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Common Fixed Point Approximation for Asymptotically Nonexpansive Mapping in Hyperbolic Space with Application

Tehreem Ishtiaq [1], Afshan Batool [1], Aftab Hussain [2,*] and Hamed Alsulami [2]

[1] Department of Mathematical Sciences, Fatima Jinnah Women University, Rawalpindi 46000, Pakistan; teh.ishtiaq2@gmail.com (T.I.); afshan.batool@fjwu.edu.pk (A.B.)
[2] Department of Mathematics, King Abdulaziz University, Jeddah, Saudi Arabia; hhaalsalmi@kau.edu.sa
* Correspondence: aniassuirathka@kau.edu.sa

Abstract

This study presents a common fixed-point iteration process that includes two asymptotically nonexpansive self-mappings in a hyperbolic space and their delta convergence. To support our results, we provide an example with a comparison table and sufficient conditions for a modified iteration scheme to have strong convergence to approximate the fixed point.

Keywords: modified iteration; common fixed point; nonexpansive mapping; convergence; hyperbolic space

MSC: 47H09; 47H10

1. Introduction

Nonlinear functional analysis is closely linked to fixed-point theory, which provides a robust framework for the examination of numerous nonlinear problems that are challenging to address through conventional analytical methods. Fixed-point approximation techniques are particularly valuable in this domain, especially when determining exact solutions is unattainable.

In many areas of applied science, a common and effective strategy is to convert a nonlinear problem into an equivalent fixed-point problem defined by an appropriate operator. This approach is valuable because solving the operator equation directly provides a solution to the original problem. A well-known example of this idea is the Banach Contraction Principle (BCP) [1], which ensures the convergence of the basic iterative process known as Picard iteration. This method is the cornerstone of many iterative techniques. Iterative methods play a central role in approximating the fixed points of nonlinear mappings within Hilbert and Banach spaces [2]. Kirk [3] also introduced the concept of asymptotically nonexpansive mappings. An important extension of this framework arises in hyperbolic spaces, a class of metric spaces characterized by the presence of convex structures. The notion of hyperbolic space has been developed in several ways due to the variety of convex structures that can be defined in such spaces. Kohlenbach [4] showed that Banach spaces can, in fact, be viewed as special instances of hyperbolic spaces. The authors in [5] established existence and uniqueness results for relevant operator equations, demonstrating the effectiveness of the fixed-point approach in generalized metric contexts. Some work [6–8] highlights the structural advantages of W-hyperbolic spaces in studying iterative schemes and provides new insights into the stability and behaviour of three-step algorithms. Recently, Agarwal et al. [9] explored fixed-point theorems and established convergence results for

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monotone, nearly asymptotically nonexpansive mappings in partially ordered hyperbolic metric spaces (2019). Khan, Fukhar-Ud-Din, and Ahmad Khan [10] proposed an implicit iterative algorithm for two finite families of nonexpansive mappings in hyperbolic spaces, establishing strong and weak convergence results under suitable conditions. Questions related to hyperbolic groups, one of the primary subjects of study in geometric group theory, have motivated and dominated the study of hyperbolic spaces. The nonlinear class of hyperbolic spaces provides a rich geometrical structure and a comprehensive abstract theoretical framework for metric fixed-point theory. Fixed-point theory and approximation methods have been extended to include hyperbolic spaces [2].

Let T be a nonempty subset of a Banach space $\mathcal{B}$, and let $X : T \to T$ be a mapping. A sequence $\{t_a\}$ in T is said to be an *approximating fixed-point sequence* if

$$\lim_{a\to\infty} \|t_a - Xt_a\| = 0.$$

The mapping X is said to be Lipschitzian if, for each $a \in \mathbb{N}$, there exists a positive number k_a such that

$$\|X^a p - X^a q\| \leq k_a \|p - q\| \quad \text{for all } p, q \in T.$$

A Lipschitzian mapping X is called uniformly k-Lipschitzian if $k_a = k$ for all $a \in \mathbb{N}$, and *asymptotically non-expansive* if $k_a \geq 1$ for all a with $\lim_{a\to\infty} k_a = 1$. Clearly, every non-expansive mapping X (i.e., $\|Xp - Xq\| \leq \|p - q\|$ for all $p, q \in T$) is asymptotically non-expansive.

Goebel and Kirk [3] introduced the class of asymptotically non-expansive mappings as an important generalization of non-expansive mappings. Schu [11] initiated the study of convergence of the iteration process to a fixed point of an asymptotically nonexpansive mapping using an improved version of the Mann scheme:

$$p_{a+1} = (1 - \gamma_a)p_a + \gamma_a X^a p_a.$$

Many authors have studied iterative methods for determining fixed points in asymptotically nonexpansive mappings. Şahin and Başarır established strong convergence results for a modified S-iteration process applied to asymptotically quasi-nonexpansive mappings in CAT(0) spaces. Khan analyzed the convergence of iterative sequences with errors for asymptotically quasi-nonexpansive mappings and demonstrated their applicability to various operator equations. Qihou investigated iterative sequences for asymptotically quasi-nonexpansive mappings, providing foundational convergence results in nonlinear analysis. Kaczor proved weak convergence properties for almost orbits of asymptotically nonexpansive commutative semigroups in metric settings. They developed fixed-point iteration processes for non-Lipschitzian asymptotically quasi-nonexpansive mappings, establishing new convergence criteria [4]. Their results provided a foundation for analyzing the stability and convergence behavior of iterative schemes in both linear and nonlinear settings.

Building on these developments, several researchers have extended such methods to hyperbolic and uniformly convex spaces for a wider class of nonlinear mappings. In this context, our proposed scheme contributes to the ongoing effort by demonstrating strong convergence under generalized contractive conditions in hyperbolic spaces. Tan and Xu [12] examined the convergence of the modified Ishikawa iterative algorithm:

$$\begin{cases} p_1 \in T, \\ p_{a+1} = (1-\gamma_a)p_a + \gamma_a X^a q_a, \\ q_a = (1-\sigma_a)p_a + \sigma_a X^a p_a. \end{cases} \tag{1}$$

where γ_a and $\sigma_a \in [0,1]$. Recently, Agarwal et al. [9], in an attempt to achieve a faster convergence rate, proposed a modified iteration called the S-iteration technique:

$$\begin{cases} p_1 \in T, \\ p_{a+1} = (1-\gamma_a)X^a p_a + \gamma_a X^a q_a, \\ q_a = (1-\sigma_a)p_a + \sigma_a X^a p_a. \end{cases} \tag{2}$$

Research on iterative approximation in hyperbolic metric spaces has grown significantly over the last ten years. The literature defines many types of hyperbolic spaces, with Kohlenbach's concept being the most frequently used definition [13].

