, we obtain the result. $\square$

Example 1. Choosing α_1 is a straight line with the function $g(u) = u + 2$ and $h(v) = v^2$ in (10) and (20), so the parameterizations of the translation surface and its focal surface are written as

$$M : \varphi(u, v) = (u, v, u + v^2 + 2),$$

$$M^* : \varphi^*(u, v) = \left(u - v^2 - \frac{1}{2}, -2v^3, u + 2v^2 + \frac{5}{2} \right),$$

respectively. One can visualize them by using the GLMakie package of Julia (see Figure 1).

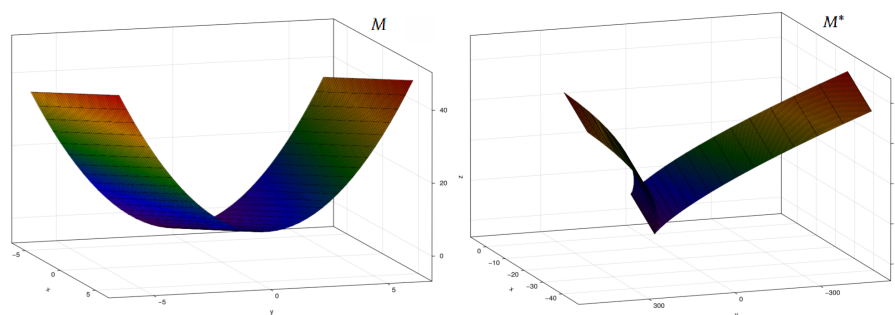


Figure 1. Three-dimensional view of a generalized cylinder ($\alpha_1(u)$ is a straight line) and its focal surface.

We now consider the special case of the function $g(u) = au + b$ when $a = 0$. In this case, $g(u) = b$ is constant and hence $g'(u) = 0$. Consequently, the generating curve $\alpha_1(u)$ becomes a straight line parallel to the x -axis, and the corresponding translation surface M reduces to a right cylinder whose rulings are parallel to the x -axis. Substituting $a = 0$ into (20), we obtain the parametrization of the focal surface M^*

$$\varphi^*(u, v) = \left(u, v - h'(v) \frac{1 + h'^2(v)}{h''(v)}, b + h(v) + \frac{1 + h'^2(v)}{h''(v)} \right), \quad (33)$$

and the unit normal vector field N^* of the focal surface M^* as

$$N^* = \frac{1}{W^*}(0, -\lambda_5, \lambda_4)$$

where

$$\begin{aligned}\lambda_4(v) &= \frac{-3h'^2(v)h''^2(v) + h'(v)h'''(v)(1 + h'^2(v))}{h''^2}, \\ \lambda_5(v) &= \frac{3h'(v)h''^2(v) - h'''(v)(1 + h'^2(v))}{h''^2(v)}.\end{aligned}$$

and

$$W^{*2} = \lambda_4^2 + \lambda_5^2.$$

Example 2. Assume, $g(u) = 1$ and $h(v) = \cos v$ in (10) and (33). The surface M and its focal surface M^* are represented as

$$M : \varphi(u, v) = (u, v, \cos v + 1),$$

$$M^* : \varphi^*(u, v) = \left(u, v - \tan v(1 + \sin^2 v), 1 - \frac{2\sin^2 v}{\cos v}\right),$$

respectively. When they are drawn, the resulting views are as follows (see Figure 2).

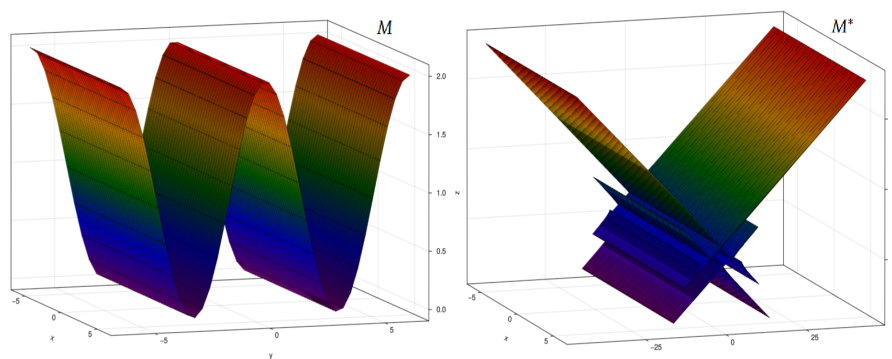


Figure 2. Three-dimensional view of a right cylinder ($g'(u) = 0$) and its focal surface.

