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On Non-Archimedean Fuzzy Metric Free Topological Groups

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Abstract

We construct the free group over a non-Archimedean fuzzy metric space $(X, M, \wedge)$ in the sense of George and Veeramani where $\wedge$ is the minimum t-norm. The two main tools used are the concept of a scheme (for every non-empty subset S of $\mathbb{N}$ of even cardinal, a permutation φ on S is a scheme for S if it is idempotent, with no fixed points and, additionally, $i < j < \varphi(i) < \varphi(j)$ does not hold for every $i, j \in S$), and the notion of a fuzzy prenorm on a fuzzy topological group. As a consequence of our results, we prove that every non-Archimedean fuzzy metric space $(X, M, \wedge)$ in the sense of George and Veeramani is isometric to a closed subspace of a non-Archimedean fuzzy metric free (Abelian) group and also that every metric space (X, d) is uniformly isomorphic to a closed subspace of a non-Archimedean fuzzy metric free (Abelian) group. Our results also apply to non-Archimedean fuzzy metric spaces in the sense of Kramosil and Michálek.

Keywords: non-Archimedean fuzzy metric space; free group; fuzzy free (Abelian) group; scheme; prenorm

MSC: 54E35; 54H11; 54A40; 22A05

1. Introduction and Motivation

There are several approaches to the notion of a fuzzy metric space in the literature, one of the topics in fuzzy structures which has deserved a wide development in the last decades. In this framework, Kramosil and Michálek introduced in [1] a concept of fuzzy metric space that extends the notion of a Menger space [2]. Later metric fuzziness in the sense of Kramosil and Michálek was modified by George and Veeramani who defined and studied a new class of fuzzy metric spaces [3,4].

The present paper is situated within the ambit of fuzzy topological groups, a field of study that has garnered significant attention from the mathematical community. The importance of topological groups lies in their multifaceted relationship with other areas of mathematics, as evidenced by their pivotal role in resolving Hilbert's fifth problem (see [5]). Furthermore, these groups have been instrumental in fostering connections between mathematics and other disciplines, such as physics and computer science (see [6–8]).

In this context, the relevance of the category of free topological groups when going in depth into the study of topological groups is apparent. In the field of topological algebra, a free topological group is a foundational concept that seamlessly integrates the principles of a free group from abstract algebra with the framework of a topological space. Free topological groups were developed in order to apply the methods of abstract algebra to topological spaces and solve specific problems in topological group theory. The theory was

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Academic Editor: Miroslav Ćirić

Received: 26 September 2025
Revised: 24 November 2025
Accepted: 25 November 2025
Published: 28 November 2025

Citation: Bors, C.; Sanchis, M. On Non-Archimedean Fuzzy Metric Free Topological Groups. *Axioms* **2025**, *14*, 878. https://doi.org/10.3390/axioms14120878

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founded by Markov in his seminal paper [9]. The fundamental question he sought to answer was: For which topological spaces X is there a topological group such that X is a closed subspace? The topology of a group's subsets is constrained by the algebraic operations in it, making this a non-trivial problem. Subsequently, Graev [10] defined a natural metric on the algebraic free group over a metric space (X, d). This metric extends the metric on X in such a way that the extension is compatible with the group structure. It is a well-known fact that, for a metric space (X, d), the topology induced by Graev's metric coincides with Markov's topology. Graev's contributions have significantly enhanced the accessibility of free topological groups on metric spaces, thereby stimulating further research in this field. Alternative constructions were proposed by Kakutani [11] and Samuel [12].

Despite their abstract definition, free topological groups have surprisingly concrete and powerful applications. For example, Kakutani employed the free Abelian topological group to demonstrate that the (covering) dimension of a compact metric space X is equal to the dimension of its free Abelian group $A(X)$ (this was a significant result that demonstrated the power and utility of the concept for solving classical problems in dimension theory. This tool has also been used to show that every topological group can be seen as a quotient of a zero-dimensional topological group [13] and every Tychonoff space is homeomorphic to a closed subspace of a Hausdorff topological group [9,10]. The paper [14] is a cornerstone for understanding how free topological groups serve as universal objects, a concept that has profound implications in geometric and asymptotic group theory. As demonstrated by the previous examples, there are numerous ways in which the properties and concepts of the theory of free topological groups, as well as those of locally convex spaces (see, for example, [15]), can be effectively translated and utilized for other classes of objects. Indeed, in the study of the inner structure of general Tychonoff spaces, it has become customary to employ free Tychonoff groups as instruments for achieving substantial general results and as a rich source of examples with distinctive features ([16], Chapter 7).

Our starting point is the so-called Graev's Extension Theorem [17]: If X is a set with a fixed point $e \in X$, then every metric d on X has an *extension* to a metric $\widehat{d}$ to the abstract free group $F(X)$ (respectively, to the abstract free Abelian group $A(X)$) with identity e such that the following properties hold:

(i) $\widehat{d}$ is a bi-invariant metric, and
(ii) is maximal among all bi-invariant metrics on $F(X)$ (respectively, on $A(X)$) extending d.

The aim of this paper is to study the existence of the non-Archimedean fuzzy metric free (Abelian) topological group over a non-Archimedean fuzzy metric space $(X, M, \wedge)$ (in the sense of George and Veeramani) with $\wedge$ the minimum t-norm. Among the various consequences that emerge from this study, it follows that a topological space X is non-Archimedean fuzzy metrizable if and only if X is isometric to a closed subset of a non-Archimedean fuzzy metrizable topological group.

Our free object is analogous to the classical metric free topological group, yet it is constructed in the more extensive category of fuzzy metric topological groups. The *fuzzy* aspect signifies that the membership of points is not a simple *yes/no question*, but rather a matter of degree. In this study, the concept of *closeness* is examined through a multifaceted lens, incorporating both a fuzzy metric and algebraic structures. The construction can be conceptualized as the development of a fuzzy metric topological envelope surrounding the algebraic skeleton of a free-like group. The classical free group is situated within this as a crisp core.

The motivation behind constructing free groups over non-Archimedean fuzzy metric spaces is a specialized theoretical pursuit that aims to merge the rigorous structure of algebra with the mathematical modeling of uncertainty. It shows that the category of non-Archimedean fuzzy metric spaces is robust enough to support fundamental algebraic

constructions, while strengthening the theoretical foundation of fuzzy analysis and topology. They expand the mathematical category of non-Archimedean fuzzy metric spaces, making it compatible with universal algebraic constructions.

In the construction of the non-Archimedean fuzzy metric free topological group, an adaptation of Graev's technique is employed. Specifically, we consider a non-Archimedean fuzzy metric on a set X which we extend to the (abstract) free group $F(X)$ and to the (abstract) free Abelian group $A(X)$ over X.

Now is a good time to review what a free group over a set X is. Let X be an arbitrary nonempty set and fix an element $e \in X$. Consider

$$X^{-1} = \left\{ x^{-1} : x \in X \right\}$$

where x^{-1} is just a formal expression if $x \neq e$ and $e = e^{-1}$. We denote by $F_a(X)$ the set of all *words* over the alphabet $X \cup X^{-1}$. A word is said to be *reduced* if it does not contain two consecutive symbols of the form xx^{-1} or $x^{-1}x$, nor any letter equal to $e = e^{-1}$. It is a well-known fact that each word in $F_a(X)$ possesses a unique reduced form.

The paper is organized as follows. We present the preliminaries in Section 2. Section 3 is devoted to the construction of the non-Archimedean fuzzy metric free (Abelian) topological group over a non-Archimedean fuzzy metric space $(X, M, \wedge)$. The conclusions are given in Section 4. It is assumed that the reader has a firm grasp on the algebraic theory of free groups. The terminology and notation employed herein are consistent with established conventions. For further inquiries regarding algebraic groups, the reader is referred to [18], and for topological notions and topological groups to [16,19].

2. Preliminaries

We now introduce the notions and terminology we shall be concerned with. Recall that a binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a *continuous t-norm* [20] if $([0,1], *)$ is a commutative topological semigroup with unity, such that the operation $*$ is compatible with the usual order on the unit interval $[0,1]$. The next notion is essential to our development.