Suppose that a hyperbolic space $(\mathcal{B}, d, \Omega)$ is a triplet where $(\mathcal{B}, d)$ is a metric space and a mapping $\Omega : \mathcal{B} \times \mathcal{B} \times [0,1] \to \mathcal{B}$ fulfills the following:

1. $d(g, \Omega(p,q,\gamma)) \leq (1-\gamma)d(g,p) + \gamma d(g,q)$,
2. $d(\Omega(p,q,\gamma), \Omega(p,q,\sigma)) = |\gamma - \sigma| d(p,q)$,
3. $\Omega(p,q,\gamma) = \Omega(q,p,1-\gamma)$,
4. $d(\Omega(p,s,\gamma), \Omega(q,u,\gamma)) \leq (1-\gamma)d(p,q) + \gamma d(s,u)$.

Where $\gamma_a, \sigma_a \in [0,1]$ and all $p, q, s, u, g \in B$.

Example

Define $X : T \to T$ by

$$X(z) = e^{i\theta}t \qquad (\theta \in \mathbb{R}).$$

1. $|e^{i\theta}t| = |t| < 1$.
2. X preserves the Poincaré distance:

$$\frac{|X(p) - X(q)|}{|1 - X(p)\overline{X(q)}|} = \frac{|e^{i\theta}p - e^{i\theta}q|}{|1 - p\bar{q}|} = \frac{|p-q|}{|1-p\bar{q}|}.$$

3. Fixed point: solve $X(t) = t \Rightarrow (e^{i\theta} - 1)t = 0$. So $t = 0$ is a fixed point (if $e^{i\theta} = 1$, then X is the identity and every point is fixed).

Therefore, X is a simple non-expansive map on a hyperbolic space that clearly has a fixed point. Saluja [14] modified Equations (1) and (2) introduced by Khan et al. [15] to approximate a common fixed point in hyperbolic space as follows:

$$\begin{cases} p_1 \in T, \\ p_{a+1} = \Omega(X^a p_a, X^a q_a, \gamma_a), \\ q_a = \Omega(p_a, X^a p_a, \sigma_a). \end{cases} \tag{3}$$

where Ω is a hyperbolic space, X is a self-mapping, $\gamma_a, \sigma_a \in [0,1]$, and all $p, q \in \Omega$.

Researchers studied iterative schemes to improve convergence to common fixed points. Dass and Debaita [16] initiated the study of two iterative mapping procedures. The iterative processes of Saluja [14] and Khan et al. [15] in hyperbolic spaces have also been updated.

The modified scheme defined by JA in hyperbolic space is as follows:

$$\begin{cases} p_1 \in T, \\ q_a = \Omega(p_a, Y^a(X^a p_a), \sigma_a), \\ p_{a+1} = \Omega(Y^a q_a, X^a q_a, \gamma_a). \end{cases} \tag{4}$$

Based on the above motivation, we define a new modified TI scheme to approximate the common fixed point of two mappings that converge faster:

$$\begin{cases} p_1 \in T, \\ q_a = \Omega(X^a p_a, Y^a p_a, \gamma_a), \\ r_a = \Omega(Y^a q_a, Y^a(X^a q_a), \sigma_a), \\ p_{a+1} = X^a r_a. \end{cases} \tag{5}$$

where $X, Y : T \to T$ are asymptotically quasi-nonexpansive mappings, Ω is a hyperbolic space, $\gamma_a, \sigma_a \in [0,1]$, and all $p, q \in \Omega$.

2. Preliminaries

This section requires recalling some previously acquired knowledge, and it is necessary to understand the underlying concepts and ideas to fully comprehend our main findings.

Definition 1 ([5]). *A subset T of a hyperbolic space $(\mathcal{B}, d, \Omega)$ is called convex if $\Omega(p, q, \gamma) \in T$ for all $p, q \in T$ and $\gamma \in [0,1]$.*

Definition 2 ([17]). *A hyperbolic space $(\mathcal{B}, d, \Omega)$ is called uniformly convex if, for every $p, q, r \in \mathcal{B}$, $n > 0$, and $\epsilon \in (0,2]$, there exists $\Gamma \in (0,1]$ such that*

$$d\left(\Omega\left(p, q, \frac{1}{2}\right), r\right) \leq (1-\Gamma)n,$$

whenever $d(p,r) \leq n$, $d(q,r) \leq n$, and $d(p,q) \geq \epsilon n$.

Definition 3 ([16,18]). *In a hyperbolic space, consider a bounded sequence $\{p_a\}$. Its asymptotic radius with respect to a point p is*

$$r(p, \{p_a\}) = \lim_{n \to \infty} \sup d(p_a, p).$$

(i) The asymptotic radius of $\{p_a\}$ with respect to a subset T is

$$r(T, \{p_a\}) = \inf\{r(p, \{p_a\}) : p \in T\}.$$

(ii) The asymptotic center of $\{p_a\}$ in T is

$$A(\{p_a\}) = \{p \in T : r(p, \{p_a\}) \leq r(q, \{p_a\}) \text{ for all } q \in T\}.$$

An asymptotic map has unique fixed points and asymptotic centers.