3.2. In Case the Second Curve α_2 Is a Straight Line (M Is a Generalized Cylinder)

Assume that the translation surface M is created by the curves $\alpha_1(u)$ and $\alpha_2(v)$ and the curve $\alpha_2(v)$ corresponds to a straight line. By substituting $h''(v) = 0$ ($h(v) = cv + d$, $c, d = \text{const.}, c \neq 0$) in (16) and (17) and carrying out computations analogous to those in Section 3.1, we obtain the parametrization of the focal surface as

$$\varphi^*(u, v) = \left(u - g'(u)\frac{1 + c^2 + g'^2(u)}{g''(u)(1 + c^2)}, v - c\frac{1 + c^2 + g'^2(u)}{g''(u)(1 + c^2)}, cv + d + g(u) + \frac{1 + c^2 + g'^2(u)}{g''(u)(1 + c^2)}\right). \quad (34)$$

The coefficients of the first fundamental form are represented by

$$\begin{aligned}E^* &= \lambda_6^2 + \lambda_7^2 + \lambda_8^2, \\ F^* &= \lambda_7 + \lambda_8 c, \\ G^* &= 1 + c^2,\end{aligned} \quad (35)$$

where

$$\begin{aligned}\lambda_6(u) &= \frac{g'(u)(-3g'(u)g''^2(u) + g'''(u)(1 + c^2 + g'^2(u)))}{g''^2(u)(1 + c^2)}, \\ \lambda_7(u) &= \frac{c(-2g'(u)g''^2(u) + g'''(u)(1 + c^2 + g'^2(u)))}{g''^2(u)(1 + c^2)}, \\ \lambda_8(u) &= \frac{(3 + c^2)g'(u)g''^2(u) - g'''(u)(1 + c^2 + g'^2(u))}{g''^2(u)(1 + c^2)}.\end{aligned}\quad (36)$$

Then, we obtain unit normal vector field as

$$N^* = \frac{1}{W^*}(-\lambda_8 + c\lambda_7, -c\lambda_6, \lambda_6), \quad (37)$$

where

$$\begin{aligned}W^{*2} &= E^*G^* - F^{*2}, \\ &= \lambda_6^2(1 + c^2) + (-\lambda_8 + c\lambda_7)^2.\end{aligned}$$

The second partial derivatives are

$$\begin{aligned}\varphi_{uu}^* &= ((\lambda_6)_u, (\lambda_7)_u, (\lambda_8)_u), \\ \varphi_{uv}^* &= (0, 0, 0), \\ \varphi_{vv}^* &= (0, 0, 0).\end{aligned}\quad (38)$$

Therefore, the coefficients l^* , m^* , and n^* are obtained as

$$\begin{aligned}l^* &= \frac{(-\lambda_8 + c\lambda_7)(\lambda_6)_u - c\lambda_6(\lambda_7)_u + \lambda_6(\lambda_8)_u}{W^*}, \\ m^* &= n^* = 0.\end{aligned}\quad (39)$$

Theorem 8. Suppose that M^* is the focal surface of M given with the parametrization (34). Then, the Gaussian curvature of the focal surface M^* vanishes, namely the focal surface M^* is flat.

Proof. Using (7), (35) and (39), we yield the desired result. $\square$

Corollary 4. Suppose that M^* is the focal surface of M given with the parametrization (34). Then, the focal surface M^* is minimal if and only if the second fundamental form coefficient l^* vanishes.

Proof. Using (8), (35) and (39), the mean curvature of the focal surface M^* is obtained as

$$H^* = \frac{l^*(1 + c^2)}{2W^{*2}}.$$

Hence, it is clear that M^* is minimal $\Leftrightarrow l^* = 0$. This completes the proof. $\square$

Theorem 9. Let M^* be the focal surface of M given with the parametrization (34) in Euclidean 3-space. Then, M^* is minimal if and only if the differentiable function $g(u)$ is presented by

$$g(u) = \pm \sqrt{\frac{c_1 - (1 + c^2)^2(u + c_2)^2}{1 + c^2}} + c_3, \quad (40)$$

where c_i , $i = 1, 2, 3$ are real constants.