Definition 1 ([3]). *A triple $(X, M, *)$ is a fuzzy metric space (in the sense of George and Veeramani) if X is a non-empty arbitrary set, $*$ is a continuous t-norm and M is a fuzzy set on $X \times X \times]0, +\infty[$ which satisfies for all $x, y, z \in X$ and $t, s > 0$:*

(GV1) $M(x, y, t) > 0$;
(GV2) $M(x, x, t) = 1$ *for all* $x \in X$ *and* $t > 0$, *and* $M(x, y, t) < 1$ *for all* $x \neq y$ *and* $t > 0$;
(GV3) $M(x, y, t) = M(y, x, t)$;
(GV4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$;
(GV5) $M(x, y, _) :]0, +\infty[\to [0, 1]$ *is continuous.*

The pair $(M, *)$ is called a fuzzy metric on X and $(X, M, *)$ a fuzzy metric space in the sense of George and Veeramani. It is well known that if $*$ is a continuous t-norm on a set X, then $* \leq \wedge$ (as usual, $\wedge$ denotes the minimum t-norm) and that $M(x, y, _)$ is a non-decreasing function for all $x, y \in X$. Since our spaces are fuzzy metric spaces in the sense of George and Veeramani, it will be essential for our construction to work with t-norms without zero divisors, that is, with t-norms $*$ that satisfy the condition that $x * y \neq 0$ whenever x and y are different from zero. Notice that the minimum t-norm and the product t-norm $\cdot$ do not have zero divisors, but the Łukasiewicz t-norm, defined as $*_{Ł}(x, y) = \max\{x + y - 1, 0\}$ $(x, y \in [0, 1])$, does. Moreover, in Section 3 an example is presented that demonstrates the failure of the natural extension of Graev's method in the fuzzy context to produce satisfactory results for the product t-norm. Given

these facts, our selection of the minimum t-norm is motivated by the Representation Theorem for Continuous t-norms which states that any continuous t-norm can be uniquely represented as an ordinal sum of three fundamental continuous t-norms: the minimum t-norm, the product t-norm and the Łukasiewicz t-norm. The theorem essentially says that any continuous t-norm is built by combining these three fundamental t-norms on disjoint subintervals of $[0,1]$ (see [21,22]).

Given a fuzzy metric $(M,*)$ on X, for each $x \in X$ the family of open balls $\{B_M(x,r,t) : x \in X, 0 < r < 1, t > 0\}$, where $B_M(x,r,t) = \{y \in X : M(x,y,t) > 1-r\}$, forms a base of neighborhoods at x for a topology τ_M on X, the so-called topology generated by $(M,*)$. A topological space (X,τ) is said to admit a compatible fuzzy metric if there is a fuzzy metric $(M,*)$ on X such that $\tau = \tau_M$.

The class of metric spaces coincides with the class of fuzzy metric spaces. Indeed, if (X,d) is a metric space and M_d is the fuzzy set in $X \times X \times]0,+\infty)$ defined as $M_d(x,y,t) = t/(t+d(x,y))$ for all $t > 0$, then $(M_d,\wedge)$ is a fuzzy metric on X, and thus $(M_d,*)$ is a fuzzy metric on X for all continuous t-norms $*$. This is the so-called fuzzy metric induced by (X,d), or the standard fuzzy metric on (X,d). Moreover, the topology τ_{M_d} agrees with the topology τ_d induced by the metric d (see [3]). Conversely, Gregori and Romaguera showed in [23] that every fuzzy metric space has an admissible uniformity with a countable base, that is, it is a metrizable space.

In the event that the triangular inequality (GV4) of Definition 1 is replaced by

(NA) $M(x,z,\max\{t,s\}) \geq M(x,y,t) * M(y,z,s)$

for all $x,y,z \in X$ and all $t,s > 0$, then $(X,M,*)$ is called a *non-Archimedean fuzzy metric space* (see [7]). It is routine to show that condition (NA) implies condition (GV4), that is, every non-Archimedean fuzzy metric space is itself a fuzzy metric space. It is straightforward to verify that condition (NA) is equivalent to the two following conditions:

(NA1) $M(x,z,t) \geq M(x,y,t) * M(y,z,t)$, and
(NA2) $M(x,y,_)$ is nondecreasing for all $x,y \in X$.

Non-Archimedean fuzzy metric spaces appear in a natural way in the study of fuzzy metric spaces. For example, the completness theorem and the compactness theorem of Gromov-Hausdorff metric can be shown in this context ([24,25]). It is a well-known fact that $(M_d,\cdot)$ and $(M_d,*_{\text{Ł}})$ are non-Archimedean fuzzy metrics. When working with the minimum t-norm $\wedge$, $(M_d,\wedge)$ is a non-Archimedean fuzzy metric only when (X,d) is an ultrametric space, that is, $d(x,y) \leq \max\{d(x,z),d(z,y)\}$ for all $x,y,z \in X$ (see [26]). Throughout, for a non-Archimedean fuzzy metric space, it is understood a non-Archimedean fuzzy metric space in the sense of George and Veeramani.

A topological group (G,τ) is an abstract group G equipped with a topology τ in such a way that the mapping $(g,h) \to g \cdot h$ of $G \times G$ onto G and the mapping $g \to g^{-1}$ of G onto G are continuous. Here, $\cdot$ is the group operation on G and g^{-1} stands for the inverse of g.

Let us briefly review a couple of other main tools in our paper. We say that a topological space (X,τ) is fuzzy metrizable if and only if it admits a compatible fuzzy metric. Moreover, the class of fuzzy metrizable topological groups [27] is that of all topological groups which are fuzzy metrizable as topological spaces. Consequently, $(G,\cdot,M,\wedge)$ is a fuzzy metric group if the following notions coexist: $(G,M,\wedge)$ is a fuzzy metric space and, meanwhile, $(G,\cdot,\tau_M)$ is a topological group. If no confusion arises, we write $(G,M,\wedge)$ instead of $(G,\cdot,M,\wedge)$.

Analogous to the classical case, the *extension* of a non-Archimedean fuzzy metric is defined as follows.

Definition 2. *Let $(N, \wedge)$ be a non-Archimedean fuzzy metric on a set X. If $X \subseteq Y$, then an extension of $(N, \wedge)$ to Y is a non-Archimedean fuzzy metric $(M, \wedge)$ on Y such that $N(x, y, t) = M(x, y, t)$ for all $x, y \in X$ and $t > 0$.*

Note that the previous definition is equivalent to saying that $(M, \wedge)$ is an extension of $(N, \wedge)$ to Y if and only if $(N, \wedge)$ is the restriction of $(M, \wedge)$ to $X \times X \times]0, +\infty[$; in symbols, $M_{|_X} = N$.

Finally, we introduce the concept of a non-Archimedean fuzzy free metric topological group over a non-Archimedean fuzzy metric space $(X, N, *)$.

Definition 3. *Let $(X, N, \wedge)$ be a non-Archimedean fuzzy metric subspace of a non-Archimedean fuzzy metric topological group $(G, M, \wedge)$ with identity e such that $e \in X$. Then $(G, M, \wedge)$ is called the non-Archimedean fuzzy metric topological group over $(X, N, \wedge)$ if the following conditions hold:*

(i) *G is algebraically generated by X, in symbols, $G = \langle X \rangle$;*
(ii) *$(M, \wedge)_{|_X} = (N, \wedge)$;*
(iii) *$(M, \wedge)$ is minimal among all bi-invariant non-Archimedean fuzzy metrics on G whose restriction to X coincides with $(N, \wedge)$.*

To get the non-Archimedean fuzzy Abelian metrizable free topological group, all we have to do is consider G in the previous definition as Abelian. The non-Archimedean fuzzy metric free (Abelian) topological group is usually denoted by $F(X)$ ($A(X)$). We shall prove in Theorem 8 that both $F(X)$ and $A(X)$ exist.

3. Extending Non-Archimedean Fuzzy Metrics from X to $F(X)$

For every non-empty set X, consider $F_a(X)$ with identity $e \in X$. As stated in the introduction, we shall use Graev's technique of extending metrics. Therefore, our first step is to extend $(M, \wedge)$ to a non-Archimedean fuzzy metric on the subset $\widetilde{X} = X \cup X^{-1}$ of $F_a(X)$.

Given a non-Archimedean fuzzy metric $(M, \wedge)$ on a non-empty set X, then we consider the fuzzy set $M^\circ \colon \widetilde{X} \times \widetilde{X} \times]0, +\infty[\to]0, 1]$ defined as

(1) $M^\circ(x, y, t) = M^\circ(x^{-1}, y^{-1}, t) = M(x, y, t)$ and
(2) $M^\circ(x^{-1}, y, t) = M^\circ(x, y^{-1}, t) = M(x, e, t) \wedge M(y, e, t)$,

for all $x, y \in X$ and all $t > 0$.