Definition 4 ([19]). *Let $\{p_a\}$ be a bounded sequence in T, where T is a nonempty closed convex subset of a uniformly convex hyperbolic space $(\mathcal{B}, d, \Omega)$. Suppose $A(\{p_a\}) = \{n\}$ and $r(\{p_a\}) = k$. If $\{n_b\}$ is a sequence in T such that*

$$\lim_{b\to\infty} r(n_b, \{p_a\}) = k,$$

then

$$\lim_{b\to\infty} n_b = n.$$

Definition 5 ([19]). *A sequence $\{p_k\}$ is said to be Δ-convergent to a point $p \in T$ if every subsequence of $\{p_k\}$ has a unique asymptotic center in T. We write*

$$\Delta\text{-}\lim_{k\to\infty} p_k = p.$$

Lemma 1 ([10]). *Let $(\mathcal{B}, d, \Omega)$ be a uniformly convex hyperbolic space. Let $\{r_a\} \subset (0,1)$ and let sequences $\{u_a\}$ and $\{v_a\}$ exist in $\mathcal{B}$. Suppose*

$$\limsup_{a\to\infty} d(p_a, p) \le i, \quad \limsup_{a\to\infty} d(q_a, p) \le i, \quad \text{and} \quad \limsup_{a\to\infty} d(\Omega(p_a, q_a, r_a), p) = i$$

for some $i \ge 0$. Then

$$\lim_{a\to\infty} d(p_a, q_a) = 0.$$

Lemma 2 ([20]). *Let $\{p_a\}$, $\{q_a\}$, and $\{r_a\}$ be non-negative sequences such that*

$$\Gamma_{a+1} \le p_a\Gamma_a + q_a, \quad a \in \mathbb{N}.$$

If $p_a \ge 1$, $\sum_{a=1}^{\infty}(p_a - 1) < \infty$, and $q_a < \infty$, then

$$\lim_{a\to\infty} \Gamma_a \quad \text{exists.}$$

Lemma 3 ([10]). *Let $(\mathcal{B}, d, \Omega)$ be a uniformly convex hyperbolic space. Suppose $s \in \mathcal{B}$ and $\{v_a\} \subset [\epsilon, 1-\epsilon]$ for any $\epsilon \in (0,1)$. Let $\{q_a\}$ and $\{r_a\}$ be sequences in $\mathcal{B}$ such that for $a \in \mathcal{N}$*

$$\limsup_{a\to\infty} d(q_a, s) \le i, \quad \limsup_{a\to\infty} d(r_a, s) \le i, \quad \text{and} \quad \limsup_{a\to\infty} d(\Omega(q_a, r_a, v_a), s) = i, \forall i = 1$$

for any $i > 0$. Then

$$\lim_{a\to\infty} d(q_a, r_a) = 0.$$

Lemma 4. *Let $(\mathcal{B}, d, \Omega)$ be a complete uniformly convex hyperbolic space. Then every bounded sequence $\{p_a\} \subset T$ has a unique asymptotic center.*

Definition 6. *A sequence $\{p_a\}$ in a hyperbolic space $(\mathcal{B}, d, \Omega)$ is said to Δ-**converge** to a point $q \in T$ if q is the unique asymptotic center for every subsequence $\{p_a\}$ of $\{p_n\}$. In this case, q is called the Δ-**limit** of $\{p_a\}$.*

3. Main Results

This section presents and examines a modified iterative TI-approach for approximating the common fixed points of two mappings in a hyperbolic space that are asymptotically nonexpansive.

Lemma 5. *Let T be a nonempty uniform convex subset of a uniformly convex hyperbolic space $(\mathcal{B}, d, \Omega)$. A mapping of $X, Y : T \to T$ has a sequence t_a fulfills the conditions*

(i) $\sum_{a=1}^{\infty}(t_a - 1) < \infty$

(ii) $F = F(X) \cap F(Y) \neq \emptyset$.

If a sequence is defined by (5), then limits $d(p_a, s)$ and $d(q_a, s)$ exist.

Proof. From the definition, we observe

$$\begin{aligned} d(q_a, r) &= d(\Omega(X^a p_a, Y^a p_a, \gamma_a), s) \\ &\leq (1-\gamma_a) d(X^a p_a, s) + \gamma_a d(Y^a p_a, s) \\ &\leq (1-\gamma_a) t_a d(p_a, s) + \gamma_a t_a d(p_a, s) \\ &= t_a d(p_a, s), \end{aligned}$$

and

$$\begin{aligned} d(r_a, s) &= d(\Omega(Y^a q_a, Y^a(X^a q_a), \sigma_a), s) \\ &\leq (1-\sigma_a) d(Y^a q_a, s) + \sigma_a d(Y^a(X^a q_a), Y^a(X^a s)) \\ &\leq (1-\sigma_a) t_a d(q_a, s) + \sigma_a t_a d(X^a q_a, X^a s) \\ &\leq (1-\sigma_a) t_a d(q_a, s) + \sigma_a t_a^2 d(q_a, s) \\ &= (t_a - \sigma_a t_a + \sigma_a t_a^2) d(q_a, s) \\ &= (1 + (t_a - 1) + \sigma_a t_a (t_a - 1)) d(q_a, s), \end{aligned}$$

and

$$\begin{aligned} d(p_{a+1}, s) &= d(r_a, s) \\ &\leq t_a d(r_a, s). \end{aligned}$$

Therefore

$$d(d(r_a, s) \leq (1 + j_a) d(q_a, s).$$

where $j_a = (t_a - 1) + \sigma_a t_a (t_a - 1)$. Using the hypothesis that $\sum_{a=1}^{\infty} (t_a - 1) < \infty$, we conclude from Lemma 1 $\lim_{a \longrightarrow \infty} d(p_a, s)$ at this point. Let $\lim_{a \longrightarrow \infty} d(p_a, s) = i$ where $i \geq 0$. Using this inequality, we have

$$\begin{aligned} d(p_{a+1}, s) &\leq (1 - \gamma + a) t_a d(p_a, s) + \gamma + a t_a d(q_a, s) \\ &= d(p_a, s) + \gamma + a t_a (d(q_a, s) - d(p_a, s)). \end{aligned}$$

Now we take the lim sup on both sides, we get

$$\begin{aligned} i &= i + limsup \gamma_a (d(q_a, s) - i) \\ &\leq i. \end{aligned}$$

Therefore $\limsup \gamma_a d(q_a, s) \leq i$. So,

$$limsup \gamma_a (d(q_a, s) - i) = 0.$$

Thus $\gamma_a \in [\epsilon, 1 - \epsilon]$, it proves that a limit exists. □

Theorem 1. *Let T be a nonempty uniform convex subset of a uniformly convex hyperbolic space $(\mathcal{B}, d, \Omega)$. A mapping $X, Y : T \to T$ has a sequence t_a fulfills the conditions*

(i) $\sum_{a=1}^{\infty}(t_a - 1) < \infty$

(ii) $F = F(X) \cap F(Y) \neq \emptyset$.