Proof. From Corollary 4, remember that if the focal surface M^* is minimal, $l^* = 0$. Namely,

$$(-\lambda_8 + c\lambda_7)(\lambda_6)_u - c\lambda_6(\lambda_7)_u + \lambda_6(\lambda_8)_u = 0. \quad (41)$$

Hence, we write

$$\lambda_6^2 \left(\left(\frac{\lambda_8}{\lambda_6} \right)_u - c \left(\frac{\lambda_7}{\lambda_6} \right)_u \right) = 0. \quad (42)$$

Therefore, by substituting (36) into (42), the differential equation

$$3g'(u)g''^2(u) = g'''(u)(1 + c^2 + g'^2(u))$$

holds and the solution (40) is obtained by using the GLMakie package of Julia. This completes the proof. $\square$

Theorem 10. Let M^* be the focal surface of M given with the parametrization (34) in Euclidean 3-space. Then,

(i) u —parameter curves of the focal surface M^* are asymptotic curves if and only if the differentiable function $g(u)$ is given by

$$g(u) = \pm \sqrt{\frac{c_1 - (1 + c^2)^2(u + c_2)^2}{1 + c^2}} + c_3,$$

where $c_i, i = 1, 2, 3$ are real constants.

(ii) v —parameter curves of the focal surface M^* are always asymptotic curves.

Proof. Assume that M^* is the focal surface of M given with the parametrization (34) in $\mathbb{E}^3$.

(i) Using Equations (37) and (38), we obtain $\langle \varphi_{uu}^*, N^* \rangle = 0$ if and only if (41) holds. Thus, we obtain the result.

(ii) From Equations (37) and (38), the equality $\langle \varphi_{vv}^*, N^* \rangle = 0$ always holds, which means that v -parameter curves are always asymptotic. $\square$

Theorem 11. Let M^* be the focal surface of M given with the parametrization (34) in Euclidean 3-space. Then,

(i) u —parameter curves of the focal surface M^* are geodesic curves if and only if the following equations hold:

$$\begin{aligned} \lambda_6(\lambda_7)_u + c\lambda_6(\lambda_7)_u &= 0, \\ -\lambda_6(\lambda_6)_u + (\lambda_8)_u(-\lambda_8 + c\lambda_7) &= 0, \\ -c\lambda_6(\lambda_6)_u - (\lambda_7)_u(-\lambda_8 + c\lambda_7) &= 0. \end{aligned} \quad (43)$$

(ii) v —parameter curves of the focal surface M^* are always geodesic curves.

Proof. Assume that M^* is the focal surface of M given with the parametrization (34) in $\mathbb{E}^3$.

(i) From Equations (37) and (38), we get $\varphi_{uu}^* \wedge N^* = 0$ if and only if (43) holds. Thus, we obtain the result.

(ii) From Equations (37) and (38), the equality $\varphi_{vv}^* \wedge N^* = 0$ always holds, which means that v -parameter curves are always geodesic. $\square$

Example 3. Taking $g(u) = e^u$, and α_2 is a straight line with the function $h(v) = -v + 1$ in (10) and (34), we write the related surfaces as

$$M : \varphi(u, v) = (u, v, e^u - v + 1),$$

$$M^* : \varphi^*(u, v) = \left(u - \frac{e^{2u}}{2} - 1, \frac{e^u}{2} + \frac{1}{e^u} + v, \frac{3e^u}{2} + \frac{1}{e^u} - v + 1 \right),$$

respectively. Moreover, their figures are as follows (see Figure 3).

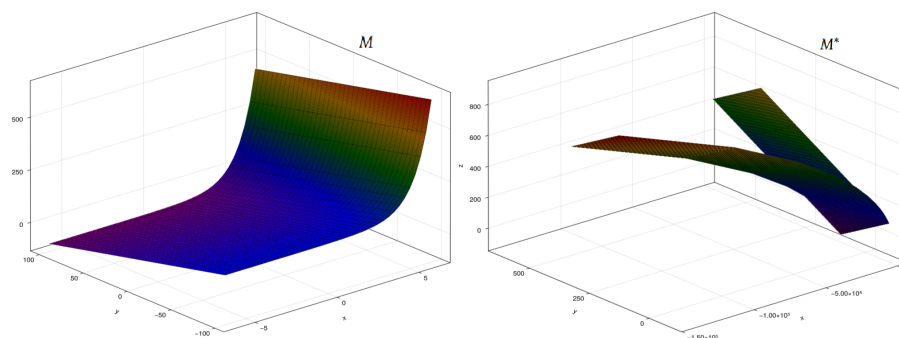


Figure 3. Three-dimensional view of a generalized cylinder ($\alpha_2(u)$ is a straight line) and its focal surface.