Example 1. *Take the metric space (X, d) where $X = \{e, a, b\}$ with the metric defined as $d(x, y) = 0$ if $x = y$ and $d(x, y) = 1$ if $x \neq y$ $(x, y \in X)$. It is an easy matter to prove that (X, d) is an ultrametric space.*

Therefore, the fuzzy metric space $(X, M_d, \wedge)$ is non-Archimedean. Choose e as the distinguished point. We have

(1) $M^\circ(a, b, t) = M^\circ(a^{-1}, b^{-1}, t) = \dfrac{t}{1 + d(a, b)} = \dfrac{t}{t + 1}$, *and*

(2) $M^\circ(a^{-1}, b, t) = M^\circ(a, b^{-1}, t) = \dfrac{t}{1 + d(a, e)} \wedge \dfrac{t}{1 + d(e, b)} = \dfrac{t}{t + 1} \wedge \dfrac{t}{t + 1} = \dfrac{t}{t + 1}$,

for all $t > 0$.

The following result is straightforward.

Proposition 1. *$(M^\circ, \wedge)$ is an non-Archimedean fuzzy metric on the subset $\widetilde{X} = X \cup X^{-1}$ of $F_a(X)$ which extends $(M, \wedge)$. Moreover, for all $u, v \in \widetilde{X}$ and $t > 0$, we have that $M^\circ(u^{-1}, v^{-1}, t) = M^\circ(v, u, t)$.*

We will now introduce several new concepts. One of the most significant is that of a scheme (see [28]), which is one of the fundamental tools when defining the functions that will allow us to obtain the extension of our non-Archimedean fuzzy metric on X. As is customary, $|H|$ stands for the cardinal of a set H and $\mathbb{N}$ for the natural numbers. As usual, a permutation on a subset S of $\mathbb{N}$ is a bijection φ from S onto S. A permutation φ is said to be idempotent (or an involution) if $\varphi^2 = \varphi$. If $\varphi(s) = s$, then s is called a fixed point (of φ).

Definition 4. *For every non-empty subset S of $\mathbb{N}$ of even cardinal, a permutation φ on S is a scheme for S if it is idempotent, with no fixed points and, additionally, $i < j < \varphi(i) < \varphi(j)$ does not hold for every $i, j \in S$.*

Schemes represent a specific class of permutations that are structurally constrained and avoid certain order patterns. They are fixed points free involutions on a subset of the natural numbers with a *non-crossing* condition. This kind of objects are sometimes called *non-crossing matchings* (see [29]). To clarify this notion, we include the following

Example 2. *Let $S = \{1, 2, 3, 4\}$. Define $\varphi \colon S \to S$ as follows:*

$$\begin{aligned} \varphi(1) &= 2, \\ \varphi(2) &= 1, \\ \varphi(3) &= 4, \\ \varphi(4) &= 3. \end{aligned}$$

It is apparent that φ is a non-fixed point idempotent permutation. Moreover, a routine check proves that φ satisfies $i < j < \varphi(i) < \varphi(j)$ does not hold for every $i, j \in S$.

In the next definition, we summarize basic and useful notions when working with (abstract) free groups. For the sake of clarity, we recall the definition of a reduced word from the introduction.

Definition 5. *Let $\mathcal{X} = u_1 u_2 \dots u_{2n}$ be an element of $F_a(X)$ with $u_i \in \widetilde{X}$ for all i. We say that:*

(a) *$\mathcal{X}$ is* reduced *if it does not contain two consecutive symbols of the form uu^{-1} or $u^{-1}u$, neither any letter equal to $e = e^{-1}$.*

(b) *$\mathcal{X}$ is* almost irreducible *if there are no consecutive symbols uu^{-1} or $u^{-1}u$ among its letters, although there may be the letter e.*

(c) *The word g obtained from $\mathcal{X}$ after applying all possible cancellations is called the* reduced form *of $\mathcal{X}$. In symbols, $[\mathcal{X}] = g$.*

By $AIR(F_a(X))$ we understand the family of all almost irreducible words in $F_a(X)$.

Note that condition (b) in the previous definition implies that the identity element can occur at most n times in an almost irreducible word whose length is $2n$. Therefore, the length of an almost irreducible word is at most twice the length of its reduced form. Each of its letters either belongs to its reduced form or is equal to e.

We need to introduce some nomenclature in order to construct our extension of the non-Archimedean fuzzy metric $(M, \wedge)$ to $F_a(X)$. If $(u_i, v_i, t_i) \in \widetilde{X} \times \widetilde{X} \times]0, +\infty[$ for $i = 1, 2, \dots m$, then we write

$$\prod_{i=1}^{m} M^{\circ}(u_i, v_i, t_i) = M^{\circ}(u_1, v_1, t_1) \wedge M^{\circ}(u_2, v_2, t_2) \wedge \cdots \wedge M^{\circ}(u_m, v_m, t_m).$$

Given a reduced word g in $F_a(X)$, we take $\mathcal{X} = u_1u_2 \dots u_{2n}$ in $F_a(X)$ such that $[\mathcal{X}] = g$. Then, for every $\varphi \in S_n$, we define

$$\Gamma_{(M,\wedge)}(\mathcal{X}, \varphi, t) = \prod_{i=1}^{2n} M^{\circ}(u_i^{-1}, u_{\varphi(i)}, t).$$

As we will see later, the following definition is the key to achieving the desired extension. The symbol $l(\mathcal{X})$ is used to represent the length of a word $\mathcal{X}$.

Definition 6. *For every $g \in F(X)$, with $g \neq e$, and for all $t > 0$, set*

$$N_{(M,\wedge)}(g, t) = \sup\{\Gamma_{(M,\wedge)}(\mathcal{X}, \varphi, t) : [\mathcal{X}] = g,\ l(\mathcal{X}) = 2n,\ \varphi \in S_n,\ n \in \mathbb{N}^+\}$$

and $N_{(M,\wedge)}(g, t) = 1$ if $g = e$.

Before continuing, let's look at an example where we work with the previous definition.

Example 3. *As above, let $X = \{e, a, b\}$. Consider the metric space (X, d) with d the discrete metric on X defined as $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ otherwise. Now, we take the non-Archimedean fuzzy metric space $(X, M, \wedge)$ where $(M, \wedge)$ is the non-Archimedean fuzzy metric induced by d. We choose e as the distinguished point in X so that $F(X)$ is the free group of two symbols.*

We calculate

$$N_{(M,\wedge)}(g, t) = \sup\{\Gamma_{(M,\wedge)}(\mathcal{X}, \varphi, t) : [\mathcal{X}] = g,\ l(\mathcal{X}) = 2n,\ \varphi \in S_n,\ n \in \mathbb{N}^+\}$$

for the reduced word $g = ab$.

Let $t > 0$. Taking into account the definition of $(M, \wedge)$, we have

$$\prod_{i=1}^{2n} M^{\circ}(u_i^{-1}, u_{\varphi(i)}, t) \leq \frac{t}{t+1}.$$

Next, let $\mathcal{X} = g = ab = u_1u_2$ and let $\varphi\colon \{1,2\} \to \{1,2\}$ defined as $\varphi(1) = 2$ and $\varphi(2) = 1$. Then

$$\Gamma_{(M,\wedge)}(\mathcal{X}, \varphi, t) = \prod_{i=1}^{2} M^{\circ}(u_i^{-1}, u_{\varphi(i)}, t) = \frac{t}{t+1}.$$

We have just shown that

$$N_{(M,\wedge)}(g, t) = \sup\{\Gamma_{(M,\wedge)}(\mathcal{X}, \varphi, t) : [\mathcal{X}] = g,\ l(\mathcal{X}) = 2n,\ \varphi \in S_n,\ n \in \mathbb{N}^+\} = \frac{t}{t+1}.$$

The properties enumerated in the following proposition are substantiated by the proof of Theorem 3.2. in [28].

Proposition 2. *Let $g \in F(X)$ with $g \neq e$. Then for every word $\mathcal{X} = u_1u_2...u_{2n}$ in $F_a(X)$ whose reduced form is g and every $\varphi \in S_n$, we have:*

(1) If there is $i \leq 2n$ such that $u_i^{-1} = u_{i+1}$ and $\varphi(i) = i + 1$, then one can find a new word $\mathcal{X}'$ obtained by eliminating u_iu_{i+1} from $\mathcal{X}$, and a new scheme φ' defined by restricting φ to $\{1, \dots, i-1, i+2, \dots, 2n\}$.

(2) If there are $i \leq 2n$ such that $u_i^{-1} = u_{i+1}$ and $\varphi(i) \neq i + 1$, then we can assume $r = \varphi(i)$ and $r' = \varphi(i+1)$. In this situation, the definition of a scheme tells us that $\{r, r'\} \cap \{i, i+1\} = \varnothing$.

Then, if we eliminate u_iu_{i+1}, we get a new word $\mathcal{X}'$ and a new scheme φ' which matches φ except for $\varphi'(r) = r'$.