Then a sequence p_a defined by (5) has strong convergence.

Proof. Assume that a sequence p_a converges to a common fixed point of X and Y, then $d(p_a, h) \longrightarrow 0$. Let $\lim_{a\longrightarrow\infty} infd(p_a, H) = 0$. Now using the above Lemma 5,

$$d(d(r_a, s) \leq (1 + j_a)d(p_a, s)$$

applying inf over $s \in H$, so

$$d(d(r_a, H) \leq (1 + j_a)d(p_a, H). \tag{6}$$

Thus we know that $\lim_{a\longrightarrow\infty} d(p_a, H)$ converges, so it is clear $d(p_a, H) \longrightarrow 0$. Therefore, there exists a positive integer N; therefore,

$$d(p_a, H) < \frac{\epsilon}{4}, \quad \forall \quad a \geq N.$$

Here it is $infd(p_a, s) : s \in H < \frac{\epsilon}{4}$ also we here easily find $s \in H$, so that using Equation (6), we observe

$$\begin{aligned} d(p_{a+b}, p_a) &\leq d(p_{a+b}, p) + d(p, p_a) \\ &\leq (1 + j_{a+b-1})(1 + j_{a+b-2}) \ldots (1 + s_N)(p_a, s) + (1 + j_{a-1})(1 + j_{a-2}) \ldots (1 + s_N)(p_a, s). \end{aligned}$$

Thus, it is stated that $\sum_{a=1}^{\infty} j_a < \infty$, the infinite product $(1 + s_a)$ converges to a positive real number l.

Therefore

$$d(p_{a+b}, p_a) \leq 2ld(p_a, s) < l\epsilon.$$

This proves that it is a Cauchy sequence and converges. □

Theorem 2. *Let T be a nonempty closed convex subset of a uniformly convex hyperbolic space $(\mathcal{B}, d, \Omega)$. A mapping $X, Y : T \to T$ has a sequence t_a fulfill the conditions*

(i) $\sum_{a=1}^{\infty}(t_a - 1) < \infty$

(ii) $F = F(X) \cap F(Y) \neq \emptyset$.

There exists $0 < \epsilon \leq \frac{1}{2}$, such that for every a $\gamma_a, \sigma_a \in [\epsilon, 1 - \epsilon]$.
Using (5), thus,

$$\lim_{a\longrightarrow\infty} d(q_a, Xq_a) = \lim_{a\longrightarrow\infty} d(q_a, Yq_a) = \lim_{a\longrightarrow\infty} d(r_a, Xr_a) = \lim_{a\longrightarrow\infty} d(r_a, Yr_a) = 0.$$

If there is any limit, then the sequences converge strongly.

Proof. From the Lemma 5

$$\lim_{a\longrightarrow\infty} d(q_a, s) = \lim_{a\longrightarrow\infty} d(r_a, s) = i.$$

Hence $\lim_{a\longrightarrow\infty} d(q_a, s) = c$ by the Lemma 5 we have,

$$\lim_{a\longrightarrow\infty} d(\Omega(X^a p_a, Y^a p_a, \gamma_a), s) = i. \tag{7}$$

Now, by applying the limit sup to the above inequality, we get

$$d(X^a p_a) \leq t_a d(p_a, s)$$

and

$$d(Y^a p_a) \leq t_a d(p_a, s),$$

so we get, $\limsup_{a\longrightarrow\infty} d(X^a p_a) \leq i$ and $\limsup_{a\longrightarrow\infty} sup d(Y^a p_a) \leq i$.

Through the Equation (7) and Lemma 4, it becomes $d(X^a p_a, Y^a p_a) \longrightarrow 0$.

Now also from

$$\lim_{a\longrightarrow\infty} d(r_a, s) = \lim_{a\longrightarrow\infty} d(\Omega(Y^a q_a, Y^a(X^a q_a), \sigma_a), s) = i$$

and

$$\begin{aligned} \limsup_{a\longrightarrow\infty} d(Y^a q_a) &\leq i \\ \limsup_{a\longrightarrow\infty} d(Y^a(X^a q_a) &\leq i \end{aligned}$$

Now again using Lemma 4, we have

$$\lim_{a\longrightarrow\infty} d((Y^a q_a), Y^a(\delta^a q_a)) = 0. \tag{8}$$

Now

$$\begin{aligned} d(q_a, Y^a q_a) &= \lim_{a\longrightarrow\infty} d(\Omega(X^a p_a, Y^a p_a, \gamma_a), Y^a q_a) \\ &\leq (1-\gamma_a) d(X^a p_a, Y^a q_a) + \gamma_a d(Y^a p_a, Y^a q_a) \\ &= \gamma_a d(X^a p_a, Y^a q_a). \end{aligned}$$

Hence $\lim_{a\longrightarrow\infty} d(X^a p_a, Y^a q_a) \longrightarrow 0$.

$$\lim_{a\longrightarrow\infty} d(q_a, Y^a q_a) = 0, \tag{9}$$

and through

$$\begin{aligned} d(q_a, X^a r_a) &= \lim_{a\longrightarrow\infty} d(\Omega(X^a p_a, Y^a p_a, \gamma_a), X^a r_a) \\ &\leq (1-\gamma_a) d(Y^a p_a, X^a r_a) \end{aligned}$$

So we get

$$\lim_{a\longrightarrow\infty} d(q_a, X^a r_a) = 0. \tag{10}$$

Now assume

$$d(r_a, Y^a q_a) = d(\Omega(Y^a q_a, Y^a(X^a q_a), \sigma_a), Y^a q_a)$$

$$\leq (1-\sigma_a) d(Y^a q_a, Y^a q_a) + \sigma_a d(Y^a(X^a q_a), Y^a q_a)$$

so,

$$d(r_a, Y^a q_a) = 0. \tag{11}$$

In the same case, we can also prove that

$$d(r_a, X^a q_a) = 0. \tag{12}$$

Furthermore,

$$d(q_a, X^a p_a) \leq d(q_a, Y^a q_a) + d(Y^a q_a, X^a q_a).$$

Now using Equations (8) and (9), we obtain $d(q_a, X^a q_a) \longrightarrow 0$. Therefore, by the Equation (12)

$$d(q_a, r_a) \longrightarrow 0. \tag{13}$$

From the aforementioned claims, the facts that follow can be written as

$$\lim_{a \longrightarrow \infty} d(X^a q_a, X^a r_a) = \lim_{a \longrightarrow \infty} d(q_{a+1}, X^a r_a) = 0.$$