Lastly, assume that the translation surface M is generated by the curves $\alpha_1(u)$ and $\alpha_2(v)$, and suppose that $\alpha_2(v)$ is a straight line parallel to the y -axis. In this case, the associated function is constant, say $h(v) = d$ with $d \in \mathbb{R}$, and hence $h'(v) = 0$. Geometrically, the v -direction contributes a family of parallel rulings, and therefore the resulting translation surface M reduces to a right cylinder whose rulings are parallel to the y -axis. By substituting $c = 0$ (equivalently, $h(v) = d$) into (34), we obtain the parametrization of the focal surface M^*

$$\varphi^*(u, v) = \left(u - g'(u) \frac{1 + g'^2(u)}{g''(u)}, v, d + g(u) + \frac{1 + g'^2(u)}{g''(u)} \right), \quad (44)$$

and the unit normal vector field as

$$N^* = \frac{1}{W^*} (-\lambda_{10}, 0, \lambda_9),$$

where

$$\begin{aligned} \lambda_9(u) &= \frac{g'(u)(-3g'(u)g''^2(u) + g'''(u)(1 + g'^2(u)))}{g''^2(u)}, \\ \lambda_{10}(u) &= \frac{3g'(u)g''^2(u) - g'''(u)(1 + g'^2(u))}{g''^2(u)}. \end{aligned}$$

Also,

$$W^{*2} = \lambda_9^2 + \lambda_{10}^2.$$

Example 4. Assume that $g(u) = u^4$ and $h(v) = -1$ in (10) and (44). Then, we write the parameterizations of the translation surface M and its focal surface M^* as

$$M : \varphi(u, v) = (u, v, u^4 - 1),$$

$$M^* : \varphi^*(u, v) = \left(\frac{2u - 16u^7}{3}, v, \frac{7u^4}{3} + \frac{1}{12u^2} - 1 \right).$$

Thus, their visualizations are as follows (see Figure 4).

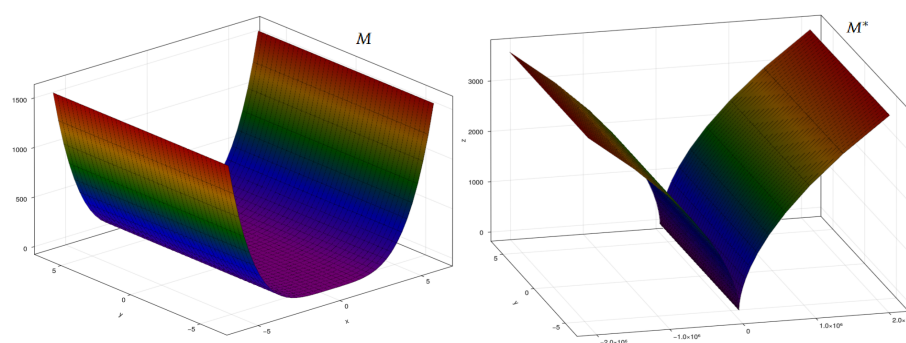


Figure 4. Three-dimensional view of a right cylinder ($h'(v) = 0$) and its focal surface.

4. Conclusions

Focal surfaces have been a subject of continued interest in differential geometry due to their rich structure and applications in geometric modeling and visualization. In the literature, researchers have studied focal surfaces in connection with various surface types such as canal surfaces, ruled surfaces, and surfaces given by Monge patches. However, focal surfaces associated specifically with translation surfaces have received relatively limited attention. Most studies have focused on the geometric properties of translation surfaces themselves, including their curvature, minimality, and applications in architectural design and structural mechanics.

In this study, we addressed this gap by systematically investigating the focal surfaces of translation surfaces in Euclidean 3-space. Starting with the general parametrization of translation surfaces generated by two planar curves, we derived explicit expressions for their associated focal surfaces based on the principal curvatures. Special cases, including when the generating curves are straight lines or when the translation surface forms a right cylinder, were analyzed in detail. We provided conditions under which the focal surfaces are flat or minimal and examined the behavior of their parameter curves in terms of geodesicity and asymptoticity. Furthermore, we illustrated our results through several examples and visualizations.

This work contributes to the theoretical understanding of the interplay between the geometry of a translation surface and the structure of its focal counterparts. As a direction for future research, one may extend these findings to higher-dimensional Euclidean spaces or investigate analogous constructions in non-Euclidean geometries. Additionally, considering the role of focal surfaces in geometric design, future work may explore computational methods for generating and visualizing focal surfaces from various families of translation surfaces with applications in computer-aided design and engineering.

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