(3) *If there are $i \leq 2n$ such that $u_{i-1}^{-1} = u_{i+1}$, $u_i = e$ and $\varphi(i) \in \{i-1, i+1\}$, then we can assume, without loss of generality, that $\varphi(i) = r' = i+1$, $\varphi(i-1) = r < i-1$ and $\varphi(i+1) = r''$. Then we can eliminate u_{i-1} and u_{i+1} to get a new word $\mathcal{X}'$ and a new scheme φ' which coincides with φ on $\{1, \ldots, r-1, r+1, \ldots, i-2, i+2, \ldots, 2n\}$ and $\varphi'(r) = i$.*

(4) *If there are $i \leq 2n$ such that $u_{i-1}^{-1} = u_{i+1}$, $u_i = e$ and $\varphi(i) \notin \{i-1, i+1\}$, then we can assume, without loss of generality, that $\varphi(i-1) = r < i-1$ and $i+1 < \varphi(i+1) = r'' < \varphi(i) = r'$. Then, eliminating u_{i-1} and u_{i+1} and translating $u_i = e$ just before $u_{\varphi(i)}$, we obtain a new word $\mathcal{X}'$ and a new scheme φ' which coincides with φ on $\{1, \ldots, r-1, r+1, \ldots, i-2, i, i+2, \ldots, r''-1, r''+1, \ldots, 2n\}$ and $\varphi'(r) = r''$.*

It should be noted that cases (1) and (2) of the previous theorem are self-explanatory. Nevertheless, it would be advisable to provide examples to illustrate cases (3) and (4).

Example 4. *For the purpose of exemplifying Case (3), let us consider $X = \{e, a, b, c, d, f, k, l, m\}$ and $S_5 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Define $\varphi\colon S_5 \to S_5$ as follows:*

$$\begin{aligned} &\varphi(1) = 2, \quad \varphi(2) = 1, \quad \varphi(3) = 6, \quad \varphi(4) = 5, \\ &\varphi(5) = 4, \quad \varphi(6) = 3, \quad \varphi(7) = 8, \quad \varphi(8) = 7, \\ &\varphi(9) = 10, \quad \varphi(10) = 9. \end{aligned}$$

Consider the word $\mathcal{X} = abcdfk^{-1}eklm$ in $F_a(X)$. Then we can eliminate k^{-1} and k to get a new word $\mathcal{X}' = abcdfelm$ and a new scheme φ' with $\varphi'(1) = 2$, $\varphi'(2) = 1$, $\varphi'(3) = 6$, $\varphi'(4) = 5$, $\varphi'(5) = 4$, $\varphi'(6) = 3$, $\varphi'(7) = 8$ and $\varphi'(8) = 7$.

The subsequent illustration is intended to provide a representation of Case (4). Let $X = \{e, a, b, c, d, f, k, l, m, p, q\}$ and define $\varphi\colon S_6 \to S_6$ as follows:

$$\begin{aligned} &\varphi(1) = 2, \quad \varphi(2) = 1, \quad \varphi(3) = 6, \quad \varphi(4) = 5, \\ &\varphi(5) = 4, \quad \varphi(6) = 3, \quad \varphi(7) = 12, \quad \varphi(8) = 11, \\ &\varphi(9) = 10, \quad \varphi(10) = 9, \quad \varphi(11) = 8, \quad \varphi(12) = 7. \end{aligned}$$

Consider the word $\mathcal{X} = abcdfk^{-1}eklmpq$ in $F_a(X)$. Then, eliminating k^{-1} and k and translating e just before q, we obtain a new word $\mathcal{X}' = abcdflmpeq$ and a new scheme φ' such that $\varphi'(1) = 2$, $\varphi'(2) = 1$, $\varphi'(3) = 8$, $\varphi'(4) = 5$, $\varphi'(5) = 4$, $\varphi'(6) = 7$, $\varphi'(7) = 6$, $\varphi'(8) = 3$, $\varphi'(9) = 10$ and $\varphi'(10) = 9$.

The next result is useful in the subsequent development.

Proposition 3. *If $g \in F(X) \setminus \{e\}$, then there exist $\mathcal{X}_g \in AIR(F_a(X))$, with $l(\mathcal{X}_g) = 2n$ $(n \geq 1)$, whose reduced form is g, and a scheme $\varphi_g \in S_n$ which does not depend on t, so that for all $t > 0$ we have*

$$N_{(M,\wedge)}(g, t) = \Gamma_{(M,\wedge)}(\mathcal{X}_g, \varphi_g, t).$$

Proof. Consider a word $\mathcal{X} = u_1u_2 \ldots u_{2n}$ on the alphabet $\widetilde{X}$ whose reduced form is g. Let $\varphi \in S_n$.

First, we will show that if we apply a few cancellations to $\mathcal{X}$, we can get a word $\mathcal{X}' \in AIR(F_a(X))$, with $l(\mathcal{X}') = 2m$, $[\mathcal{X}'] = g$, and a scheme $\varphi' \in S_m$ such that

$$\Gamma_{(M,\wedge)}(\mathcal{X}', \varphi', t) \geq \Gamma_{(M,\wedge)}(\mathcal{X}, \varphi, t),$$

for all $t > 0$.

To see this, notice that if $\mathcal{X}$ is a reduced word, then the proof is done. Therefore, we can assume that $\mathcal{X}$ is not reduced. Then there must be $i \leq 2n$ such that $u_i^{-1} = u_{i+1}$ or $u_{i-1}^{-1} = u_{i+1}$ and $u_i = e$. There are a few cases to consider. Each of these cases is related to one of the cases of Proposition 2. Henceforth, $\mathcal{X}'$ and φ' will denote the outcomes obtained through the application of Proposition 2 to $\mathcal{X}$ and φ, respectively.

Case 1. If $u_i^{-1} = u_{i+1}$ and $\varphi(i) = i+1$, it is obvious that $\Gamma_{(M,\wedge)}(\mathcal{X}', \varphi', t) \geq \Gamma_{(M,\wedge)}(\mathcal{X}, \varphi, t)$, for all $t > 0$.

Case 2. Suppose $u_i^{-1} = u_{i+1}$ and $\varphi(i) \neq i+1$. Choose r and r' as in Case 2 of Proposition 2. Then, for every $t > 0$, we have

$$M^\circ(u_r^{-1}, u_{r'}, t) \geq M^\circ(u_r^{-1}, u_i, t) \wedge M^\circ(u_i, u_{r'}, t) = M^\circ(u_r, u_i^{-1}, t) \wedge M^\circ(u_{i+1}^{-1}, u_{r'}, t).$$

Thus, $\Gamma_{(M,\wedge)}(\mathcal{X}', \varphi', t) \geq \Gamma_{(M,\wedge)}(\mathcal{X}, \varphi, t)$ for all $t > 0$.

Case 3. If $u_{i-1}^{-1} = u_{i+1}$, $u_i = e$ and $\varphi(i) \in \{i-1, i+1\}$, it is easy to show that, for all $t > 0$,

$$\Gamma_{(M,\wedge)}(\mathcal{X}, \varphi, t) - \Gamma_{(M,\wedge)}(\mathcal{X}', \varphi', t) = M^\circ(u_r, u_{r'}, t) \wedge M^\circ(u_{r'}, e, t) - M^\circ(u_r, e, t) \leq 0.$$

where r and r' are chosen like they were in Case 3 of Proposition 2. Therefore, $\Gamma_{(M,\wedge)}(\mathcal{X}', \varphi', t) \geq \Gamma_{(M,\wedge)}(\mathcal{X}, \varphi, t)$ for all $t > 0$.

Case 4. If $u_{i-1}^{-1} = u_{i+1}$, $u_i = e$ and $\varphi(i) \notin \{i-1, i+1\}$, an argument similar to the one used in the previous case shows that

$$\Gamma_{(M,\wedge)}(\mathcal{X}', \varphi', t) \geq \Gamma_{(M,\wedge)}(\mathcal{X}, \varphi, t)$$

for all $t > 0$.

Now, the previous study of possible cases allows us to repeat the cancellations explained in Proposition 2 until we obtain a pair $(\mathcal{X}', \varphi')$, such that $\mathcal{X}' \in AIR(F_a(X))$. Note that there are finitely many such pairs, so one of them, say $(\mathcal{X}_g, \varphi_g)$, satisfies the equality $\Gamma_{(M,\wedge)}(\mathcal{X}_g, \varphi_g, t) = N_{(M,\wedge)}(g, t)$ and Proposition 3 is proved.

Finally, it should be noted that the conclusions derived from the previous analysis are independent of the value of t. Indeed, for a fixed t, the new word and the new scheme are obtained by successively applying the cancellations described in Proposition 2. This process does not involve the value of t. As a result, we obtain an almost irreducible word and a scheme that do not depend on t. □

The ensuing notion is essential in order to achieve our objective.