Hence $\lim_{a \longrightarrow \infty} d(q_a, q_{a+1}) = 0$, so

$$d(q_a, X^a p_a) \leq d(q_a, q_{a+1}) + d(q_{a+1}, X^a q_a),$$

Thus, it entails that $\lim_{a \longrightarrow \infty} d(q_a, X^a p_a) = 0$, additionally

$$d(q_a, r_a) \leq d(q_a, q_{a+1}) + d(q_{a+1}, r_a) \longrightarrow 0 \tag{14}$$

$$d(q_a, Y^a q_a) \leq d(q_a, r_a) + d(r_a, Y^a q_a) \longrightarrow 0 \tag{15}$$

although

$$d(r_a, X^a r_a) \leq d(r_a, Y^a r_a) + d(Y^a r_a, X^a r_a) \longrightarrow 0. \tag{16}$$

So, we conclude

$$d(r_a, r_{a+1}) \leq d(r_a, q_{a+1}) + d(q_{a+1}, r_{a+1}).$$

Now using (13) and (14), we deduce

$$\begin{aligned} d(q_a, Xq_a) &\leq d(q_a, q_{a+1}) + d(q_{a+1}, X^{a+1} q_{a+1}) + d(X^{a+1} q_{a+1}, X^{a+1} q_a) + d(X^{a+1} q_a, Xq_a) \\ &\leq d(q_a, q_{a+1}) + d(q_{a+1}, X^{a+1} q_{a+1}) + t_{a+1} d(q_a, q_{a+1}) + t - 1 d(X^a q_a, q_a). \end{aligned}$$

After applying the limit $a \longrightarrow \infty$, the sequence converges and tends towards 0. Now in similar manner $a \longrightarrow \infty\ d(r_a, Xr_a) \longrightarrow 0$. We also observed that $d(q_a, Yq_a) \longrightarrow 0$, and $d(Yq_a, Yr_a) \longrightarrow 0$.

Now we assume that it is demi-convex. Hence

$$d(q_{a_i}, q') \leq d(q_{a_i}, Xd(q_{a_i}) + d(d(q_{a_i}, q')), \tag{17}$$

Equation (17) $\longrightarrow 0$ as $i \longrightarrow \infty$. This implies that $q' \in F(\Omega)$, $q_a \longrightarrow q'$ and $d(q_a, r_a) \longrightarrow 0$. $\Longrightarrow r_a \longrightarrow q'$. It is proved that it converges strongly and q' lies in $F(X) \cap F(Y)$. This completes this proof. □

Theorem 3. *Let T be a non-empty closed convex subset of a uniformly convex hyperbolic space $(\mathcal{B}, d, \Omega)$. A mapping of $X, Y : T \to T$ be asymptotically nonexpansive with $F(X) \neq \emptyset$ and $F(Y) \neq \emptyset$, and let $\{t_a\}$ be a sequence with $t_a \geq 1$ satisfying $\sum_{n=1}^{\infty}(t_a^2 - 1) < \infty$. For an arbitrary initial point $p_1 \in T$, define the sequence $\{p_a\}$ by the iterative process (1.5) then $\{p_a\}$ Δ-converges to a point in $F(X) \cap F(Y)$.*

Proof. From Lemma 5, we have $d(p_a, Xp_a) \to 0$ and $d(p_a, Yp_a) \to 0$ as $a \to \infty$. Since $\{p_a\}$ is bounded, Lemma 4 guarantees that every bounded sequence has a unique asymptotic center.

Let $\{r_a\}$ be an arbitrary subsequence of $\{p_a\}$. Since $\{p_a\}$ is bounded, therefore $\{r_a\}$ is also bounded. Denote the asymptotic centers of the sequence and the subsequence as $A(\{p_a\})p$ and $A(\{r_a\}) = r$, respectively. Our purpose is to prove that rp and that r is a common fixed point of X and Y.

First, we show $r \in F(X)$. From the asymptotic nonexpansive property of X, it follows that $\lim_{a\to\infty} d(X^k r_a, X^{k+1} r_a) = 0$ for any $k \in \mathbb{N}$. For any positive integers u and v, we derive the following inequality:

$$\begin{aligned} d(X^u r, r_a) &\leq d(X^u r, X^u r_a) + \sum_{k=0}^{u-1} d(X^{k+1} z_a, X^k r_a) \\ &\leq k_u\, d(r, r_a) + \sum_{k=0}^{u-1} d(X^{k+1} r_a, X^k r_a). \end{aligned}$$

Taking the limit as $a \to \infty$ for a fixed u, we obtain

$$\gamma(X^u r, \{r_a\}) = \limsup_{a\to\infty} d(X^a r, r_a) \leq k_u\, \gamma(r, \{r_a\}).$$

Now, taking the limit as $u \to \infty$ and $k_u \to 1$, we find

$$\limsup_{u\to\infty} \gamma(X^u r, \{r_a\}) \leq \gamma(r, \{r_a\}).$$

However, since r is the unique asymptotic center of $\{r_a\}$, we have $\gamma(r, \{r_a\}) \leq \gamma(X^u r, \{r_a\})$ for every u. This implies

$$\lim_{m\to\infty} \gamma(X^u r, \{r_a\}) = \gamma(r, \{r_a\}).$$

Using Definition 4, this states $X^u r \to r$, and hence $r \in F(X)$. An identical argument shows that $r \in F(Y)$, so $r \in F(Y) \cap F(X)$.

It remains to show that, p. Suppose, for contradiction, that $r \neq p$. Since $r \in F(X) \cap F(Y)$, Lemma 5 implies that $\lim_{a\to\infty} d(p_a, r)$ exists. Using the uniqueness of asymptotic centers, we deduce the following chain of inequalities:

$$\begin{aligned} \limsup_{a\to\infty} d(r_a, r) &< \limsup_{a\to\infty} d(r_a, p) \\ &\leq \limsup_{a\to\infty} d(p_a, p) \\ &< \limsup_{a\to\infty} d(p_a, p) \\ &= \limsup_{a\to\infty} d(r_a, r). \end{aligned}$$

This is a contradiction, as the first and last terms are identical. Therefore, $r = p$. Since the subsequence $\{r_a\}$ was arbitrary and $A(\{r_a\}) = \{p\}$ for all such subsequences, we conclude that $\{p_a\}$ Δ-converges to p, which is a common fixed point of X and Y. □

4. Numerical Example

To thoroughly evaluate the applicability of the iterative scheme (1.3) in approximating fixed points of generalized nonexpansive mappings. The following example demonstrates the effectiveness of the proposed method.