Definition 7. *Let $(G, M, \wedge)$ be a fuzzy topological group . A function T from $G\times]0, +\infty[$ to $]0, 1]$ is said to be an invariant fuzzy prenorm on the group G if it satisfies the following properties:*

(1) $T(e, t) = 1$ *for all* $t > 0$,

(2) $T(gh, t) \geq T(g, t) \wedge T(h, t)$ *for all* $g, h \in G, t > 0$,

(3) $T(h^{-1}gh, t) = T(g, t)$ *for all* $g, h \in G, t > 0$.

The above concept is sometimes referred to as a fuzzy norm, which can lead to confusion with the concept of a norm in a fuzzy normed space. Let us look at a classical example (see Section 3.2 in [30]).

Example 5. (Radius fuzzy prenorm) *Let $(G,\cdot)$ be a group with identity element e. Fix a parameter $0<\delta<1$ and define the function $T: G\times]0,+\infty[\to]0,1]$ by:*

$$T(g,t)=\begin{cases}1 & \text{if } g=e,\\ \delta & \text{if } g\neq e \text{ and } t\geq 1,\\ \delta^2 & \text{if } g\neq e \text{ and } t<1.\end{cases}$$

Let us check that T is a prenorm. Condition (1) in Definition 7 is clear. We show Condition (2). Notice that

$$T(gh,t)\geq T(g,t)\geq T(g,t)\wedge T(h,t)$$

for all $t>0$. This proves Condition (2).

We now prove Condition (3), that is, $T(h^{-1}gh,t)=T(g,t)$ for all $t>0$. To do this, we can assume, without loss of generality, that $g\neq e$. In this situation, $h^{-1}gh\neq e$ so that $T(h^{-1}gh,t)=T(g,t)$.

We will prove that the function $N_{(M,\wedge)}$ defined above is an invariant fuzzy prenorm on the group $F(X)$. To do this, we need the following definition.

Definition 8. *The juxtaposition of two schemes $\varphi\in S_n$ and $\psi\in S_m$ is the scheme $\sigma_{\varphi,\psi}\in S_{n+m}$ which coincides with φ when restricted to $\{1,\dots,2n\}$ and such that $\sigma_{\varphi,\psi}(2n+i)=\psi(i)$ for all $i\in\{1,\dots,2m\}$.*

Proposition 4. *The function $N_{(M,\wedge)}$ defined above is an invariant fuzzy prenorm on the group $F(X)$.*

Proof. It is immediate that $N_{(M,\wedge)}(e,t)=1$ and $0<N_{(M,\wedge)}(g,t)\leq 1$, for all $g\in F(X)$ and all $t>0$.

In order to prove that $N_{(M,\wedge)}(gh,t)\geq N_{(M,\wedge)}(g,t)\wedge N_{(M,\wedge)}(h,t)$, for all $g,h\in F(X)$ and $t>0$, consider two words $\mathcal{X}=u_1u_2\dots u_{2n}$ and $\mathcal{Y}=v_1v_2\dots v_{2m}$ in $F_a(X)$ whose reduced forms are, respectively, g and h. Let $\varphi\in S_n$ and $\psi\in S_m$ be two schemes such that $N_{(M,\wedge)}(g,t)=\Gamma_{(M,\wedge)}(\mathcal{X},\varphi,t)$ and $N_{(M,\wedge)}(h,t)=\Gamma_{(M,\wedge)}(\mathcal{Y},\psi,t)$. Take $\mathcal{Z}=\mathcal{X}\mathcal{Y}=u_1\dots u_{2n}v_1\dots v_{2m}$ and let $\sigma_{\varphi,\psi}$ be the juxtaposition of φ and ψ. It is straightforward that $[\mathcal{Z}]=gh$ and

$$\Gamma_{(M,\wedge)}(\mathcal{Z},\sigma,t)=\Gamma_{(M,\wedge)}(\mathcal{X},\varphi,t)\wedge\Gamma_{(M,\wedge)}(\mathcal{Y},\psi,t)=N_{(M,\wedge)}(g,t)\wedge N_{(M,\wedge)}(h,t).$$

Thus, $N_{(M,\wedge)}(gh,t)\geq N_{(M,\wedge)}(g,t)\wedge N_{(M,\wedge)}(h,t)$.

We finish the proof by showing condition (3) of Definition 7. Assume, without loss of generality, that $h=u\in\widetilde{X}$. Choose a word $\mathcal{X}=u_1u_2\dots u_{2n}\in F_a(X)$ whose reduced form is g and a scheme $\varphi\in S_n$ such that $N_{(M,\wedge)}(g,t)=\Gamma_{(M,\wedge)}(\mathcal{X},\varphi,t)$. Let $\mathcal{Z}=u^{-1}\mathcal{X}u$, and consider a new scheme σ which relates u^{-1} with u and such that $\sigma(i+1)=\varphi(i)$ for all $i\in\{1,\dots,2n\}$. It is clear that $[\mathcal{Z}]=u^{-1}gu$. Then

$$N_{(M,\wedge)}(u^{-1}gu,t)\geq\Gamma_{(M,\wedge)}(\mathcal{Z},\sigma,t)=\Gamma_{(M,\wedge)}(\mathcal{X},\varphi,t)=N_{(M,\wedge)}(g,t).$$

If we replace u^{-1} by u, u by u^{-1} and g by $u^{-1}gu$ in the previous expression, we have $N_{(M,\wedge)}(g,t)\geq N_{(M,\wedge)}(u^{-1}gu,t)$. This completes the proof. □

An important property of the prenorm $N_{(M,\wedge)}$ is given by

Proposition 5. *The following equality holds for all $x, y \in X$ and $t > 0$:*

$$N_{(M,\wedge)}(x^{-1}y, t) = M(x, y, t) = N_{(M,\wedge)}(yx^{-1}, t).$$

Proof. Consider two elements $x, y \in X$. Notice that the equality

$$N_{(M,\wedge)}(x^{-1}y, t) = N_{(M,\wedge)}(xx^{-1}yx^{-1}, t) = N_{(M,\wedge)}(yx^{-1}, t)$$

follows directly from Proposition 4. Thus, we only have to prove that $N_{(M,\wedge)}(x^{-1}y, t) = M(x, y, t)$. We will consider two cases.

Case 1. $x = y$. In this case, $M(x, x, t) = 1$ for all $t > 0$. Thus, for all $t > 0$,

$$\Gamma_{(M,\wedge)}(\mathcal{X}, \varphi, t) = N_{(M,\wedge)}(x^{-1}x, t) = N_{(M,\wedge)}(e, t) = 1,$$

with $\mathcal{X} = x^{-1}x$ and $\varphi \in S_1$ a scheme with $\varphi(1) = 2$.

Case 2. $x \neq y$. Consider $g = x^{-1}y$. Applying Proposition 3, there is a scheme $\varphi \in S_m$ and $\mathcal{X} \in AIR(F_a(x))$ with $l(\mathcal{X}) = 2m \leq 4$ and $[\mathcal{X}] = g$ such that:

$$\Gamma_{(M,\wedge)}(\mathcal{X}, \varphi, t) = N_{(M,\wedge)}(g, t)$$

for all $t > 0$. In the case we are working with, m can be either 1 or 2, and the possible words for $\mathcal{X}$ are $x^{-1}y$, $ex^{-1}ey$, $x^{-1}eye$ or $ex^{-1}ye$.

If $\mathcal{X} = x^{-1}y$, then $m = 1$ and, there is only one scheme φ_1 with $\varphi_1(1) = 2$ which allows us to obtain

$$\begin{aligned}\Gamma_{(M,\wedge)}(\mathcal{X}, \varphi_1, t) &= M^{\circ}(x, y, t) \wedge M^{\circ}(y, x, t)\\ &= M(x, y, t) \wedge M(y, x, t) = M(x, y, t),\end{aligned}$$

for all $t > 0$.