We presently give a specific instance and show how convergence to a certain point occurs mathematically and graphically using our defined iterative scheme.

Example 1. *Consider T = B(0;1), this T is a ball with center 0 and radius 1 in $\mathcal{R}^2$. Let us define self-maps X and Y as $X(p_1, p_2) = (sinp_1, sinp_2)$ and $Y(p_1, p_2) = (p_1^2, p_2^2)$, as it is clear $p, q \in T$ and take $p = (p_1, p_2)$ and $q = (q_1, q_2)$*

Let be $q_1 < p_1$ and $q_2 < p_2$, therefore, we have

$$\begin{aligned} d(Y^a p, Y^a q) &= ||Y^a p - Y^a q|| \\ &= ||(p_1^{2a}, p_2^{2a}) - (q_1^{2a}, q_1^{2a})|| \\ &= [(p_1^{2a} - q_1^{2a})^2 + (p_2^{2a} - q_2^{2a})^2]^{\frac{1}{2}} \\ &= [|p_1 - q_1|^2\{p_1^{2a-1} + q_1 p_1^{2a-2} + \cdots + q_1^{2a-1}\}^2 + |p_2 - q_2|^2\{p_2^{2a-1} + q_2 p_2^{2a-2} + \cdots + q_2^{2a-1}\}^2]^{\frac{1}{2}} \\ &\leq [|p_1 - q_1|^2\{2ap_1^{2a-1}\}^2 + [|p_2 - q_2|^2\{2ap_2^{2a-1}\}^2]^{\frac{1}{2}} \end{aligned}$$

Assume $y_a = max\{1, 2ap_1^{2a-1}\}$ and $h_a = max\{1, 2ap_2^{2a-1}\}$. Let take $I_a = max\{y_a, h_a\}$. Then it is obvious $I_a \longrightarrow 1$ as $a \longrightarrow \infty$. Thus

$$\begin{aligned} d(Y^a p, Y^a q) &= I_a[|p_1 - q_1|^2 + |p_2 - q_2|^2]^{\frac{1}{2}} \\ &= I_a|p - q|. \end{aligned}$$

As it is clear that Y is an asymptotically nonexpansive mapping and X also has the same property, (0,0) is a common fixed point of both maps.

Example 2. *Consider T = B(0;1), this T is taken as a ball with center 0 and radius 1 in $\mathcal{R}^2$. Let us define self-map X and Y as $X(p_1, p_2) = (p_1^2, 0)$ and $Y(p_1, p_2) = (0, p_2^2)$. It can be observed that neither mapping is nonexpansive. Let $p = (p_1, p_2)$ and $q = (q_1, q_2) \in T$. Then*

$$X(p_1, p_2) = (p_1^{2a}, 0) \quad and \quad Y(p_1, p_2) = (0, p_2^{2a}).$$

If $q_1 \leq p_1$, then

$$\begin{aligned} \|X^a p - X^a q\|_2 &= \|(p_1^{2a}, 0) - (0, p_2^{2a})\| \\ &= |p_1^{2a} - p_2^{2a}| \\ &= [|p_1 - q_1|^2\{p_1^{2a-1} + q_1 p_1^{2a-2} + \cdots + q_1^{2a-1}\} \\ &\leq \|p - q\|_2\{2ap_1^{2a-1}\}. \end{aligned}$$

Suppose that $y_a = max\{1, 2ap_1^{2a-1}\} \leq max\{1, 2a(1)^{2a-1}\}$. *Hence X is asymptotically nonexpansive mapping; then, for any* $p_1 \in T$ *defined a sequence* $\{p_1\}$ *is defined by*

$$p_{a+1} = \Omega(X^a p_a, Y^a q_a, \sigma_a) \quad and \quad q_a = \Omega(X^a p_a, Y^a p_a, \gamma_a).$$

Let $p_a = (p_{a_1}, p_{a_2})$, $p_{a+1} = (p_{(a+1)_1}, p_{(a+1)_2})$ *and* $q_a = (q_{a_1}, q_{a_2})$, *and* $\sigma_a = \gamma_a = \frac{1}{2}$. *Then*

$$(p_{(a+1)_1}, p_{(a+1)_2}) = \frac{1}{2}(0, p_a^{2a}) + \frac{1}{2}(0, q_a^{2a}) = \frac{1}{2}(p_a^{2a}, q_a^{2a})$$

and $(q_{a_1}, q_{a_2}) = \frac{1}{2}(p_a^{2a}, p_a^{2a})$. *Thus* $q_{a1} = \frac{1}{2}p_a^{2a}$ *and* $(p_{a+1_1}, p_{a+1_2}) = \frac{1}{2}\frac{1}{2^{2a}}(p_a^{2a}, q_a^{2a})$, *it is clear that* p_a *converges to a common fixed point (0,0). A table comparison shows that our modified TI-algorithm has a faster rate of convergence, and also graphical representation is also provided. We take* $p_1 = (\frac{3}{4}, \frac{3}{4})$ *and* $\gamma, \sigma = \frac{1}{2}$. *A convergence of the iterative scheme is obtained in the given Table 1, and Figures 1 and 2 illustrates the variation in values for different algorithms.*

Table 1. Convergence analysis and comparison of our algorithm (5) in Example 1.

k	**TI-Scheme (5)**	**JA (4)**	**Saluja**	**D**
1	(0.000722, 0.000722)	(0.566583, 0.566583)	(0.597018, 0.597018)	(0.631199, 0.631199)
2	(0.000004, 0.000004)	(0.091520, 0.091520)	(0.179742, 0.179742)	(0.344357, 0.344357)
3	(0.000000, 0.000000)	(0.000048, 0.000048)	(0.002857, 0.002857)	(0.172179, 0.172179)
4	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000008, 0.000008)	(0.086090, 0.086090)
5	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.043045, 0.043045)
6	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.000000, 0.000000)	(0.021522, 0.021522)
7	-	-	-	(0.010761, 0.010761)
8	-	-	-	(0.005381, 0.005381)
9	-	-	-	(0.002690, 0.002690)
10	-	-	-	(0.001345, 0.001345)
11	-	-	-	(0.000673, 0.000673)
12	-	-	-	(0.000336, 0.000336)
13	-	-	-	(0.000168, 0.000168)
14	-	-	-	(0.000084, 0.000084)
15	-	-	-	(0.000042, 0.000042)
16	-	-	-	(0.000021, 0.000021)
17	-	-	-	(0.000011, 0.000011)
18	-	-	-	(0.000005, 0.000005)
19	-	-	-	(0.000003, 0.000003)
20	-	-	-	(0.000001, 0.000001)
21	-	-	-	(0.000001, 0.000001)
22	-	-	-	(0.000000, 0.000000)

The proposed iterative method exhibits fast convergence as compared to the existing schemes, primarily due to its faster convergence behavior and reduced iteration count. Specifically, the proposed approach attains convergence at the second iteration, whereas the competing methods require the 4th, 5th, and even the 22nd iteration to achieve similar results. This substantial reduction in computational effort highlights the effectiveness of the method in terms of accuracy and speed.

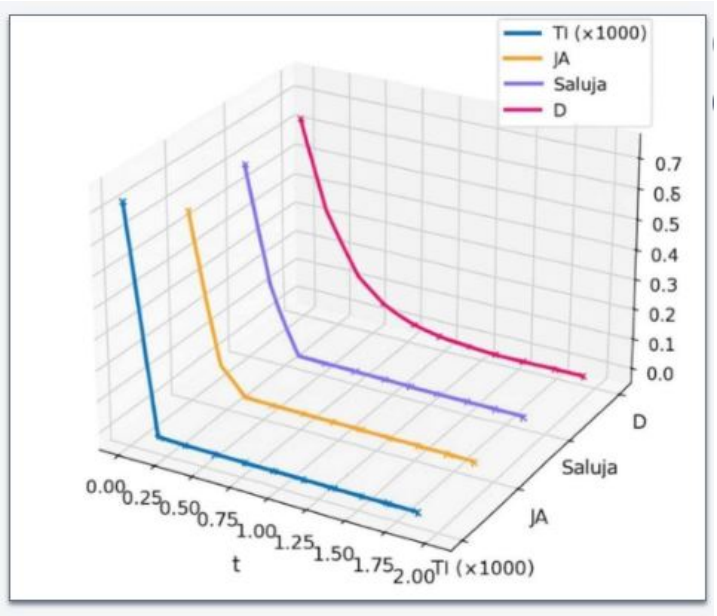

Figure 1. 3D graphical analysis of our scheme (5) corresponding to the data in Table 1.

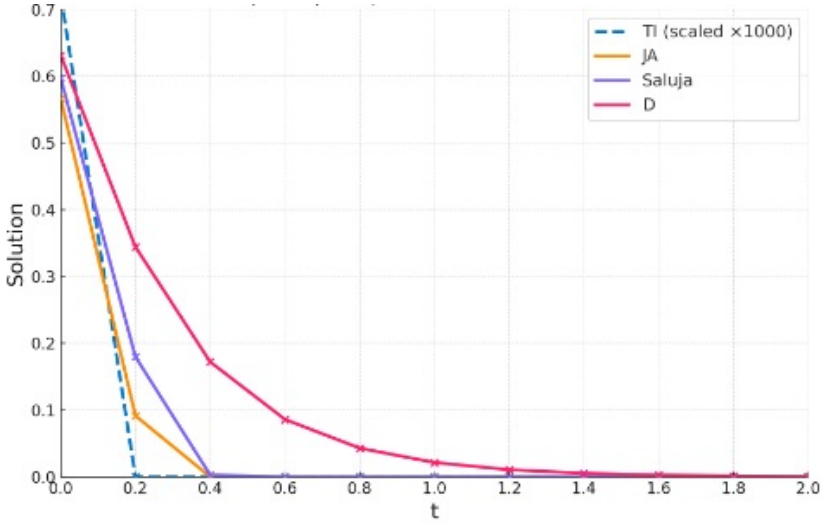

Figure 2. Graphical analysis for our scheme (5) and for Table 1.

Example 3. *Let $T = B(0,1) \subset \mathbb{R}^2$ be the closed unit ball centered at the origin and let $(a_n)_{n\geq 1}$ be a sequence with $a_n \geq 1$ for all n. For each index n define maps $X_{a_n}, Y_{a_n} : T \to T$ by*

$$X_{a_n}(p_1, p_2) = (p_1^{2a_n},\ p_2), \qquad Y_{a_n}(p_1, p_2) = (p_1,\ p_2^{2a_n}).$$

(These maps preserve T because $|p_i| \leq 1 \Rightarrow |p_i^{2a_n}| \leq 1$.)

Let $p = (p_1, p_2)$, $q = (q_1, q_2) \in T$. For $r \geq 1$ and $|u|, |v| \leq 1$ the mean value inequality gives

$$|u^r - v^r| \ = \ r\,|\xi|^{r-1}\,|u - v| \leq r\,|u - v| \qquad (\text{for some } \xi \in [u, v],\ |\xi| \leq 1).$$

Applying this with $r = 2a_n$ yields the uniform bound (independent of p, q):

$$|p_1^{2a_n} - q_1^{2a_n}| \leq 2a_n\,|p_1 - q_1|, \qquad |p_2^{2a_n} - q_2^{2a_n}| \leq 2a_n\,|p_2 - q_2|.$$

Hence

$$\|X_{a_n}(p) - X_{a_n}(q)\| = |p_1^{2a_n} - q_1^{2a_n}| \leq 2a_n \, \|p - q\|,$$

and similarly

$$\|Y_{a_n}(p) - Y_{a_n}(q)\| \leq 2a_n \, \|p - q\|.$$

Thus the Lipschitz constants for X_{a_n} and Y_{a_n} on the whole domain T are

$$L_n := 2a_n.$$

which is uniform (depends only on n, not on the points p, q).