In order to study the cases in which $\mathcal{X}$ takes the values $ex^{-1}ey$, $x^{-1}eye$ or $ex^{-1}ye$, we can assume, without loss of generality, that $\mathcal{X} = ex^{-1}ey$. In this situation, $m = 2$ and there are just two possible schemes $\varphi \in S_2$. The first one with $\varphi_2(1) = 2$ and the second one with $\varphi_3(1) = 4$. Then, for all $t > 0$, we have

(1) $\varphi_2(1) = 2$ which implies

$$\begin{aligned}\Gamma_{(M,\wedge)}(\mathcal{X}, \varphi_2, t) &= M^{\circ}(e^{-1}, x^{-1}, t) \wedge M^{\circ}(e^{-1}, y, t)\\ &= M(x, e, t) \wedge M(e, y, t) \leq M(x, y, t),\end{aligned}$$

for all $t > 0$, or

(2) $\varphi_3(1) = 4$, which implies

$$\begin{aligned}\Gamma_{(M,\wedge)}(\mathcal{X}, \varphi_3, t) &= M^{\circ}(e^{-1}, y, t) \wedge M^{\circ}(x, e, t)\\ &= M(e, y, t) \wedge M(x, e, t) \leq M(x, y, t),\end{aligned}$$

for all $t > 0$. Therefore, we get

$$\begin{aligned}N_{(M,\wedge)}(x^{-1}y, t) &= \sup\{\Gamma_{(M,\wedge)}(\mathcal{X}, \varphi, t) : [\mathcal{X}] = x^{-1}y,\ l(\mathcal{X}) = 2n,\ \varphi \in S_n,\ n \in \mathbb{N}^+\}\\ &= \Gamma_{(M,\wedge)}(x^{-1}y, \varphi_1, t) = M(x, y, t).\end{aligned}$$

This completes the proof. □

The concepts used so far can be adapted, *mutatis mutandis*, to the case of a fuzzy metric in which the t-norm we are working with is the product. In the following example, we will show that if the product t-norm had been employed in lieu of the minimum t-norm, then the previous result would not necessarily hold.

Example 6. *As in Example 3, let us consider $X = \{e, a, b\}$ and we take e as the distinguished point. Consider the fuzzy metric space $(X, M_d, \cdot)$ where $(M_d, \cdot)$ is the fuzzy metric induced by the discrete metric d on X defined as $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ otherwise, and $\cdot$ is the product t-norm.*

Consider the reduced word $g = a^{-1}b$. Let $\mathcal{X} = u_1 \cdot \cdots \cdot u_{2n}$ be an element of $F_a(X)$ whose reduced form is g and take $\varphi \in S_n$. Let $A, B \subset \{1, 2, \cdots 2n\}$ such that $u_i = a^{-1}$ for all $i \in A$ and $u_i = a$ for all $i \in B$ (the set B can be empty). Being g the reduced form of $\mathcal{X}$, the cardinality of A must be greater than the cardinality of B. Therefore, there is i with $u_i = a^{-1}$ and $\varphi(i) = j$ with $u_j = e$ or $u_j = b$. Assume, with no loss of generality, that $u_j = e$. Since φ is idempotent, $\varphi(j) = i$.

Let now $t > 0$. According to the definition of $\Gamma_{M,\cdot}(\mathcal{X}, \varphi, t)$, the terms $M^\circ(u_i^{-1}, u_{\varphi(i)}, t) = M(a, e, t) = \dfrac{t}{1+t}$ and $M^\circ(u_j^{-1}, u_{\varphi(j)}, t) = M(e, a^{-1}, t) = M(e, e, t) \circ M(a, e, t) = \dfrac{1}{1+t}$ are both present in the expression

$$\Gamma_{(M,\cdot)}(\mathcal{X}, \varphi, t) = \prod_{i=1}^{2n} M^\circ(u_i^{-1}, u_{\varphi(i)}, t)$$

which implies that

$$\Gamma_{(M,\cdot)}(\mathcal{X}, \varphi, t) = \prod_{i=1}^{2n} M^\circ(u_i^{-1}, u_{\varphi(i)}, t) \leq \left(\frac{1}{1+t}\right)^2.$$

Since $\mathcal{X}$ and φ were arbitrary, we have shown that

$$\begin{aligned} N_{(M,\cdot)}(a^{-1}b, t) &= \sup\left\{\Gamma_{(M,\wedge)}(\mathcal{X}, \varphi, t) : [\mathcal{X}] = g,\ l(\mathcal{X}) = 2n,\ \varphi \in S_n,\ n \in \mathbb{N}^+\right\} \\ &\leq \left(\frac{1}{1+t}\right)^2 \neq M(a, b, t). \end{aligned}$$

To achieve our purpose, we now define $\widehat{M}$ on $F(X) \times F(X) \times]0, +\infty[$ by

$$\widehat{M}(g, h, t) = N_{(M,\wedge)}(g^{-1}h, t),$$

for all $g, h \in F(X)$ and all $t > 0$. The following proposition is the central result of this section. Notice that the restriction of $(\widehat{M}, \wedge)$ to X coincides with $(M, \wedge)$. First, a definition.

Definition 9. *A non-Archimedean fuzzy metric $(M, \wedge)$ on an abstract group G is called bi-invariant if it is both left and right invariant, that is, $M(lg, lh, t) = M(gl, hl, t) = M(g, h, t)$ for all $l, g, h \in G$, $t > 0$.*

Proposition 6. *$(\widehat{M}, \wedge)$ is a bi-invariant non-Archimedean fuzzy metric on $F(X)$.*

Proof. The first three conditions $(GV1)$-$(GV3)$ are immediate. In order to prove (NA), notice that, since $(M, \wedge)$ satisfies $(NA2)$ so does $(\widehat{M}, \wedge)$. Thus, we only have to show $(NA1)$. To do this, take arbitrary $g, h, k \in F(X)$ and $t > 0$ and note that

$$\begin{aligned} N_{(M,\wedge)}(g^{-1}k,t) &\geq N_{(M,\wedge)}(g^{-1},t) \wedge N_{(M,\wedge)}(k,t) = \\ &= N_{(M,\wedge)}(h^{-1}g^{-1}h,t) \wedge N_{(M,\wedge)}(h^{-1}kh,t) \geq \\ &\geq N_{(M,\wedge)}(h^{-1},t) \wedge N_{(M,\wedge)}(g^{-1}h,t) \wedge N_{(M,\wedge)}(h^{-1}k,t) \wedge N_{(M,\wedge)}(h,t) \geq \\ &\geq N_{(M,\wedge)}(g^{-1}h,t) \wedge N_{(M,\wedge)}(h^{-1}k,t). \end{aligned}$$

Let us now show (GV5), namely that the function $\widehat{M}(g,h,_)\colon]0,+\infty[\to]0,1]$ is continuous for all $g,h \in F(X)$. The result is evident if $g^{-1}h = e$. Suppose the opposite, that is, choose $g,h \in F(X)$ with $g^{-1}h \neq e$. Then, by Proposition 3, there exist $\mathcal{X}_{g^{-1}h} \in F_a(X)$, with $l(\mathcal{X}_g) = 2n$ $(n \geq 1)$, whose reduced form is $g^{-1}h$, and a scheme $\varphi_{g^{-1}h} \in S_n$, which does not depend on t, such that

$$\widehat{M}(g,h,t) = \Gamma_{(M,\wedge)}(\mathcal{X}_{g^{-1}h}, \varphi_{g^{-1}h}, t)$$

for all $t > 0$.

Now, taking into account the definition of $\Gamma_{(M,\wedge)}(\mathcal{X}_{g^{-1}h}, \varphi_{g^{-1}h}, _)$, the result can be deduced readily from the definition of $(M^\circ, \wedge)$ and the fact that $(M, \wedge)$ satisfies (GV5).

We will finish the proof by showing that $(\widehat{M}, \wedge)$ is bi-invariant. By definition of $(\widehat{M}, \wedge)$,

$$\widehat{M}(lg,lh,t) = N_{(M,\wedge)}(g^{-1}l^{-1}lh,t) = N_{(M,\wedge)}(g^{-1}h,t) = \widehat{M}(g,h,t)$$

for all $l,g,h \in F(X)$ and $t > 0$. In a similar way, we obtain

$$\widehat{M}(gl,hl,t) = N_{(M,\wedge)}(l^{-1}g^{-1}hl,t) = N_{(M,\wedge)}(ll^{-1}g^{-1}hll^{-1},t) = N_{(M,\wedge)}(g^{-1}h,t) = \widehat{M}(g,h,t).$$

This completes the proof. □

One of the most interesting properties of the fuzzy metric $(\widehat{M}, \wedge)$ is

Proposition 7. *The non-Archimedean fuzzy metric $(\widehat{M}, \wedge)$ is minimal among all bi-invariant non-Archimedean fuzzy metrics on $F(X)$ whose restriction to X coincides with $(M, \wedge)$.*

Proof. Let $n \in \mathbb{N}$ and consider $g = u_1u_2\ldots u_n \in F(X)$. By Theorem 3, there is $\mathcal{X}_g \in AIR(F_a(X))$ with $l(\mathcal{X}_g) = 2n$ $(n \geq 1)$ whose reduced form is g, and a scheme $\varphi_g \in S_n$ which do not depend on t so that for all $t > 0$, we have

$$\widehat{M}(u_1u_2\ldots u_n,e,t) = N_{(M,\wedge)}(u_n^{-1}\ldots u_2^{-1}u_1^{-1}e,t) = \Gamma_{(M,\wedge)}(\mathcal{X}_g, \varphi_g, t).$$