Example 4. *Still let $T = B(0,1)$ and let $(a_n)_{n\geq 1}$ satisfy $a_n \geq 1$. Define maps $X_{a_n}, Y_{a_n} : T \to T$ by*

$$X_{a_n}(p_1, p_2) = (p_1^{2a_n}, 0), \qquad Y_{a_n}(p_1, p_2) = (0, p_2^{2a_n}).$$

As in Example 3, for any $p, q \in T$ we get the uniform estimates

$$\|X_{a_n}(p) - X_{a_n}(q)\| = |p_1^{2a_n} - q_1^{2a_n}| \leq 2a_n \, \|p - q\|,$$

and

$$\|Y_{a_n}(p) - Y_{a_n}(q)\| = |p_2^{2a_n} - q_2^{2a_n}| \leq 2a_n \, \|p - q\|.$$

Thus, the same uniform Lipschitz constants $L_n = 2a_n$.

5. Application to the Integral Equation

Let

$$T := C([a,b], \mathbb{R})$$

be the Banach space of continuous real-valued functions on $[a,b]$, equipped with the supremum norm

$$\|u\|_\infty := \sup_{t\in[a,b]} |u(t)|.$$

We consider the nonlinear integral equation

$$u(t) = f(t) + \nu \left(\int_a^b p(t,\mu) \, L_1(\mu, u(\mu)) \, d\mu \right) \left(\int_a^b p(t,\mu) \, L_2(\mu, u(\mu)) \, d\mu \right), \qquad t \in [a,b], \tag{18}$$

where

- $\nu \in \mathbb{R}$ is a constant;
- $f \in C([a,b], \mathbb{R})$;
- $p : [a,b] \times [a,b] \to \mathbb{R}$ is continuous;
- $L_i : [a,b] \times \mathbb{R} \to \mathbb{R}$ $(i = 1,2)$ are continuous in both variables.

Define the operator $f : T \to T$ by

$$(fu)(t) := f(t) + \nu \, I_1(t,u) \, I_2(t,u), \tag{19}$$

where for $i = 1, 2$,

$$I_i(t,u) := \int_a^b p(t,\mu) \, L_i(\mu, u(\mu)) \, d\mu.$$

We impose the following hypotheses:

(i) $p(t,\mu)$ is continuous on $[a,b]\times[a,b]$ and there exists a constant $P>0$ such that

$$\sup_{t\in[a,b]}\int_a^b |p(t,\mu)|\,d\mu \le P.$$

(ii) For $i=1,2$, $L_i(\mu,y)$ is continuous and satisfies a Lipschitz condition in y: there exist measurable $\ell_i(\mu)\ge 0$ with

$$|L_i(\mu,y)-L_i(\mu,z)| \le \ell_i(\mu)\,|y-z|, \qquad \forall\,\mu\in[a,b],\ \forall y,z\in\mathbb{R},$$

and

$$\widetilde{L}_i := \sup_{t\in[a,b]}\int_a^b |p(t,\mu)|\,\ell_i(\mu)\,d\mu < \infty.$$

(iii) $f\in C([a,b],\mathbb{R})$.

Lemma 6. *Under* (i)–(iii)*, the operator $f: T\to T$ is well-defined and maps continuous functions into continuous functions.*

Proof. Since p and L_i are continuous and u is continuous, the integrand

$$(t,\mu)\mapsto p(t,\mu)\,L_i(\mu,u(\mu))$$

It is continuous on a compact set. Convergence therefore gives that $I_i(t,u)$ is continuous in t. Thus $(\mathscr{f}u)(t)$, being a sum and product of continuous functions, is continuous on $[a,b]$. Hence $\mathscr{f}u\in T$. □

Lemma 7. *For any $u,v\in T$, the operator f satisfies*

$$\|fu-fv\|_\infty \le |\nu|\Big(A(u)\,\widetilde{L}_2 + B(v)\,\widetilde{L}_1\Big)\|u-v\|_\infty,$$

where

$$A(u) := \sup_{t\in[a,b]}|I_1(t,u)|, \qquad B(v) := \sup_{t\in[a,b]}|I_2(t,v)|.$$

Proof. For $t\in[a,b]$,

$$\begin{aligned}|(f(u)(t)-f(v)(t)| &= |\nu|\,|I_1(t,u)I_2(t,u)-I_1(t,v)I_2(t,v)|\\ &\le |\nu|\big(|I_1(t,u)|\,|I_2(t,u)-I_2(t,v)|+|I_2(t,v)|\,|I_1(t,u)-I_1(t,v)|\big).\end{aligned}$$

Using the Lipschitz condition in (ii):

$$|I_i(t,u)-I_i(t,v)| \le \Big(\int_a^b |p(t,\mu)|\,\ell_i(\mu)\,d\mu\Big)\|u-v\|_\infty \le \widetilde{L}_i\|u-v\|_\infty.$$

Taking the supremum over $t\in[a,b]$ gives the stated inequality. □

Theorem 4. *If*

$$|\nu|\Big(A(u)\,\widetilde{L}_2 + B(u)\,\widetilde{L}_1\Big) < 1 \qquad \textit{for all } u\in T,$$

then f is a contraction on T and the integral Equation (18) *has a unique solution in T. If the above constant cannot be made strictly less than* 1*, but*

$$\|f^n u - f^n v\|_\infty \le k_n\|u-v\|_\infty, \qquad k_n\to 1.$$

Then f is an asymptotically nonexpansive mapping.

The space $T = C([a,b],\mathbb{R})$ is a convex metric space with the standard convex combination

$$W(u,v,\lambda)(t) = (1-\lambda)u(t) + \lambda v(t), \qquad \lambda \in [0,1],$$

which satisfies the hyperbolic metric axioms. This structure is used to define iterative schemes for asymptotically nonexpansive operators.

6. Conclusions

In this study, we approximate a fixed point associated with nonexpansive mappings using an iterative scheme suggested by TI (1.5). Some convergence results are shown using the iterative technique. According to a mathematical example, the TI-iterative scheme for solving generalized mappings has a faster rate of convergence than other currently known approaches. We show extended nonexpansiveness and give convergence results in closed convex hyperbolic spaces. The main conclusions of the research have been supported by a few experimental studies. This new strategy shows a faster rate of convergence. This method can be explored further in hyperbolic spaces and other metric spaces, and for asymptotically non-expansive mappings to test its stability and efficiency in broader settings. Future work may also involve applying the scheme to real-world problems to confirm its practical effectiveness.

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