Let $(S, \wedge)$ be a bi-invariant non-Archimedean fuzzy metric on $F(X)$ extending $(M, \wedge)$. We may assume, without loss of generality, that

$$\Gamma_{(M,\wedge)}(\mathcal{X}_g, \varphi_g, t) = \prod_{i=1}^{2n} M^\circ(u_i^{-1}, u_{\varphi(i)}, t),$$

where $M^\circ(u_i^{-1}, u_{\varphi(i)}, t)$ can be either of the form $M^\circ(u_i,e,t) = M(u_i,e,t)$ or $M^\circ(u_i^{-1},u_j,t) = M(u_i,e,t) \wedge M(u_j,e,t)$, which leads to the following:

$$\begin{aligned}\widehat{M}(u_1u_2...u_n,e,t) &= \prod_{i=1}^{2n} M(u_i,e,t) = \prod_{i=1}^{2n} S(u_i,e,t)\\ &= S(u_1,e,t)\wedge S(u_2,e,t)\wedge S(u_3,e,t)\wedge ...\wedge S(u_n,e,t) =\\ &= S(u_1,e,t)\wedge S(u_2^{-1},e,t)\wedge S(u_3^{-1},e,t)\wedge ...\wedge S(u_n^{-1},e,t)\leq\\ &\leq S(u_1,u_2^{-1},t)\wedge S(u_3^{-1},e,t)\wedge ...\wedge S(u_n^{-1},e,t) =\\ &= S(u_1u_2,e,t)\wedge S(u_3^{-1},e,t)\wedge ...\wedge S(u_n^{-1},e,t)\leq\\ &\leq S(u_1u_2,u_3^{-1},t)\wedge ...\wedge S(u_n^{-1},e,t) =\\ &= S(u_1u_2u_3,e,t)\wedge ...\wedge S(u_n^{-1},e,t)\leq ...\\ &\leq S(u_1u_2...u_{n-1},u_n^{-1},t) = S(u_1u_2...u_n,e,t).\end{aligned}$$

The previous inequalities imply that for all $g,h\in F(X)$ and all $t>0$, we have

$$\widehat{M}(g,h,t) = \widehat{M}(gh^{-1},e,t)\leq S(gh^{-1},e,t) = S(g,h,t).$$

This completes the proof. □

In the following, we show that $(\widehat{M},\wedge)$ generates a topological group topology on $F(X)$ whose restriction to X coincides with the topology of the non-Archimedean fuzzy metric space $(X,M,\wedge)$. For every $t>0$, and $0<s<1$, we define

$$U_{(M,\wedge)}(t,s) = \{g\in F(X) : N_{(M,\wedge)}(g,t) > 1-s\}.$$

Notice that $N_{(M,\wedge)}(g,t) = \widehat{M}(g,e,t)$ and, consequently, $N_{(M,\wedge)}(g,_)$ is non-decreasing. The bi-invariance of $(\widehat{M},\wedge)$ allows us to prove the following fundamental result.

Proposition 8. *The family $\mathcal{N} = \{U_{(M,\wedge)}(t,s) : t>0,\ 0<s<1\}$ forms a base at the identity for a topological group topology $\mathcal{T}_{(M,\wedge)}$ on $F(X)$ whose restriction to X coincides with the topology of the space X generated by $(M,\wedge)$. Moreover, X is a closed subset of $(F(X),\mathcal{T}_{(M,\wedge)})$.*

Proof. There are five conditions to be verified (Pontryagin's conditions) in order to prove that $\mathcal{T}_{(M,\wedge)}$ is a neighborhood system at e for a group topology:

(1) For every $U_{(M,\wedge)}(t_1,s_1), U_{(M,\wedge)}(t_2,s_2)\in\mathcal{N}$, there exists $U_{(M,\wedge)}(t_3,s_3)\in\mathcal{N}$ such that $U_{(M,\wedge)}(t_3,s_3)\subseteq U_{(M,\wedge)}(t_1,s_1)\cap U_{(M,\wedge)}(t_2,s_2)$.
Indeed, take $U_{(M,\wedge)}(t_3,s_3)$ with $t_3=\min\{t_1,t_2\}$ and $s_3=\min\{s_1,s_2\}$. It is easy to check that $U_{(M,\wedge)}(t_3,s_3)$ satisfies the required condition.

(2) For every $U_{(M,\wedge)}(t,s)\in\mathcal{N}$, there exists $U_{(M,\wedge)}(t_1,s_1)\in\mathcal{N}$ such that

$$U_{(M,\wedge)}(t_1,s_1)U_{(M,\wedge)}(t_1,s_1)\subseteq U_{(M,\wedge)}(t,s).$$

To see this, we use the continuity of the t-norm to get s_1 such that $(1-s_1)\wedge(1-s_1) > (1-s)$, and we choose $t_1<t$. Now, if we consider $U_{(M,\wedge)}(t_1,s_1)$, then, for $g,h\in U_{(M,\wedge)}(t_1,s_1)$, we have

$$\begin{aligned}N_{(M,\wedge)}(gh,t) &\geq N_{(M,\wedge)}(g,t)\wedge N_{(M,\wedge)}(h,t)\geq\\ &\geq N_{(M,\wedge)}(g,t_1)\wedge N_{(M,\wedge)}(h,t_1) > (1-s_1)\wedge(1-s_1) > (1-s)\end{aligned}$$

which implies $U_{(M,\wedge)}(t_1,s_1)U_{(M,\wedge)}(t_1,s_1)\subseteq U_{(M,\wedge)}(t,s)$.

(3) For every $U_{(M,\wedge)}(t,s)\in\mathcal{N}$ and $g\in F(X)$, there exists $V_{(M,\wedge)}(t_1,s_1)\in\mathcal{N}$ such that $gV_{(M,\wedge)}(t_1,s_1)\subseteq U_{(M,\wedge)}(t,s)$ and $V_{(M,\wedge)}(t_1,s_1)g\subseteq U_{(M,\wedge)}(t,s)$.

Take $U_{(M,\wedge)}(t,s)$ and $g \in F(X)$ with $N_{(M,\wedge)}(g,t) := 1-s_1$. Obviously, $1-s_1 > 1-s$ and $\delta := s - s_1 > 0$. If we consider $V_{(M,\wedge)}(t,\delta)$, then, for all $h \in V_{(M,\wedge)}(t,\delta)$, we have

$$N_{(M,\wedge)}(gh,t) \geq N_{(M,\wedge)}(g,t) \wedge N_{(M,\wedge)}(h,t) > (1-s_1) \wedge (1-\delta) > 1-s.$$

It follows that $gV_{(M,\wedge)}(t,\delta) \subseteq U_{(M,\wedge)}(t,s)$. A similar argument shows that $V_{(M,\wedge)}(t,\delta)g$ is contained in $U_{(M,\wedge)}(t,s)$.

(4) For every $U_{(M,\wedge)}(t,s) \in \mathcal{N}$ and $g \in F(X)$, there exists $V_{(M,\wedge)}(t',s') \in \mathcal{N}$ such that $g^{-1}V_{(M,\wedge)}(t',s')g \subseteq U_{(M,\wedge)}(t,s)$.
We only have to consider $V_{(M,\wedge)}(t',s') = U_{(M,\wedge)}(t,s)$ and $h \in U_{(M,\wedge)}(t,s)$. Since $U_{(M,\wedge)}(t,s) = \{g \in F(X) : N_{(M,\wedge)}(g,t) > 1-s\}$ and $N_{(M,\wedge)}(h,t) > 1-s$, by Proposition 4 it follows that $N_{(M,\wedge)}(g^{-1}hg,t) = N_{(M,\wedge)}(h,t) > 1-s$. Therefore, $g^{-1}hg \in U_{(M,\wedge)}(t,s)$ and so $g^{-1}U_{(M,\wedge)}(t,s)g \in U_{(M,\wedge)}(t,s)$.

(5) For every $U_{(M,\wedge)}(t,s) \in \mathcal{N}$, there exists $V_{(M,\wedge)}(t',s') \in \mathcal{N}$ such that $V^{-1}_{(M,\wedge)}(t',s') \subseteq U_{(M,\wedge)}(t,s)$, where $V^{-1}_{(M,\wedge)}(t',s') = \{v^{-1} : v \in V_{(M,\wedge)}(t',s')\}$.

As $\widehat{M}(e,g,t) = \widehat{M}(g^{-1},e,t) = \widehat{M}(e,g^{-1},t)$, it follows that $U_{(M,\wedge)}(t,s) = U^{-1}_{(M,\wedge)}(t,s)$ for all $U_{(M,\wedge)}(t,s) \in \mathcal{N}$. Hence, it suffices to consider $V_{(M,\wedge)}(t',s') = U_{(M,\wedge)}(t,s)$.

We have just proved that $\mathcal{N} = \{U_{(M,\wedge)}(t,s) : t > 0,\ 0 < s < 1\}$ forms a base at the identity for a topological group topology $\mathcal{T}_{(M,\wedge)}$ on $F_a(X)$. Moreover, since the restriction of $(\widehat{M},\wedge)$ to X coincides with $(M,\wedge)$, it is apparent that $\mathcal{T}_{(M,\wedge)}|_X$ coincides with the topology induced by $(M,\wedge)$.

Next, we prove that X is closed in $(F(X),\mathcal{T}_{(M,\wedge)})$. In order to get this, suppose there is a sequence $\{z_n\} \subseteq X$ convergent to some $g \in F(X) \setminus X$ such that $g = x_1^{\epsilon_1}x_2^{\epsilon_2}...x_n^{\epsilon_n}$ with $\epsilon_i = \pm 1$ for all $i \in \{1,...,n\}$ and lead to a contradiction.

First, we prove that g does not belong to X^{-1}. Indeed, given $x \in X$, since $\{z_n\}$ does not converge to e, there exist $t_1 > 0$ and $0 < s < 1$ such that, for all $n \in \mathbb{N}$, there is $m(n) > n$ such that $M(z_{m(n)},e,t_1) \leq 1-s$. Taking into account that $M^\circ(z_n,x^{-1},t) = M(z_n,e,t) \wedge M(x,e,t)$ for all $t > 0$, we have that $M(z_{m(n)},e,t) \wedge M(x,e,t) \leq 1-s$, that is $M^\circ(z_{m(n)},x^{-1},t_1) \leq 1-s$, which implies that $\{z_n\}$ does not converge to x^{-1}.

An argument similar to the previous one allows us to find positive numbers $t_1,t_2,...,t_n$, $s_1,s_2,...,s_n,s_{n+1}$ and a subsequence $\{z_{n_j}\}$ of $\{z_n\}$ with $\widehat{M}(z_{n_j},x_i^{\epsilon_i},t_i) = M^\circ(z_{n_j},x_i^{\epsilon_i},t_i) \leq 1-s_i$, for all $i \in \{1,...,n\}$.

Now set $t = \min\{t_1,t_2,...,t_n\}$ and $s = \min\{s_1,s_2,...,s_n\}$. Then, for all $i \in \{1,...,n\}$, $\widehat{M}(z_{n_j},x_i^{\epsilon_i},t) \leq 1-s_i \leq 1-s$ holds for every letter of g, their inverses and the identity. Furthermore, there is $\mathcal{X}_n$ and $\varphi_n \in S_n$, $n \in \mathbb{N}^+$ such that for all n_j we have

$$\widehat{M}(z_{n_j},g,t) = N_{(M,\wedge)}(z_{n_j}^{-1}g,t) = \prod_{i=1}^{2n}\{M^\circ(x_i^{-1},x_{\varphi(i)},t)\} \leq 1-s.$$

This shows that the subsequence $\{z_{n_j}\}$ does not converge to g, a contradiction. Thus, X is closed in $(F(X),\mathcal{T}_{(M,\wedge)})$. □

The Abelian case follows, *mutatis mutandis*, from similar arguments. In summary, the preceding results have demonstrated the existence of the non-Archimedean fuzzy metric free (Abelian) topological group over a fuzzy metric space.

Proposition 9. *Every non-Archimedean fuzzy metric $(M,\wedge)$ on a non-empty set X extends to a non-Archimedean fuzzy metric $(\widehat{M}_A,\wedge)$ on the abstract group $F(X)$ (respectively, $A(X)$) so that*

(1) *$(\widehat{M}_A,\wedge)$ generates a topological group topology on $F(X)$ (respectively on $A(X)$).*

(2) *The restriction to X of the topology generated by $(\widehat{M}_A, \wedge)$ coincides with the topology induced by $(M, \wedge)$ on X.*
(3) *$(\widehat{M}_A, \wedge)$ is minimal among the bi-invariant non-Archimedean fuzzy metrics making $F(X)$ (respectively, $A(X)$) a topological group.*
(4) *X is closed in $(F(X), \widehat{M}_A, \wedge)$ (respectively in $(A(X), \widehat{M}_A, \wedge)$*

We close the paper with some consequences of our construction. Several applications are worth to be considered. We start with an embedding theorem, which follows from the results of the previous section. Recall that two (non-Archimedean) fuzzy metric spaces $(X, M, *)$ and $(Y, N, \star)$ are isometric if there exists a bijection f from X onto Y such that $M(x, y, t) = N(f(x), f(y), t)$ for all $x, y \in X$. Moreover, two uniform spaces $(X, \mathcal{U})$ and $(Y, \mathcal{V})$ are uniformly isomorphic if there is a bijection f from X onto Y such that both f and f^{-1} are uniformly continuous.

Proposition 10. *Every non-Archimedean fuzzy metric space is isometric to a closed subspace of a non-Archimedean fuzzy metric free (Abelian) group.*

Given a metric space, (X, d), it is a well-known fact that the uniformity induced by the metric d coincides with the uniformity induced by the standard fuzzy metric M_d associated to d. Thus, a consequence of Proposition 10 is

Corollary 1. *Every ultrametric fuzzy metric space is uniformly isomorphic to a closed subspace of a non-Archimedean fuzzy metric free (Abelian) group.*

Proposition 10 can be seen not as a simple *fuzzified* version of the classical one; it is a richer, more complex structure where the t-norm acts as the architect of the *shape*. It controls how nearness is composed and measured, and this control is essential for constructing the free group object that can isometrically contain the original space. The bi-invariance in this context is a much stronger statement, ensuring *equality* across all scales t with the proof of this property deeply intertwined with the minimum t-norm.

Recall that a topological space (X, τ) is said to be *separable* if there is a countable subset $D \subseteq X$ which is dense in X. It is a standard result that the topological free (Abelian) group $G(X)$ (in the sense of Graev) on a Tychonoff space (X, τ) is endowed with the supremum of all topological group topologies on $G(X)$ whose restriction to X is τ. Moreover, if (X, τ) is separable, then $G(X)$ is also separable: indeed, the countable subgroup generated by a countable dense subset of Y is dense in $G(Y)$ (see [31]). The previous consideration allows us to obtain the following.

Proposition 11. *If $(X, \widehat{M}, \wedge)$ is separable, then so is $(F(X), \widehat{M}, \wedge)$ (respectively, $(A(X), \widehat{M}, \wedge)$).*

Remark 1. It should be noted that the construction carried out using non-Archimedean fuzzy metrics in the sense of George and Veeramani can also be applied to Kramosil and Michálek fuzzy metric spaces (see [1]).

4. Conclusions

In this paper, we introduce the non-Archimedean fuzzy metric free (Abelian) group over a non-Archimedean fuzzy metric space $(X, M, \wedge)$ in the sense of George and Veeramani. The study of free groups has been a subject of considerable interest in the field of Topological Algebra, due to the numerous counterexamples they offer a breadth of applications in related areas of mathematics, including functional analysis, analysis of group-acting codes, etc. Therefore, the results presented in the paper provide a foundation for further

research on free groups in the fuzzy context and for exploring potential applications to several branches of fuzzy theory beyond Topological Algebra.

In light of the results of the paper and taking into account the properties studied in the crisp case, future lines of research arise. Among them, we can mention the following: (1) Study how the subsets F_n of reduced words of length at most n influence the determination of topological properties of $F(X)$, (2) Characterize when $F(X)$ verifies a given topological property $\mathcal{P}$, (3) Does it exist a non-Archimedean fuzzy metric free topological group when working with the product (respectively, the Łukaseiwizs) t-norm, and (4) It is possible to construct a fuzzy metric free topological group on an Archimedean fuzzy metric space?

It is worth noting that the mathematical structure of a fuzzy free metric topological group represents the intersection of distinct mathematical disciplines. While this specific combination is largely the domain of pure mathematics, it holds significant potential for applied sciences. For example, free groups are currently used in Non-Commutative Cryptography (e.g., protocols based on the conjugacy search problem in braid groups or free groups). Introducing *(non-Archimedean) fuzzy metric* structures allows for Biometric Cryptosystems, that is, Biometric data (fingerprints, iris scans) is inherently *fuzzy* (never identical twice). A (non-Archimedean) fuzzy metric topological group could model a key exchange protocol where the keys are elements of a free group derived from noisy biometric data. The *(non-Archimedean) fuzzy metric* allows the system to accept a key that is *close enough* within a topological threshold.

Author Contributions: Both authors have contributed equally to all aspects of the work: contextualization, methodology, and other relevant elements. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The original contributions presented in this study are included in the article. Further inquiries can be directed to the corresponding author.

Conflicts of Interest: The authors declare no conflicts of interest.

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